LINEAR OPERATORS ON QUASI-CONTINUOUS FUNCTIONS

BY
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1. **Introduction.** In this paper we study a class of linear transformations for each of which the transform of a function is a function. This special class of transformations has such applications as the smoothing of experimental data, the prediction of outputs of a physical system for various inputs, and the estimation of the velocities and accelerations of an object from observations of its positions at various times.

By the statement that f is a function, we imply that if t is a real number, then $f(t)$ is a number. By the statement that f is quasi-continuous(1), we mean that f is a function such that if t is a real number then the limits $f(t-)$ and $f(t+)$ exist. Some lemmas on quasi-continuous functions appear in §2.

DEFINITION 1.1. The statement that T is a Q operator over the interval $[a, b]$ means that T is a transformation such that

(i) if y is quasi-continuous, then Ty is a function; if $g=Ty$ and s is a real number, then we denote the number $g(s)$ by $Ty(s)$,

(ii) if y_1 is quasi-continuous and y_2 is quasi-continuous, then $T(y_1+y_2) = Ty_1+Ty_2$,

(iii) if y is quasi-continuous and k is a number, then $T(ky)=k(Ty)$,

(iv) if y is quasi-continuous, c is a real number, and $z(t)=y(t+c)$ for each real number t, then $Tz(s)=Ty(s+c)$ for each real number s, and

(v) if s is a real number, then there is a positive number B_s such that if y is quasi-continuous and $M>|y(s-t)|$ for each number t in $[a, b]$, then $|Ty(s)| \leqq MB_s$; by the norm, $|T(s)|$, of T at s we mean the greatest lower bound of the set of all such numbers B_s.

It will be observed that if T is a Q operator over the interval $[a, b]$ and $T_0y=Ty(0)$ for each quasi-continuous function y, then T_0 is a bounded linear transformation from the set of all quasi-continuous functions to the set of all numbers (i.e., T_0 is a bounded linear functional operation as defined in [2] and [3]). We give the following example of a Q operator.

EXAMPLE 1.1. Suppose that if y is quasi-continuous and s is a real number, then

$$Ty(s) = [-y(s-2) - 3y(s-1) + 76y(s) + 76y(s+1) - 3y(s+2) - y(s+3)]/144.$$

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(1) Except for the use of the word "quasi-continuous," we use the terminology and notation of [1]. In particular, "integral" is defined as in [1].

It follows that T is a Q operator over the interval $[-3, 2]$ and that $|T(s)| = 10/9$ for each real number s. Moreover, if y is a polynomial of degree 3 or less, then $Ty(s) = \int_s^{s+1} y(t)dt$. This operator is designed for use with experimental data, for which the values used for y in the formula may include errors of observation. In effect, the operator smooths the raw data, interpolates, and gives the integral of the smoothed and interpolated data; it is derived from Jenkins' modified osculatory interpolation formula [4].

In §3 we show that if T is a Q operator over the interval $[a, b]$, then Ty is quasi-continuous, of bounded variation, or continuous, according as y is quasi-continuous, of bounded variation, or continuous. In §4 we show that if T is a Q operator and y is a quasi-continuous function then Ty is the sum of two integrals.

We adopt the following notation. If y is quasi-continuous, then y_L and y_R denote the functions such that $y_L(t) = y(t-)$ and $y_R(t) = y(t+)$ for each real number t.

DEFINITION 1.2. The statement that T is a Q_1 operator over the interval $[a, b]$ means that T is a Q operator over $[a, b]$ such that if y is quasi-continuous then $Ty(s+) - Ty(s-) = 2[Ty_R(s) - Ty_L(s)]$ for each real number s; i.e., if $Ty = x$, $T(y_L) = u$ and $T(y_R) = v$, then $x(s+) - x(s-) = 2[v(s) - u(s)]$ for each real number s.

In §5 we show that a Q_1 operator T is a Q operator such that if y is a quasi-continuous function then Ty is an integral. In §6 we find conditions sufficient to assure that a Q_1 operator T has various properties which may be desirable in applications. For example, we find a condition sufficient to assure that if y is quasi-continuous then Ty has a derivative, and we exhibit a Q_1 operator T' such that $T'y$ is the derivative of Ty. In §7 we give a family of Q_1 operators, one of which is a limit of the "most powerful" smoothing operators given in [4].

2. **Lemmas concerning quasi-continuous functions.** The following results will be used later in this paper.

LEMMA 2.1. *For the function f to be quasi-continuous, it is necessary and sufficient that if $[a, b]$ is an interval and $\epsilon > 0$ then there is a step-function s such that $|f(t) - s(t)| < \epsilon$ for each number t in $[a, b]$.*

For a proof, see Lemma 4.1b of [1]; see [5] also.

LEMMA 2.2. *For the function f to be quasi-continuous it is necessary and sufficient that if $[a, b]$ is an interval and $\epsilon > 0$ then there is a subdivision $t_0, t_1, \cdots, t_n$ of $[a, b]$ such that if p and q are in one of the segments (t_i, t_{i+1}) then $|f(p) - f(q)| < \epsilon$.*

Proof is omitted.

LEMMA 2.3. *If $f_1, f_2, f_3, \cdots$ is a sequence of quasi-continuous functions which converges uniformly to a function f, then f is quasi-continuous. Moreover,* *if $f_n(t+)=f_n(t)$ for each positive integer n and each real number t, then $f_R=f$; likewise, if $f_n(t-)=f_n(t)$ for each positive integer n and each real number t, then $f_L=f$.*

Proof is omitted.

LEMMA 2.4. *If $f_1, f_2, \cdots,$ is a sequence of functions which converges uniformly to a function f, and $[a, b]$ is an interval, and $V>0$, and $V_a^b(f_n)\leqq V$ for each positive integer n, then $V_a^b(f)\leqq V$.*

For a proof, see Lemma 4.2a of [1].

LEMMA 2.5. *If f is a quasi-continuous function, then there are a quasi-continuous function g and a quasi-continuous function h such that*
(i) *$g_L=g$ and $h_R=h$,*
(ii) *$g+h=f$, and*
(iii) *if $[a, b]$ is an interval and $|f(t)|\leqq M<M_1$ for each number t in $[a, b]$, then $|g(t)|\leqq 1.5M_1$ and $|h(t)|\leqq 1.5M_1$ for each number t in $[a, b]$.*

Proof. We introduce the following notation. Suppose that $[a, b]$ is an interval, z is quasi-continuous, and D is a subdivision $t_0, t_1, \cdots, t_m$ of $[a, b]$. Then L_Dz and R_Dz denote the pair of functions such that $2L_Dz(a)=z(a)$, $2R_Dz(b)=z(b)$, and for $i=0, 1, \cdots, m-1$,

$$2(t_{i+1}-t_i)L_Dz(t) = (t_{i+1}-t)[2z(t_i+)-z(t_i)]+(t-t_i)z(t_{i+1}) \text{ if } t_i<t\leqq t_{i+1},$$

and

$$2(t_{i+1}-t_i)R_Dz(t) = (t_{i+1}-t)z(t_i)+(t-t_i)[2z(t_{i+1}-)-z(t_{i+1})] \text{ if } t_i\leqq t<t_{i+1}.$$

We make the following three observations. First, if $|z(t)|\leqq M$ for each number t in $[a, b]$, then $|L_Dz(t)|\leqq 1.5M$ and $|R_Dz(t)|\leqq 1.5M$ for $a\leqq t\leqq b$. Second, if $\epsilon>0$ and it is true that $|z(p)-z(q)|\leqq\epsilon$ if p and q are in one of the segments (t_i, t_{i+1}), then $|z(t)-L_Dz(t)-R_Dz(t)|\leqq\epsilon$ for $a\leqq t\leqq b$. Finally, $L_Dz(t-)=L_Dz(t)$ if $a<t\leqq b$, and $R_Dz(t+)=R_Dz(t)$ if $a\leqq t<b$.

