INTEGRATION OF PATHS—A FAITHFUL REPRESENTATION OF PATHS BY NONCOMMUTATIVE FORMAL POWER SERIES

BY

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In the m-dimensional affine space $\mathfrak{R}^m$ with the coordinates $\langle x_1, \cdots, x_m \rangle$, the product $\alpha \cdot \beta$ of two curves α and β is the curve α followed by β, and the inverse α^{-1} is obtained from α by changing the orientation of α. A curve is irreducible if it cannot be expressed in the form $\alpha \cdot \gamma \cdot \gamma^{-1} \cdot \beta$, where α, β, γ are curves. As in [3], we associate to each curve α the formal power series

$$\theta(\alpha) = 1 + \sum_{p=1}^{\infty} \sum \int_\alpha dx_{i_1} \cdots dx_{i_p} X_{i_1} \cdots X_{i_p}$$

in the noncommutative indeterminates $X_1, \cdots, X_m$. The main result of this paper is briefly as follows: If $\theta(\alpha) = \theta(\beta)$ for two irreducible piecewise C^1 and continuous curves α and β, then β can be obtained from α by translation.

We begin with considering curves in a differentiable manifold $\mathfrak{M}$. For the purpose of proving the main result, one may replace the manifold $\mathfrak{M}$ by the affine space $\mathfrak{R}^m$ in all arguments.

The main result relies heavily upon Fundamental Lemma which asserts that, for each irreducible curve α, there is some iterated integral not vanishing along α. This lemma is indeed the center of this paper, and the majority of the definitions and lemmas in §§1–3 are aimed at its proof.

If $\omega_1, \cdots, \omega_m$ are differentials in a differentiable manifold $\mathfrak{M}$, then we may define for a curve α in $\mathfrak{M}$, the formal power series

$$\Theta(\alpha) = 1 + \sum_{p=1}^{\infty} \sum \int_\alpha \omega_{i_1} \cdots \omega_{i_p} X_{i_1} \cdots X_{i_p}.$$

Theorem 4.2 shows that an irreducible continuous curve α is uniquely determined by its initial point and $\Theta(\alpha)$ if the rank of $\omega_1, \cdots, \omega_m$ is equal to the dimension of $\mathfrak{M}$ along the path α. A generalization of the main result is given in Theorem 4.3 which states that, if $\omega_1, \cdots, \omega_m$ are linearly independent Maurer-Cartan forms of an m-dimensional real Lie group $\mathfrak{G}$, then any irreducible contihuous curve α in $\mathfrak{G}$ is uniquely determined by $\Theta(\alpha)$ up to a left translation.

1. **Paths in a differentiable manifold.** Let $\mathfrak{M}$ be an n-dimensional C^r-differentiable manifold, $r \geqq 1$. A path α in $\mathfrak{M}$ is a map α: $[a, b] \rightarrow \mathfrak{M}$ which is

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continuous except for a finite number of jump discontinuities in the interior of the closed interval $[a, b]$. In other words, the left limit

$$\alpha(t_0-) = \lim_{t \to t_0-} \alpha(t)$$

exists for $a < t_0 \leqq b$; the right limit $\alpha(t_0+) = \lim_{t \to t_0+} \alpha(t)$ exists for $a \leqq t_0 < b$; and only for a finite number of t, $a < t < b$, the inequality $\alpha(t-) \neq \alpha(t+)$ holds. A point t of $[a, b]$ is called a value of α and is said to be an interior or end value according as it is an interior or end point of $[a, b]$. If $\alpha(t+) \neq \alpha(t-)$, then t is called a jump value of α, and $\alpha(t)$ is taken to be the pair $\alpha(t+)$, $\alpha(t-)$, both of which are called points of α. The trace of α is the set of all points of α. Two paths of disjoint traces are said to be disjoint.

It is clear that, if $\alpha(t)$ is given except for a finite number of values of t, $a \leqq t \leqq b$, then the path α is uniquely determined. Therefore we shall often treat a path without providing separate descriptions regarding its jump discontinuities.

Let $[a', b']$ be a closed subinterval of $[a, b]$. Denote by $\alpha[a', b']$ the portion of α with the parameter running from a' to b'. If the path $\alpha[a', b']$ is simple, then it is called an arc of α. We say that the arc $\alpha[a', b']$ is right to the value a', left to the value b' and near the value t, $a' < t < b'$. We also say that $\alpha[a', b']$ is near the end value a (or b) when $a' = a$ (or $b' = b$).

A path β: $[c, d] \to \mathfrak{M}$ is equivalent to α or is obtained from α by change of parameter if there exists an increasing continuous function $\tau(t)$ which maps $[c, d]$ onto $[a, b]$ with $\alpha\tau = \beta$.

DEFINITION 1.1. Given two paths α: $[a, b] \to \mathfrak{M}$ and β: $[c, d] \to \mathfrak{M}$, define $\alpha \cdot \beta$ to be the path γ: $[0, b-a+d-c] \to \mathfrak{M}$ such that $\gamma(t) = \alpha(t+a)$ for $0 \leqq t \leqq b-a$ and $\gamma(t) = \beta(t+c+b-a)$ for $b-a \leqq t \leqq b-a+d-c$. For any α: $[a, b] \to \mathfrak{M}$ define α^{-1} to be the path γ: $[0, b-a] \to \mathfrak{M}$ such that $\gamma(t) = \alpha(b-t)$ for $0 \leqq t \leqq b-a$. Each of the paths $\alpha \cdot \beta$ and α^{-1} will undergo only change of parameter when α and β are replaced by equivalent paths.

A path α: $[a, b] \to \mathfrak{M}$ is regular at the value t, if the tangent vector of α, denoted by $\dot{\alpha}(t)$, is continuous and nonvanishing near the value t. The path α is regular if it is regular at each value t, $a \leqq t \leqq b$. A regular path is thus continuous. The path α is piecewise regular, if there are $a_0, a_1, \cdots, a_k$ with $a = a_0 < a_1 < \cdots < a_k = b$ such that each path $\alpha[a_{i-1}, a_i]$, $i = 1, \cdots, k$, is regular. A piecewise regular path may have a finite number of jump discontinuities.

