PROPERTIES OF PRIMARY NONCOMMUTATIVE RINGS

BY
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Introduction. In a series of papers [7; 8; 9; 10], E. Snapper discussed extensively the properties of completely primary rings, primary rings, and associated topics for the commutative case. This paper extends some of the theory of [7] to the noncommutative case. Most of [7] extends intact for duo rings which are rings where every right ideal is a left ideal and every left ideal is a right ideal. Some of this theory for general noncommutative rings is proved here.

An example in §5 of this paper answers Snapper's tempting conjecture [7, p. 678].

1. **Preliminary discussion and definitions.** Let R be a ring with identity.

Definition 1.1. Let a be a two sided ideal of R. The union of all ideals b such that $b^n \subseteq a$ for some positive integer n is a two sided ideal of R and is called the radical of a which we shall denote by $N(a)$.

Definition 1.2. Let a be a two sided ideal of R. The set of all elements $x \in R$ such that $x^n \in a$ for some positive integer n is said to be the nil-radical of a which we shall denote by $P(a)$.

If a is 0 in the previous definitions we use the symbols N and P for the radicals of 0. Although P is not always a two sided ideal, *we shall be interested in P only when it is a two sided ideal.*

Definition 1.3. The set of all elements $x \in R$ such that $yx+1$ is a unit of R for all $y \in R$ is a two sided ideal of R and is called the *Jacobson radical* of R which we denote by J.

If $x \in J$ then $xy+1$ will also be a unit of R for all $y \in R$.

Definition 1.4. A ring R is called *duo* provided every right ideal is a left ideal and every left ideal is a right ideal.

If follows immediately from the above definition that if R is a duo ring then $Ra = aR$ for all $a \in R$, i.e., for $a, b \in R$ there exists an element $c \in R$ such that $ab = ca$. Hence in this case P is a two sided ideal.

Definition 1.5. A ring R is said to be *completely N primary* provided R/N is a division ring(1).

In this definition if we substitute for N the symbols J and P we have the definitions for a *completely J primary ring* and a *completely P primary ring* when P is a two sided ideal.

Definition 1.6. A two sided ideal q of R is said to be *completely prime*

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(1) Every Grassmann algebra with finite basis is a completely N primary duo ring.

provided $ab \in q$, $a \in R$, $b \in R$, implies at least one of these elements is contained in q.

DEFINITION 1.7. A two sided ideal q is called *right N primary* provided $ab \in q$, $b \notin q$ implies $a \in N(q)$. The ideal q is called *left N primary* provided $ab \in q$, $a \notin q$ implies $b \in N(q)$. The ideal q is said to be *N primary* provided it is both right and left N primary.

If, in Definition 1.7 we substitute the symbols P and J for N we have the definitions of right P primary, left P primary, P primary, right J primary, left J primary and J primary.

DEFINITION 1.8. A ring R is said to be a *right N primary ring* provided 0 is a right N primary ideal. Similarly, a *left N primary ring* is defined. A ring R is N primary provided it is both left and right N primary.

If, in Definition 1.8 we substitute the symbols P and J for N we have the definition for right P primary ring, left P primary ring, P primary ring, right J primary ring, left J primary ring, and J primary ring.

Suppose P is a two sided ideal. Then $N \subseteq P \subseteq J$ and it follows that N primary$\Rightarrow P$ primary$\Rightarrow J$ primary. In addition, suppose R is N primary. Then since the elements of P are nilpotent $P=N$ and P is a two sided ideal. Further, since R is then P primary, it follows from the proof of Theorem 2.2 of [1] that P, which is equal to N, is a completely prime ideal, i.e., R/N is an integral domain.

If R is a completely N primary ring certainly N is a two sided ideal which is a maximal right ideal of R. Hence $N=P=J$. In this case suppose $ab=0$, then if $a \neq 0$, b must be contained in N otherwise it would have an inverse. Similarly if $ab=0$, $b \neq 0$ then $a \in N$. Hence, completely N primary implies N primary.

If R satisfies the A.C.C. for right ideals and is P primary, one can easily show from the proof of Theorem 2.2 of [1] and from [5] that P is a two sided ideal and $P^n=0$ for some positive integer n. Hence in this case P primary$\Rightarrow N=P\Rightarrow N$ primary.

Many of the properties of N primary rings follow from the fact that divisors of zero are contained in J.

STATEMENT 1.1. If R is J primary and if $x \in R$ is a right unit then x is a unit. If xy is a right unit both x and y are units.

Proof. Suppose $xz=1$, then $(1-zx)z=0$. Hence $1-zx \in J$, which implies that $zx=1+w$ for $w \in J$. Hence zx is a unit. The proof of the second statement follows from the first.

STATEMENT 1.2. Let R be J primary and $x \neq 0$ and y be elements of R, then $xy=x$ or $yx=x$ implies y is a unit. Hence 1 is the only nonzero idempotent.

Proof. If $xy=x$, then $x(y-1)=0 \Rightarrow y-1 \in J$. Hence $y=1+z$ for $z \in J$. Thus y is a unit. The proof for $yx=x$ is the same. To prove the second part suppose $x^2=x$, then x is a unit. Consequently $x=(x^{-1}x)x=x^{-1}x^2=x^{-1}x=1$.

