SOME PROPERTIES OF PARTITIONS (2)

BY

A. O. L. ATKIN AND S. M. HUSSAIN

7. Atkin and Swinnerton-Dyer [7] have given a complete account of Dyson's rank functions for the cases $q=5$ and $q=7$. We shall adopt throughout the notation of [7] together with some of our own, and require to refer so constantly to [7] that we have decided to regard this paper as a sequel, numbering the sections, theorems, and equations, consecutively. Thus any reference in this paper to §§1 to 6, Theorems 1 to 5, and Lemmas 1 to 8, is to the result so numbered in [7].

Our main object in this paper is to carry out the corresponding investigation for $q=11$, in particular to obtain the values of the $r_{bc}(d)$ for $q=11$. These values are considerably more complicated than those for $q=5$ and $q=7$. Their interest lies largely in the methods by which they were obtained; as in [7] the complete set of values had to be obtained empirically before the proof of the correctness of any one was possible. We shall describe these methods in §9. §8 contains some preliminary results for the functions $P(a)$ with $q=11$, and in §10 we state and prove the main results in Theorem 6. §11 contains the evaluation of $\sum_{n=0}^{\infty} y^n p(11n+6)$ in a form similar to Ramanujan's results for $q=5$ and $q=7$, although less elegant. In the table of notation and list of references at the end we only give the notation and references introduced in this paper and not that of [7]. We observe finally that some letters occur twice in the text in different senses, but the contexts are so different as to give no danger of confusion.

8. It is not practicable to use the $P(a)$ themselves directly for $q=11$; instead we introduce certain rational functions of the $P(a)$ which we regard as fundamental, and develop the relations between the $P(a)$ for $q=11$ systematically in terms of these new functions. In fact we write

$$(8.1)\qquad \begin{aligned} r &= -y^2P(1)/P(3)P(5), & u &= yP(3)/P(2)P(4),\\ s &= -yP(2)/P(1)P(5), & v &= yP(5)/P(3)P(4).\\ t &= P(4)/P(1)P(2), \end{aligned}$$

By (4.6) to (4.10) and Lemma 8 we have

$$(8.2)\qquad \begin{aligned} t-s &= su/r = P^2(3)/P^2(1)P(4) = [2g(1)-g(2)+1]/P^2(0),\\ u-t &= tv/s = -P^2(5)/P^2(2)P(3) = [-2g(2)+g(4)-1]/P^2(0),\\ v-u &= ur/t = -y^3P^2(1)/P^2(4)P(5) = [2g(4)+g(3)]/P^2(0),\\ r-v &= vs/u = -yP^2(2)/P^2(3)P(1) = [2g(3)+g(5)]/P^2(0),\\ s-r &= rt/v = -yP^2(4)/P^2(5)P(2) = [2g(5)+g(1)]/P^2(0). \end{aligned}$$

Received by the editors February 15, 1957.

From (8.2) we obtain

$$
\begin{aligned}
3g(2) + g(5) + 1 &= (-r + s + t - 2u + v)P^2(0),\\
g(1) - g(2) - 2g(5) + 1 &= (r - 2s + t)P^2(0),\\
-2g(1) - g(3) + g(5) - 2 &= (-r + 2s - 2t + u)P^2(0),\\
g(1) + 2g(3) + g(4) + 1 &= (r - s + t - u)P^2(0),\\
-g(3) - 2g(4) &= (u - v)P^2(0).
\end{aligned}
\tag{8.3}
$$

It is easily seen that only three of the relations between r, s, t, u, and v, in (8.2) are independent. It is also clear that any equation involving r, s, t, u, and v, derived from (8.2), remains valid if these quantities are permuted cyclically. Accordingly we exhibit only one equation for each symmetrical set of five equations in (8.4) and (8.5) below, which are immediate deductions from (8.2), and will be required in §§9 and 10 respectively for the systematic simplification of expressions involving r, s, t, u, and v.

$$
\begin{aligned}
rs/t &= r - s - v,\\
rt/s &= r + u,\\
rt/v &= s - r,\\
rs/u &= -s + rt/u,\\
rv/u &= r - v - rt/u,\\
rtv/su &= r - rt/u,\\
stv/u^2 &= -r - t + rt/u.
\end{aligned}
\tag{8.4}
$$

$$
\begin{aligned}
r^2s &= -r^2v - rsu - uvs + vrt,\\
r^2t &= -r^2v - uvs + vrt,\\
r^2u &= -tur + vrt,\\
rst &= rsu - vrt.
\end{aligned}
\tag{8.5}
$$

We have also the single equations

$$rs + st + tu + uv + vr = 0, \tag{8.6}$$

$$g(1) - g(2) + g(3) + g(4) + g(5) = 0, \tag{8.7}$$

and the congruence

$$4r - 2s + t + 5u + 3v \equiv 1/P^2(0) \pmod{11}. \tag{8.8}$$

This completes the preliminary work for §§9 and 10, but it is convenient to obtain here some further results required in §11. We write

$$\alpha = r/s,\ \beta = s/t,\ \gamma = t/u,\ \delta = u/v,\ \epsilon = v/r, \tag{8.9}$$

and

(8.10) $\lambda = \alpha\gamma + \beta\delta + \gamma\epsilon + \delta\alpha + \epsilon\beta, \quad \mu = \alpha^2\gamma + \beta^2\delta + \gamma^2\epsilon + \delta^2\alpha + \epsilon^2\beta.$

The relations of (8.2) become

$$\alpha + \delta + \beta\gamma = 0, \qquad 1/\alpha = 1 - \beta - \epsilon = 1 + \gamma\delta,$$

together with all the equations obtainable by permuting these cyclically, and we have

$$\alpha\beta\gamma\delta\epsilon = 1.$$

We shall use $\sum$ to denote a sum obtained by permuting the typical term cyclically.

Now from $\alpha+\delta+\beta\gamma=0$ and $1=\alpha-\alpha\beta-\epsilon\alpha$ we obtain respectively $2\sum\alpha+\sum\alpha\beta=0$ and $\sum\alpha-2\sum\alpha\beta=5$, whence

$$\sum \alpha = 1, \qquad \sum \alpha\beta = -2.$$

Next $\sum\alpha\beta\gamma\delta=\sum 1/\epsilon=\sum(1-\alpha-\delta)=3$.

Finally $\sum(\alpha\beta\gamma + \alpha\gamma\delta) = \sum(-\alpha^2 - \alpha\delta + 1 - \alpha) = -(\sum\alpha)^2 + 2\sum\alpha\beta + 2\sum\alpha\gamma - \sum\alpha\beta + 5 - \sum\alpha = \lambda - 1$. Collecting the above results it appears that α, β, γ, δ, and ϵ, are the roots of the quintic equation

$$z^5 - z^4 - 2z^3 + z^2 + 3z - 1 + \lambda(z^3 - z^2) = 0. \tag{8.11}$$

There are four independent relations between α, β, γ, δ, and ϵ. We can express any three rationally in terms of the other two, which are themselves connected by a relation typified by one of

$$\begin{aligned} \alpha\gamma(\alpha + \gamma) - (\alpha + \gamma) + 1 &= 0, \\ (\alpha\beta)^2 - \alpha\beta(\alpha + \beta) + 2\alpha\beta - (\alpha + \beta) &= 0. \end{aligned} \tag{8.12}$$

We now prove that any cyclically symmetric function of α, β, γ, δ, and ϵ, can be expressed rationally in terms of λ and μ. In fact we have