Unless otherwise specified, a path hereafter will be taken to be at least piecewise regular.

DEFINITION 1.2. Let B be a set of mutually disjoint paths $\beta_1, \cdots, \beta_m$. Let α: $[a, b] \to \mathfrak{M}$ be a path satisfying the following conditions:

(a) There are paths $\alpha[a_1, b_1]$, $\alpha[a_2, b_2]$, $\cdots$, $\alpha[a_k, b_k]$, $a \leqq a_1 < b_1 \leqq a_2 < \cdots \leqq a_k < b_k \leqq b$, each of which is equivalent to one of the paths

$\beta_1, \cdots, \beta_m, \beta_1^{-1}, \cdots, \beta_m^{-1}$;

(b) If $\alpha(t)$, $a \leqq t \leqq b$, is a point of β_j, $j=1, \cdots, m$, then t is a value of one of $\alpha[a_i, b_i]$, $i=1, \cdots, k$.

Then we say that α runs through B along the array of paths $\alpha[a_1, b_1], \cdots, \alpha[a_k, b_k]$. If B consists of only one path β, then we simply say that α runs through β.

It is easy to see that a simple path α runs through another simple path β if and only if the trace of α contains that of β.

Let α: $[a, b] \rightarrow \mathfrak{M}$ and β: $[c, d] \rightarrow \mathfrak{M}$ be two equivalent paths. Then there is one and only one increasing function $\tau(t)$ from $[c, d]$ onto $[a, b]$ with $\alpha\tau = \beta$. Moreover, $d\tau(t)/dt$ is bounded below by a positive number and has only a finite number of jump discontinuities for $c \leqq t \leqq d$.

DEFINITION 1.3. A value t of a continuous path α is normal relative to another continuous path β, if, for each value s of β with $\beta(s) = \alpha(t)$, there exist two arcs respectively of α and β near t and s such that both the arcs have the same trace. If a value t of α is not normal relative to β, then t is said to be abnormal relative to β. If the point $\alpha(t)$ is not a point of β, then the value t of α is said to be normal relative to β.

LEMMA 1.1. *The set of the values of a simple path* α: $[a, b] \rightarrow \mathfrak{M}$ *normal relative to another simple path* β *is open and everywhere dense in* $[a, b]$.

Proof. It is clear that the set S of the values of α normal relative to β is open in $[a, b]$. Let $[a_1, b_1]$ be a subinterval of $[a, b]$. If $[a_1, b_1]$ contains no normal values of α relative to β, then each $\alpha(t)$, $a_1 \leqq t \leqq b_1$, is a point of β, and the trace of the arc $\alpha[a_1, b_1]$ is contained in that of β. Therefore β runs through $\alpha[a_1, b_1]$ which is absurd. Hence S is everywhere dense in $[a, b]$, and the lemma is proved.

DEFINITION 1.4. An interior value s of a path α is called a reducible value of α if there exists an arc left to s and another arc right to s, both of which have the same trace; i.e., one of the arcs is equivalent to the inverse of the other. The path α is irreducible if it has no reducible values. An interior value s of a path α is called a knotty value if it is neither a reducible value nor a jump value of α and if there exists no arc near s.

It is evident that α is regular neither at a knotty value nor at a reducible value.

The following assertion is evident.

LEMMA 1.2. *If* c *is a reducible value of* α: $[a, b] \rightarrow \mathfrak{M}$, *then there are two unique values* c' *and* c'' *of* α, $a \leqq c' < c < c'' \leqq b$, *such that* $\alpha[c, c'']$ *is equivalent to* $(\alpha[c', c])^{-1}$ *and* $c' - a$ *is not a reducible value of the path* $\alpha[a, c'] \cdot \alpha[c'', b]$ *provided* $a < c'$ *and* $c'' < b$.

2. Integration of paths.

Let $f_i(t)$, $i=1, \cdots, p$, be real (or complex) valued functions piecewise continuous in an interval I which contains points a and b.

Define the iterated integral $\int_a^b f_1(t)dt \cdots f_p(t)dt$, $p \geqq 2$, by the inductive formula

$$\int_a^b f_1(t)dt \cdots f_p(t)dt = \int_a^b \left[\int_a^\tau f_1(t)dt \cdots f_{p-1}(t)dt \right] f_p(\tau)d\tau.$$

Such integrals are well defined for $p \geqq 1$.

If $s \in I$, then, for $p \geqq 1$,

$$\begin{aligned} \int_a^s f_1(t)dt \cdots f_p(t)dt = & \int_b^s f_1(t)dt \cdots f_p(t)dt + \cdots \\ & + \int_a^b f_1(t)dt \cdots f_i(t)dt \int_b^s f_{i+1}(t)dt \cdots f_p(t)dt + \cdots \\ & + \int_a^b f_1(t)dt \cdots f_p(t)dt. \end{aligned} \tag{2.1}$$

The above formula may be easily obtained by induction on p. The case $p=1$ is trivial. We show the case $p>1$ by differentiating the both sides of (2.1) with respect to s and applying the induction hypothesis.

LEMMA 2.1. *The product* $\int_a^b f_1(t)dt \cdots f_i(t)dt \int_a^b f_{i+1}(t)dt \cdots f_p(t)dt$ *is a linear combination of the integrals* $\int_a^b f_{i_1}(t)dt \cdots f_{i_p}(t)dt$, $(i_1, \cdots, i_p)$ *running over all permutations of* $(1, \cdots, p)$.