If R is a completely N primary ring then of course $N=J$. However, there exist rings which are not completely N primary and where $N=J$. (See [7, p. 668]).

STATEMENT 1.3. Let R be a ring with identity. Then the following conditions are equivalent.

(1) The Jacobson radical J is a maximal right ideal.

(2) The set of nonunits of R coincide with J.

(3) The set of nonunits of R is a two sided ideal.

(4) R contains a unique maximal right ideal.

Proof. If J is a maximal right ideal and $x \notin J$, then $J+xR=R$. This implies that there exists $y \in J$, $z \in R$ such that $y+xz=1$. Hence $xz=1-y$, which implies that xz is a unit and therefore x is a right unit. Similarly x is a left unit since J is also a maximal left ideal. Thus (1)$\Rightarrow$(2) while it is trivial that (2)$\Rightarrow$(3)$\Rightarrow$(4). To show that (4)$\Rightarrow$(1) let q be the unique maximal right ideal. We will show that $q \subseteq J$. If $x \in q$ then $xy+1 \notin q$ for all $y \in R$. Suppose $xy+1$ is not a right unit, then $(xy+1)P \subset R$ and hence $xy+1$ would be contained in maximal right ideal, namely q, which is impossible. Hence $xy+1$ is a right unit for all y. Thus $x \in J$.

2. **Relationship between R and R/N.** Let R be a ring with identity and radicals N, P and J as defined in §1. Let H denote the natural homomorphism of R onto R/N. If q is any subset of R let $\bar{q}$ denote the subset qH of R/N. Thus $\overline{R}=R/N$.

2a. *Units.* One can easily see that H maps units of R on units of R/N. The following statement shows the converse to be true.

STATEMENT 2.1. If m is a two sided ideal of R the following conditions are equivalent.

(1) An element $x \in R$ is a unit if and only if the coset of x modulo m is a unit of R/m.

(2) $m \subseteq J$.

Proof. (1)$\Rightarrow$(2). If $x \in m$ then certainly $xy+1$ mod m is a unit. Hence $xy+1$ is a unit of R for all $y \in R$. Thus $x \in J$. To show now that (2)$\Rightarrow$(1). Let x be an element of R whose coset is a unit of R/m. Then $xy=1+u$ for $u \in J$. Consequently x is a right unit of R. Similarly $zx=1+w$ for $w \in J$ and x is a left unit.

Hence *unit* $\Longleftrightarrow$ *unit* mod N *and unit ideal* $\Longleftrightarrow$ *unit ideal* mod N.

2b. *Relatively prime.*

DEFINITION 2.1. Two right ideals q_1 and q_2 are termed *relatively prime* provided $q_1+q_2=R$. Two elements x and y are *relatively prime* if $xR+yR=R$.

Clearly if $q_1+q_2=R$ then $\bar{q}_1+\bar{q}_2=\overline{R}$ and from 2a the converse follows immediately. Hence *relatively prime* $\Longleftrightarrow$ *relatively prime* mod N where we used only the fact that $N \subseteq J$.

2c. *Divisors and associates.*

DEFINITION 2.2. If x and y are nonzero elements of R, then x is termed either a *left factor* of y or a *right divisor* of y if $yR \subseteq xR$, i.e., $y = xa$. Similarly we define *right factor* and *left divisor*. The element x is a *factor* of y if it is both a right and left factor of y.

DEFINITION 2.3. Two elements x and y of R are called *right associates* if $xR = yR$, *left associates* if $Rx = Ry$, and associates if they are both right and left associates.

Clearly if x and y are right (left) associates then $\bar{x}$ and $\bar{y}$ are right (left) associates. Hence *right (left) associates* $\Rightarrow$ *right (left) associates* mod N. If two elements are right (left) associates, they are clearly both zero or nonzero and either both left (right) divisor of zero or not left (right) divisors of zero.

STATEMENT 2.2. If R is a J primary ring and if x and y are nonzero right associates, then $x = yv$ where v is a unit of R. If x and y are left associates then $x = vy$ where v is a unit of R.

Proof. If $xR = yR$ then $x = yv$ and $y = xw$. Consequently $x = xwv$ and from Statements 1.1 and 1.2 we conclude that v is a unit.

2d. *Proper divisors.*

DEFINITION 2.4. An element $x \in R$ is called a *proper right divisor* of $y \in R$ if $yR \subset xR$, and a *proper left divisor* of $y \in R$ if $Ry \subset Rx$. The element x is a *proper divisor* if it is both a proper left and right divisor.

Clearly *proper right (left) divisor* $\Rightarrow$ *proper right (left) divisor* mod N. In addition, we have the following converse for a J primary ring.

STATEMENT 2.3. Let m be an ideal contained in J and let $x \in R$, $x \notin m$. Let R/m be J primary. Then, if $y \in R$, the following are equivalent.

(1) y is a proper right (left) divisor of x.

(2) y is a right (left) divisor of x and $\bar{y}$ is a proper (left) right divisor of $\bar{x}$.