LEMMA 9. *Any expression of the form* $\sum\alpha^l\beta^m\gamma^n\delta^p\epsilon^q$, *where* l, m, n, p, *and* q, *are positive, negative, or zero, integers, is equal to*

$$Q_1(\lambda) + \mu Q_2(\lambda),$$

where $Q_1(\lambda)$ *and* $Q_2(\lambda)$ *are polynomials in* λ *with integral coefficients.*

Since $\alpha\beta\gamma\delta\epsilon=1$, we may suppose without loss of generality that l, m, n, p, and q, are greater than or equal to zero. Assume now that the lemma is true for all values of l, m, n, p, and q, with $l+m+n+p+q\leqq k$, where $k\geqq 1$, and consider any $\sum\alpha^l\beta^m\gamma^n\delta^p\epsilon^q$ with $l+m+n+p+q=k+1$. If any two of α, β, γ, δ, and ϵ, adjacent in the cyclic order, have nonzero indices, we can express $\sum\alpha^l\beta^m\gamma^n\delta^p\epsilon^q$ as a linear combination of similar sums with $l+m+n+p+q\leqq k$ by using $\beta\gamma=-\alpha-\delta$, and the four other similar equations; and so by the induction hypothesis it is equal to $Q_1(\lambda)+\mu Q_2(\lambda)$.

Consider now $\sum\alpha^l\gamma^n$ where $l\geqq 2$, $n\geqq 2$. Since $\lambda\sum\alpha^{l-1}\gamma^{n-1}$ is equal to $\sum\alpha^l\gamma^n$ plus a linear combination of cyclically symmetric sums *in which adjacent letters occur*, we have

$$\sum \alpha^l\gamma^n = Q_1'(\lambda) + \mu Q_2'(\lambda) + \lambda[Q_1''(\lambda) + \mu Q_2''(\lambda)] = Q_1(\lambda) + \mu Q_2(\lambda).$$

Next we observe that, for $l\geqq 3$, $\sum\alpha^l\gamma = \sum\alpha\sum\alpha^{l-1}\gamma - \sum\gamma^2\alpha^{l-1}$ plus a linear combination of cyclically symmetric sums in which adjacent letters occur, and similarly for $\sum\alpha\gamma^n$: also $\sum\alpha^2\gamma=\mu$, $\sum\alpha\gamma^2=-\mu-3$, $\sum\alpha\gamma=\lambda$. Finally $\sum\alpha^l$ can clearly be expressed as $Q(\lambda)$ by (8.11).

Thus if the lemma is true for $l+m+n+p+q\leqq k$ it is true for all l, m, n, p, and q, with $l+m+n+p+q=k+1$. But the lemma is clearly true for $k=1$, and hence it is true for all values of k by the strong form of mathematical induction.

In particular μ^2 can be expressed in terms of λ and μ; in fact it is easily verified that

$$\mu^2 + 3\mu + \lambda^3 - 8\lambda^2 + 11\lambda + 34 = 0. \tag{8.13}$$

In our application of Lemma 9 we require to express some function $F(y)$, known to be a linear combination of sums $\sum\alpha^l\beta^m\gamma^n\delta^p\epsilon^q$, in terms of λ and μ. Since the lowest powers of y in the expansions of λ and μ are y^{-2} and y^{-3} respectively, we assume a form with $Q_1(\lambda)$ of degree h and $Q_2(\lambda)$ of degree $h-2$ if $y^{2h}F(y)$ is a Taylor series, and $Q_1(\lambda)$ of degree h and $Q_2(\lambda)$ of degree $h-1$ if $y^{2h+1}F(y)$ is a Taylor series. The actual values of $Q_1(\lambda)$ and $Q_2(\lambda)$ are then obtained by equating coefficients.

9. In attempting to obtain empirically the values of the $r_{bc}(d)$ for $q=11$ it was clearly convenient to avoid the terms involving $\sum(m, 0)$ which occur in a manner known from (6.7); also the forms of the $r_{bc}(d)$ for $q=5$ and $q=7$, together with the congruences for $\Phi_{11}(b)$ given by Theorem 3, suggested that the values of $r_{bc}(5)$, for example, would involve a factor $P(4)/P(2)P(5)$. We accordingly defined $R_{bc}(d)$, the "normalized" form of $r_{bc}(d)$ for $q=11$, as follows:

$$\begin{aligned}
R_{01}(0) &= P(1)[r_{01}(0) + 1 + 3y^2\textstyle\sum (2, 0)/P(0)],\\
R_{12}(0) &= P(1)[r_{12}(0) - y^2\textstyle\sum (2, 0)/P(0)],\\
R_{34}(4) &= P(2)[r_{34}(4) + y^3\textstyle\sum (4, 0)/P(0)],\\
R_{45}(4) &= P(2)[r_{45}(4) - 2y^3\textstyle\sum (4, 0)/P(0)],\\
R_{01}(7) &= P(3)[r_{01}(7) - y^3\textstyle\sum (5, 0)/P(0)],\\
R_{12}(7) &= P(3)[r_{12}(7) + 2y^3\textstyle\sum (5, 0)/P(0)],\\
R_{23}(7) &= P(3)[r_{23}(7) - y^3\textstyle\sum (5, 0)/P(0)],\\
R_{23}(9) &= P(4)[r_{23}(9) + y^2\textstyle\sum (3, 0)/P(0)],\\
R_{34}(9) &= P(4)[r_{34}(9) - 2y^3\textstyle\sum (3, 0)/P(0)],
\end{aligned}$$

$$\begin{aligned}
R_{45}(9) &= P(4)[r_{45}(9) + y^3 \textstyle\sum (3, 0)/P(0)],\\
R_{12}(10) &= P(5)[r_{12}(10) - 1/y - \textstyle\sum (1, 0)/P(0)],\\
R_{23}(10) &= P(5)[r_{23}(10) + 2/y + 2\textstyle\sum (1, 0)/P(0)],\\
R_{34}(10) &= P(5)[r_{34}(10) - 1/y - \textstyle\sum (1, 0)/P(0)],
\end{aligned}$$

and, for all other values of b and c with $c=b+1$,

$$\begin{aligned}
R_{bc}(0) &= P(1)r_{bc}(0),\\
R_{bc}(1) &= P(2)P(3)r_{bc}(1)/P(5),\\
R_{bc}(2) &= P(1)P(4)r_{bc}(2)/P(3),\\
R_{bc}(3) &= P(1)P(3)r_{bc}(3)/P(2),\\
R_{bc}(4) &= P(2)r_{bc}(4),\\
R_{bc}(5) &= P(2)P(5)r_{bc}(5)/P(4),\\
R_{bc}(6) &= yr_{bc}(6),\\
R_{bc}(7) &= P(3)r_{bc}(7),\\
R_{bc}(8) &= P(4)P(5)r_{bc}(8)/yP(1),\\
R_{bc}(9) &= P(4)r_{bc}(9),\\
R_{bc}(10) &= P(5)r_{bc}(10),
\end{aligned}$$

and, for all remaining values of b and c, by the relations

$$\begin{aligned}
R_{bc}(d) + R_{ce}(d) &= R_{be}(d),\\
R_{cb}(d) &= - R_{bc}(d).
\end{aligned}$$