Proof. We use induction on p. The cases $p=1$, $i=0$ and $i=p$ are trivial. For $p \geqq 2$, $0<i<p$, set

$$g(\tau) = \int_a^\tau f_1(t)dt \cdots f_i(t)dt \int_a^\tau f_{i+1}(t)dt \cdots f_p(t)dt.$$

Then by the induction hypothesis the lemma holds for both

$$h_1(\tau) = \int_a^\tau f_1(t)dt \cdots f_{i-1}(t)dt \int_a^\tau f_{i+1}(t)dt \cdots f_p(t)dt$$

and

$$h_2(\tau) = \int_a^\tau f_1(t)dt \cdots f_i(t)dt \int_a^\tau f_{i+1}(t)dt \cdots f_{p-1}(t)dt.$$

Hence

$$g(b) = \int_a^b g'(t)dt = \int_a^b h_1(t)f_i(t)dt + \int_a^b h_2(t)f_p(t)dt$$

is a linear combination of the integrals $\int_a^b f_{i_1}(t)dt \cdots f_{i_p}(t)dt$.

If $|f_i(t)| \leqq M$, $a \leqq t \leqq b$, $i=1, \cdots, p$, then the inequality

$$\left|\int_a^b f_1(t)dt \cdots f_p(t)dt\right| \leqq M^p(b-a)^p/p!, \qquad p \geqq 1, \tag{2.2}$$

can be easily verified by induction on p.

LEMMA 2.2. *Let $f_i(t)$, $g_i(t)$, $i=1, \cdots, p$, be piecewise continuous with $|f_i(t)| \leqq M$, $|g_i(t)| \leqq M$, $|g_i(t)-f_i(t)| \leqq \delta$ for $a \leqq t \leqq b$. Then, for $p \geqq 1$,*

$$\left|\int_a^b g_1(t)dt \cdots g_p(t)dt - \int_a^b f_1(t)dt \cdots f_p(t)dt\right| \leqq \delta M^{p-1}(b-a)^p/(p-1)!$$

Proof. The case $p=1$ is trivial. Using the induction hypothesis, we have, for $p \geqq 2$ and $a \leqq \tau \leqq b$,

$$\begin{aligned}
&\left| g_p(\tau)\int_a^\tau g_1(t)dt \cdots g_{p-1}(t)dt - f_p(\tau)\int_a^\tau f_1(t)dt \cdots f_{p-1}(t)dt\right| \\
&\qquad \leqq \left|\int_a^\tau g_1(t)dt \cdots g_{p-1}(t)dt - \int_a^\tau f_1(t)dt \cdots f_{p-1}(t)dt\right| \, |g_p(\tau)| \\
&\qquad\quad + \left|\int_a^\tau f_1(t)dt \cdots f_{p-1}(t)dt\right| \, |g_p(\tau) - f_p(\tau)| \\
&\qquad \leqq \delta M^{p-1}(\tau-a)^{p-1}/(p-2)! + \delta M^{p-1}(\tau-a)^{p-1}/(p-1)! \\
&\qquad = p\delta M^{p-1}(\tau-a)^{p-1}/(p-1)!.
\end{aligned}$$

Hence

$$\begin{aligned}
&(p-1)!\left|\int_a^b g_1(t)dt \cdots g_p(t)dt - \int_a^b f_1(t)dt \cdots f_p(t)dt\right| \\
&\qquad \leqq \int_a^b p\delta M^{p-1}(t-a)^{p-1}dt = \delta M^{p-1}(b-a)^p \cdot
\end{aligned}$$

For a system of local coordinates $\mathbf{x} = \langle x_1, \cdots, x_n\rangle$ about $\mathfrak{p} \in \mathfrak{M}$, denote by $\mathbf{x}(\mathfrak{p})$ the coordinates of $\mathfrak{p}$. In the local coordinate system a differential π takes the form $\sum f_i(\mathbf{x})dx_i$. The differential π is said to be C^s if all $f_1(\mathbf{x}), \cdots, f_n(\mathbf{x})$ are C^s for any choice of the local coordinate system $\mathbf{x}$. We require π to be at least C^0.

For any path α: $[a, b] \to \mathfrak{M}$, the value of

$$\pi(\dot{\alpha}(t)) = \sum f_i(\mathbf{x}(\alpha(t)))dx_i(\alpha(t))/dt$$

is independent of the choice of $\mathbf{x}$. Let $\pi_1, \cdots, \pi_p$ be differentials in $\mathfrak{M}$, and α: $[a, b] \to \mathfrak{M}$ a path. Define

$$\int_\alpha \pi_1 \cdots \pi_p = \int_a^b \pi_1(\dot{\alpha}(t))dt \cdots \pi_p(\dot{\alpha}(t))dt.$$

We observe that the above integral does not change its value when the path undergoes a change of parameter. Moreover

$$\int_{\alpha^{-1}} \pi_1 \cdots \pi_p = \int_b^a \pi_1(\dot{\alpha}(t))dt \cdots \pi_p((\dot{\alpha}(t))dt.$$

For any two paths α and β in $\mathfrak{M}$, the following are direct consequences of (2.1):

$$\int_{\alpha\cdot\beta} \pi_1 \cdots \pi_p = \int_\beta \pi_1 \cdots \pi_p + \cdots + \int_\alpha \pi_1 \cdots \pi_i \int_\beta \pi_{i+1} \cdots \pi_p + \cdots + \int_\alpha \pi_1 \cdots \pi_p; \tag{2.2}$$

$$\int_{\alpha\cdot\alpha^{-1}} \pi_1 \cdots \pi_p = 0. \tag{2.3}$$

Let $\omega_1, \cdots, \omega_m$ be differentials in $\mathfrak{M}$, and $X_1, \cdots, X_m$ distinct noncommutative indeterminates. Define as in [3], the exponential homomorphism and antihomomorphism Θ and Θ^* such that

$$\Theta(\alpha) = 1 + \sum_{p=1}^{\infty} \sum \int_\alpha \omega_{i_1} \cdots \omega_{i_p} X_{i_1} \cdots X_{i_p}$$

and

$$\Theta^*(\alpha) = 1 + \sum_{p=1}^{\infty} \sum \int_\alpha \omega_{i_p} \cdots \omega_{i_1} X_{i_1} \cdots X_{i_p}.$$

It follows immediately from (2.2) and (2.3) that $\Theta(\alpha\cdot\beta) = \Theta(\alpha)\Theta(\beta)$, $\Theta^*(\alpha\cdot\beta) = \Theta^*(\beta)\Theta^*(\alpha)$, $\Theta(\alpha^{-1}) = [\Theta(\alpha)]^{-1}$ and $\Theta^*(\alpha^{-1}) = [\Theta^*(\alpha)]^{-1}$.