Now suppose that $[a, b]$ is an interval and $|f(t)|\leqq M<M_1$ if $a\leqq t\leqq b$. Let ϵ denote a positive number less than M_1-M. We now construct a sequence $f_1, f_2, f_3, \cdots$ of functions and a sequence $D_1, D_2, D_3, \cdots$ of subdivisions of $[a, b]$ in the following manner. Let f_1 denote the function f, and let D_1 denote a subdivision $t_0, t_1, \cdots, t_m$ of $[a, b]$ such that if p and q are in one of the segments (t_i, t_{i+1}) then $|f_1(p)-f_1(q)|<\epsilon/2$. For each positive integer n, let $f_{n+1}=f_n-L_{D_n}f_n-R_{D_n}f_n$, where D_n denotes a subdivision $t_0, t_1, \cdots, t_m$ of $[a, b]$ such that if p and q are in one of the segments (t_i, t_{i+1}) then $|f_n(p)-f_n(q)|\leqq\epsilon/2^n$. Suppose that $a\leqq t\leqq b$. Then $|f_1(t)|\leqq M$, and therefore $|L_{D_1}f_1(t)|\leqq 1.5M$ and $|R_{D_1}f_1(t)|\leqq 1.5M$; moreover, $|f_2(t)|=|f_1(t)-L_{D_1}f_1(t)-R_{D_1}f_1(t)|\leqq\epsilon/2$. By induction, if n is an integer greater than 1, and $a\leqq t\leqq b$, then $|L_{D_n}f_n(t)|\leqq 1.5\epsilon/2^{n-1}$, $|R_{D_n}f_n(t)|\leqq 1.5\epsilon/2^{n-1}$, and

$$\left| f_1(t) - \sum_{p=1}^{n} L_{D_p} f_p(t) - \sum_{p=1}^{n} R_{D_p} f_p(t) \right|$$
$$= \left| f_{n+1}(t) \right| = \left| f_n(t) - L_{D_n} f_n(t) - R_{D_n} f_n(t) \right| \leqq \epsilon/2^n.$$

But $\sum_{p=1}^{\infty} L_{D_p} f_p$ converges uniformly in $[a, b]$ to a function g; and if $a \leqq t \leqq b$, then $|g(t)| \leqq \sum_{p=1}^{\infty} |L_{D_p} f_p(t)| \leqq 1.5(M+\epsilon) < 1.5M_1$. Similarly, $\sum_{p=1}^{\infty} R_{D_p} f_p$ converges uniformly in $[a, b]$ to a function h, and if $a \leqq t \leqq b$ then $|h(t)| < 1.5M_1$. Moreover, if $a \leqq t \leqq b$ then $g(t)+h(t)=f(t)$. From Lemma 2.3, it follows that if $a < t \leqq b$ then $h(t-)$ exists and $g(t-)=g(t)$, and that if $a \leqq t < b$ then $g(t+)$ exists and $h(t+)=h(t)$. This completes the proof.

LEMMA 2.6. *Suppose that $[t_0, t_1]$ is an interval, $\epsilon > 0$, and f is a function such that*

(i) *if $t_0 \leqq t \leqq t_1$, then the derivative $f'(t)$ exists, and*

(ii) *if s_1 and s_2 are in $[t_0, t_1]$, then $|f'(s_1) - f'(s_2)| < \epsilon$. Then*

$$|(f(t_1) - f(t_0))/(t_1 - t_0) - f'(t)| < \epsilon(2)^{1/2}$$

for each number t in $[t_0, t_1]$.

Proof is omitted, since this lemma can be obtained by applying the theorem of the mean to the real part and the imaginary part of f.

LEMMA 2.7. *If f has a quasi-continuous derivative f', then f' is continuous.*

Proof is omitted, since the lemma follows readily from well-known results and can be derived from Lemma 2.6.

LEMMA 2.8. *Suppose that $[a, b]$ is an interval, y is a function which is bounded in $[a, b]$, and x is a function whose derivative, x', is continuous in $[a, b]$. If $\int_a^b y(t)dx(t) = I$ or $\int_a^b y(t)x'(t)dt = I$, then $\int_a^b y(t)dx(t) = \int_a^b y(t)x'(t)dt$.*

Proof is omitted, since this lemma follows with little difficulty from Lemma 2.6.

LEMMA 2.9. *Suppose that f is a function whose derivative, f', is of bounded variation in the interval $[a, b]$, h is a real number other than zero, and $[c, d]$ is an interval such that if t is in $[c, d]$ then t and $t+h$ are in $[a, b]$. If $g(t) = [f(t+h) - f(t)]/h$ for each number t in $[c, d]$, then g is of bounded variation in $[c, d]$, and $V_c^d(g) \leqq V_a^b(f')$.*

Proof. Suppose that $t_0, t_1, \cdots, t_n$ is a subdivision of $[c, d]$, and let S denote the sum $\sum_{i=0}^{n-1} |g(t_{i+1}) - g(t_i)|$. Now

$$h[g(t_{i+1}) - g(t_i)] = \int_0^h 1 df(t_{i+1} + t) - \int_0^h 1 df(t_i + t)$$
$$= \int_0^h [f'(t_{i+1} + t) - f'(t_i + t)]dt.$$

It follows that if $h>0$, then

$$h\,|\,g(t_{i+1}) - g(t_i)\,| \leqq \int_0^h |\,f'(t_{i+1}+t) - f'(t_i+t)\,|\,dt$$

and

$$hS \leqq \int_0^h \sum_{i=0}^{n-1} |\,f'(t_{i+1}+t) - f'(t_i+t)\,|\,dt \leqq \int_0^h V_a^b(f')dt = hV_a^b(f'),$$

so that $S \leqq V_a^b(f')$. By a similar argument, if $h<0$, then $S \leqq V_a^b(f')$. Hence g is of bounded variation in $[c, d]$, and $V_c^d(g) \leqq V_a^b(f')$. This completes the proof.

3. **Some properties of Q operators.** In this section we suppose that there are given an interval $[a, b]$ and a Q operator T over $[a, b]$.

LEMMA 3.1a. *Suppose that s is a real number and y and z are quasi-continuous functions such that $y(s-t)=z(s-t)$ if $a \leqq t \leqq b$. Then $Ty(s)=Tz(s)$.*

Proof. Let $w=y-z$. By (ii) and (iii) of Definition 1.1, $Ty(s)-Tz(s) = Tw(s)$. But $w(s-t)=0$ if $a \leqq t \leqq b$; and by (v) of Definition 1.1, if $\epsilon>0$ then $|Tw(s)| \leqq \epsilon|T(s)|$; hence $Tw(s)=0$, or $Ty(s)=Tz(s)$. This completes the proof.