Next, using the congruent form of $\prod(1-x^r)^{-1}$ given by Theorem 3 and the values of $S(b)$ derived from (6.7), we obtained congruent forms of $R_{bc}(d)$ for all b, c, and d, which were simplified by (8.3) and (8.4). It was found that every $R_{bc}(d)$ with $d\neq 6$ was congruent (mod 11) to the product of $P^3(0)$ and a linear combination of the functions r, s, t, u, v, rt/u, su/v, tv/r, ur/s, and vs/t; while every $R_{bc}(6)$ was congruent to the product of $P^3(0)$ and a linear combination of rt, su, tv, ur, and vs. By inspection of these results we found set of congruent relations(1) between the $R_{bc}(d)$ for different values of d, as follows:

$$\begin{aligned}
R_{35}(0) &\equiv R_{23}(1) \equiv R_{13}(2) - R_{45}(2) - P(0) \equiv R_{02}(3) \equiv R_{12}(4) - P(0)\\
&\equiv R_{35}(5) - R_{12}(5) \equiv R_{45}(7) \equiv R_{02}(8) - R_{34}(8) - P(0)\\
&\equiv R_{23}(9) - R_{45}(9) \equiv R_{35}(10) - R_{02}(10) - P(0) \equiv R_{25}(6)/t \pmod{11},
\end{aligned} \tag{9.1}$$

(1) The appearance of $P(0)$ in (9.1) to (9.9) and (9.14) below arises from an occasional use of (8.8).

$$
\begin{aligned}
(9.2)\qquad R_{24}(0) &\equiv R_{12}(1) + R_{45}(1) \equiv R_{02}(2) + R_{45}(2) \equiv R_{24}(3) - R_{45}(3) - P(0) \\
&\equiv R_{23}(4) + P(0) \equiv R_{34}(5) + P(0) \equiv R_{23}(7) - R_{01}(7) + P(0) \\
&\equiv R_{23}(8) - R_{01}(8) - R_{45}(8) + P(0) \\
&\equiv R_{02}(9) + R_{23}(9) + R_{45}(9) - P(0) \equiv R_{45}(10) \equiv R_{15}(6)/t \pmod{11},
\end{aligned}
$$

$$
\begin{aligned}
(9.3)\qquad R_{45}(0) &\equiv R_{34}(1) - R_{45}(1) \equiv R_{23}(2) - R_{45}(2) \\
&\equiv R_{12}(3) + R_{45}(3) - R_{34}(3) + P(0) \equiv R_{01}(4) \\
&\equiv R_{45}(5) - R_{01}(5) - P(0) \equiv R_{01}(7) - R_{23}(7) + R_{34}(7) - P(0) \\
&\equiv R_{02}(8) - R_{23}(8) \equiv - R_{01}(9) + R_{23}(9) - R_{45}(9) + P(0) \\
&\equiv R_{34}(10) - R_{12}(10) \equiv R_{34}(6)/t \pmod{11},
\end{aligned}
$$

$$
\begin{aligned}
(9.4)\qquad & R_{01}(0) - R_{45}(0) + 3R_{15}(0) + P(0) \equiv R_{24}(1) + R_{05}(1) + R_{23}(1) + P(0) \\
&\equiv 2R_{12}(2) + 3R_{24}(2) + R_{45}(2) + P(0) \equiv 2R_{15}(3) + R_{01}(3) + P(0) \\
&\equiv R_{12}(4) + R_{34}(4) + R_{15}(4) - P(0) \equiv R_{25}(5) + R_{03}(5) \equiv R_{05}(7) \\
&\equiv R_{01}(8) + R_{45}(8) - P(0) \equiv R_{24}(9) + R_{23}(9) + P(0) \equiv R_{14}(10) \\
&\equiv - R_{23}(6)/r \pmod{11},
\end{aligned}
$$

$$(9.5)\qquad R_{45}(4) \equiv R_{02}(5) - R_{45}(5) + P(0) \equiv R_{13}(6)/t \pmod{11},$$

$$(9.6)\qquad R_{13}(2) \equiv R_{01}(9) + R_{34}(9) - R_{45}(9) - P(0) \equiv R_{34}(6)/u \pmod{11},$$

$$(9.7)\qquad R_{15}(1) \equiv R_{13}(7) \equiv R_{24}(6)/v - P(0) \pmod{11},$$

$$(9.8)\qquad R_{14}(0) \equiv R_{24}(3) - P(0) \equiv - R_{35}(6)/s \pmod{11},$$

$$(9.9)\qquad R_{12}(8) \equiv R_{25}(10) - P(0) \equiv - R_{34}(6)/r \pmod{11}.$$

The comparative simplicity of these congruences and the observation that the first few terms in the power series expansions of congruent quantities were identical and not merely congruent (mod 11) led us to conjecture that (9.1) to (9.9) were in fact *identities*.

We obtained next the following five independent congruent relations for the $R_{bc}(6)$:

$$(9.10)\qquad -vR_{01}(6) + tR_{02}(6) - (v + t)R_{03}(6) + vR_{04}(6) + vR_{05}(6) \equiv 0 \pmod{11},$$

$$(9.11)\qquad sR_{01}(6) - tR_{02}(6) - sR_{05}(6) \equiv 0 \pmod{11},$$

$$(9.12)\qquad -uR_{01}(6) - uR_{02}(6) + tR_{03}(6) + (2u - t)R_{05}(6) \equiv 0 \pmod{11},$$

$$(9.13)\qquad -vR_{01}(6) + (v - s)R_{02}(6) + sR_{03}(6) - uR_{04}(6) + uR_{05}(6) \equiv 0 \pmod{11},$$

$$(9.14)\qquad (r - s)R_{01}(6) - (r + t)R_{02}(6) + tR_{03}(6) + sR_{05}(6) \equiv - rtP(0) \pmod{11}.$$

These again appeared to be identities; finally, by solving (9.10) to (9.14) as a set of simultaneous linear equations and substituting the values of the

$R_{bc}(6)$ so found in (9.1) to (9.9), we obtained a complete set of conjectural values of the $R_{bc}(d)$ for $q=11$. These values, and the proof that they are in fact correct, are given in Theorem 6 below.

It may be observed here that there are no linear identities between the $r_{bc}(d)$ for a *given* value of d when $q=11$. There are however certain congruence relations; if we write

$$\begin{aligned} f(d) &= r_{23}(d) - 4r_{34}(d) + 4r_{45}(d), \\ g(d) &= r_{12}(d) - 4r_{23}(d) + 6r_{34}(d) - 4r_{45}(d), \\ h(d) &= r_{01}(d) - 4r_{12}(d) + 5r_{23}(d) - 4r_{45}(d), \end{aligned} \tag{9.15}$$

we have

$$\begin{aligned} f(0) &\equiv 0 \pmod{11}, \\ g(1) - f(1) &\equiv 0 \pmod{11}, \\ 2h(2) - g(2) - 4f(2) &\equiv 0 \pmod{11}, \\ 3h(3) + 2g(3) + 2f(3) &\equiv 0 \pmod{11}, \\ 3h(4) + 2g(4) - f(4) &\equiv 0 \pmod{11}, \\ h(5) + f(5) &\equiv 0 \pmod{11}, \\ g(6) &\equiv 0 \pmod{11}, \\ 2h(6) + f(6) &\equiv 0 \pmod{11}, \\ 4h(7) + g(7) - 6f(7) &\equiv 0 \pmod{11}, \\ h(8) - g(8) - f(8) &\equiv 0 \pmod{11}, \\ h(9) + g(9) - f(9) &\equiv 0 \pmod{11}, \\ 4h(10) - 4g(10) - f(10) &\equiv 0 \pmod{11}. \end{aligned} \tag{9.16}$$

All of these are in fact equivalent to

$$(9.2) - (9.3) - 5(9.1) \equiv 0 \pmod{11},$$

in the obvious notation.