3. Fundamental lemma.

LEMMA 3.1. *Let a path α run through a set B of mutually disjoint simple paths $\beta_1, \cdots, \beta_m$. For each i, $i = 1, \cdots, m$, take an open set $\mathfrak{C}_i$ containing a point of β_i and disjoint to the traces of all β_j, $j \neq i$. Then there exist C^{r-1} differentials $\omega_1, \cdots, \omega_m$ such that*

(a) *each ω_i vanishes outside $\mathfrak{C}_i$;*

(b) *if ω_i does not vanish at $\alpha(t)$, then $\alpha(t)$ is a point of β_i;*

(c) $\int_{\beta_i} \omega_i > 0$.

Proof. Let β_i be regular at some interior value s with $\beta_i(s) \in \mathfrak{C}_i$. Choose a system of local coordinates $\mathbf{x}$ about $\beta_i(s)$ such that $x_j(\beta_i(s)) = 0$, $j = 1, \cdots, n$, and $[dx_j(\beta_i(t))/dt]_{t=s} = \delta_{1j}$, where δ_{ij} is the Kronecker symbol. Take sufficiently small positive numbers c and μ satisfying the following conditions:

(a) The cube of all points $\mathfrak{p}$ with $|x_j(\mathfrak{p})| \leq 2\mu$, $j = 1, \cdots, n$, is contained in $\mathfrak{C}_i$.

(b) For $|\xi| \leq \mu$, the equation $\xi = x_1(\beta_i(t))$ may be solved uniquely for t

with $|t-s|<c$; and $t=t(\xi)$ is a C^1 function of ξ, $-\mu\leqq\xi\leqq\mu$.

(c) For $t(-\mu)\leqq t\leqq t(\mu)$, we have $|x_j(\beta_i(t))|\leqq\mu$, $j=1, \cdots, n$.

(d) If $|x_j(\alpha(t))|<\mu$, $j=1, \cdots, n$, then $\alpha(t)$ is a point of the arc

$$\beta_i[t(-\mu), t(\mu)].$$

It is clear that there exists a C^∞ function $g(x)$ such that $g(x)=0$ for $|x|\geqq\mu$, $g(x)\geqq 0$ for $|x|<\mu$, and $g(0)>0$. Construct the differential ω_i which takes the form $g(x_1)\cdots g(x_n)dx_1$ within the cube of all points $\mathfrak{p}$, $|x_j(\mathfrak{p})|<\mu$, $j=1, \cdots, n$, and vanishes outside the cube. If ω_i does not vanish at $\alpha(t)$, then $\alpha(t)$ lies within the cube and is thus a point of β_i. Moreover,

$$\int_{\beta_i}\omega_i = \int_{t(-\mu)}^{t(\mu)} g(x_1(\beta_i(t)))\cdots g(x_n(\beta_i(t)))dx_1(\beta_i(t)) > 0.$$

Hence the lemma is proved.

COROLLARY.

$$\int_{\beta_j}\omega_{i_1}\cdots\omega_{i_p} = 0 \textit{ if some } i_q\neq j.$$

LEMMA 3.2. *Let a path* α: $[a, b]\to\mathfrak{M}$ *be divided into arcs* $\alpha[a_0, a_1], \cdots, \alpha[a_{i-1}, a_i], \cdots, \alpha[a_{q-1}, a_q]$, $a=a_0<a_1<\cdots<a_q=b$, *and let each* $\alpha[a_{i-1}, a_i]$ *run through the set B of disjoint simple paths* $\beta_1, \cdots, \beta_m$ *along the arcs* $\gamma_{i\,1}, \cdots, \gamma_{i\,k_i}$, *such that*

(a) $\gamma_{i\,j}$ *and* $\gamma_{i\,j+1}$ *are disjoint*, $j=1, \cdots, k_i-1$;

(b) *either* $\gamma_{i-1\,k_{i-1}}$, $\gamma_{i\,1}$ *or* $\gamma_{i\,k_i}$, $\gamma_{i+1\,1}$ *are disjoint*, $i=2, \cdots, q-1$;

(c) *for each* i, $i=1, \cdots, q-1$, $\alpha[a_{i-1}, a_i]$ *and* $\alpha[a_i, a_{i+1}]$ *are respectively disjoint to some* $\gamma_{i+1\,\nu}$ *and* $\gamma_{i\,\lambda}$, $1\leqq\lambda\leqq k_i$, $1\leqq\nu\leqq k_{i+1}$.

