ON POSITIVE-DEFINITE INTEGRAL KERNELS AND A RELATED QUADRATIC FORM

BY

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1. **Introduction and preliminaries.** Let ρ be a continuous complex function of two real variables on the interval $J=[a, b]$. Let C be the Banach space of continuous complex functions on $[a, b]$ with the norm

$$\|f\|_\infty = \underset{a \leq t \leq b}{\text{l.u.b.}} |f(t)|,$$

and B the dual space of C. B can be interpreted as the space of all complex Radon measures on $[a, b]$, where the measure μ acts on the function f to give $\int f(t)d\mu(t)$ (all integrals are taken on $[a, b]$ unless otherwise stated), and the dual norm on B can be interpreted as $|||\mu|||=$total variation of μ. A continuous operator R from B to C is defined by $R\mu(t)=\int \rho(t, s)d\mu(s)$. We shall assume that ρ is *positive-definite*, in the sense that $\int\int \rho(s, t)d\mu(s)d\bar{\mu}(t)>0$ for all nonzero μ in B.

Let H be $L_2[a, b]$ and let S be the restriction of R to those measures of the form $dm=h(s)ds$, $h\in H$. The function ρ may also be interpreted as the covariance function of a Gaussian process defined with respect to: the probability space Ω of all complex functions on J, the Borel field generated by finite dimensional cylinder sets, and a measure P induced by ρ upon the latter. If ρ satisfies certain mild differentiability conditions (Loeve, [4]; Doob, [2]), the outer measure P^* of C in Ω is unity, and both the measure and the stochastic process may be considered as defined on C. Thus the functions of C may be considered as elements of a probability space. Statistical estimation theory has been applied to them (Grenander, [3]; Slepian, [5]). If one takes the values of such a function $f(t)$ at a finite sequence σ of points $t_1^\sigma<t_2^\sigma< \cdots <t_{n(\sigma)}^\sigma$ in J, and forms the expression

$$\mu^\sigma(f) = \sum_{j=1}^{n(\sigma)} f(t_j^\sigma)\bar{a}_j^\sigma(f) = \sum_{j=1}^{n(\sigma)} \sum_{k=1}^{n(\sigma)} \rho(t_j^\sigma, t_k^\sigma)a_k^\sigma(f)\bar{a}_j^\sigma(f),$$

where the a_j^σ are defined by

$$f(t_k^\sigma) = \sum_{j=1}^{n(\sigma)} \rho(t_k^\sigma, t_j^\sigma)a_j^\sigma(f),$$

(existence and uniqueness being a consequence of the positive-definiteness of ρ), then one has an essential component of an $n(\sigma)$-dimensional approxima-

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tion to the "maximum-likelihood ratio" which one might wish to consider in estimating some parameter of the process.

Now from other considerations (Grenander, [3, pp. 221–222]) one can show that the set of functions f for which the $\mu^\sigma(f)$ remain bounded as σ is refined, is of P-measure zero. It is, however, of interest to characterize these functions analytically. We shall see that they are precisely the range of $S^{1/2}$ (Theorem 1).

We also shall investigate the range of R, that is, the collection of those f in C for which the equation $R\mu = f$ has a solution in B. Now, a simple necessary and sufficient condition that f be in the range of S is that

$$\sum_{n=1}^{\infty} \frac{|f_n|^2}{\lambda_n^2} < \infty,$$

where the λ_n are the eigenvalues of S (with proper multiplicity), i.e., $\lambda_n\phi_n(t) = \int\rho(t, s)\phi_n(s)ds$ for some ϕ_n in H, and the f_n are the Fourier coefficients $f_n = \int f(s)\bar{\phi}_n(s)ds$ with respect to a complete orthonormal set of eigenfunctions of S. (The special role played by Lebesgue measure on J could be played equally well by any Radon measure whose support is the entire interval J; but if this measure is nonatomic, then a homeomorphism of J will transform us back to the case of Lebesgue measure again.)