10. THEOREM 6. *For $q=11$ we have, writing $D=P(0)/(rt+su+tv+ur+vs)$,*

$$R_{01}(0) = D(3u^2 + 5rt + su + tv - 3ur + vs), \tag{10.1}$$

$$R_{01}(1) = D(-2s^2 - 2rt + 3su + tv - 2ur - vs), \tag{10.2}$$

$$R_{01}(2) = D(-3r^2 - 5rt - tv + 3ur), \tag{10.3}$$

$$R_{01}(3) = D(-4u^2 + rt - su + tv - 6ur + vs), \tag{10.4}$$

$$R_{01}(4) = D(-3rt - su + ur + 3vs), \tag{10.5}$$

$$R_{01}(5) = D(2v^2 + rt - 3ur + vs), \tag{10.6}$$

$$R_{01}(6) = D(rsu + stv - 4tur - 3uvs + 3vrt), \tag{10.7}$$

$$R_{01}(7) = D(-s^2 - rt + 4su + 2tv - 5ur + 2vs), \tag{10.8}$$

$$R_{01}(8) = D(-t^2 - rt + 5tv - ur + 2vs), \tag{10.9}$$

$$R_{01}(9) = D(5rt + tv - 2ur - vs), \tag{10.10}$$

$$R_{01}(10) = D(-3rt - tv + 4ur - 2vs). \tag{10.11}$$

$$R_{12}(0) = D(-u^2 - 2rt - 3ur + vs), \tag{10.12}$$

$$R_{12}(1) = D(-s^2 + rt - 2su + 2ur - vs), \tag{10.13}$$

$$R_{12}(2) = D(5rt + tv - 2ur - vs), \tag{10.14}$$

$$R_{12}(3) = D(4u^2 - su - tv + 4ur), \tag{10.15}$$

$$R_{12}(4) = D(2rt - su + tv - ur + 2vs), \tag{10.16}$$

$$R_{12}(5) = D(2v^2 - rt + su + 2ur), \tag{10.17}$$

$$R_{12}(6) = D(-2rsu - stv + 4tur - vrt), \tag{10.18}$$

$$R_{12}(7) = D(2s^2 + rt - 4su - tv + 5ur + vs), \tag{10.19}$$

$$R_{12}(8) = D(t^2 - su - 3tv + ur), \tag{10.20}$$

$$R_{12}(9) = D(-rt - su + 3ur), \tag{10.21}$$

$$R_{12}(10) = D(-t^2 + rt + 4tv - 3ur - 2vs). \tag{10.22}$$

$$R_{23}(0) = D(-2rt + su + 5ur - vs), \tag{10.23}$$

$$R_{23}(1) = D(rt - 2su - 2ur + vs), \tag{10.24}$$

$$R_{23}(2) = D(3r^2 - rt - su + 2ur + vs), \tag{10.25}$$

$$R_{23}(3) = D(-4u^2 - rt + tv - ur), \tag{10.26}$$

$$R_{23}(4) = D(rt - su - tv + ur - 4vs), \tag{10.27}$$

$$R_{23}(5) = D(-3v^2 + tv - ur), \tag{10.28}$$

$$R_{23}(6) = D(-3tur + 2uvs - vrt), \tag{10.29}$$

$$R_{23}(7) = D(-s^2 + 3su + tv - 4ur - 2vs), \tag{10.30}$$

$$R_{23}(8) = D(2rt + 2tv - ur - vs), \tag{10.31}$$

$$R_{23}(9) = D(-r^2 - su - 3ur), \tag{10.32}$$

$$R_{23}(10) = D(2t^2 + rt + su - 6tv + 2ur + 3vs). \tag{10.33}$$

$$R_{34}(0) = D(4rt - su - 3ur - 2vs), \tag{10.34}$$

$$R_{34}(1) = D(s^2 - 2rt + su + ur + vs), \tag{10.35}$$

$$R_{34}(2) = D(-4r^2 - 3rt + su - 2ur + vs), \tag{10.36}$$

$$R_{34}(3) = D(3u^2 + 2rt + su + ur - vs), \tag{10.37}$$

$$R_{34}(4) = D(-v^2 - 3rt + su + tv - ur + 3vs), \tag{10.38}$$

(10.39) $R_{34}(5) = D(rt - su - tv + ur - 4vs),$

(10.40) $R_{34}(6) = D(rsu - stv + tur + 3vrt),$

(10.41) $R_{34}(7) = D(-rt - su + 3ur),$

(10.42) $R_{34}(8) = D(-3rt + tv + ur),$

(10.43) $R_{34}(9) = D(2r^2 - rt + su + tv + 2ur + vs),$

(10.44) $R_{34}(10) = D(-t^2 - 2rt - su + 4tv - 2ur + vs).$

(10.45) $R_{45}(0) = D(-3rt - su + ur + 3vs),$

(10.46) $R_{45}(1) = D(s^2 + rt + 2su - 2vs),$

(10.47) $R_{45}(2) = D(3r^2 + 2rt + ur - 2vs),$

(10.48) $R_{45}(3) = D(-u^2 - 2rt - 3ur + vs),$

(10.49) $R_{45}(4) = D(2v^2 + 2rt + 2su + ur - 3vs),$

(10.50) $R_{45}(5) = D(2v^2 - rt + tv - ur + 5vs),$

(10.51) $R_{45}(6) = D(rsu + stv - 2uvs - 3vrt),$

(10.52) $R_{45}(7) = D(rt - 2su - 2ur + vs),$

(10.53) $R_{45}(8) = D(t^2 + 2rt + su - 2tv - ur + vs),$

(10.54) $R_{45}(9) = D(-r^2 - rt + su - ur - vs),$

(10.55)(2) $R_{45}(10) = D(2rt + 2ur - 3vs).$

The proof is similar to those of Theorems 4 and 5. By (2.13), Lemma 6, (6.1) to (6.4), (6.7), and (8.3), we have the following set of eleven simultaneous equations for $r_{01}(d)$ $(d=0$ to $10)$,

$$\begin{aligned} &P(4)r_{01}(0)/P(2) + yP(3)r_{01}(4)/P(4) + yr_{01}(6) - y^2P(1)r_{01}(7)/P(5) \\ &\qquad - yP(5)r_{01}(9)/P(3) - yP(2)r_{01}(10)/P(1) \\ &\quad = (-r + s + t - 2u + v)P(0) \\ &\qquad - 3y^2P(4)\sum(2, 0)/P(0)P(2) - P(4)/P(2), \end{aligned} \tag{10.56}$$

$$\begin{aligned} &-P(2)r_{01}(0)/P(1) + P(4)r_{01}(1)/P(2) + yP(3)r_{01}(5)/P(4) + yr_{01}(7) \\ &\qquad - y^2P(1)r_{01}(8)/P(5) - yP(5)r_{01}(10)/P(3) \\ &\quad = 3y^2P(2)\sum(2, 0)/P(0)P(1) \\ &\qquad + P(2)/P(1) - y^4\sum(5, 0)/P(0) + 3yP(0)/P(3), \end{aligned} \tag{10.57}$$

(2) We may remark here that the statement of Theorem 6 clarifies the nature of the identities (9.1) to (9.9). Each of the equal expressions in (9.1), for example, is equal to $D(rt-2su-2ur+vs)$, and it is clear that for every $d\neq 6$ some linear combination of $R_{bc}(d)$ and $P(0)$ will be equal to this. An isolated identity such as (9.5) contains a term Dv^2 which can only occur for $d=4$ and $d=5$. There are of course several possible complete sets of identities similar to (9.1) to (9.9).