If differentials $\omega_1, \cdots, \omega_m$ *are constructed as in Lemma* 3.1, *then some*

$$\int_\alpha \omega_{i_1}\cdots\omega_{i_p}\neq 0.$$

Proof. Denote by Θ the exponential homomorphism with respect to $\omega_1, \cdots, \omega_m$. Then

$$\Theta(\beta_i) = 1 + \int_{\beta_i}\omega_i X_i + \int_{\beta_i}\omega_i\omega_i X_i^2 + \cdots$$

is a power series in the indeterminate X_i alone, and so is

$$\Theta(\beta_i^k) = \Theta(\beta_i)^k = 1 + k\mu_i X_i + \cdots,$$

where $\mu_i=\int_{\beta_i}\omega_i$. If a path $\alpha[a', b']$, $a\leqq a'<b'\leqq b$, is disjoint to each β_i, $i=1, \cdots, m$, then $\Theta(\alpha[a', b'])=1$. Consequently

$$\Theta(\alpha) = \Theta(\gamma_{1\,1})\cdots\Theta(\gamma_{1\,k_1})\cdots\Theta(\gamma_{q\,1})\cdots\Theta(\gamma_{q\,k_q}).$$

We define the paths $\gamma_1, \cdots, \gamma_q$ in the following manner:

If $\gamma_{i\,k_i}$ and $\gamma_{i+1\,1}^{-1}$, $1 \leqq i \leqq q-1$, are equivalent; let l be the integer such that $\gamma_{i\,k_i-j+1}$ and $\gamma_{i+1\,j}^{-1}$ are equivalent for $j=1, \cdots, l$ but not for $j=l+1$. It follows from (c) that $k_i - l \geqq 1$ and $l+1 \leqq k_{i+1}$. Set $\gamma_i = \gamma_{i\,1} \cdots \gamma_{i\,k_i-l}$ and $\gamma_{i+1} = \gamma_{i+1\,l+1} \cdots \gamma_{i+1\,k_{i+1}}$. In this case we have obviously

$$\Theta(\gamma_{i1}) \cdots \Theta(\gamma_{ik_i})\Theta(\gamma_{i+1\,1}) \cdots \Theta(\gamma_{i+1\,k_{i+1}}) = \Theta(\gamma_i)\Theta(\gamma_{i+1}).$$

If, for any j, $1 \leqq j \leqq q$, the above procedure does not apply, we set

$$\gamma_j = \gamma_{j\,1} \cdots \gamma_{j\,k_j}.$$

It follows that $\Theta(\alpha) = \Theta(\gamma_1) \cdots \Theta(\gamma_q) = \Theta(\gamma_1 \cdots \gamma_q)$. Now $\gamma_1 \cdots \gamma_q$ is evidently equivalent to a product $\beta_{i_1}^{l_1} \cdots \beta_{i_p}^{l_p}$ with $l_\lambda \neq 0$, $\lambda = 1, \cdots, p$, and $i_1 \neq i_2 \neq \cdots \neq i_p$. Thus

$$\begin{aligned}\Theta(\alpha) &= \Theta(\beta_{i_1}^{l_1}) \cdots \Theta(\beta_{i_p}^{l_p}) \\ &= (1 + l_1\mu_{i_1}X_{i_1} + \cdots) \cdots (1 + l_p\mu_{i_p}X_{i_p} + \cdots).\end{aligned}$$

The coefficient of the term $X_{i_1} \cdots X_{i_p}$ in the power series $\Theta(\alpha)$ is $l_1\mu_{i_1} \cdots l_p\mu_{i_p} \neq 0$. Hence $\int_\alpha \omega_{i_1} \cdots \omega_{i_p} \neq 0$.

LEMMA 3.3. *An irreducible path* α: $[a, b] \rightarrow \mathfrak{M}$ *can be divided into arcs*

$$\alpha[a_0, a_1], \cdots, \alpha[a_{q-1}, a_q], \qquad a = a_0 < \cdots < a_q = b,$$

possessing the following properties:

(a) *Each arc* $\alpha[a_{i-1}, a_i]$ *is regular.*

(b) *If the value* a_i, $1 \leqq i \leqq q-1$, *is not knotty, the two adjacent arcs* $\alpha[a_{i-1}, a_i]$ *and* $\alpha[a_i, a_{i+1}]$ *have no point in common other than possibly the point* $\alpha(a_i)$.

(c) *No two consecutive values* a_{i-1} *and* a_i *are both knotty.*

(d) *If* $\alpha(t+)$ *or* $\alpha(t-)$ *is one of* $\alpha(a_i+)$, $\alpha(a_i-)$, $i=0, \cdots, q$, *then* t *must be one of* $a_0, \cdots, a_q$.

Proof. Let $\alpha[c_1, c_2]$ and $\alpha[c_2, c_3]$ be any two regular arcs of α with c_2 not a knotty value of α. There exist two arcs $\alpha[c', c_2]$ and $\alpha[c_2, c'']$, $c_1 < c' < c_2 < c'' < c_3$, which have at most the point $\alpha(c_2)$ in common. Then any two adjacent arcs among $\alpha[c_1, c']$, $\alpha[c', c_2]$, $\alpha[c_2, c'']$, $\alpha[c'', c_3]$ have not more than one point in common. Since the path α can be easily divided into a finite number of regular arcs, it is therefore obvious that α can be further divided into arcs $\alpha[b_0, b_1], \cdots, \alpha[b_{k-1}, b_k]$, $a = b_0 < \cdots < b_k = b$, satisfying the conditions (a), (b), (c) of the lemma. Let $a_0, a_1, \cdots, a_q$, taken in order, be all the values of α such that either $\alpha(a_i+)$ or $\alpha(a_i-)$ is one of the points $\alpha(b_j+)$, $\alpha(b_j-)$, $j=0, \cdots, k$. Then, the values $b_0, \cdots, b_k$ are among the values $a_0, \cdots, a_q$, and consequently the division of the path α into arcs $\alpha[a_0, a_1], \cdots, \alpha[a_{q-1}, a_q]$ satisfies the conditions (a), (b), (c). Suppose that, for the value t, either $\alpha(t+)$ or $\alpha(t-)$ is some $\alpha(a_i+)$ or $\alpha(a_i-)$. If a_i is a jump value of α, then a_i, at which α is not regular, must be one of $b_0, \cdots, b_k$. If a_i is not a jump value, then $\alpha(a_i\pm)=\alpha(a_i)$ is one of $\alpha(b_0+), \cdots, \alpha(b_k+), \alpha(b_0-), \cdots, \alpha(b_k-)$. In both cases, t is one of $a_0, \cdots, a_q$. Hence the condition (d) is also satisfied, and the lemma is proved.