Of course, there is a necessary and sufficient condition for f to be in the range of R, due to Hahn and Banach [1]: that there exist an $M > 0$ such that

$$\left| \sum_{k=1}^{n} f(t_k)\eta_k \right| \leq M \max_{t \in J} \left| \sum_{k=1}^{n} \rho(t, t_k)\eta_k \right|,$$

where the integer n and the complex numbers $\eta_1, \eta_2, \cdots, \eta_n$ are arbitrary.

For many purposes, however, such criteria as the above are difficult to apply. In what follows, we approach the question otherwise: by examining the operators $S^{1/2}$ and $R^{1/2}$.

More notation: we shall write $\| \;\|$ for the mean-square norm in H. It is well known that S is a completely continuous, positive-definite operator on H. We note also that each eigenfunction is continuous, as is clear from the equation

$$\lambda_n\phi_n(t) = \int \rho(t, s)\phi_n(s)ds;$$

and Mercer's theorem tells us that $\sum_n \lambda_n\phi_n(s)\bar{\phi}_n(t)$ converges uniformly to $\rho(s, t)$ (all sums are to run from 1 to ∞ unless otherwise indicated). The eigenfunctions are complete in H. Furthermore, their linear combinations are uniformly dense in C. For suppose this were not the case; then, by the Hahn-Banach theorem there would be some nonzero μ in B with $\int\phi_n(t)d\mu(t) = 0$ for all n. But then

$$\iint \rho(s, t) d\mu(s) d\bar{\mu}(t) = \iint \sum_n \lambda_n \phi_n(s) \bar{\phi}_n(t) d\mu(s) d\bar{\mu}(t)$$
$$= \sum_n \lambda_n \iint \phi_n(s) \bar{\phi}_n(t) d\mu(s) d\bar{\mu}(t)$$
$$= \sum \lambda_n \left| \int \phi_n(s) d\mu(s) \right|^2 = 0,$$

contradicting the positive-definiteness assumption.

2. **Convergence of approximating quadratic forms.** We define $x(t)$, for $a \leqq t \leqq b$, as the element $\sum_n \bar{\phi}_n(t)(\lambda_n)^{1/2}\phi_n$ in H ($x(t)$ is, for each t, a function on $[a, b]$); a legitimate definition, since $\sum_n |\bar{\phi}_n(t)(\lambda_n)^{1/2}|^2$ is finite, and, in fact, is just $\rho(t, t)$. Then clearly $(x(s), x(t)) = \rho(t, s)$. Furthermore, the $x(t)$ span H; for $x(t)\phi_n(t)$ is a continuous function from $[a, b]$ to H, and if we take its Bochner integral, we get

$$\int x(t)\phi_n(t) dt = \int \sum_m (\lambda_m)^{1/2} \bar{\phi}_m(t) \phi_m \phi_n(t) dt$$
$$= \sum (\lambda_m)^{1/2} \phi_m \int \bar{\phi}_m(t) \phi_n(t) dt = (\lambda_n)^{1/2} \phi_n.$$

We consider the $\mu^\sigma(f)$ of §1:

$$\overset{\sigma}{\mu}(f) = \sum_{j=1}^{n(\sigma)} f(\overset{\sigma}{t}_j) \overset{\sigma}{\bar{a}}_j(f) = \sum_{j=1}^{n(\sigma)} \sum_{k=1}^{n(\sigma)} \rho(\overset{\sigma}{t}_j, \overset{\sigma}{t}_k) \overset{\sigma}{a}_k(f) \overset{\sigma}{\bar{a}}_j(f).$$

Then $\mu^\sigma(f)$ is clearly $\geqq 0$; let $\mu(f)$ be the l.u.b. (possibly infinite) of all $\mu^\sigma(f)$.

K^σ is defined as the subspace of H spanned by $x(t_1^\sigma), \cdots, x(t_{n(\sigma)}^\sigma)$, and $x^\sigma(f)$ is defined as $\sum_{j=1}^{n(\sigma)} a_j^\sigma(f) x(t_j^\sigma)$. Notice that $\mu^\sigma(f)$ is precisely $\|x^\sigma(f)\|^2$.