Proof. Obviously (2)$\Rightarrow$(1). If (1) holds then $x = yz$ where z is not a unit. Hence from 2.1 we conclude that $\bar{x} = \bar{y}\bar{z}$ where $\bar{z}$ is not a unit of R/m, while $x \notin m$ implies that $\bar{x}$ is not the zero element of R/m. Statement 2.2 implies that $\bar{x}$ and $\bar{y}$ are not associates and hence $\bar{y}$ is a proper divisor of $\bar{x}$.

2e. *Irreducibles and fundamental irreducibles.*

DEFINITION 2.5. An element x is called irreducible provided $x = yz$ implies that either y or z is a unit.

From this definition, it follows for a J primary ring or a duo ring that a unit is necessarily an irreducible.

STATEMENT 2.4. If m is an ideal contained in J and x is irreducible in R/m, then x is irreducible in R.

Proof. If $x = yz$ then $\bar{x} = \bar{y}\bar{z}$ in R/m. Hence since $\bar{x}$ is irreducible we can assume that $\bar{y}$ is a unit. From Statement 2.1 then y is a unit. Hence x is irreducible.

This shows that irreducible mod $N \Rightarrow$ irreducible. In view of the direction of this implication we made the following definition.

DEFINITION 2.6. An element $x \in R$ is called a fundamental irreducible if its coset $\bar{x}$ is irreducible in R/N.

Accordingly the fundamental irreducibles are irreducible and the units are fundamental irreducibles for J primary or duo rings.

2f. *Completely prime ideals and prime elements in duo rings.*

STATEMENT 2.5. A two sided ideal p is completely prime if and only if $N\subseteq p$ and p/N is a completely prime ideal of R/N.

Proof. If $x\in N$ we have $x^n=0\in p$ for some positive integer n. Hence $x\in p$ which implies that $N\subseteq p$. If now $N\subseteq p$ then $\bar{R}/\bar{p}\cong R/p$. From this we can conclude that p is completely prime in R if and only if $\bar{p}$ is completely prime in $\bar{R}$.

If N is a completely prime ideal then N is the only nil completely prime ideal and if N is not completely prime than R contains no completely prime nilpotent ideals. If $N=P$ and N is not completely prime, then R contains no nil completely prime ideals.

We make the following definition for duo rings.

DEFINITION 2.7. An element x of a duo ring R is termed prime if $xR=Rx$ is a completely prime ideal.

This definition is equivalent to the statement that x is prime if $x=yz$ implies that either y or z is an associate of x.

STATEMENT 2.6. If R is a J primary duo ring then every nonzero prime element is irreducible.

Proof. If x is prime and $x=yz$ then either $yR=xR$ or $zR=xR$ which implies by Statement 2.2 that either y or z must be a unit.

If a two sided ideal p is a *maximal right* ideal then R/p is a division ring. We call such an ideal a *right maximal two sided ideal.* It is obvious that a two sided ideal p is right maximal if and only if (1). $N\subseteq p$ and (2). $\bar{p}$ is right maxinal in $\bar{R}$.

2g. *Radical N of a two sided ideal.* If a is a two sided ideal then as in §1 let $N(a)$ denote the union of all nilpotent two sided ideals mod a. Clearly $N(a+N)=N(a)+N=N(a)$, where N is $N(0)$.

STATEMENT 2.7. If a and b are two sided ideals of R, then[2]:

(1) $N(\bar{a})=\{N(a)\}^-$,

(2) $N(\bar{a})\subseteq N(\bar{b})\Leftrightarrow N(a)\subseteq N(b)$,

(3) $N(\bar{a})=N(\bar{b})\Leftrightarrow N(a)=N(b)$,

(4) $\bar{a}=\bar{b}\Rightarrow N(a)=N(b)$.

Proof. (1). If q is a two sided ideal of R and $\bar{q}$ is contained in $N(\bar{a})$, then $\bar{q}^h\subseteq a+N$. Hence $q\subseteq N(a+N)=N(a)$. Conversely, if $\bar{q}\subseteq\{N(a)\}^-$ then $q\subseteq N+N(a)=N(a)$. Hence $q^h\subseteq a$ and therefore $\bar{q}^h\subseteq\bar{a}$ which shows that $\bar{q}\subseteq N(\bar{a})$.

(2). If $N(a)\subseteq N(b)$ then $\{N(a)\}^-\subseteq\{N(b)\}^-$ and by (1) $N(\bar{a})\subseteq N(\bar{b})$. Conversely, if $N(\bar{a})\subseteq N(\bar{b})$ let $q\subseteq N(a)$. Then $q^h\subseteq a$ and hence $\bar{q}^h\subseteq\bar{a}$. It follows

[2] In certain places to simplify printing, the bar over a symbol is replaced by $\{\ \}^-$, i.e., if R is a ring with radical N and q is a subset of R then the image of q under the natural homomorphism from R to R/N is denoted by either $\bar{q}$ or $\{q\}^-$.

that $\bar{q}\subseteq N(\bar{a})\subseteq N(\bar{b})=\{N(b)\}^{-}$ and hence $q\subseteq N+N(b)=N(b)$. Conversely, $N(a)\subseteq N(b)$ and (2) is proved.

It follows immediately that (2)⇒(3)⇒(4).