$$(10.58)\quad \begin{aligned} &-P(5)r_{01}(0)/P(3) - P(2)r_{01}(1)/P(1) + P(4)r_{01}(2)/P(2) \\ &\quad + yP(3)r_{01}(6)/P(4) + yr_{01}(8) - y^2P(1)r_{01}(9)/P(5) \\ &\quad = 3y^2P(5)\sum(2,0)/P(0)P(3) + P(5)/P(3) - 3P(0)P(3)/P(1)P(2), \end{aligned}$$

$$(10.59)\quad \begin{aligned} &-P(5)r_{01}(1)/P(3) - P(2)r_{01}(2)/P(1) + P(4)r_{01}(3)/P(2) \\ &\quad + yP(3)r_{01}(7)/P(4) + yr_{01}(9) - y^2P(1)r_{01}(10)/P(5) \\ &\quad = y^4P(3)\sum(5,0)/P(0)P(4) + P(0)P(3)P(4)/P(1)P(2)P(5), \end{aligned}$$

$$(10.60)\quad \begin{aligned} &-yP(1)r_{01}(0)/P(5) - P(5)r_{01}(2)/P(3) - P(2)r_{01}(3)/P(1) \\ &\quad + P(4)r_{01}(4)/P(2) + yP(3)r_{01}(8)/P(4) + yr_{01}(10) \\ &\quad = 3y^3P(1)\sum(2,0)/P(0)P(5) + yP(1)/P(5) \\ &\quad - 3yP(0)P(1)P(5)/P(2)P(3)P(4), \end{aligned}$$

$$(10.61)\quad \begin{aligned} &r_{01}(0) - yP(1)r_{01}(1)/P(5) - P(5)r_{01}(3)/P(3) - P(2)r_{01}(4)/P(1) \\ &\quad + P(4)r_{01}(5)/P(2) + yP(3)r_{01}(9)/P(4) = -3y^2\sum(2,0)/P(0) + 1, \end{aligned}$$

$$(10.62)\quad \begin{aligned} &r_{01}(1) - yP(1)r_{01}(2)/P(5) - P(5)r_{01}(4)/P(3) - P(2)r_{01}(5)/P(1) \\ &\quad + P(4)r_{01}(6)/P(2) + yP(3)r_{01}(10)/P(4) = 0, \end{aligned}$$

$$(10.63)\quad \begin{aligned} &P(3)r_{01}(0)/P(4) + r_{01}(2) - yP(1)r_{01}(3)/P(5) - P(5)r_{01}(5)/P(3) \\ &\quad - P(2)r_{01}(6)/P(1) + P(4)r_{01}(7)/P(2) \\ &\quad = -3y^2P(3)\sum(2,0)/P(0)P(4) \\ &\quad - P(3)/P(4) + 3P(0)P(3)P(4)/P^2(2)P(5) + yP(0)P(2)/P(3)P(5), \end{aligned}$$

$$(10.64)\quad \begin{aligned} &P(3)r_{01}(1)/P(4) + r_{01}(3) - yP(1)r_{01}(4)/P(5) - P(5)r_{01}(6)/P(3) \\ &\quad - P(2)r_{01}(7)/P(1) + P(4)r_{01}(8)/P(2) \\ &\quad = -y^3P(2)\sum(5,0)/P(0)P(1) \\ &\quad - y^2P(0)P(1)P(2)/P(4)P^2(5), \end{aligned}$$

$$(10.65)\quad \begin{aligned} &P(3)r_{01}(2)/P(4) + r_{01}(4) - yP(1)r_{01}(5)/P(5) - P(5)r_{01}(7)/P(3) \\ &\quad - P(2)r_{01}(8)/P(1) + P(4)r_{01}(9)/P(2) \\ &\quad = -y^3P(5)\sum(5,0)/P(0)P(3) - P(0)/P(2), \end{aligned}$$

$$(10.66)\quad \begin{aligned} &P(3)r_{01}(3)/P(4) + r_{01}(5) - yP(1)r_{01}(6)/P(5) - P(5)r_{01}(8)/P(3) \\ &\quad - P(2)r_{01}(9)/P(1) + P(4)r_{01}(10)/P(2) = 0. \end{aligned}$$

We now multiply the equations (10.56) to (10.66) by 1, $yP(1)/P(5)$, $y/P(2)$, $P(3)/P(2)$, $yP(1)/P(3)$, $P(4)/P(2)$, $y/P(1)$, $y^2/P(5)$, $y^2/P(4)$, $P(4)/P(1)$, and $y^2/P(3)$, respectively, and replace $r_{01}(d)$ by $R_{01}(d)$, using the definition of $R_{01}(d)$ in §9; we use (8.4) without explicit mention. These transformed equations, (10.56 bis) to (10.66 bis), rearranged so that their right hand sides are zero, are now regarded as a set of eleven simultaneous equations, linear in $R_{01}(d)$ ($d=0$ to 10), *which have a unique solution*(3). Accordingly after each equation (10.56 bis) to (10.66 bis) we shall write down the result of substituting the values of $R_{01}(d)$ given by (10.1) to (10.11) in the left hand side, and show that on simplification this result reduces to zero; (10.1) to (10.11) will then be established. We have

$$\text{(10.56 bis)} \quad \begin{aligned} &tR_{01}(0) + uR_{01}(4) + R_{01}(6) + rR_{01}(7) - vR_{01}(9) + sR_{01}(10) \\ &\qquad - (-r + s + t - 2u + v)P(0) = 0. \end{aligned}$$

$$\begin{aligned} (\text{L.H.S.})/D &= (-s^2r + 3u^2t) + (9rsu - stv - 9tur + uvs) + (-4rst + 2tuv \\ &\quad + uvr + 3vrs) + 0 + (-s^2u + 3u^2r) + (-4r^2u - 3s^2v + 4t^2r + u^2s - 2v^2t) \\ &= (-s^2r + 3u^2t) + (9rsu - stv - 9tur + uvs) + (-4rsu + 4vrt + 2tur - 2stv \\ &\quad + uvs - tur + 3vrt - 3uvs) + 0 + (s^2r - rsu + vrt - 3u^2t - 3stv + 3tur) \\ &\quad + (-4vrt + 4tur - 3rsu + 3uvs + 4stv - 4vrt + tur - rsu - 2uvs + 2stv) \\ &= 0. \end{aligned}$$

$$\text{(10.57 bis)} \quad \begin{aligned} &sR_{01}(0) + (r + u)R_{01}(1) - (s + u)R_{01}(5) - rR_{01}(7) \\ &\qquad + (-r + s + u)R_{01}(8) + rR_{01}(10) + 3rP(0) = 0. \end{aligned}$$

$$\begin{aligned} (\text{L.H.S.})/D &= (-s^2r - t^2s - 2v^2u) + (rsu + 6stv - 4tur - 4vrt) + (3rst + 6tuv \\ &\quad - 4vrs) + (-t^2u) + (-s^2u - 2v^2s) + (11r^2u + 2s^2v + t^2r + 6u^2s) \\ &= (-s^2r - t^2s - 2v^2u) + (rsu + 6stv - 4tur - 4vrt) + (3rsu - 3vrt + 6tur \\ &\quad - 6stv - 4vrt + 4uvs) + (t^2s + rsu - stv + tur) + (s^2r - rsu + vrt + 2v^2u \\ &\quad - 2uvs + 2tur) + (11vrt - 11tur + 2rsu - 2uvs + stv - vrt + 6tur - 6rsu) \\ &= 0 \end{aligned}$$

It will be seen that in the two equations (10.56 bis) and (10.57 bis) above we have grouped the terms resulting from the substitution of the values of $R_{01}(d)$ in six sets, containing terms cyclically similar to r^2v, rsu, rst, r^2s, r^2t, and r^2u, respectively; substituting the values given by (8.5) of the terms in the last four sets shows that the result is zero. This process is entirely automatic, and we shall not carry it through for the remaining nine equations, confining ourselves to writing down the transformed equations.

$$\text{(10.58 bis)} \quad \begin{aligned} &-vR_{01}(0) - vR_{01}(1) + uR_{01}(2) + uR_{01}(6)/t + (u - v)R_{01}(8) \\ &\qquad + (-u + v)R_{01}(9) + 3uP(0) = 0. \end{aligned}$$

$$\text{(10.59 bis)} \quad \begin{aligned} &(-t + u)R_{01}(1) + (s - t)R_{01}(2) + tR_{01}(3) + uR_{01}(7) + uR_{01}(9) \\ &\qquad + (s + u)R_{01}(10) + (s - t + u)P(0) = 0. \end{aligned}$$

(3) This may be seen by proving that a determinant is nonzero (cf. Hussain, [8]), but it is easier to observe that the equations are in fact the necessary and sufficient conditions that $\sum_0^{d=10} x^d R_{01}(d)$ be the quotient of two given power series.