THEOREM 3.1. *If s is a real number, then $|T(s)| = |T(0)|$.*

Proof. Suppose that s is a real number. Now if y is quasi-continuous and $z(t)=y(t-s)$ for each real number t, then $z(s-t)=y(-t)$ for each number t in $[a, b]$; by Lemma 3.1a and (iv) of Definition 1.1, $Tz(s)=Ty(0)$. It follows from (v) of Definition 1.1 that $|T(s)| = |T(0)|$. This completes the proof.

REMARK. In view of Theorem 3.1, we shall hereafter refer to the norm of T as $|T|$; i.e., if s is a real number, then $|T| = |T(s)|$.

THEOREM 3.2. *If $a<c<b$, then there are a Q operator T_1 over $[a, c]$ and a Q operator T_2 over $[c, b]$ such that*

(i) *if y is quasi-continuous then $Ty=T_1y+T_2y$, and*

(ii) $|T_1|+|T_2|=|T|$.

Proof. If y is quasi-continuous and s is a real number, then we define numbers $T_1y(s)$ and $T_2y(s)$ in the following manner. Let $u(s-t)=0$ if $t>c$ and $u(s-t)=y(s-t)$ if $t \leqq c$ and let $T_1y(s)=Tu(s)$. Let $v(s-t)=y(s-t)$ if $t>c$ and $v(s-t)=0$ if $t \leqq c$; and let $T_2y(s)=Tv(s)$. By Definition 1.1, T_1 is a Q operator over $[a, c]$, and T_2 is a Q operator over $[c, b]$; moreover, if y is quasi-continuous, then $T_1y+T_2y=Ty$.

We now show that $|T_1|+|T_2| \geqq |T|$. Suppose that $\epsilon>0$, s is a real number and y is a quasi-continuous function such that $|Ty(s)|>|T|-\epsilon$ and $|y(s-t)| \leqq 1$ for each number t in $[a, b]$. Now $|T_1|+|T_2| \geqq |T_1y(s)| + |T_2y(s)| \geqq |T_1y(s)+T_2y(s)| = |Ty(s)| > |T|-\epsilon$. Hence $|T_1| = T_1 + |T_2| \geqq |T|$.

Finally we show that $|T_1|+|T_2| \leqq |T|$. Suppose that $\epsilon>0$, s is a real number, and z is a quasi-continuous function such that $T_1z(s)=|T_1z(s)| > |T_1|-\epsilon$, $T_2z(s)=|T_2z(s)|>|T_2|-\epsilon$, and $|z(s-t)|\leqq 1$ if t is in $[a, b]$. Now $|T| \geqq |Tz(s)| = |T_1z(s)+T_2z(s)| = T_1z(s)+T_2z(s) > |T_1|+|T_2|-2\epsilon$. Hence $|T| \geqq |T_1|+|T_2|$. This completes the proof.

Lemma 3.3a. *There is a number sequence c_1, c_2, c_3 such that if s is a real number and y is a step-function such that $y(s-p)=y(s-q)$ for each pair p, q of numbers between a and b, then $Ty(s)=c_1y(s-b)+c_2y(s-b+)+c_3y(s-a)$; moreover, $|T| \geqq |c_1|+|c_2|+|c_3|$.*

Proof. Let functions u, v, w be defined as follows:

$$u(-t) = 1 \text{ if } t \geqq b \text{ and } u(-t) = 0 \text{ if } t < b;$$

$$v(-t) = 1 \text{ if } a < t < b \text{ and } v(-t) = 0 \text{ if } t \geqq b \text{ or } t \leqq a; \text{ and}$$

$$w(-t) = 1 \text{ if } t \leqq a \text{ and } w(-t) = 0 \text{ if } t > a.$$

Let $c_1=Tu(0)$, $c_2=Tv(0)$, and $c_3=Tw(0)$. Now if s is a real number, $a\leqq t\leqq b$, and y is a step-function such that $y(s-p)=y(s-q)$ for each pair p, q of numbers between a and b, then $y(s-t)=y(s-b)u(-t)+y(s-b+)v(-t)+y(s-a)w(-t)$, and therefore $Ty(s)=c_1y(s-b)+c_2y(s-b+)+c_3y(s-a)$.

Now let d_1, d_2, d_3 denote a number sequence such that $|d_1|=|d_2|=|d_3|=1$ and $c_1d_1\geqq 0$, $c_2d_2\geqq 0$, and $c_3d_3\geqq 0$. For each real number t, let $z(t)=d_1u(t)+d_2v(t)+d_3w(t)$. Now if $a\leqq t\leqq b$, then $|z(-t)|=1$, so that $|Tz(0)|\leqq|T|$; but $Tz(0)=c_1d_1+c_2d_2+c_3d_3=|c_1|+|c_2|+|c_3|$; so $|T|\geqq|c_1|+|c_2|+|c_3|$. This completes the proof.

Lemma 3.3b. *Suppose that $t_0, t_1, \cdots, t_{2n}$ is a subdivision of $[a, b]$. There is a number sequence $c_0, c_1, \cdots, c_{2n}$ such that if s is a real number and y is a step-function such that $y(s-p)=y(s-q)$ for each pair p, q of numbers between t_{2i} and t_{2i+2}, $i=0, 1, \cdots, n-1$, then $Ty(s)=\sum_{i=0}^{2n} c_iy(s-t_i)$; moreover, $|T| \geqq \sum_{i=0}^{2n}|c_i|$.*

Proof is omitted, since this lemma follows from Theorem 3.2 and Lemma 3.3a.

Lemma 3.3c. *If y is a step-function and $[c, d]$ is an interval, then Ty is of bounded variation in $[c, d]$, and $V_c^d(Ty)\leqq|T|\cdot V_{c-b}^{d-a}(y)$.*

Proof. Let $s_0, s_1, \cdots, s_m$ denote a subdivision of $[c, d]$. Let $t_0, t_1, \cdots, t_{2n}$ denote a subdivision of $[a, b]$ such that if j is one of the integers $0, 1, \cdots, m$, and i is one of the integers $0, 1, \cdots, n-1$, and p and q are numbers between t_{2i} and t_{2i+2}, then $y(s_j-p)=y(s_j-q)$. By Lemma 3.3b, there is a number sequence $c_0, c_1, \cdots, c_{2n}$ such that $Ty(s_j)=\sum_{i=0}^{2n} c_iy(s_j-t_i)$ for $j=0, 1, \cdots, m$, and $|T|\geqq\sum_{i=0}^{2n}|c_i|$. Now if j is one of the integers $0, 1, \cdots, m-1$, then $Ty(s_j+1)-Ty(s_j)=\sum_{i=0}^{2n} c_i[y(s_{j+1}-t_i)-y(s_j-t_i)]$. Hence

$$\begin{aligned}\sum_{j=0}^{m-1} \left| Ty(s_{j+1}) - Ty(s_j) \right| &= \sum_{j=0}^{m-1} \left| \sum_{i=0}^{2n} c_i[y(s_{j+1} - t_i) - y(s_j - t_i)] \right| \\ &\leqq \sum_{j=0}^{m-1} \sum_{i=0}^{2n} \left| c_i \right| \cdot \left| y(s_{j+1} - t_i) - y(s_j - t_i) \right| \\ &\leqq \sum_{i=0}^{2n} \left| c_i \right| \cdot V_{c-t_i}^{d-t_i}(y) \\ &\leqq \left| T \right| \cdot V_{c-b}^{d-a}(y).\end{aligned}$$

The lemma now follows at once.

