FLEXURE[1]

BY

L. M. MILNE-THOMSON[2]

The importance of the problem of the flexure of a prismatic beam may be measured by the considerable amount of research which the problem has attracted. To cite only a few names [1] of contributors to the theory, Morris, Sokolnikoff, Stevenson and Wigglesworth can be mentioned. Stevenson in particular reduced the problem to the determination of six canonical flexure functions, but a uniform process for obtaining these is lacking.

This paper presents a uniform method, depending essentially on a special formulation of the boundary condition, which leads via conformal mapping of the cross-section on a unit circle and Cauchy's formula, to a single flexure function. In terms of this single flexure function, which gives the shearing stress without further calculation, displacement, torsion and center of flexure may be found directly. The method is presented here for a simply-connected cross-section, but should be capable of extension to a doubly-connected cross-section which can be mapped on a circular annulus. It is remarkable that the whole effect of the loading is expressed by a single complex parameter.

1. We consider a beam of homogeneous material in the form of a prism or cylinder bounded by planes perpendicular to the generating lines. The beam is fixed at one end, the root, and is in equilibrium when acted upon by a transverse force applied in the plane of the other end. Body force is ignored, and the lateral surface of the beam is free of applied force.

Take the origin O in the plane of the root and three mutually perpendicular axes of reference Ox, Oy, OR, the axis OR being parallel to the generating lines of the beam and the other two axes being so chosen that the set forms a right handed system (Fig. 1). The designation OR is used instead of the more usual Oz to avoid confusion with the complex variable $z=x+iy$ which will be used subsequently. On the other hand there will be no confusion if we employ the customary notation $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$ for the stress components.

The transverse force, or load, W is applied in the plane of the end section $R=L$, where L is the length of the beam. This load W will be supposed applied not as a concentrated force but by a distribution of shear (stresses)

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(2) Visiting Professor of Applied Mathematics, Brown University, Providence 12, Rhode Island. Now at the Mathematics Research Center, United States Army, University of Wisconsin.

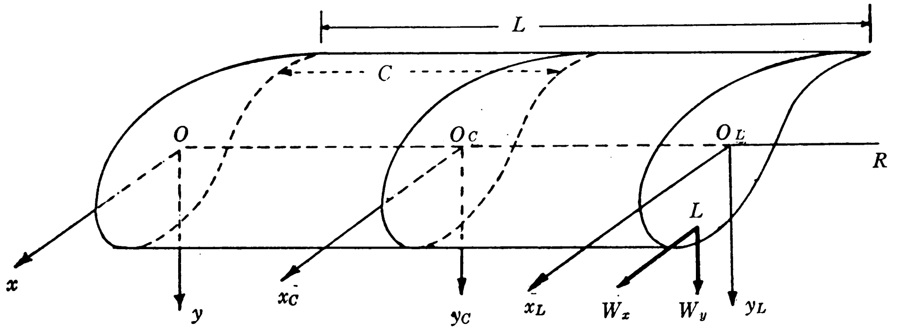

FIG. 1

over the plane $R=L$, the components being denoted by $\widehat{xz}_L$, $\widehat{yz}_L$ parallel to the x- and y-axes.

Take as base point O_L the point in which OR meets the plane $R=L$. Then the above shear system will be statically equivalent to forces W_x, W_y through O_L and a moment M about OR where

(1) $$W_x = \int \widehat{xz}_L dS, \quad W_y = \int \widehat{yz}_L dS, \qquad M = \int (x\cdot\widehat{yz}_L - y\cdot\widehat{xz}_L)dS$$

the integrals being taken over the face $R=L$. Supposing these forces to be equivalent to the force W acting at the point $L_0(x_0, y_0, L)$, or the load-point, we must have