(10.60 bis) $$rR_{01}(0) - vR_{01}(2) + (r - v)R_{01}(3) + (r + u)R_{01}(4) + (u - v)R_{01}(8) - rR_{01}(10) + (-3r - 3u + 3v)P(0) = 0.$$

(10.61 bis) $$tR_{01}(0) - (r + u)R_{01}(1) - (r + t)R_{01}(3) - tR_{01}(4) + (-r + t - u)R_{01}(5) + uR_{01}(9) = 0.$$

(10.62 bis) $$(-t + u)R_{01}(1) + uR_{01}(2) + (t - u)R_{01}(4) + tR_{01}(5) + tR_{01}(6)/s - uR_{01}(10) = 0.$$

(10.63 bis) $$-uR_{01}(0) - uR_{01}(2) - rR_{01}(3) + (r + u)R_{01}(5) + R_{01}(6) - (r + u)R_{01}(7) + (-r + 3t + 3ut/s)P(0) = 0.$$

(10.64 bis) $$uR_{01}(1) - sR_{01}(3) + (s + u)R_{01}(4) - R_{01}(6) + sR_{01}(7) - (s + u)R_{01}(8) + (r + rs/v)P(0) = 0.$$

(10.65 bis) $$(-s + t)R_{01}(2) + tR_{01}(4) + (-r + s)R_{01}(5) - (r + t)R_{01}(7) + sR_{01}(8) + tR_{01}(9) + tP(0) = 0.$$

(10.66 bis) $$-sR_{01}(3) + (r - s)R_{01}(5) + rR_{01}(6)/v + rR_{01}(8) + sR_{01}(9) + (r - s)R_{01}(10) = 0.$$

(It should be observed that terms such as u^2rs/t, arising from $uR_{01}(6)/t$, must be reduced to third degree polynomials in r, s, t, u, and v; this is always possible by one of the first three relations of (8.4) in the cases which occur.) This completes the proof of (10.1) to (10.11).

It would be unduly tedious to carry out the proof of (10.12) to (10.55) in the same detail; we shall therefore write out the equations corresponding to (10.56 bis) to (10.66 bis) in a form suitable for rapid verification if desired. These equations are:

(10.67) $$r[R_{12}(7) - P(0)] + s[R_{12}(10) + 2P(0)] + t[R_{12}(0) - P(0)] + uR_{12}(4) - vR_{12}(9) + R_{12}(6) = 0.$$

(10.68) $$r[R_{12}(1) - R_{12}(7) - R_{12}(8) + R_{12}(10) - P(0)] + s[R_{12}(0) - R_{12}(5) + R_{12}(8)] + u[R_{12}(1) - R_{12}(5) + R_{12}(8) - P(0)] = 0.$$

(10.69) $$u[R_{12}(2) + R_{12}(8) - R_{12}(9) - P(0)] - v[R_{12}(0) + R_{12}(1) + R_{12}(8) - R_{12}(9)] + uR_{12}(6)/t = 0.$$

(10.70) $$s[R_{12}(2) + R_{12}(10) - 2P(0)] + t[-R_{12}(1) - R_{12}(2) + R_{12}(3) + 2P(0)] + u[R_{12}(1) + R_{12}(7) + R_{12}(9) + R_{12}(10) - 3P(0)] = 0.$$

(10.71) $$r[R_{12}(0) + R_{12}(3) + R_{12}(4) - R_{12}(10)] + u[R_{12}(4) + R_{12}(8)] - v[R_{12}(2) + R_{12}(3) + R_{12}(8)] - rvP(0)/s = 0.$$

$$-r[R_{12}(1)+R_{12}(3)+R_{12}(5)]+t[R_{12}(0)-R_{12}(3)-R_{12}(4)+R_{12}(5)] - u[R_{12}(1)+R_{12}(5)-R_{12}(9)]=0. \tag{10.72}$$

$$t[-R_{12}(1)+R_{12}(4)+R_{12}(5)-P(0)] + u[R_{12}(1)+R_{12}(2)-R_{12}(4)-R_{12}(10)]+tR_{12}(6)/s=0. \tag{10.73}$$

$$r[-R_{12}(3)+R_{12}(5)-R_{12}(7)+2P(0)] - u[R_{12}(0)+R_{12}(2)-R_{12}(5)+R_{12}(7)+P(0)] + R_{12}(6)+ruP(0)/s=0. \tag{10.74}$$

$$s[-R_{12}(3)+R_{12}(4)+R_{12}(7)-R_{12}(8)]+u[R_{12}(1)+R_{12}(4)-R_{12}(8)] - R_{12}(6)-2(r+rs/v)P(0)=0. \tag{10.75}$$

$$-r[R_{12}(5)+R_{12}(7)]+s[-R_{12}(2)+R_{12}(5)+R_{12}(8)] + t[R_{12}(2)+R_{12}(4)-R_{12}(7)+R_{12}(9)-2P(0)]=0. \tag{10.76}$$

$$r[R_{12}(5)+R_{12}(8)+R_{12}(10)]-s[R_{12}(3)+R_{12}(5)-R_{12}(9)+R_{12}(10)] + rR_{12}(6)/v-stP(0)/u=0. \tag{10.77}$$

$$r[R_{23}(7)+P(0)]+s[R_{23}(10)-2P(0)]+t[R_{23}(0)+2P(0)] + u[R_{23}(4)-P(0)]-vR_{23}(9)+R_{23}(6)=0. \tag{10.78}$$

$$r[R_{23}(1)-R_{23}(7)-R_{23}(8)+R_{23}(10)]+s[R_{23}(0)-R_{23}(5)+R_{23}(8)] + u[R_{23}(1)-R_{23}(5)+R_{23}(8)+2P(0)]=0. \tag{10.79}$$

$$-rP(0)+u[R_{23}(2)+R_{23}(8)-R_{23}(9)-P(0)] - v[R_{23}(0)+R_{23}(1)+R_{23}(8)-R_{23}(9)-P(0)]+uR_{23}(6)/t=0. \tag{10.80}$$

$$s[R_{23}(2)+R_{23}(10)+P(0)]-t[R_{23}(1)+R_{23}(2)-R_{23}(3)+P(0)] + u[R_{23}(1)+R_{23}(7)+R_{23}(9)+R_{23}(10)+3P(0)]=0. \tag{10.81}$$

$$r[R_{23}(0)+R_{23}(3)+R_{23}(4)-R_{23}(10)]+u[R_{23}(4)+R_{23}(8)] - v[R_{23}(2)+R_{23}(3)+R_{23}(8)]=0. \tag{10.82}$$