THEOREM 3.3. *If y is quasi-continuous, then Ty is quasi-continuous; if y is of bounded variation, then Ty is of bounded variation; if y is continuous, then Ty is continuous.*

Proof. Suppose first that y is quasi-continuous. Let $y_1, y_2, y_3, \cdots$ denote a sequence of step-functions which converges uniformly to y. By (v) of Definition 1.1, $Ty_1, Ty_2, Ty_3, \cdots$ converges uniformly to Ty. Now if $[c, d]$ is an interval and n is a positive integer, then by Lemma 3.3c, Ty_n is of bounded variation in $[c, d]$. Hence each of the functions Ty_i is quasi-continuous; by Lemma 2.3, Ty is quasi-continuous.

Suppose now that y is of bounded variation, and that $[c, d]$ is an interval, and $V \geqq V_{c-b}^{d-a}(y)$. Let $y_1, y_2, y_3, \cdots$ denote a sequence of step-functions converging uniformly to y such that $V_{c-b}^{d-a}(y_n) \leqq V$, $n=1, 2, 3, \cdots$. Then $Ty_1, Ty_2, Ty_3, \cdots$ is a sequence of functions converging to Ty uniformly, and $V_c^d(Ty_n) \leqq |T| V$ for $n=1, 2, 3, \cdots$. By Lemma 2.4, $V_c^d(Ty) \leqq |T| V$; hence Ty is of bounded variation.

Suppose, finally, that y is continuous and that $[c, d]$ is an interval. Let ϵ denote a positive number, and let δ denote a positive number such that if $c-b \leqq p < q \leqq d-a$ and $q-p<\delta$, then $|y(p)-y(q)| < \epsilon$. Now suppose that $c \leqq s_1 < s_2 \leqq d$ and $s_2 - s_1 < \delta$. If $a \leqq t \leqq b$, then $|y(s_1-t)-y(s_2-t)| < \epsilon$; so $|Ty(s_1)-Ty(s_2)| < \epsilon |T|$. Hence Ty is continuous. This completes the proof.

COROLLARY 3.3a. *If T_1 is a Q operator over the interval $[a_1, b_1]$ and T_2 is a Q operator over the interval $[a_2, b_2]$, and $T_3 y = T_1(T_2 y)$ for each quasi-continuous function y, then T_3 is a Q operator over the interval $[a_1+a_2, b_1+b_2]$, and $|T_3| \leqq |T_1| \cdot |T_2|$.*

Proof. By Theorem 3.3, if y is quasi-continuous, then so is $T_2 y$; therefore $T_3 y$ is a function. It can readily be verified that T_3 has the properties listed as (ii), (iii), and (iv) in Definition 1.1. Let s denote a real number, suppose that y is quasi-continuous, and that $M \geqq |y(s-t)|$ if $a_1+a_2 \leqq t \leqq b_1+b_2$. Now if $a_1 \leqq s_1 \leqq b_1$ and $a_2 \leqq t \leqq b_2$, then $a_1+a_2 \leqq s_1+t \leqq b_1+b_2$; hence $|y(s-s_1-t)| \leqq M$ if $a_1 \leqq s_1 \leqq b_1$ and $a_2 \leqq t \leqq b_2$, so that $|T_2 y(s-s_1)| \leqq |T_2| M$ if $a_1 \leqq s_1 \leqq b_1$, and therefore $|T_3 y(s)| \leqq |T_1| \cdot |T_2| M$. Hence T_3 is a Q operator over $[a_1+a_2,$

$b_1+b_2]$, and $|T_3| \leqq |T_1| \cdot |T_2|$. This completes the proof.

EXAMPLE 3.1. We give an example of Corollary 3.3a for which $|T_3| < |T_1| \cdot |T_2|$. Suppose that if y is quasi-continuous and s is a real number, then $T_1y(s) = [y(s-)+y(s)]/2$ and $T_2y(s) = [y(s)-y(s+)]/2$. Then $T_3y(s) = [y(s)-y(s+)]/4$. Hence $|T_1| = 1$, $|T_2| = 1$, and $|T_3| = 1/2$.

4. **Integral representation of Q operators.** In this section we suppose that there are given an interval $[a, b]$ and a Q operator over $[a, b]$. We introduce the following notation:

$$J(t) = 0 \text{ if } t < 0 \text{ and } J(t) = 1 \text{ if } t \geqq 0,$$

$$L(s) = TJ_L(s) \text{ and } R(s) = TJ_R(s) = TJ(s) \text{ for each real number } s,$$

$$x_1(t) = 2L(t) - L(t-) \text{ and } x_2(t) = 2R(t) - R(t+) \text{ for each real number } t.$$

We note in passing that $L(t) = [x_1(t-)+x_1(t)]/2$ and $R(t) = [x_2(t)+x_2(t+)]/2$.

LEMMA 4.1a. *$R(s) = L(s) = L(a) = 0$ if $s<a$; and $R(s) = L(s) = R(b)$ if $s>b$. Moreover, $R(s-) = L(s-)$ and $R(s+) = L(s+)$ for each real number s.*

Proof. If $s<a$ and $a \leqq t \leqq b$, then $s-t<0$, so that $J_L(s-t) = J_R(s-t) = J_L(a-t) = 0$, and therefore $L(s) = R(s) = L(a) = 0$. If $s>b$ and $a \leqq t \leqq b$, then $s-t>0$, so that $J_L(s-t) = J_R(s-t) = J_R(b-t) = 1$, and therefore $L(s) = R(s) = R(b)$.

Suppose now that s is a real number. Let $s_1, s_2, s_3, \cdots$ denote a decreasing sequence which converges to s. Let $a_1, a_2, a_3, \cdots$ denote a number sequence such that if p is a positive integer then $|a_p| = 1$ and $a_p[R(s_p)-L(s_p)] \geqq 0$. For each positive integer n, let f_n denote the step-function such that if t is a real number then $f_n(t) = \sum_{p=1}^{n} a_p[J_R(s_p+t) - J_L(s_p+t)]$. Then $Tf_n(0) = \sum_{p=1}^{n} |R(s_p)-L(s_p)|$. But if $a \leqq t \leqq b$, then $|f_n(-t)| \leqq 1$, whence $|Tf_n(0)| \leqq |T|$. Hence $\sum_{p=1}^{n} |R(s_p)-L(s_p)|$ converges, and therefore $|R(s_p)-L(s_p)| \to 0$ as $p \to +\infty$. It follows that $R(s+) = L(s+)$. By a similar argument, if s is a real number, then $R(s-) = L(s-)$. This completes the proof.