LEMMA 3.4. *Let B be a finite set of mutually disjoint simple paths. Then any path α has an arc γ such that each path of B either runs through γ or is disjoint to γ.*

Proof. Without loss of generality, the path α: $[a, b]\rightarrow\mathfrak{M}$ may be assumed to be simple. According to Lemma 1.1, the set of values of α normal relative to a path of B is open and everywhere dense in $[a, b]$. Therefore some interior values s of α is normal relative to each path of B. Since B consists of only a finite number of paths, it follows from Definition 1.3 that there is an arc γ of α near the value s such that each path of B either runs through γ or is disjoint to γ. Hence the lemma is proved.

LEMMA 3.5 (FUNDAMENTAL LEMMA). *If α is an irreducible piecewise regular path in an n-dimensional C^r differentiable manifold $\mathfrak{M}$, $1\leqq r\leqq\infty$, then there exist m differentials $\omega_1, \cdots, \omega_m$, such that some $\int_\alpha\omega_{i_1}\cdots\omega_{i_p}\neq 0$. To be more exact, for the path α, there exist distinct points $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ in $\mathfrak{M}$ such that, given any neighborhoods $\mathfrak{C}_i$ of $\mathfrak{p}_i$, $i=1, \cdots, m$, we can construct C^{r-1} differentials ω_i vanishing in $\mathfrak{M}-\mathfrak{C}_i$, $i=1, \cdots, m$, with $\Theta(\alpha)\neq 1$ and $\Theta^*(\alpha)\neq 1$, where Θ and Θ^* are respectively the exponential homomorphism and anti-homomorphism with respect to $\omega_1, \cdots, \omega_m$.*

Proof. Let the path α: $[a, b]\rightarrow\mathfrak{M}$ be divided into arcs $\alpha[a_0, a_1], \cdots, \alpha[a_{q-1}, a_q]$, $a=a_0<\cdots<a_q=b$, according to Lemma 3.3. Let B be any set of mutually disjoint simple regular paths $\beta_1, \cdots, \beta_m$ such that each arc $\alpha[a_{i-1}, a_i]$ runs through B along the arcs $\gamma_{i\,1}, \cdots, \gamma_{i\,k_i}$. Then the condition (a) of Lemma 3.2 holds, for $\alpha[a_{i-1}, a_i]$ is simple; and the condition (b) of Lemma 3.2 also holds owing to the conditions (b) and (c) of Lemma 3.3. In order to prove this lemma, it is therefore sufficient to construct a set B in such a way that, for each i, $i=1, \cdots, q-1$, $\alpha[a_{i-1}, a_i]$ and $\alpha[a_i, a_{i+1}]$ are respectively disjoint to some $\gamma_{i+1\,\nu}$ and $\gamma_{i\,\lambda}$. Such a set B will be said to be admissible with respect to the division of α into the arcs $\alpha[a_0, a_1], \cdots, \alpha[a_{q-1}, a_q]$. It is clear that B will remain to be admissible with respect to the same division of α into the arcs if each β_i is replaced by an arc of β_i. We shall carry out the construction of an admissible set B with respect to the division of α by induction on q.

The case $q=1$ is trivial. For $q\geqq 2$, the path $\alpha[a_0, a_{q-1}]$ is divided into the $q-1$ arcs $\alpha[a_0, a_1], \cdots, \alpha[a_{q-2}, a_{q-1}]$ according to Lemma 3.3, and thus the induction hypothesis can be applied. Let B_1 be an admissible set of paths with respect to this division of $\alpha[a_0, a_{q-1}]$. None of the traces of $\alpha[a_{q-2}, a_{q-1}]$ and $\alpha[a_{q-1}, a_q]$ is contained in the other; otherwise, α would be reducible at the value a_{q-1}. Thus there are an arc γ of $\alpha[a_{q-2}, a_{q-1}]$ disjoint to $\alpha[a_{q-1}, a_q]$

and an arc γ' of $\alpha[a_{q-1}, a_q]$ disjoint to $\alpha[a_{q-2}, a_{q-1}]$. By applying Lemma 3.4, the arcs γ and γ' may be chosen such that every $\alpha[a_{i-1}, a_i]$, $i=1, \cdots, q-2$, as well as each path of B_1 either runs through or is disjoint to each of γ and γ'. We carry out stepwise the following replacements and supplements in B_1:

(a) If a path β of B_1 runs through γ, we replace β by γ in B_1.

(b) If γ is disjoint to each path of B_1, we supplement B_1 with the path γ.

(c) We further replace β' of B_1 by γ', if β' runs through γ', and supplement B_1 with γ' if γ' is disjoint to each path of B_1.

Thus a new set B of paths is constructed from B_1. It is not difficult to see that B is admissible with respect to the division of α. Hence the lemma is proved.

DEFINITION 3.1. Let α, β, γ, α_1, α_2 be paths in $\mathfrak{M}$ such that α and β are respectively equivalent to $\alpha_1\cdot\gamma\cdot\gamma^{-1}\cdot\alpha_2$ and $\alpha_1\cdot\alpha_2$. Then α is said to be reduced to β, and the process of reducing α to β is called a reduction. If α^* is an irreducible path obtained from a path α by applying successively a finite number of reductions, then α^* is called an irreducible path of α. We take the empty path for the irreducible path of $\gamma\cdot\gamma^{-1}$.