2h. *Residue class rings.* If q is a two sided ideal of R then $N(R/q)$ is equal to $N(q)/q$, i.e., $N(R/q)$ consists of the images of $N(q)$ under the natural homomorphism from R to R/q. We denote again by $\{R/q\}^{-}$ the residue class ring of R/q modulo its radical $N(R/q)$, i.e., $\{R/q\}^{-}=(R/q)/(N/q)$ $=[R/q]/[N(q)/q]$. Since $q\subseteq N(q)\subseteq R$ then $R/N(q)\cong[R/q]/[N(q)/q]$ $=\{R/q\}^{-}$. Since $N\subseteq N(q)\subseteq R$ then $R/N(q)\cong[R/N]/[N(q)/N]\cong\bar{R}/\{N(q)\}^{-}$. Hence we have $\{R/q\}^{-}\cong R/N(q)\cong\bar{R}/\{N(q)\}^{-}$. In particular we see that if b is a nilpotent two sided ideal, then $N(b)=N$ and $\{R/b\}^{-}\cong R/N$. Thus the residue class ring R/b modulo its radical is isomorphic to R modulo its radical.

3. **Right primary ideals.** We defined a two sided ideal q to be right P primary provided $ab\in q$ and $b\notin q$ implies $a^{n}\in q$ for some positive integer n. In order to discuss the right primary ideals we impose three conditions, namely:

(i) $P(q)$ is a two sided ideal where q is any right P primary two sided ideal(3).

(ii) $P=N$.

(iii) The nontrivial completely prime two sided ideals of R/N are maximal right ideals.

3a. *Right primary nil ideals.* It is easy to discuss the right P primary nil two sided ideals. For if q is such an ideal then $P(q)$ is P and in this case P is a completely prime ideal. Consequently if P is not completely prime there are no right P primary nil two sided ideals. If however P is completely prime R may have many such primary ideals. In fact if R is completely P primary every nil two sided ideal is right P primary.

3b. *Not nil primary ideals.*

STATEMENT 3.1. Let R be a ring satisfying (i), (ii), and (iii). Then a nontrivial not nil two sided ideal p is a completely prime ideal if and only if p is a maximal right ideal. A not nil two sided ideal q is P primary if and only if $P(q)$ is a completely prime ideal.

Proof. Let p be a nontrivial not nil completely prime two sided ideal of R. If p is completely prime we conclude from 2f that $\bar{p}$ is completely prime and nontrivial by 2a. Hence $\bar{p}$ is a maximal right ideal of $\bar{R}$. We conclude that p is a two sided ideal which is a maximal right ideal by 2f. This proves the first part of the statement.

It follows from [1] and (i) that if q is a P primary two sided ideal then $P(q)$ is a completely prime ideal. If $P(q)=R$ then $q=R$ since $R\in 1$ and in this case q is completely prime. If $P(q)\subset R$ then $P\subseteq P(q)\subset R$. Consequently $P(q)$ is a maximal right ideal. Hence $R/P(q)$ is a division ring which implies that q is P primary(4).

(3) This is true if R satisfies the A.C.C. for right ideals or if R is duo.

(4) In this case all the right (left) P primary two sided ideals are P primary.

We can now easily show that if R satisfies the conditions (i), (ii), (iii), then *right P primary* $\Rightarrow$ *right P primary* mod N. For if q is a right P primary two sided ideal then $P(q)$ is completely prime and by 2f and 2g it follows that $\{P(q)\}^- = P(\bar{q})$ is a completely prime ideal. Then since $\bar{R}$ will necessarily satisfy conditions (i), (ii), (iii), it follows that $\bar{q}$ will be right P primary.

Definition 3.1. An element x contained in a duo ring R is said to be P primary provided $xR = Rx$ is P primary.

From the preceding discussion in this section we can conclude that *an element x contained in a duo ring R which satisfies* (i), (ii), (iii), *is P primary and not nilpotent if and only if $\bar{x}$ is a P primary, nonzero element of $\bar{R}$.*

Note that if $\bar{q}$ is a nonzero completely prime ideal of $\bar{R}$ we can only conclude that q is P primary in R. (See [7, p. 673].)

For our case, when q is a P primary, not nil, nontrivial, two sided ideal of R, then $P(q) = p$ is a maximal right ideal of R. Hence from $2h$, we have $\{R/q\}^- \cong R/p \cong \{R/p\}^-$ and therefore $\{R/q\}^-$ is a division ring. Consequently if R is a ring satisfying (i), (ii), (iii) and if q is a P primary two sided ideal which is not nil or trivial with P radical p, then R/q is a completely P primary ring whose residue class is isomorphic to $R/p \cong \bar{R}/\bar{p}$.

This section depended heavily on conditions (i), (ii), (iii). One should note that if these conditions are satisfied in R then they will be satisfied in R/n where n is a two sided ideal contained in $P = N$.

4. **Factorization in duo rings.** In this section R will be a duo ring with identity. For such a ring we have the following property. If $x \in R$ then for every element $y_1, y_2 \in R$ there exists $z_1, z_2 \in R$ such that $xy_1 = z_1x$ and $y_2x = xz_2$. Most of the theorems for commutative rings on factorization in primary rings will be valid for N primary duo rings(5).

We want to discuss the factorization of a duo ring R when R/N is a unique factorization domain whose nontrivial completely prime ideals are maximal.