LEMMA 4.1b. *If s is a real number, then*

$$L(s) = \int_a^{b+} J_L(s-t)dx_1(t)$$

and

$$R(s) = \int_{a-}^{b} J_R(s-t)dx_2(t).$$

Proof. By Lemma 4.1a, $x_1(t) = x_2(t) = x_1(a) = 0$ if $t<a$; and $x_1(t) = x_2(t) = x_2(b)$ if $t>b$. Moreover, by Theorem 3.3, R and L are of bounded variation, and therefore x_1 and x_2 are of bounded variation. Suppose that s is a real number and p is a positive number such that $p>s-b$ and $p>a-s$. Then $\int_a^{b+} J_L(s-t)dx_1(t) = \int_{a-p}^{b+p} J_L(s-t)dx_1(t)$. By the integration by parts formula,

$$\int_{a-p}^{b+p} J_L(s-t)dx_1(t) = J_L(s-b-p)x_1(b+p) - J_L(s-a+p)x_1(a-p) - \int_{a-p}^{b+p} x_1(t)dJ_L(s-t),$$

or

$$\int_{a}^{b+} J_L(s-t)dx_1(t) = [x_1(s-) + x_1(s)]/2 = L(s).$$

By a similar argument,

$$\int_{a-}^{b} J_R(s-t)dx_2(t) = [x_2(s) + x_2(s+)]/2 = R(s).$$

THEOREM 4.1. *Suppose that y is quasi-continuous and s is a real number. Let g and h denote quasi-continuous functions such that $g_L=g$, $h_R=h$, and $g+h=y$. Then*

$$Ty(s) = \int_{a}^{b+} g(s-t)dx_1(t) + \int_{a-}^{b} h(s-t)dx_2(t).$$

Proof. If g is a step-function, it follows from Lemma 4.1b that $Tg(s)=\int_a^{b+}g(s-t)dx_1(t)$; if g is not a step-function, it follows from Lemmas 2.3 and 4.1b of the present paper and Lemma 4.1a of [1] that $Tg(s)=\int_a^{b+}g(s-t)dx_1(t)$. Similarly, $Th(s)=\int_{a-}^{b}h(s-t)dx_2(t)$. The theorem now follows from (ii) of Definition 1.1.

REMARK 4.1. Upon comparing Theorem 4.1 with (v) of Definition 1.1, one might suppose that $x_1(b+)=x_1(b)$ and $x_2(a-)=x_2(a)$. To show that this need not be so, we give the following example. For each quasi-continuous function y and real number s, let $Ty(s)=[y(s)+y(s-1)]/2$, so that T is a Q operator over $[0, 1]$. For this example,

$$x_1(t) = \begin{cases} 0 & \text{if } t \leqq 0, \\ 1/2 & \text{if } 0 < t \leqq 1, \\ 1 & \text{if } t > 1. \end{cases} \qquad x_2(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1/2 & \text{if } 0 \leqq t < 1, \\ 1 & \text{if } t \geqq 1. \end{cases}$$

THEOREM 4.2. *Suppose that $[c, d]$ is an interval and u_1 and u_2 are functions of bounded variation such that*

(i) $u_1(t)=u_2(t)=u_1(c)=0$ *if* $t<c$,

(ii) $u_1(t)=u_2(t)=u_2(d)$ *if* $t>d$, *and*

(iii) $u_1(t-)=u_2(t-)$ *and* $u_1(t+)=u_2(t+)$ *for each real number* t.

If s is a real number and g and h are quasi-continuous functions such that $g_L=g$ and $h_R=h$, let $Ug(s)$, $Uh(s)$, and $Uy(s)$, where $y=g+h$, denote the numbers $\int_c^{d+}g(s-t)du_1(t)$, $\int_{c-}^{d}h(s-t)du_2(t)$, and $Ug(s)+Uh(s)$, respectively. Then U is a

Q operator over $[c, d]$, *and* $|U| \leqq 1.5[V_c^{d+}(u_1)+V_{c-}^d(u_2)]$.

Proof. We show first that if y is quasi-continuous, then there is just one function which is Uy. Suppose that y is quasi-continuous and that g, h, p, q is a sequence of quasi-continuous functions such that $g_L=g$, $h_R=h$, $p_L=p$, $q_R=q$, $g+h=y$, and $p+q=y$. Suppose that $f=g-p$; then $f_L=f$. But $f=q-h$, and therefore $f_R=f$; so f is continuous. Now if s is a real number, then $\int_{-\infty}^{+\infty}f(s-t)du_1(t)-\int_{-\infty}^{+\infty}f(s-t)du_2(t)=\int_{-\infty}^{+\infty}f(s-t)d[u_1(t)-u_2(t)]$. Since f is continuous, u_1-u_2 is of bounded variation, and u_1-u_2 is zero except for a countable subset of $[c, d]$, it follows that $\int_{-\infty}^{+\infty}f(s-t)d[u_1(t)-u_2(t)]=0$, so that $\int_c^{d+}f(s-t)du_1(t)=\int_{c-}^d f(s-t)du_2(t)$, or $\int_c^{d+}g(s-t)du_1(t)-\int_c^{d+}p(s-t)du_1(t)=\int_{c-}^d q(s-t)du_2(t)-\int_{c-}^d h(s-t)du_2(t)$, or $Ug(s)-Up(s)=Uq(s)-Uh(s)$. Hence $Ug(s)+Uh(s)=Up(s)+Uq(s)$; i.e., there is just one function which is Uy. Moreover, if $|y(s-t)| \leqq M<M_1$ for each number t in $[c, d]$, then by Lemma 2.5 the functions p and q may be chosen so that $|p(s-t)| \leqq 1.5M_1$ and $|q(s-t)| \leqq 1.5M_1$ for each number t in $[c, d]$; so by Theorem 2.1 of [1], $|Uy(s)| \leqq 1.5M_1[V_c^{d+}(u_1)+V_{c-}^d(u_2)]$. The theorem now follows from Definition 1.1 and the properties of integrals (cf. Theorem 2.1 of [1]).

REMARK 4.2. Compare Theorem 4.1 with the representation given in [2]; for an expression for the norm $|T|$, see [6].

5. Q_1 **operators.** In this section, we show that if T is a Q_1 operator and y is a quasi-continuous function, then Ty is an integral; and we obtain a condition sufficient for the product of two Q_1 operators to be a Q_1 operator.

THEOREM 5.1. *For T to be a Q_1 operator over the interval $[a, b]$ it is necessary and sufficient that there is a function x such that*

(i) *x is of bounded variation, $x(t)=0$ if $t \leqq a$, $x(t)=x(b)$ if $t>b$, and*

(ii) *if y is quasi-continuous and s is a real number, then*

$$Ty(s)=\int_b^a y(s-t)\,dx(t).$$

Proof. A. Suppose that T is a Q_1 operator over $[a, b]$, and let R, L, x_1, and x_2 be defined as in §4. By Definition 1.2, $R(s+)-R(s-)=2[R(s)-L(s)]$ for each real number s; and by Lemma 4.1a, $R(s-)=L(s-)$, so that $2R(s)-R(s+)=2L(s)-L(s-)$, or $x_2(s)=x_1(s)$ for each real number s. It follows from Theorem 3.3 that x_1 is of bounded variation, from Lemma 4.1a that $x_1(t)=0$ if $t \leqq a$ and $x_1(t)=x_1(b)$ if $t>b$, and from Theorem 4.1 that $Ty(s)=\int_a^b y(s-t)dx_1(t)$ for each quasi-continuous function y and real number s.