THEOREM 3.1. *A piecewise regular path α in $\mathfrak{M}$ has one and only one irreducible path α^* up to equivalence.*

Proof. It follows easily from Lemma 1.2 that α has at least one irreducible path α^*. Suppose that α' be another irreducible path of α. Let $\omega_1, \cdots, \omega_m$ be any C^{r-1} differentials in $\mathfrak{M}$, and Θ, the corresponding exponential homomorphism. Then $\Theta(\alpha)=\Theta(\alpha^*)=\Theta(\alpha')$, and $\Theta(\gamma)=1$ where $\gamma=\alpha\cdot\alpha'^{-1}$. If γ is empty, then both α^* and α' are empty, and the theorem holds. Assume that γ is not empty. As a consequence of Fundamental Lemma, γ is reducible and its irreducible path is empty. Since both α^* and α' are irreducible, the path γ has only one reducible value. By applying Lemma 1.2, we conclude that α^* and α' are equivalent. Hence the theorem is proved.

4. The uniqueness theorems.

LEMMA 4.1. *If α: $[a, b]\to\mathfrak{R}^m$ is an irreducible piecewise regular continuous path in the m-dimensional real affine space $\mathfrak{R}^m$ with coordinates $\langle x_1, \cdots, x_m\rangle$, then some $\int_\alpha dx_{i_1}\cdots dx_{i_p}\neq 0$.*

Proof. According to Fundamental Lemma, there are, in $\mathfrak{R}^m$, differentials $\omega_1, \cdots, \omega_q$, not necessarily distinct, such that, for the path α: $[a, b]\to\mathfrak{R}^m$, $\int_\alpha\omega_1\cdots\omega_q\neq 0$ where the differentials $\omega_i=\sum f_{ij}(\mathbf{x})dx_j$, $i=1, \cdots, q$, can be made to vanish outside some bounded open set $\mathfrak{C}$. By Weierstrass Approximation Theorem, each $f_{ij}(\mathbf{x})$ can be approximated by a polynomial $\bar{f}_{ij}(\mathbf{x})$ to any desired degree in the set $\mathfrak{C}$. Set $\bar{\omega}_i=\sum\bar{f}_{ij}(\mathbf{x})dx_j$. In the light of Lemma 2.2, $\left|\int_\alpha\bar{\omega}_1\cdots\bar{\omega}_q-\int_\alpha\omega_1\cdots\omega_q\right|$ can be made arbitrarily close to zero. Consequently, there exist polynomials $\bar{f}_{ij}(\mathbf{x})$ such that $\int_\alpha\bar{\omega}_1\cdots\bar{\omega}_q\neq 0$, which implies that some $\int_\alpha\bar{f}_{1j_1}(\mathbf{x})dx_{j_1}\cdots\bar{f}_{qj_q}(\mathbf{x})dx_{j_q}\neq 0$. At last we conclude that there exist monomials $g_1(\mathbf{x}), \cdots, g_q(\mathbf{x})$ with

$$\int_\alpha g_1(\mathbf{x})dx_{j_1} \cdots g_q(\mathbf{x})dx_{j_q} \neq 0. \tag{4.1}$$

It suffices to show that the integral in (4.1) is a linear combination of elementary integrals along α, by which we mean integrals of the type $\int_\alpha dx_{i_1} \cdots dx_{i_p}$. We shall carry out the proof by induction on q. It follows from Lemma 2.1 that the product of any finite number of elementary integrals along $\alpha[a, t]$ is a linear combination of elementary integrals along $\alpha[a, t]$. Since $x_i(\alpha(t)) = \int_{\alpha[a,t]} dx_i + x_i(\alpha(a))$, the integral $\int_{\alpha[a,t]} g_1(\mathbf{x})dx_{j_1}$ is a linear combination of elementary integrals along $\alpha[a, t]$. Our assertion therefore holds for $q=1$. For $q \geqq 2$,

$$\int_\alpha g_1(\mathbf{x})dx_{j_1} \cdots g_q(\mathbf{x})dx_{j_q}$$
$$= \int_a^b \left[\int_{\alpha[a,t]} g_1(\mathbf{x})dx_{j_1} \cdots g_{q-1}(\mathbf{x})dx_{j_{q-1}} \right] g_q(\mathbf{x}(\alpha(t)))dx_{j_q}(\alpha(t))$$

and, according to the induction hypothesis and Lemma 2.1, the integrand of the integral at the right side of the above equation is a linear combination of elementary integrals along $\alpha[a, t]$. Hence the integral in (4.1) is a linear combination of elementary integrals along α, and consequently some

$$\int_\alpha dx_{i_1} \cdots dx_{i_p} \neq 0.$$

THEOREM 4.1 (UNIQUENESS THEOREM). *Let $\mathfrak{R}^m$ be the m-dimensional real affine space with coordinates $\langle x_1, \cdots, x_m \rangle$. Let θ be the exponential homomorphism with respect to $dx_1, \cdots, dx_m$. Then one of the two irreducible piecewise regular continuous paths α and β can be obtained from the other by translation of $\mathfrak{R}^m$ and change of parameter if and only if $\theta(\alpha) = \theta(\beta)$.*

Proof. Since any translation in $\mathfrak{R}^m$ and any change of parameter of α do not alter $\theta(\alpha)$, the necessity of the theorem is clear. Without loss of generality, let α: $[0, 1] \to \mathfrak{R}^m$ and β: $[0, 1] \to \mathfrak{R}^m$ be the two irreducible continuous paths in consideration. Denote by T the translation of $\mathfrak{R}^m$ with $T\beta(1) = \alpha(1)$. Then $\gamma = \alpha \cdot T\beta^{-1}$ is again a continuous path. Since $\theta(\gamma) = \theta(\alpha)\theta(T\beta^{-1}) = \theta(\alpha)\theta(\beta)^{-1} = 1$, the path γ: $[0, 2] \to \mathfrak{R}^m$ must be reducible at the value 1. It follows from Lemma 1.2 that there are values s and t of γ such that $(\gamma[s, 1])^{-1}$ and $\gamma[1, t]$ are equivalent, and moreover s is not a reducible value of the path $\gamma[0, s] \cdot \gamma[t, 2]$ provided $0 < s < t < 2$. Since, by Lemma 4.1, the irreducible path of γ is empty, we conclude that $s=0$ and $t=1$, i.e. α and $T\beta$ are equivalent. Hence the theorem is proved.