$$-r[R_{23}(1)+R_{23}(3)+R_{23}(5)] + t[R_{23}(0)-R_{23}(3)-R_{23}(4)+R_{23}(5)-P(0)] - u[R_{23}(1)+R_{23}(5)-R_{23}(9)]=0. \tag{10.83}$$

$$t[-R_{23}(1)+R_{23}(4)+R_{23}(5)+2P(0)] + u[R_{23}(1)+R_{23}(2)-R_{23}(4)-R_{23}(10)]+tR_{23}(6)/s=0. \tag{10.84}$$

$$-r[R_{23}(3)-R_{23}(5)+R_{23}(7)+P(0)] - u[R_{23}(0)+R_{23}(2)-R_{23}(5)+R_{23}(7)]+R_{23}(6)=0. \tag{10.85}$$

$$\begin{aligned}(10.86)\quad & s[-R_{23}(3) + R_{23}(4) + R_{23}(7) - R_{23}(8) + P(0)] \\ & \qquad + u[R_{23}(1) + R_{23}(4) - R_{23}(8)] - R_{23}(6) - suP(0)/v = 0.\end{aligned}$$

$$\begin{aligned}(10.87)\quad & -r[R_{23}(5) + R_{23}(7) - P(0)] + s[-R_{23}(2) + R_{23}(5) + R_{23}(8) - P(0)] \\ & \qquad + t[R_{23}(2) + R_{23}(4) - R_{23}(7) + R_{23}(9) + 2P(0)].\end{aligned}$$

$$\begin{aligned}(10.88)\quad & r[R_{23}(5) + R_{23}(8) + R_{23}(10) + P(0)] \\ & \qquad - s[R_{23}(3) + R_{23}(5) - R_{23}(9) + R_{23}(10)] \\ & \qquad + rR_{23}(6)/v + 2stP(0)/u = 0.\end{aligned}$$

$$\begin{aligned}(10.89)\quad & r[R_{34}(7) - P(0)] + s[R_{34}(10) + P(0)] + t[R_{34}(0) - P(0)] \\ & \qquad + u[R_{34}(4) + P(0)] - vR_{34}(9) + R_{34}(6) = 0.\end{aligned}$$

$$\begin{aligned}(10.90)\quad & r[R_{34}(1) - R_{34}(7) - R_{34}(8) + R_{34}(10)] + s[R_{34}(0) - R_{34}(5) + R_{34}(8)] \\ & \qquad + u[R_{34}(1) - R_{34}(5) + R_{34}(8) - P(0)] = 0.\end{aligned}$$

$$\begin{aligned}(10.91)\quad & u[R_{34}(2) + R_{34}(8) - R_{34}(9) + 2P(0)] \\ & \qquad - v[R_{34}(0) + R_{34}(1) + R_{34}(8) - R_{34}(9) + 2P(0)] \\ & \qquad + uR_{34}(6)/t + 2rP(0) = 0.\end{aligned}$$

$$\begin{aligned}(10.92)\quad & s[R_{34}(2) + R_{34}(10)] - t[R_{34}(1) + R_{34}(2) - R_{34}(3)] \\ & \qquad + u[R_{34}(1) + R_{34}(7) + R_{34}(9) + R_{34}(10) - P(0)] = 0.\end{aligned}$$

$$\begin{aligned}(10.93)\quad & r[R_{34}(0) + R_{34}(3) + R_{34}(4) - R_{34}(10) - P(0)] + u[R_{34}(4) + R_{34}(8)] \\ & \qquad - v[R_{34}(2) + R_{34}(3) + R_{34}(8)] = 0.\end{aligned}$$

$$\begin{aligned}(10.94)\quad & -r[R_{34}(1) + R_{34}(3) + R_{34}(5) - P(0)] \\ & \qquad + t[R_{34}(0) - R_{34}(3) - R_{34}(4) + R_{34}(5) + 2P(0)] \\ & \qquad - u[R_{34}(1) + R_{34}(5) - R_{34}(9)] = 0.\end{aligned}$$

$$\begin{aligned}(10.95)\quad & t[-R_{34}(1) + R_{34}(4) + R_{34}(5)] \\ & \qquad + u[R_{34}(1) + R_{34}(2) - R_{34}(4) - R_{34}(10) - P(0)] \\ & \qquad + tR_{34}(6)/s + vP(0) = 0.\end{aligned}$$

$$\begin{aligned}(10.96)\quad & -r[R_{34}(3) - R_{34}(5) + R_{34}(7)] - u[R_{34}(0) + R_{34}(2) - R_{34}(5) + R_{34}(7)] \\ & \qquad + R_{34}(6) = 0.\end{aligned}$$

$$\begin{aligned}(10.97)\quad & s[-R_{34}(3) + R_{34}(4) + R_{34}(7) - R_{34}(8)] \\ & \qquad + u[R_{34}(1) + R_{34}(4) - R_{34}(8) + P(0)] - R_{34}(6) = 0.\end{aligned}$$

$$\begin{aligned}(10.98)\quad & -r[R_{34}(5) + R_{34}(7) + 2P(0)] + s[-R_{34}(2) + R_{34}(5) + R_{34}(8) + 2P(0)] \\ & \qquad + t[R_{34}(2) + R_{34}(4) - R_{34}(7) + R_{34}(9) - 2P(0)] = 0.\end{aligned}$$

$$\begin{aligned}(10.99)\quad & r[R_{34}(5) + R_{34}(8) + R_{34}(10) - P(0)] \\ & \qquad - s[R_{34}(3) + R_{34}(5) - R_{34}(9) + R_{34}(10) + P(0)] \\ & \qquad - rR_{34}(6)/v + tP(0) = 0.\end{aligned}$$

$$rR_{45}(7) + sR_{45}(10) + tR_{45}(0) + u[R_{45}(4) - P(0)] + v[-R_{45}(9) + P(0)] + R_{45}(6) = 0. \tag{10.100}$$

$$r[R_{45}(1) - R_{45}(7) - R_{45}(8) + R_{45}(10)] + s[R_{45}(0) - R_{45}(5) + R_{45}(8)] + u[R_{45}(1) - R_{45}(5) + R_{45}(8)] = 0. \tag{10.101}$$

$$u[R_{45}(2) + R_{45}(8) - R_{45}(9) - P(0)] - v[R_{45}(0) + R_{45}(1) + R_{45}(8) - R_{45}(9) - P(0)] + uR_{45}(6)/t - rP(0) = 0. \tag{10.102}$$

$$s[R_{45}(2) + R_{45}(10)] - t[R_{45}(1) + R_{45}(2) - R_{45}(3)] + u[R_{45}(1) + R_{45}(7) + R_{45}(9) + R_{45}(10)] = 0. \tag{10.103}$$

$$r[R_{45}(0) + R_{45}(3) + R_{45}(4) - R_{45}(10) + 2P(0)] + u[R_{45}(4) + R_{45}(8)] - v[R_{45}(2) + R_{45}(3) + R_{45}(8)] = 0. \tag{10.104}$$

$$-r[R_{45}(1) + R_{45}(3) + R_{45}(5) + 2P(0)] + t[R_{45}(0) - R_{45}(3) - R_{45}(4) + R_{45}(5) - P(0)] - u[R_{45}(1) + R_{45}(5) - R_{45}(9)] = 0. \tag{10.105}$$

$$t[-R_{45}(1) + R_{45}(4) + R_{45}(5) - 2P(0)] + u[R_{45}(1) + R_{45}(2) - R_{45}(4) - R_{45}(10) + 2P(0)] + tR_{45}(6)/s - 2vP(0) = 0. \tag{10.106}$$

$$r[R_{45}(3) - R_{45}(5) + R_{45}(7)] + u[R_{45}(0) + R_{45}(2) - R_{45}(5) + R_{45}(7)] - R_{45}(6) = 0. \tag{10.107}$$

$$s[-R_{45}(3) + R_{45}(4) + R_{45}(7) - R_{45}(8)] + u[R_{45}(1) + R_{45}(4) - R_{45}(8) - 2P(0)] - R_{45}(6) = 0. \tag{10.108}$$

$$-r[R_{45}(5) + R_{45}(7) - P(0)] + s[-R_{45}(2) + R_{45}(5) + R_{45}(8) - P(0)] + t[R_{45}(2) + R_{45}(4) - R_{45}(7) + R_{45}(9) + P(0)] = 0. \tag{10.109}$$

$$r[R_{45}(5) + R_{45}(8) + R_{45}(10) + P(0)] - s[R_{45}(3) + R_{45}(5) - R_{45}(9) + R_{45}(10)] + rR_{45}(6)/v = 0. \tag{10.110}$$

This completes the proof of Theorem 6.