B. Suppose that x is a function such that (i) and (ii) of the theorem are true. By Theorem 4.2, T is a Q operator over $[a, b]$. Let L and R be defined as in §4. Now if s is a real number, then $R(s)=[x(s)+x(s+)]/2$ and $L(s)=[x(s-)+x(s)]/2$, whence $2R(s)-R(s+)=2L(s)-L(s-)=x(s)$; since $R(s+)=L(s+)$ and $R(s-)=L(s-)$, it follows that $R(s+)-R(s-)=2[R(s)-L(s)]$ and $L(s+)-L(s-)=2[R(s)-L(s)]$. Now if y is a step-function and s is a real number, then there are a number sequence $a_1, a_2, \cdots, a_{2n}$ and a real-number sequence $t_1, t_2, \cdots, t_n$ such that

$$y(s-t) = \sum_{p=1}^{n} [a_{2p-1}J_R(s-t_p-t) + a_{2p}J_L(s-t_p-t)] \text{ if } a \leqq t \leqq b;$$

hence $Ty(s) = \sum_{p=1}^{n} [a_{2_p-1}R(s-t_p) + a_{2_p}L(s-t_p)]$, so that $Ty(s+) - Ty(s-) = 2[Ty_R(s) - Ty_L(s)]$. It now follows from Lemmas 2.1, 3.3c, and 2.3 that if y is quasi-continuous and s is a real number, then $Ty(s+) - Ty(s-) = 2[Ty_R(s) - Ty_L(s)]$. Hence T is a Q_1 operator over $[a, b]$. This completes the proof.

REMARK 5.1. If T is a Q_1 operator over an interval $[a, b]$ then there is just one function x such that

(i) $x(a) = 0$ and

(ii) if y is quasi-continuous and s is a real number, then $Ty(s) = \int_{-\infty}^{+\infty} y(s-t)dx(t)$. Moreover, x is of bounded variation.

THEOREM 5.2. *Suppose that U is a Q_1 operator over the interval $[a, b]$, V is a Q_1 operator over the interval $[c, d]$, and $Ty = U(Vy)$ for each quasi-continuous function y. Let u and v denote the functions such that $u(a) = v(c) = 0$ and $Uy(s) = \int_{-\infty}^{+\infty} y(s-t)du(t)$ and $Vy(s) = \int_{-\infty}^{+\infty} y(s-t)dv(t)$ if y is quasi-continuous and s is a real number. If $u_L = u_R$ or $v_L = v_R$, then T is a Q_1 operator over $[a+c, b+d]$.*

Proof. By Corollary 3.3a, T is a Q operator over $[a+c, b+d]$. Suppose that $u_L = u_R$ or $v_L = v_R$. Let R and L denote the functions TJ_R and TJ_L, respectively. Now $VJ_L = (v_L + v)/2$ and $VJ_R = (v + v_R)/2$; hence if s is a real number, then $2L(s) = \int_a^b [v_L(s-t) + v(s-t)]du(t)$ and $2R(s) = \int_a^b [v(s-t) + v_R(s-t)]du(t)$, so that $2[R(s) - L(s)] = \int_a^b [v_R(s-t) - v_L(s-t)]du(t)$. Since v is of bounded variation, so is $v_R - v_L$; and if $v_R \neq v_L$, then there is a countable real-number set K such that $v_R(t) \neq v_L(t)$ if and only if t is in K. Since u is of bounded variation, it follows from Theorem 3.1 of [1] and Lemma 4.2a of [1] that

$$\begin{aligned} 4[R(s) - L(s)] &= \sum_{(t)} [v_R(s-t) - v_L(s-t)][u(t) - u(t-)] \\ &\quad + \sum_{(t)} [v_R(s-t) - v_L(s-t)][u(t+) - u(t)] \\ &= \sum_{(t)} [v_R(s-t) - v_L(s-t)][u_R(t) - u_L(t)]. \end{aligned}$$

But by hypothesis, $v_R = v_L$ or $u_R = u_L$; hence $R = L$.

Now $2L = U(v_L + v)$, where U is a Q_1 operator; hence if s is a real number, then $2[L(s+) - L(s-)] = 4[Uv_R(s) - Uv_L(s)]$. But $Uv_R - Uv_L = U(v + v_R) - U(v_L + v) = 2(R - L) = 0$. Hence if s is a real number then $L(s+) - L(s-) = 0$; and by Lemma 4.1a, $R(s+) = L(s+)$ and $R(s-) = L(s-)$; hence $R(s+) - R(s-) = 2[R(s) - L(s)] = 0$. By the argument in part B of the proof of Theorem 5.1, it follows that T is a Q_1 operator over $[a+c, b+d]$. This completes the proof.

6. Q_1 **operators having specified properties.** In this section we suppose that T is a Q_1 operator over an interval $[a, b]$, and that x is the function of bounded variation such that $x(t)=0$ if $t\leqq a$, $x(t)=x(b)$ if $t>b$, and $Ty(s) =\int_a^b y(s-t)dx(t)$ if y is quasi-continuous and s is a real number. We shall obtain conditions on x which are sufficient to assure that T has various ones of the properties described in the following definition.

DEFINITION 6.1. (i) The statement that T is symmetric means that if y is quasi-continuous and $z(t)=y(-t)$ for each real number t, then $Tz(s) =Ty(-s)$ for each real number s.

(ii) The statement that T has property B means that if y is quasi-continuous, then Ty is of bounded variation.

(iii) The statement that T has property C means that if y is quasi-continuous, then Ty is continuous.

(iv) The statement that T has property D means that (a) if y is quasi-continuous, then Ty has a derivative, and (b) there is a positive number N such that if y is quasi-continuous, s is a real number, and $M\geqq|y(s-t)|$ for each number t in $[a, b]$, and $g=Ty$, then $|g'(s)|\leqq MN$.

(v) If K is a set of quasi-continuous functions, then the statement that the members of K are invariant under T means that if y is in K then $Ty=y$.

THEOREM 6.1. *For T to be symmetric, it is necessary and sufficient that $x(-t)-x(0)=x(0)-x(t)$ for each real number t.*

Proof. A. Suppose that T is symmetric. If y is quasi-continuous, and $z(t) =y(-t)$ for each real number t, then $Tz(0)=Ty(0)$, or $\int_{-\infty}^{+\infty}z(-t)dx(t) =\int_{-\infty}^{+\infty}y(-t)dx(t)$, or $\int_{-\infty}^{+\infty}[y(t)-y(-t)]dx(t)=0$. Hence if f is an odd quasi-continuous function, then $\int_{-\infty}^{+\infty}f(t)dx(t)=0$. Suppose that if $p>0$ then

$$f_p(t) = \begin{cases} 0 \text{ if } t < -p, \text{ or } t = 0 \text{ or } t > p, \\ -1 \text{ if } -p \leqq t < 0, \\ 1 \text{ if } 0 < t \leqq p. \end{cases}$$

Since f_p is an odd function, it follows that $\int_{-\infty}^{+\infty}f_p(t)dx(t)=0$. Integration by parts gives the equation $\int_{-\infty}^{+\infty}x(t)df_p(t)=0$. By Theorem 3.1 of [1],

$$[x(-p-) - x(0)] + [x(-p) - x(0)] - [x(0-) - x(0)] - [x(0+) - x(0)] + [x(p) - x(0)] + [x(p+) - x(0)] = 0$$

for each positive number p. Consequently, upon considering a sequence of positive numbers $p_1, p_2, p_3, \cdots$ which converges to zero, we conclude that $x(0-)-x(0)=x(0)-x(0+)$. By a similar argument, if $p>0$, then $x(-p-) -x(0)=x(0)-x(p+)$; hence $x(-p)-x(0)=x(0)-x(p)$.