COROLLARY. *If, for two piecewise regular continuous paths α and β in $\mathfrak{R}^m$, $\theta(\alpha) = \theta(\beta)$ then the irreducible path of β can be obtained from the irreducible path of α by translation and change of parameter.*

The author has conjectured in §3 of [2] that the exponential homomorphism of the path group into the noncommutative formal power series ring is an isomorphism. The above theorem gives an affirmative answer to that conjecture.

THEOREM 4.2. *Let Θ be the exponential homomorphism with respect to the C^{r-1} differentials $\pi_1, \cdots, \pi_m$ in the n dimensional real C^r manifold $\mathfrak{M}$, $r \geqq 2$. If the differentials $\pi_1, \cdots, \pi_m$, $m \geqq n$, are of rank n along two irreducible piecewise regular continuous paths α and β in $\mathfrak{M}$ such that α and β have the same initial point and $\Theta(\alpha) = \Theta(\beta)$, then β may be obtained from α by change of parameter.*

Proof. Let α: $[0, a] \to \mathfrak{M}$ and β: $[0, b] \to \mathfrak{M}$ be the given paths, and construct the paths $\bar{\alpha}$: $[0, a] \to \mathfrak{R}^m$ and $\bar{\beta}$: $[0, b] \to \mathfrak{R}^m$ such that $x_i(\bar{\alpha}(t)) = \int_0^t \pi_i(\dot{\alpha}(s))ds$ and $x_i(\bar{\beta}(t)) = \int_0^t \pi_i(\dot{\beta}(s))ds$, $i = 1, \cdots, m$. Then both $\bar{\alpha}$ and $\bar{\beta}$ are piecewise regular and continuous with $\bar{\alpha}(0) = \bar{\beta}(0)$ and $\theta(\bar{\alpha}) = \theta(\bar{\beta})$. Consequently there exists an increasing continuous function $\tau(t)$, $0 \leqq t \leqq b$ such that $\tau(0) = 0$, $\tau(b) = a$ and $\bar{\alpha}\tau(t) = \bar{\beta}(t)$. We are going to show that $\alpha\tau(t) = \beta(t)$ for $0 \leqq t \leqq b$.

The equation $\alpha\tau(t) = \beta(t)$ obviously holds for $t = 0$. If it holds for $0 \leqq t < s \leqq b$, then $\alpha\tau(s) = \beta(s)$ by continuity. It remains to prove that, if $\alpha\tau(s) = \beta(s) = \mathfrak{p}$, $0 \leqq s < b$, then $\alpha\tau(t) = \beta(t)$, $s \leqq t < s + \delta$, for some $\delta > 0$. Without loss of generality, we may suppose that $\pi_1, \cdots, \pi_n$ are linearly independent about the point $\mathfrak{p}$. For a local coordinate system y about $\mathfrak{p}$, we may write $\pi_i = \sum_{j=1}^n f_{ij}(y)dy_j$, $i = 1, \cdots, n$. Then, for $s \leqq t < s + \delta'$, δ' being some positive number,

$$\begin{aligned} \sum_{j=1}^n f_{ij}(y(\beta(t)))dy_j(\beta(t))/dt &= \pi_i(\dot{\beta}(t)) \\ &= dx_i(\bar{\beta}(t))/dt = dx_i(\bar{\alpha}\tau(t))/dt \\ &= \sum_{j=1}^n f_{ij}(y(\alpha\tau(t)))dy_j(\alpha\tau(t))/dt. \end{aligned}$$

In other words, for $s \leqq t < s + \delta'$, both $y = y(\beta(t))$ and $y = y(\alpha\tau(t))$ are solutions of the system of differential equations

$$\sum_{j=1}^n f_{ij}(y)dy_j/dt = \pi_i(\dot{\beta}(t)), \qquad i = 1, \cdots, n, \tag{4.2}$$

with the initial condition $y = y(\mathfrak{p})$ when $t = s$. Since the determinant $|f_{ij}(y)| \neq 0$ about $y = y(\mathfrak{p})$, we obtain, by solving (4.2) for dy_i/dt,

$$dy_i/dt = \sum_{j=1}^n g_{ij}(y)\pi_j(\dot{\beta}(t)), \qquad i = 1, \cdots, n, \tag{4.3}$$

where each $g_{ij}(y)$ is C^{r-1} about $y=y(\mathfrak{p})$. According to the fundamental existence theorem of differential equations, (4.3) possesses a unique solution about $t=s$ with the given initial condition, and so does the system (4.2). Hence $y(\beta(t))=y(\alpha\tau(t))$ i.e. $\beta(t)=\alpha\tau(t)$, $s\leqq t<s+\delta$, for some $\delta>0$, and the theorem is proved.

THEOREM 4.3. *Let Θ be the exponential homomorphism with respect to a base $\omega_1, \cdots, \omega_m$ of the Maurer-Cartan forms of a real Lie group $\mathfrak{G}$. Then one of two irreducible piecewise regular continuous paths α and β in $\mathfrak{G}$ can be obtained from the other by left translation and change of parameter if and only if $\Theta(\alpha)=\Theta(\beta)$.*

Proof. Observe that, if T is a left translation in $\mathfrak{G}$, then $\Theta(T\alpha)=\Theta(\alpha)$ because $\omega_1, \cdots, \omega_m$ are left invariant. Thus we only need prove the sufficiency. Let T be the left translation carrying the initial point of α to that of β. Since $\Theta(T\alpha)=\Theta(\alpha)=\Theta(\beta)$, we apply the preceding theorem and assert that β may be obtained from $T\alpha$ by change of parameter. Hence the theorem is proved.

Finally we remark that Theorems 4.1–4.3 are also valid when, in the assertions, the exponential homomorphisms θ and Θ are replaced by the corresponding exponential antihomomorphisms θ^* and Θ^*.

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