Definition 4.1. A duo ring A with identity is called a *unique factorization domain* if the following two conditions are satisfied.

(1) Every nonzero element of A is a product of a finite number of irreducible elements.

(2) If $a_1a_2 \cdots a_s = b_1b_2 \cdots b_t$ where $a_1, a_2, \cdots, a_s$ and $b_1, b_2, \cdots, b_t$ are nontrivial irreducibles, then $s = t$ and for a suitable ordering of the subscripts a_i is an associate of b_i, for $i = 1, 2, \cdots, s$.

For a duo ring the following three properties are valid.

(1) If x is irreducible then xv and vx are irreducible where v is a unit.

(2) If x is irreducible and v is a unit then $vx = xw$ implies that w is a unit.

(3) If $x = ab$ and $b = vd$ where a, b, and d are irreducible then $x = uad$ where u is a unit.

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(5) The factorization theorems for commutative rings are given in [7, pp. 674–678].

LEMMA 4.1. *A duo ring A which is a unique factorization domain is an integral domain.*

Proof. Let $a_1 \cdots a_n$ be a product of irreducibles elements of A. We shall show that $a_1 \cdots a_n \neq 0$. If $n=1$ this statement follows from the fact that 0 is reducible, i.e., $0=00$. We assume that the statement is valid for less than n irreducibles. If $a_1 \cdots a_n = 0$ then none of the irreducibles is zero. For in that case less than the product of n irreducibles would be zero. Furthermore $a_1 a_1 \cdots a_n = 0 \cdot a_1 = 0 = a_1 \cdots a_n$ which contradicts Definition 4.1. Thus $a_1 \cdots a_n \neq 0$. Since every two elements x and y of A can be factored into irreducibles certainly $xy \neq 0$ if $x \neq 0$ and $y \neq 0$.

If A is a unique factorization domain then since it is an integral domain it follows that an irreducible element is prime. For suppose $bc \in aR = Ra$ where a is irreducible. Then $bc = ad$ and hence since a is irreducible a must appear in the factorization of either b or c. Say $c = g_1 g_2 \cdots g_i a g_{i+2} \cdots g_n$. Then since $g_i a = ah_i$, $i = 1, 2, \cdots, n$ where the h_i must be irreducible since A is a unique factorization domain, we have $c = af$, i.e., c is contained in aR.

We refer to a duo ring which is at the same time a principal ideal ring and an integral domain as a *duo principal ideal domain.*

THEOREM 4.1. *A duo ring A is a unique factorization domain whose completely prime ideals are maximal if and only if A is a principal ideal domain.*

Proof. One can using the classical method of [4, pp. 114–122] and the fact that R is duo show quite easily that a duo principal ideal domain is a unique factorization domain(6). Certainly the completely prime ideals are maximal. For from Statement 2.6 we can conclude that prime elements are irreducible. In a duo principal ideal ring an irreducible element will generate a maximal ideal.

To prove the converse let the duo ring A be a unique factorization domain whose prime ideals are maximal. We first show that if x and y are two elements of an ideal q, then any greatest common divisor of x and y belongs to q(7). If x or y is zero, this is obvious. Let the g.c.d. of $x \neq 0$ and $y \neq 0$ be d and then $x = x_1 d$ and $y = y_1 d$. If either x_1 or y_1 is a unit, obviously $d \in q$. Hence we can assume that x_1 and y_1 are not units and hence $x_1 = a_1 \cdots a_n$ and $y_1 = b_1 \cdots b_m$ are the factorizations of x_1 and y_1 into nontrivial irreducibles. None of the a_i, $i = 1, 2, \cdots, n$ could be associates of the b_i, $i = 1, 2, \cdots, m$ since d is the g.c.d. of x and y. Then since the irreducible elements are prime the ideals $a_i R$ and $b_j R$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$ are maximal. Consequently the a_i and b_j are relatively prime. Hence x_1 and y_1 are relatively

(6) This statement also follows from [3, p. 34]. For similar elements of duo rings are associates.

(7) One can easily show using the classical methods of [4, Chapter IV] that every two elements have a g.c.d. which is unique up to the multiplication by a unit.

prime. Hence there exists λ_1 and λ_2 in A such that $\lambda_1 x+\lambda_2 y=1$. From this it follows that $\lambda_1 x+\lambda_2 y=d$. Therefore $d\in q$. We shall now show that every ideal of A is principal. If an ideal q is zero it is principal. If q is not the zero ideal we can choose from the element in q an element x with the least number of irreducible factors. We prove that $q=xR$. Let $y\in q$ and let d be the g.c.d. of x and y. Then $x=x_1 d$ and $y=y_1 d$ and we know that $d\in q$. Hence d is a divisor of x with the same number of nontrivial irreducibles. Consequently $dR=xR$ and $y\in xR$. This proves our theorem.

PRINCIPAL LEMMA FOR DUO RINGS. *Let x be an element of the duo ring R and let $x=a_1 \cdots a_m$ mod N where $a_1, \cdots, a_m$ are elements of R which are relatively prime in pairs. Then there exists elements $b_1, \cdots, b_m$ such that:*

(1) $b_i = a_i$ mod N *for* $i = 1, \cdots, m$ *and hence* $b_1, \cdots, b_m$ *are also relatively prime in pairs.*

(2) $$x = b_1 \cdots b_m.$$

The proof of this lemma follows closely the proof of the similar lemma for the commutative case (see [7, p. 672]) and is therefore omitted.