LEMMA 1. *If $\sigma \ll \tau$, then the projection of $x^\tau(f)$ on K^σ is $x^\sigma(f)$.*

Proof. We need only show that $x^\tau(f) - x^\sigma(f)$ is orthogonal to K^σ. Now since $\sigma \ll \tau$, there is for each j from 1 to $n(\sigma)$ some $k(j)$ between 1 and $n(\tau)$ with $t_j^\sigma = t_{k(j)}^\tau$. Then

$$(\overset{\tau}{x}(f) - \overset{\sigma}{x}(f), x(\overset{\sigma}{t}_j)) = \sum_{k=1}^{n(\tau)} \overset{\tau}{a}_k(f) \rho(\overset{\sigma}{t}_j, \overset{\tau}{t}_k) - \sum_{l=1}^{n(\sigma)} \overset{\sigma}{a}_l(f) \rho(\overset{\sigma}{t}_j, \overset{\sigma}{t}_l)$$
$$= \sum_{k=1}^{n(\tau)} \overset{\tau}{a}_k(f) \rho(\overset{\tau}{t}_{k(j)}, \overset{\tau}{t}_k) - \sum_{l=1}^{n(\sigma)} \overset{\sigma}{a}_l(f) \rho(\overset{\sigma}{t}_j, \overset{\sigma}{t}_l)$$
$$= f(\overset{\tau}{t}_{k(j)}) - f(\overset{\sigma}{t}_j) = 0.$$

For g in H,

$$S^{1/2} g(t) = \sum_n (\lambda_n)^{1/2} (g, \phi_n) \phi_n(t).$$

LEMMA 2. *For any g in H, the expansion for $S^{1/2}g$ in terms of the ϕ_n is uniformly convergent, and $S^{1/2}$ is a continuous map: $H\to C$.*

Proof.

$$\left|\sum_{n=n_0}^{\infty}(\lambda_n)^{1/2}(g,\phi_n)\phi_n(t)\right|^2 \leqq \left(\sum_{n=n_0}^{\infty}\lambda_n|\phi_n(t)|^2\right)\left(\sum_{n=n_0}^{\infty}|(g,\phi_n)|^2\right),$$

by the Schwartz inequality.

$$\sum_{n=n_0}^{\infty}\lambda_n|\phi_n(t)|^2 \leqq \rho(t,t),$$

so the product goes to zero as n increases, uniformly with respect to t. Further,

$$\left|\sum_n(\lambda_n)^{1/2}(g,\phi_n)\phi_n(t)\right|^2 \leqq \sum_n\lambda_n|\phi_n(t)|^2\sum_n|(g,\phi_n)^2|$$
$$= \rho(t,t)\|g\|^2 \to 0 \text{ uniformly in } t \text{ as } \|g\|\to 0.$$

THEOREM 1. *If $\sigma\ll\tau$, then $\mu^\sigma(f)\leqq\mu^\tau(f)$. Furthermore $\mu(f)$ is finite if and only if $f=S^{1/2}g$ for some $g\in H$; and in this event, $\mu(f)=\|g\|^2$.*

Proof. The first statement follows from Lemma 1, since $\sigma\ll\tau$ implies $K^\sigma<K^\tau$, so that $\mu^\sigma(f)=\|x^\sigma(f)\|^2\leqq\|x^\tau(f)\|^2=\mu^\tau(f)$.