11. We now write

$$f(z) = \prod (1 - z^r). \tag{11.1}$$

Thus

$$f(y) = P(0)P(1)P(2)P(3)P(4)P(5),$$
$$f(y^{11}) = P(0),$$
$$1/f(x) = \sum p(n)x^n.$$

Further we write

$$(11.2)\qquad \begin{aligned} a &= -\,x^{-4}P(2)/P(1), \quad b = x^{-5}P(4)/P(2), \quad c = x^{2}P(3)/P(4), \\ d &= -\,x^{-3}P(5)/P(3), \quad e = -\,x^{10}P(1)/P(5). \end{aligned}$$

It is easily verified that

$$(11.3)\qquad \begin{aligned} a^{11} &= -\,\alpha^{-5}\beta^{2}\gamma\delta^{-2}, \quad b^{11} = -\,\beta^{-5}\gamma^{2}\delta\epsilon^{-2}, \quad c^{11} = -\,\gamma^{-5}\delta^{2}\epsilon\alpha^{-2}, \\ d^{11} &= -\,\delta^{-5}\epsilon^{2}\alpha\beta^{-2}, \quad e^{11} = -\,\epsilon^{-5}\alpha^{2}\beta\gamma^{-2}. \end{aligned}$$

By Lemma 6 with $q=11$ we have

$$(11.4)\qquad x^{-5}f(x)/f(y^{11}) = a + b + c + d + e + 1.$$

In (11.4) we replace x by $\omega_r x$ where ω_r ($r=1$ to 11) are the eleventh roots of unity, and multiply together the eleven resulting equations, obtaining

$$(11.5)\qquad y^{-5}f^{12}(y)/f^{12}(y^{11}) = \prod_{r=1}^{11}(a\omega_r^{-4} + b\omega_r^{-5} + c\omega_r^{2} + d\omega_r^{-3} + e\omega_r^{10} + 1).$$

Now as ω_r runs through the eleventh roots of unity so does ω_r^4, so that the product on the right hand side of (11.5) is equal to

$$\prod_{r=1}^{11}(a\omega_r^{5} + b\omega_r^{2} + c\omega_r^{-3} + d\omega_r^{10} + e\omega_r^{4} + 1),$$

and is thus unchanged if a, b, c, d, and e, are interchanged cyclically. The product is thus a linear combination of terms $\sum a^l b^m c^n d^p e^q$, $\sum$ denoting as in §8 a sum of five terms obtained by permuting the typical term cyclically, and considering the left hand side of (11.5) such terms as occur can only involve x in terms of $y=x^{11}$. Thus if $a^l b^m c^n d^p e^q$ occurs we must have

$$(11.6)\qquad -4l - 5m + 2n - 3p + 10q \equiv 0 \pmod{11}.$$

Hence

$$\begin{aligned} -(a^l b^m c^n d^p e^q)^{11} \\ = (\alpha\beta\gamma\delta\epsilon)^{l-5m+4n-4p+8q}\alpha^{-5l-2n+p+2q}\beta^{-5m-2p+q+2l}\gamma^{-5n-2q+l+2m}\delta^{-5p-2l+m+2n} \\ \cdot\epsilon^{-5q-2m+n+2p}, \end{aligned}$$

where the indices of α, β, γ, δ, and ϵ, on the right hand side are multiples of 11 by (11.6). Thus every term occurring in the right hand side of (11.5) is of the form $\alpha^{l'}\beta^{m'}\gamma^{n'}\delta^{p'}\epsilon^{q'}$, where l', m', n', p', and q', are positive, negative or zero integers, and such terms occur in symmetrical sets of five terms each. Hence by Lemma 9 we have

$$y^{-5}f^{12}(y)/f^{12}(y^{11}) = A + B\lambda + C\mu + D\lambda^2 + E\lambda\mu,$$

where A, B, C, D, and E, are constants, and comparing the coefficients of powers of y on each side it may be seen that

$$(11.7)\qquad y^{-5}f^{12}(y)/f^{12}(y^{11}) = \lambda\mu - 17\lambda^2 - 108\mu + 346\lambda - 131.$$

We may note here that

$$(11.8)\qquad a^{11} + b^{11} + c^{11} + d^{11} + e^{11} = \lambda\mu - 6\lambda^2 + 82\lambda - 9\mu - 297,$$

and that $y^{-5}f^{12}(y)/f^{12}(y^{11}) - \sum a^{11} - 1$ is divisible by 11.

We obtain finally the value of $\Phi(6)$ (cf. §4). $\Phi(6)$ is the coefficient of x^6 in $1/f(x)$ regarded as a polynomial of degree 10 in x with coefficients involving x in terms of $y = x^{11}$. Thus $y^{-4}f^{12}(y)\Phi(6)/f^{11}(y^{11})$ is the coefficient of x^0 in $y^{-5}f^{12}(y)/[f^{12}(y^{11})(a+b+c+d+e+1)]$. This is a cyclically symmetric polynomial of degree 10 in a, b, c, d, and e; and the terms which give the coefficient of x^0 occur only in symmetrical sets of five expressible as $\sum \alpha^{l'}\beta^{m'}\gamma^{n'}\delta^{p'}\epsilon^{q'}$, as before. (This is *not* true for the coefficient of any power of x other than x^0; the five terms of $\sum a^2b$, for example, do not appertain to the same power of x.) Equating coefficients in the power series expansions as before we obtain

$$(11.9)\qquad f^{12}(y)\Phi(6)/y^4f^{11}(y^{11}) = 11(\lambda - 9)^2 + 121(2 - \lambda - \mu).$$

NOTATION.

$r, s, t, u, v.$	Defined by (8.1)
$\alpha, \beta, \gamma, \delta, \epsilon.$	Defined by (8.9)
$\lambda, \mu.$	Defined by (8.10)
$R_{bc}(d).$	Defined in §9.
$D = P(0)/(rt+su+tv+ur+vs).$	Defined in §10.
$a, b, c, d, e.$	Defined by (11.2)
$f(z).$	Defined by (11.1)

$\sum$ is used in connection with $\alpha, \beta, \gamma, \delta, \epsilon$, and a, b, c, d, e, to denote a sum obtained by permuting the typical term cyclically.

All other notation is that of [7].

REFERENCES

7. A. O. L. Atkin and P. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. (3) vol. 4 (1953) pp. 84–106.

8. S. M. Hussain, *Evaluation of a determinant of eleventh order in the theory of partitions* (unpublished).

UNIVERSITY COLLEGE,
DURHAM, ENGLAND
UNIVERSITY OF THE PANJAB,
LAHORE, WEST PAKISTAN