If R is a duo ring where R/N is a principal ideal domain then R satisfies the conditions (i), (ii), (iii) of §3 and therefore $N=P$, the nontrivial prime ideal of R/N are maximal, and $P(q)$ is an ideal where q is an ideal of R.

THEOREM 4.2. *Let R be a duo ring with identity and let R/N be a principal ideal domain then:*

(1) *Every not nilpotent element x of R can be factored as $x=va_1 \cdots a_n$, where v is a unit and where $a_1, \cdots, a_n$ are P primary, not nilpotent, nonunits, which are relatively prime in pairs.*

(2) *If $va_1 \cdots a_n=ub_1 \cdots b_m$, where v and u are units, $a_1, \cdots, a_n$ are P primary nonunits which are relatively prime in pairs, and where the same is true for $b_1, \cdots, b_m$; then $n=m$ and for a suitable rearrangement of the subscripts a_i is associated with b_i, $i=1, 2, \cdots, n$. If $n>1$, the elements $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ are not nilpotent.*

Proof. We first show that (2) is valid. Since R is duo it follows that $(a_1R)\cdot(a_2R) \cdots (a_nR)=(b_1R)\cdot(b_2R) \cdots (b_mR)$. In addition since these are relatively prime in pairs and since R is duo it follows that $(a_1R)\cap(a_2R)\cap \cdots \cap(a_nR)=(b_1R)\cap(b_2R)\cap \cdots \cap(b_mR)$. From [2] it follows that these ideals in some order are equal and $n=m$. If $n>1$, none of the a_i's or b_i's can be nilpotent. For if a_1 is nilpotent and a_2 and a_1 are relatively prime then 2a implies that a_2 is a unit which is not the case.

To prove (1) let x be a not nilpotent element of R. Then $\bar{x}$ is a nonzero element of $\overline{R}$ and hence $\bar{x}$ can be factored as $\bar{x}=\bar{v}\bar{a}_1^{h_1}\cdot\bar{a}_2^{h_2} \cdots \bar{a}_n^{h_n}$ where $\bar{v}$ is a unit of $\overline{R}$ and $\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n$ are irreducible, not associated, nonunits of $\overline{R}$. Consequently $a_1^{h_1}, a_2^{h_2}, \cdots, a_n^{h_n}$ are P primary, not zero, nonunits of R which are relatively prime in pairs. It follows then from the principal lemma for duo rings that $x=ub_1 \cdots b_n$ where $v=u \bmod N$, and $a_i=b_i \bmod N$, $i=1, 2, \cdots, n$. We conclude from 3b that $b_1, \cdots, b_n$ are P primary and not nilpotent elements of R and from 2a and 2b that u is a unit of R and $b_1, \cdots, b_n$ are relatively prime in pairs.

We shall now discuss some of the implications of this theorem for a duo ring R where R/P is a principal ideal domain.

4a. *The primary not nilpotent elements of* R. From §3, since the conditions (i), (ii), and (iii) are satisfied, and element $a \in R$ is P primary, not nilpotent nonunit if and only if a is a P primary, nonzero, nonunit of R. Therefore $\bar{a}=\bar{v}\bar{\pi}^h$ where $\bar{v}$ is a unit of $\bar{R}$ and $\bar{\pi}$ is a nontrivial irreducible element of $\bar{R}$. Thus we have *an element $a \in R$ is a P primary, nonnilpotent, nonunit, if and only if $a=v\pi^h+d$ where v is a unit and π is a nontrivial fundamental irreducible while $h \geqq 1$ and $d \in P$.* If $a=v_1\pi_1^{h_1}+d_1=v_2\pi_2^{h_2}+d_2$ then $\bar{v}_1\bar{\pi}_1^{h_1}=\bar{v}_2\bar{\pi}_2^{h_2}$. Hence $h_1=h_2$ and $\pi_1R=\pi_2R$. Conversely if $a=v_1\pi_1^{h_1}+d_1$ and $\pi_2=v'\pi_1+d'$ then $a=v_2\pi_2^{h_1}+d_2$ for suitable v_2 and d_2. *Thus if $a=v\pi^h+d$ then $a=v_1(v_2\pi+d_2)^h+d_1$ for all units v_2 of R and all $d_2 \in P$ where v_1 and d_1 are suitably chosen and these include all such representations.* The nontrivial fundamental irreducibles $v\pi+d$ for all v and d are called the *fundamental irreducibles of the P primary element a.*

4b. *The not nilpotent elements of* R. Let x be a not nilpotent element of R. Then $x=va_1 \cdot a_2 \cdots a_n$ where v is a unit and the a_i are P primary. Then if $v_i\pi_i+d_i$ is a nontrivial fundamental irreducible of a_i of multiplicity h_i, we say that $v_i\pi_i+d_i$ is a nontrivial fundamental irreducible of x of multiplicity h_i. From Theorem 4.2 and §4a the nontrivial fundamental irreducibles of x and there unique multiplicities are determined by x and do not depend upon any factorization of x. Consequently we have that two not nilpotent elements of R are relatively prime if and only if they have no nontrivial fundamental irreducibles in common.