Next suppose $f=S^{1/2}g$ for some g in H. Then $(f,\phi_n)=(\lambda_n)^{1/2}(g,\phi_n)$. Let $\gamma_n=(g,\phi_n)$. Then $(g,x(t))=(\sum_n\gamma_n\phi_n,\ \sum_n\bar{\phi}_n(t)(\lambda_n)^{1/2}\phi_n)=\sum_n(f,\phi_n)\phi_n(t)$. The map $t\to x(t)$ is continuous from $[a,b]$ to H; therefore $(g,x(t))$ is continuous in t. But the equality just derived shows that it equals $f(t)$ almost everywhere; therefore, since f is continuous, $(g,x(t))=f(t)$ for all t in $[a,b]$. Then, denoting by P_σ the projection operator from H onto K^σ, we get $(P_\sigma g,\ x(t_j^\sigma))=(g,\ x(t_j^\sigma))=f(t_j^\sigma)=(x^\sigma(f),\ x(t_j^\sigma))$, so that $P_\sigma g=x^\sigma(f)$, and $\mu^\sigma(f)=\|x^\sigma(f)\|^2=\|P_\sigma g\|^2\leqq\|g\|^2$. In fact, $\mu(f)=\|g\|^2$, because $\bigcup_\sigma K^\sigma$ is dense in H, so that, as σ gets finer and finer, P_σ converges strongly to the identity operator in H, and therefore $\|P_\sigma x\|^2\to\|x\|^2$.

On the other hand, suppose $\mu(f)$ is finite. Then $\|x^\sigma(f)\|^2$ is bounded, so that, by the weak compactness of a sphere in Hilbert space, there is some g in H and some cofinal sequence $\{\sigma_\alpha\}$ such that $x^{\sigma_\alpha}(f)$ converges weakly to g. Then, for any y in K^σ, we have

$$(P_\sigma g,y) = (g,y) = \lim_\alpha(x^{\sigma_\alpha}(f),y)$$
$$= \lim_\alpha(P_\sigma x^{\sigma_\alpha}(f),y) = \lim_\alpha(x^\sigma(f),y) = (x^\sigma(f),y), \text{ so that } P_\sigma g = x^\sigma(f).$$

Now denote by t^σ the closest point of σ to t. Then

$$|(x^\sigma(f),x(t))-f(t)| \leqq |(x^\sigma(f),x(t^\sigma))-f(t^\sigma)|$$
$$+\|x^\sigma(f)\|\|x(t^\sigma)-x(t)\| + |f(t^\sigma)-f(t)|.$$

$f(t^{\sigma_\alpha})\to f(t)$ as α increases, and similarly $x(t^\sigma)\to x(t)$. $\|x^\sigma(f)\|$ is bounded by $(\mu(f))^{1/2}$. And $(x^\sigma(f),\ x(t^\sigma))$ is precisely $f(t^\sigma)$. Therefore $(g,\ x(t))=f(t)$. But write g in the form $\sum_{n=1}^\infty \gamma_n\phi_n$. Then

$$(g, x(t)) = \left(\sum_n \gamma_n\phi_n,\ \sum_{m=1}^\infty \bar{\phi}_m(t)(\lambda_m)^{1/2}\phi_m\right) = \sum_n \gamma_n(\lambda_n)^{1/2}\phi_n(t).$$

Thus, we have $f=S^{1/2}g$. Alternately phrased, $\mu(f)<\infty$ if and only if

$$\sum_n \frac{|(f,\ \phi_n)|^2}{\lambda_n} < \infty.$$

REMARK 1. Considering f now as an element of a probability space C, we may interpret $\psi_n(f)=\lambda_n^{-1/2}(f,\ \phi_n)=\lambda_n^{-1/2}f_n$ as random variables, which are easily seen to be Gaussian. Furthermore,

$$\begin{aligned}\lambda_n^{-1/2}\lambda_m^{-1/2}\int_C f_n\bar{f}_m dP(f) &= \lambda_n^{-1/2}\lambda_m^{-1/2}\int_J ds\int_J dt\phi_n(s)\bar{\phi}_m(t)\int_C f(s)\bar{f}(t)dP(f)\\ &= \lambda_n^{-1/2}\lambda_m^{-1/2}\int_J ds\int_J dt\rho(t,\ s)\phi_n(s)\bar{\phi}_m(t) = \delta_{mn}.\end{aligned}$$

Thus the $\psi_n(f)$ are uncorrelated, and, since Gaussian, therefore independent. They all have variance equal to unity. By the standard convergence theorems it is then easy to conclude the fact mentioned in §1, namely, that the set of f for which $\sum_n |\psi_n(f)|^2<\infty$ is a set of zero probability.