B. Suppose that $x(-t)-x(0)=x(0)-x(t)$ for each real number t. Suppose that y is quasi-continuous, $z(t)=y(-t)$ for each real number t, and s is a real number. Then

$$
\begin{aligned}
Tz(s) &= \int_{-\infty}^{+\infty} z(s-t)dx(t) = \int_{+\infty}^{-\infty} z(s+t)dx(-t) \\
&= \int_{+\infty}^{-\infty} y(-s-t)d[x(-t)-x(0)] \\
&= -\int_{+\infty}^{-\infty} y(-s-t)d[x(t)-x(0)] = \int_{-\infty}^{+\infty} y(-s-t)dx(t) = Ty(-s).
\end{aligned}
$$

This completes the proof.

THEOREM 6.2. *Suppose that if* $[c, d]$ *is an interval then there is a positive number* N *such that if* $s_0, s_1, \cdots, s_m$ *is a subdivision of* $[c, d]$ *and* $t_0, t_1, \cdots, t_n$ *is a subdivision of* $[a-d, b-c]$, *then*

$$
\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \left| x(s_{i+1}+t_{j+1}) - x(s_{i+1}+t_j) - x(s_i+t_{j+1}) + x(s_i+t_j) \right| \leqq N.
$$

Then T *has property* B.

Proof. Suppose that y is quasi-continuous, $[c, d]$ is an interval, and $M \geqq |y(s-t)|$ if s is in $[c, d]$ and t is in $[a, b]$. Let $s_0, s_1, \cdots, s_m$ denote a subdivision of $[c, d]$, and suppose that $\epsilon > 0$. Now if s is a real number, then $Ty(s) = \int_{-\infty}^{+\infty} y(s-t)dx(t) = \int_{-\infty}^{+\infty} y(-t)dx(s+t)$; hence if I denotes the sum $\sum_{i=0}^{m-1} |Ty(s_{i+1}) - Ty(s_i)|$, then $I = \sum_{i=0}^{m-1} \left| \int_{-\infty}^{+\infty} y(-t)d[x(s_{i+1}+t) - x(s_i+t)] \right|$. Let $t_0, t_1, \cdots, t_n$ denote a subdivision of $[a-d, b-c]$ such that if i is one of the integers $0, 1, \cdots, m-1$, then

$$
\begin{aligned}
\Bigg| \int_{-\infty}^{+\infty} & y(-t)d[x(s_{i+1}+t) - x(s_i+t)] \\
&- 2^{-1}\sum_{j=0}^{n-1} [y(-t_j) + y(-t_{j+1})][x(s_{i+1}+t_{j+1}) - x(s_i+t_{j+1}) \\
&- x(s_{i+1}+t_j) + x(s_i+t_j)] \Bigg| < \epsilon/m.
\end{aligned}
$$

Now $|y(-t)| \leqq M$ if $a-d \leqq t \leqq b-c$; hence $I < MN + \epsilon$, and consequently $V_c^d(Ty) \leqq MN$. This completes the proof.

COROLLARY 6.2a. *If* x *has a derivative which is of bounded variation, then* T *has property* B.

Proof. Suppose that $[c, d]$ is an interval. If D is a subdivision $s_0, s_1, \cdots, s_m$ of $[c, d]$ and E is a subdivision $t_0, t_1, \cdots, t_n$ of $[a-d, b-c]$, let $S(D, E)$ denote the sum $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |x(s_{i+1}+t_{j+1}) - x(s_i+t_{j+1}) - x(s_{i+1}+t_j) + x(s_i+t_j)|$. Suppose that D and E are such subdivisions and that $\epsilon > 0$. By hypothesis, x' is of bounded variation and (by Lemma 2.7) is therefore continuous; moreover, $x'(t) = 0$ if $t < a$ or $t > b$. Hence there is a positive number δ such that if

p and q are real numbers and $|p-q|<\delta$, then $|x'(p)-x'(q)|<\epsilon$. Let $u_0, u_1, \cdots, u_r$ denote a refinement E' of E such that $u_{j+1}-u_j<\delta$, $j=0, 1, \cdots, r-1$. Now

$$S(D, E') = \sum_{i=0}^{m-1}\sum_{j=0}^{r-1}\left|\frac{x(s_{i+1}+u_{j+1})-x(s_{i+1}+u_j)}{u_{j+1}-u_j} - \frac{x(s_i+u_{j+1})-x(s_i+u_j)}{u_{j+1}-u_j}\right|(u_{j+1}-u_j);$$

and by Lemma 2.6,

$$\begin{aligned} S(D, E') &< 2[(b-c)-(a-d)]\epsilon(2)^{1/2} \\ &\quad + \sum_{i=0}^{m-1}\sum_{j=0}^{r-1} |x'(s_{i+1}+u_j) - x'(s_i+u_j)|\,(u_{j+1}-u_j) \\ &< 2[(b-c)-(a-d)]\epsilon(2)^{1/2} + \sum_{j=0}^{r-1}(u_{j+1}-u_j)V_{-\infty}^{+\infty}(x') \\ &< [(b-c)-(a-d)][2\epsilon(2)^{1/2} + V_{-\infty}^{+\infty}(x')]. \end{aligned}$$

But since E' is a refinement of E, it follows that $S(D, E) \leqq S(D, E')$. Consequently, $S(D, E) \leqq [(b-c)-(a-d)]\,V_{-\infty}^{+\infty}(x')$. By Theorem 6.2, T has property B. This completes the proof.

THEOREM 6.3. *For T to have property* C, *it is necessary and sufficient that x is continuous.*

Proof. Suppose that T has property C; then TJ_L is continuous and TJ_R is continuous. Since $TJ_L=(x_L+x)/2$ and $TJ_R=(x+x_R)/2$, it follows that x is continuous. Suppose now that x is continuous and that $\epsilon>0$. Since $x(t)=0$ if $t\leqq a$ and $x(t)=x(b)$ if $t>b$, there is a positive number δ such that if p and q are real numbers and $|p-q|<\delta$, then $|x(p)-x(q)|<\epsilon$. Now if s_1 and s_2 are two real numbers and $|s_1-s_2|<\delta$, and $g(t)=x(s_1+t)-x(s_2+t)$ for each real number t, then $|g(t)|<\epsilon$ for each real number t, and $V_{-\infty}^{+\infty}(g)\leqq 2V_a^b(x)$. Suppose that y is quasi-continuous, s_1 is a real number, and $[c, d]$ is an interval containing s_1. If s_2 is in $[c, d]$, then $|Ty(s_1)-Ty(s_2)| = |\int_{a-d}^{b-c}y(-t)d[x(s_1+t) -x(s_2+t)]|$; so by Lemma 4.2a of [1], $Ty(s_2)\to Ty(s_1)$ as $s_2\to s_1$. That is to say, Ty is continuous, or T has property C. This completes the proof.