4c. *Irreducible elements of* R. We shall show that if *x is an irreducible not nilpotent element of R then x is a P primary element of R.* From §3 all we need show is that $\bar{x}$ is a nonzero P primary element of $\bar{R}$. If $\bar{x}$ is not primary then $\bar{x}=\bar{a}\bar{b}$ where $\bar{a}$ and $\bar{b}$ are nonunits which are relatively prime. Then from the principal lemma for duo rings we have $x=ab$ where a and b are nonunits which is a contradiction. Hence $x=v\pi^h+d$.

5. **Factorization in general noncommutative rings where R/P is a principal ideal domain.** Let R be a ring with identity where R/P is a principal ideal domain. If u is a right unit of R then by 2a. $\bar{u}$ is a right unit of R/P. Consequently $\bar{u}$ is also a left unit of R/P and again by 2a u is a left unit of R. Consequently every right or left unit of R is a unit. In this way the units of R are trivial irreducibles. From [3, p. 34] we know that every nonzero element $\bar{a}$ of R/P which is not a unit may be written as $\bar{b}_1 \cdots \bar{b}_n$, where $\bar{b}_i$ are nontrivial irreducible; and if $\bar{a}=\bar{c}_1 \cdots \bar{c}_m$, where $\bar{c}_i$ are nontrivial irreducible then $m=n$ and the $\bar{b}_i$'s and $\bar{c}_i$'s may be arranged into similar pairs in R/P.

Suppose $\bar{a}$ is a nonnilpotent, nonunit of R/P and $\bar{a}=\bar{b}_1 \cdots \bar{b}_n$ where $\bar{b}_i$ are nontrivial irreducible in R/P. Suppose $a=c_1c_2 \cdots c_m$ where c_i are nonunits, then $\bar{a}=\bar{c}_1\bar{c}_2 \cdots \bar{c}_m$ and consequently $m \leqq n$. Thus we have:

THEOREM 5.1. *If R is a ring with identity where R/P is a principal ideal domain then every nonnilpotent, nonunit element of R can be expressed as a product of nontrivial irreducible elements. The number of irreducible elements in such a product will be less than or equal to the unique number of irreducible elements in R/P whose product is equal to $\bar{a}$.*

It would be tempting to conjecture as in [7, p. 678] that the number of irreducible elements is unique in R. However, this is not the case as the following example shows.

EXAMPLE. Consider the following commutative ring. Let R be the set of all elements $a+bx$ where $a, b \in I$, the ring of integers. Equality is defined by: $a_1+b_1x=a_2+b_2x$ if and only if $a_1=a_2$ and $b_1=b_2$. Addition is defined by $(a_1+b_1x)+(a_2+b_2x)=(a_1+a_2)+(b_1+b_2)x$. Define $x^2=0$. Then multiplication is defined by the usual polynomial multiplication, i.e., $(a_1+b_1x)(a_2+b_2x) =a_1a_2+(a_1b_2+a_2b_1)x$. Hence $xR=N$ is the radical of R. In R we have:

(1) $16=2\cdot 2\cdot 2\cdot 2$,

(2) $16=(4-x)(4+x)$.

We assert in (1) and (2) that 16 is a product of nonnilpotent, nonunits which are irreducible in R. Obviously this is true for (1). In (2), $(4+x)$ is obviously not nilpotent and is a nonunit by 2a. We shall now show that $(4\pm x)$ is irreducible. Suppose:

(3) $(a_1+b_1x)(a_2+b_2x)=(4\pm x)$ where (a_1+b_1x) and (a_2+b_2x) are nonnilpotent, nonunits.

Then a_1 and a_2 are not zero, nor can they be ± 1. Equation (3) implies that $a_1a_2=4$, which in turn implies that $a_1=a_2=\pm 2$. From this equation (3) implies that $\pm 2(b_1+b_2)=\pm 1$, which is impossible.

Thus, as this example displays, the *number* of irreducible elements in a factorization in R is not unique.

6. **Polynomial ring over a duo ring.** Suppose R is a duo ring with identity. Let $R[x]$ denote the ring of polynomials with coefficients in R where $ax=xa$ for all a contained in R. The ring $R[x]$ will not necessarily be duo but does have the following property. If f is any polynomial there exist polynomials g and h such that $af=ga$ and $fa=ah$ for all $a\in R$. The degree of a nonzero polynomial f in $R[x]$ is the exponent of the highest power of x which occurs in f with a nonzero coefficient. The degree of f is denoted by $D(f)$.

For each ring R in this section we shall denote the set of nilpotent elements by $P(R)$.

LEMMA 6.1. *If R is a duo ring with identity the following conditions are equivalent:*

(1) *The radical* $P(R)=0$.

(2) *If* f *is any nonzero polynomial and* g *is a regular*(8) *polynomial of* $R[x]$, *then* $D(fg) \geqq D(f)$.

Hence if $P(R)=0$ *and* f *is regular then* $D(f)$ *is the minimum degree of the regular polynomials of* $fR[x]$ *and* $R[x]f$.