REMARK 2. We are also in a position to say something about the dependence of $\mu(f)$ upon the interval J. In fact, let J' be any subinterval of J with nonempty interior; define eigenvalues and eigenfunctions

$$\lambda_n'\phi_n'(t) = \int_{J'} \rho(t,\ s)\phi_n'(s)ds, \qquad t\in J'$$

and let $\mu_{J'}(f)$ be defined by restricting σ to J'. Then we have

THEOREM 2. *$\mu_{J'}(f)\leqq\mu(f)$; so that if $\mu(f)<\infty$ then also $\mu_{J'}(f)<\infty$ and hence*

$$\sum_n \frac{|(f,\ \phi_n')'|^2}{\lambda_n'} < \infty,$$

where

$$(f,\ \phi_n')' = \int_{J'} f(t)\bar{\phi}_n'(t)dt.$$

Proof. The first statement follows from Lemma 1; the second, by considering the arguments for Theorem 1 rephrased in terms of $L_2(J')$.

REMARK 3. If K_t is the closed linear manifold in H spanned by the $x(s)$,

$a \leqq s \leqq t \leqq b$, and if π_t is the projection onto K_t, then $\mu_{[a,t]}(f) = \|\pi_t g\|^2$, where $f = S^{1/2}g$, $g \in H$. The π_t form a resolution of the identity operator on H, and $\|\pi_t(g)\|$ is a left-continuous (and, of course, monotonic nondecreasing) function of t. That it is continuous on the left for each t in J follows from the continuity of ρ: firstly, for $s \geqq s_0$, $\|\pi_s x(s_0)\|$ is constant; and as $s \uparrow s_0$, $\|\pi_s x(s_0)\| \uparrow \|\pi_{s_0} x(s_0)\|$, since

$$\begin{aligned} |\rho(s_0, s)| &= |(x(s_0), x(s))| = |(\pi_s x(s_0), x(s))| \\ &\leqq \|\pi_s x(s_0)\| \|x(s)\| \leqq \|x(s_0)\| \|x(s)\| \end{aligned}$$

and the extreme members of the above chain both approach $\|x(s_0)\|^2$ as $s \uparrow s_0$. Let now g be any element in H. Given $\epsilon > 0$, one can find a finite linear combination $\sum_1^n a_i x(s_i)$ with $a \leqq s_i \leqq s_0$ such that $\|\sum_1^n a_i x(s_i) - \pi_{s_0} g\| < \epsilon/2$. If one then picks δ so small that $\|\pi_s x(s_i) - \pi_{s_0} x(s_i)\| < [\sum |a_i|^{-1}]\epsilon/2$ for $|s - s_0| < \delta$ and for each $i = 1, 2, \cdots, n$; one can insure that $\|(\pi_s - \pi_{s_0})\pi_{s_0} g\| = \|\pi_s g - \pi_{s_0} g\| < \epsilon$, and thus show the left-continuity of $\|\pi_s g\|$ at s_0.

REMARK 4. If ρ' is another kernel satisfying the same assumptions as ρ, S' the corresponding operator on H, and $\mu'^{\sigma}(f)$ the corresponding approximating sums, and if it happens that $S' \geqq cS$ for some constant $c > 0$, then boundedness of the $\mu^{\sigma}(f)$ for any particular f implies boundedness of the $\mu'^{\sigma}(f)$. For example, let

$$\rho(s, t) = \int_{-\infty}^{\infty} e^{i(s-t)} dF(\lambda)$$

and

$$\rho'(s, t) = \int_{-\infty}^{\infty} e^{i(s-t)} dF'(\lambda), \qquad dF' \geqq c dF.$$

The truth of the remark follows from the next lemma.