THEOREM 6.4. *For T to have property* D, *it is necessary and sufficient that x has a derivative which is of bounded variation. Moreover, if T has property* D, *y is quasi-continuous, and $g=Ty$, then g' is continuous, and $g'(s)=\int_a^b y(s-t)\, dx'(t)$ for each real number s.*

Proof. A. Suppose that T has property D. If y is quasi-continuous and $g=Ty$, let Uy denote the function g'. It follows from Definition 1.1 and (iv) of

Definition 6.1 that U is a Q operator over $[a, b]$. By Theorem 3.3, if y is quasi-continuous, then Uy is quasi-continuous; and by Lemma 2.7, Uy is continuous; i.e., if y is quasi-continuous and $g=Ty$, then g' is continuous. In particular, UJ_R and UJ_L are continuous; so by Lemma 4.1a, $UJ_R=UJ_L$, and it follows that U is a Q_1 operator over $[a, b]$. Let u denote the function UJ. If y is quasi-continuous and s is a real number, then $Uy(s)=\int_a^b y(s-t)du(t)$, But if $R=TJ_R$, then $u=R'$; and since R has a derivative, R is continuous so that $x=R$, whence $x'=R'=u$. Consequently, if y is quasi-continuous and s is a real number, then $Uy(s)=\int_a^b y(s-t)dx'(t)$. Moreover, since U is a Q_1 operator, x' is of bounded variation.

B. Suppose that x has a derivative which is of bounded variation. By Lemma 2.7, x' is continuous; and we observe that $x'(t)=0$ if $t\leqq a$ or $t\geqq b$. Suppose that $\epsilon>0$, and let h denote a real number other than zero such that if $p<q$ and $q-p\leqq|h|$ then $|x'(p)-x'(q)|<\epsilon$. For each real number t, let $g_h(t)$ denote the number $[x(t+h)-x(t)]/h$. By Lemma 2.6, $|g_h(t)-x'(t)|<2\epsilon$ for each real number t; and by Lemma 2.9, the total variation of g_h is less than or equal to the total variation of x'. Now if y is quasi-continuous and s is a real number, then $[Ty(s+h)-Ty(s)]/h=\int_{-\infty}^{+\infty}y(-t)dg_h(s+t)$; and by Lemma 4.2a of [1], $[Ty(s+h)-Ty(s)]/h\rightarrow\int_{-\infty}^{+\infty}y(-t)dx'(s+t)=\int_a^b y(s-t)dx'(t)$ as $h\rightarrow 0$. If y is quasi-continuous and s is a real number, let $T'y(s)$ denote the number $\int_a^b y(s-t)dx'(t)$. Then T' is a Q_1 operator over $[a, b]$, such that if y is quasi-continuous and $g=Ty$, then $g'=T'y$. Hence T has property D. This completes the proof.

THEOREM 6.5. *Suppose that n is a positive integer. For all polynomials of degree n or less to be invariant under T, it is necessary and sufficient that $x(b)=1$ and $\int_a^b t^p dx(t)=0$ for $p=1, 2, \cdots, n$. If T is symmetric, and n is an even integer, and all polynomials of degree n or less are invariant under T, then all polynomials of degree $n+1$ or less are invariant under T.*

Proof is omitted, since the theorem readily follows if for each polynomial y and real number s we consider the Maclaurin expansion of $y(s-t)$ in powers of t.

7. **A family of smoothing operators and differentiating operators.** In this section we suppose that n is a positive integer and $[a, b]$ is an interval of unit length. We shall exhibit a Q_1 operator T on $[a, b]$ such that

(i) all polynomials of degree 2 or less are invariant under T, and

(ii) if y is quasi-continuous, then Ty has an nth derivative.

Let polynomials u, v, and w be defined as follows. If t is a real number, then

$$2u(t) = 1 + [2t - (a + b)]\sum_{p=0}^{n}\binom{2p}{p}(t - a)^p(b - t)^p,$$

$$v(t) = (n + 2)\binom{2n + 3}{n + 1}(t - a)^{n+1}(b - t)^{n+1},$$

and

$$w(t) = (n+2)\binom{2n+3}{n+1}(t-a)^{n+2}(b-t)^{n+1}.$$

Let $A=(2n+5)[2ab+(n+1)/(2n+3)]$ and $B=(a+b+A)/2$. Let x denote the function such that $x(t)=0$ if $t<a$, $x(t)=1$ if $t>b$, and $x(t)=u(t)+Bv(t)-Aw(t)$ if $a\leqq t\leqq b$. If y is quasi-continuous and s is a real number, let $Ty(s)$ denote the number $\int_a^b y(s-t)dx(t)$, and let $T'y(s)=\int_a^b y(s-t)dx'(t)$.

By some rather tedious manipulation, it can be seen that x has a continuous nth derivative which is of bounded variation, and that $x(a)=0$, $x(b)=1$, $\int_a^b t\,dx(t)=0$, and $\int_a^b t^2dx(t)=0$, so that polynomials of degree 2 or less are invariant under T. Moreover, if $a=-1/2$, then T is symmetric, so that polynomials of degree 3 or less are invariant under T. If y is quasi-continuous and $g=Ty$, then $g'=T'y$.

In particular, if $n=3$ and $a\leqq t\leqq b$, then

$$\begin{aligned} x(t) = 0.5 + 0.5(2t-a-b)[1 &+ 2(t-a)(b-t) + 6(t-a)^2(b-t)^2 \\ &+ 20(t-a)^3(b-t)^3 - 140(22+99ab)(t-a)^4(b-t)^4] \\ &+ 315(a+b)(t-a)^4(b-t)^4, \end{aligned}$$

and

$$\begin{aligned} x'(t) = 140\,(t-a)^3(b-t)^3[(45+198ab) &+ 9(a+b)(a+b-2t) \\ &- (198+891ab)(t-a)(b-t)]. \end{aligned}$$

In particular, if $n=3$, $a=-1/2$, and $-1/2\leqq t\leqq 1/2$, then

$$x(t) = 0.5 + t[1 + 2(.25-t^2) + 6(.25-t^2)^2 + 20(.25-t^2)^3 + 385(.25-t^2)^4],$$

and

$$x'(t) = 315(.25-t^2)^3[11(.25-t^2)-2],$$

and T is a limit of the "most powerful" smoothing operators (i.e., operators with minimum smoothing coefficients) described, e.g., in [4]. For this instance the operators T and T' have been used with quite satisfactory results on a digital computer with experimental data, the integrals being approximated by an approximating sum as in [1], with the subdivision $-.50$, $-.45$, $-.40$, $\cdots$, $.45$, $.50$ of the interval $[-1/2, 1/2]$.

If $n=3$, $a=0$, and $0\leqq t\leqq 1$, then

$$\begin{aligned} x(t) = 0.5 + 0.5(2t-1)[1 &+ 2t(1-t) + 6t^2(1-t)^2 + 20t^3(1-t)^3 \\ &- 3080t^4(1-t)^4] + 315t^4(1-t)^4, \end{aligned}$$

and

$$x'(t) = 2520t^3(1 - t)^3(3 - 12t + 11t^2);$$

in this instance T is not symmetric, and the application of the operators T and T' to experimental data does not give results as satisfactory as those obtained with the symmetric operator previously described.

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