Proof. (2)⇒(1). Let a be a nonzero nilpotent element of R and say $a^s=0$, $a^{s-1}\neq 0$ where $s\geqq 2$. If b is any regular element of R let $f=a^{s-1}x-c$ and $g=a^{s-1}x+b$, where $ca^{s-1}x=a^{s-1}xb$. Then g is regular and $fg=-cb$ which has degree zero. This contradicts (2).

We shall now show that (1)⇒(2). Suppose $P(R)=0$. Let $f=a_nx^n+\cdots+a_1x+a_0$ and $g=b_sx^s+\cdots+b_1x+b_0$, where $a_n\neq 0$. In addition suppose $D(fg)<n$. We need only show that g is not regular. We shall prove that $a_nb_s=a_n^2b_{s-1}=\cdots=a_n^{s+1}b_0=0$. Certainly $a_nb_s=0$ since $D(fg)<n$, and hence we shall make the induction hypothesis that $a_nb_s=\cdots=a_n^{s-i+1}b_i=0$ for some i, $i=1, 2, \cdots, s$. Since R is a duo ring there exists a polynomial h of degree n such that $a_n^{s-i+1}f=ha_n^{s-i+1}$ where the leading coefficient of h is a_n. Then $(ha_n^{s-i+1}g)=ha_n^{s-i+1}(b_{i-1}x^{i-1}+\cdots+b_0)$ where $D(ha_n^{s-i+1}g)=D(a_n^{s-i+1}fg)\leqq D(fg)<D(f)=n$. Thus $a_n^{s-1+1}b_{i-1}$, which is the coefficient of x^{n+i-1} in ha_n^{s-i+1}, must be zero. This completes the inductive portion of the proof. Consequently the statement is true for all integers i. Hence $a_nb_s=\cdots=a_n^{s+1}b_0=0$. It follows that $a_n^{s+1}b_s=\cdots=a_n^{s+1}b_0=0$ and hence since $a_n\neq 0$ and $P(R)=0$, we have $a_n^{s+1}g=0$. Thus g is not regular.

LEMMA 6.2. *Let* f *be a polynomial in* $R[x]$ *where* R *is a duo ring. If* $fg=0$, *where* g *is also a polynomial in* $R[x]$, *there exists an element* $c\neq 0$ *of* R *such that* $f(x)c=0$.

The proof of this lemma is very similar to the proof for the commutative case and is therefore omitted. (See [6, p. 34].)

Since a polynomial f is contained in $P(R[x])$ if and only if its coefficients are contained in $P(R)$, we have that $P(R[x])$ is a two sided ideal of $R[x]$ and that $\sum \bar{a}_i\bar{x}^i=0$ if and only if each $\bar{a}_i=0$. (The single bar denotes the image under the natural homomorphism from $R[x]$ to $R[x]/P(R[x])$). Hence we can consider $R[x]/P(R[x])$ as the polynomial ring $(R/P)[x]$ where P is the radical $P(R)$.

THEOREM 6.1. *A polynomial* $f=a_nx^n+\cdots+a_1x+a_0$ *is a unit of* $R[x]$, *where* R *is a duo ring, if and only if* a_0 *is a unit of* R *and the other coefficients are nilpotent elements of* R.

Proof. If a_0 is a unit and $a_1, \cdots, a_n$ are nilpotent, then f is a unit of $(R/P)[x]$. Consequently by 2a f is a unit of $R[x]$. Conversely, if f is a unit of

(8) An element g is regular if it is neither a left nor right divisor of zero.

$R[x]$, then $\bar{f}=\bar{a}_n\bar{x}^n+\cdots+\bar{a}_0$ is a unit of $(R/P)[x]$ and hence $\bar{f}(R/P)[x]$ contains a regular polynomial of degree zero. We conclude from Lemma 6.1, since $P(R/P)=0$, that f has degree zero. Consequently $\bar{a}_n=\cdots=\bar{a}_1=\bar{0}$ and $\bar{a}_0$ is a unit of R/P. Hence $a_n, \cdots, a_1$ are nilpotent and from 2a a_0 is a unit of R.

THEOREM 6.2. *The Jacobson radical $J(R[x])$ and the radical $P(R[x])$ of a polynomial ring $R[x]$, where R is a duo ring, are the same.*

Proof. We need only show that $J(R[x])\subseteq P(R[x])$. If $f=a_nx^n+\cdots+a_1x+a_0\in J(R[x])$, then $xf+1=a_nx^{n+1}+\cdots+a_1x^2+a_0x+1$ is a unit. Hence $a_n, \cdots, a_0$ are nilpotent.

THEOREM 6.3. *If R is a duo ring the following statements are equivalent.*
(1) *R is a right P primary ring.*
(2) *$R[x]$ is a right P primary ring.*
(3) *$R[x]$ is a right J primary ring.*

Proof. Obviously, by Theorem 6.2, (2) and (3) are equivalent. (2) immediately implies (1). We shall now show that (1)$\Rightarrow$(2). Suppose $fg=0$ and $g\neq 0$. Then, by Lemma 6.2, there exists a nonzero element c in R such that $fc=0$. Hence the coefficients of R are left divisor of zero and therefore are contained in $P(R)$. Consequently f is contained in $P(R[x])$.

From the preceding discussion it would seem natural to investigate algebraic extensions of R where R is a completely N primary ring or a duo completely N primary ring.

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