LEMMA 3. *Let A, B be bounded operators on H, c a positive constant, and $A \geqq cB \geqq 0$. Then the range of $A^{1/2}$ contains that of $B^{1/2}$.*

Proof. Define T_0 from the range of $A^{1/2}$ to that of $B^{1/2}$ by $T_0(A^{1/2}x) = B^{1/2}x$. Since $c^2\|B^{1/2}x\|^2 \leqq \|A^{1/2}x\|^2$, this definition makes sense, and, in fact, T_0 is bounded by c^{-1}. T_0 can therefore be extended to a bounded operator T on all of H. Now, $TA^{1/2} = B^{1/2}$, so $A^{1/2}T^* = B^{1/2}$, and the range of $A^{1/2}$ clearly contains that of $B^{1/2}$.

3. **The operator** $R^{1/2}$. One can define a square root for R, as follows: given μ in B, let $\gamma_n = (\lambda_n)^{1/2} \int \bar{\phi}_n(s) d\mu(s)$. Then

$$\sum_n |\gamma_n|^2 = \sum \lambda_n \int\int \bar{\phi}_n(s)\phi_n(t) d\mu(s) d\bar{\mu}(t) = \int\int \rho(t, s) d\mu(s) d\bar{\mu}(t),$$

and we define $R^{1/2}\mu$ to be the element $\sum_n \gamma_n \phi_n$ in H. So $R^{1/2}$ is a linear map from B to H. Let B_0 denote the unit sphere of B.

LEMMA 4. *R is continuous from B_0 in the weak topology to C in the norm topology (and hence also from B in the norm topology to C in the norm topology).*

Proof. If μ_α is weakly convergent to μ, and $\mu_\alpha, \mu \in B_0$, then $\mu_\alpha - \mu \in 2B_0$ and $\mu_\alpha - \mu$ converges weakly to zero. So it will suffice to consider the case where $\mu_\alpha \in B_0$ and μ_α converges weakly to zero.

$$\left| \int \rho(s, t) d\mu_\alpha(t) \right| \leqq \sum_{n=1}^{n_0} \lambda_n \|\phi_n\|_\infty \left| \int \bar{\phi}_n(t) d\mu_\alpha(t) \right| + \left| \int \sum_{n=n_0+1}^{\infty} \lambda_n \phi_n(s) \bar{\phi}_n(t) d\mu_\alpha(t) \right|.$$

By choosing n_0 large, we can make

$$\left| \sum_{n=n_0+1}^{\infty} \lambda_n \phi_n(s) \bar{\phi}_n(t) \right| < \epsilon/2,$$

so that

$$\left| \int \rho(s, t) d\mu_\alpha(t) \right| \leqq \sum_{n=1}^{n_0} \lambda_n \|\phi_n\|_\infty \int \bar{\phi}_n(t) d\mu_\alpha(t) + \epsilon/2 \, |||\mu_\alpha||| .$$

Then, for sufficiently large α, we have

$$\sum_{n=1}^{n_0} \lambda_n \|\phi_n\|_\infty \left| \int \bar{\phi}_n(t) d\mu_\alpha(t) \right| < \epsilon/2,$$

so that $\left| \int \rho(s, t) d\mu_\alpha(t) \right| < \epsilon$ for such α.

THEOREM 3. *$R^{1/2}$ is continuous from B_0 in the weak topology to H in the norm topology (and hence also from B in the norm topology to H in the norm topology). The equality*

$$(h, R^{1/2}\mu) = \int S^{1/2} h(s) d\bar{\mu}(s)$$

holds for all $h \in H$ and $\mu \in B$. If d_μ is of the form $g ds$ with g in H, then $S^{1/2}g = R^{1/2}\mu$. And finally, $S^{1/2}R^{1/2} = R^{1/2}R^{1/2} = R$.

Proof. As in proving Lemma 4, it suffices in the continuity proof to show that if μ_α converges weakly to 0 while $|||\mu_\alpha|||$ remains bounded by 1, then $R^{1/2}\mu_\alpha$ converges in norm to zero. Now,

$$\begin{aligned} \|R^{1/2}\mu_\alpha\|^2 &= \sum_n \lambda_n \left| \int \bar{\phi}_n(s) d\mu_\alpha(s) \right|^2 \\ &= \sum_{n=1}^{n_0} \lambda_n \left| \int \bar{\phi}_n(s) d\mu_\alpha(s) \right|^2 + \int\int \sum_{n=n_0+1}^{\infty} \lambda_n \bar{\phi}_n(s) \phi_n(t) d\mu(s) d\bar{\mu}(t), \end{aligned}$$

and, proceeding as in Lemma 4, we get this small by first choosing n_0 large and then choosing α large.

The equality $(h, R^{1/2}\mu) = \int S^{1/2}h(s)d\bar{\mu}(s)$ can be seen as follows: in Lemma 2 it was pointed out that $\sum_n (\lambda_n)^{1/2}(h, \phi_n)\phi_n(t)$ converges uniformly. Then $\int \sum_n (\lambda_n)^{1/2}(h, \phi_n)\phi_n(s)d\bar{\mu}(s) = \sum_n \gamma_n(h, \phi_n)$, where $R^{1/2}\mu = \sum \gamma_n\phi_n$; this is the desired result.

If $d\mu = gdt$, then $(S^{1/2}g, \phi_n) = (\lambda_n)^{1/2}\int \bar{\phi}_n(s)g(s)ds$, but also $(R^{1/2}\mu, \phi_n) = (\lambda_n)^{1/2}\int \bar{\phi}_n(s)d\mu(s)$, so that $S^{1/2}g = R^{1/2}\mu$.

To see that $S^{1/2}R^{1/2} = R$: first, observe that $S^{1/2}R^{1/2}|H = S^{1/2}S^{1/2}|H = S = R|H$. Now, by Lemma 2 and the first part of this theorem, $S^{1/2}R^{1/2}$ is continuous from B_0 in the weak topology to C in the norm topology. Lemma 4 asserts the same for R. Finally, $H \cap B_0$ is weakly dense in B_0, since in particular the finite linear combinations of characteristic functions of Borel sets are so. Therefore $S^{1/2}R^{1/2}$ and R agree on B_0, and hence on all of B.

COROLLARY. *If $f = S^{1/2}g$, $g \in H$, and also $f = R\mu$, $\mu \in B$, then* $\int f(s)d\bar{\mu}(s) = \|g\|^2$.

THEOREM 4. *The range of R is a proper subset of the range of $S^{1/2}$.*

Proof. That it is a subset is clear from the equality $S^{1/2}R^{1/2} = R$. To see that it is a proper subset: the image under $R^{1/2}$ of B_0 is a compact subset of H, since B_0 is weakly compact. Now, the range of R is the image under $S^{1/2}$ of the range of $R^{1/2}$. Since $S^{1/2}$ has no nullspace, its range would thus equal that of R only if the range of $R^{1/2}$ were all of H. But $R^{1/2}(B_0)$ is compact in H, hence nowhere dense. Then $\bigcup_{n=1}^{\infty} (R_0)^{1/2}(nB_0)$ is a first category set in H, and certainly not all of H.

BIBLIOGRAPHY

1. S. Banach, *Theorie des operation linéaires*, Warsaw, 1932, Chapter IV, §2.

2. J. L. Doob, *Stochastic processes depending on a continuous parameter*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 107–110.

3. U. Grenander, *Stochastic processes and statistical inference*, Ark. Mat. vol. 1, no. 17 (1950) pp. 195–277.

4. M. Loève (a note by) in P. Levy, *Processus stochastiques et mouvement Brownien*, Paris 1948, p. 316.

5. D. Slepian, *Estimation of signal parameters in the presence of noise*, Trans. IRE Professional Gp. on Information Theory, PGIT-3, March 1954, pp. 68–89.

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