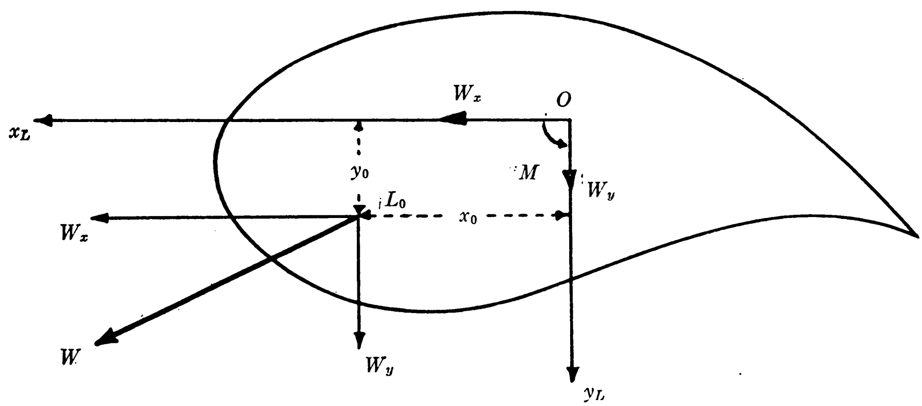

FIG. 2

(2) $$W^2 = W_x^2 + W_y^2,$$

(3) $$M = x_0 W_y - y_0 W_x.$$

Now consider the equilibrium of the material between the planes $R=C$ and $R=L$. Denote by $(\widehat{xz}_C, \widehat{yz}_C, \widehat{zz}_C)$ the components of the stress vector, at the plane $R=C$, exerted by the material for which $R>C$ upon the material for which $R<C$. The corresponding components of the stress vector exerted by the material for which $R<C$ upon the material for which $R>C$ will be $(-xz_C, -yz_C, -zz_C)$ and this latter stress distribution must balance the loan applied to the end $R=L$. Therefore, integrating over the area of the sectiod $R=C$,

$$(4) \qquad W_x = \int \widehat{xz}_C dS, \qquad W_y = \int \widehat{yz}_C dS, \qquad M = \int (x\cdot\widehat{yz}_C - y\cdot\widehat{xz}_C)dS,$$

where W_x, $W_y M$ are the same as in (1) and

$$(5) \qquad \int \widehat{zz}_C dS = 0.$$

Further by taking moments about the lines through O_C, the point where OR meets $R=C$, parallel to the x- and y-axes, we have

$$(6) \qquad (L-C)W_y + \int y\cdot\widehat{zz}_C dS = 0, \qquad (L-C)W_x + \int x\cdot\widehat{zz}_C dS = 0.$$

The conditions that the lateral surface shall be free of applied force are

$$(7) \qquad l\widehat{xx} + m\widehat{yx} = 0, \qquad l\widehat{xy} + m\widehat{yy} = 0 \text{ at the lateral surface,}$$

$$(8) \qquad l\widehat{xz} + m\widehat{yz} = 0 \text{ at the lateral surface,}$$

where $(l, m, 0)$ are the direction cosines of the outward normal at the arc element ds of the boundary of a cross-section of the beam so that

$$(9) \qquad l = \frac{\partial y}{\partial s}, \qquad m = -\frac{\partial x}{\partial s}\cdot$$

Equations (1) to (9) express the conditions of the general problem.

We now observe that the stress components $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ enter only in conditions (7). It is proposed to study St. Venant's problem [2] which arises if we assume antiplane stress, that is to say, if the load is applied in such a way that

$$(10) \qquad \widehat{xx} = \widehat{yy} = \widehat{xy} = 0.$$

Thus St. Venant's problem is expressed by equations (1) to (10).

Since there are no body forces by hypothesis, the stress equilibrium equations [3] reduce to

$$(11) \qquad \frac{\partial\widehat{xz}}{\partial R} = 0, \qquad \frac{\partial\widehat{yz}}{\partial R} = 0, \qquad \frac{\partial\widehat{xz}}{\partial x} + \frac{\partial\widehat{yz}}{\partial y} + \frac{\partial\widehat{zz}}{\partial R} = 0.$$

The first two of these have the important consequence that $\widehat{xz}$ and $\widehat{yz}$ are independent of R. The last equation of (11) then shows that $\widehat{zz}$ must be a linear function of R, for the first two terms of this equation do not contain R. Thus, since $\widehat{zz}_L=0$, we must have

$$\widehat{zz} = (R - L)f(x, y), \tag{12}$$

where $f(x, y)$ is independent of R.

Equations (7) to (10) show that, at the boundary of a cross-section,

$$\widehat{xz}\frac{\partial y}{\partial s} - \widehat{yz}\frac{\partial x}{\partial s} = 0.$$

This condition implies the vanishing of the imaginary part of $(\widehat{xz}-i\widehat{yz})\partial z/\partial s$, where $z=x+iy$. Therefore we have the boundary condition

$$(\widehat{xz} - i\widehat{yz})\frac{\partial z}{\partial s} - (\widehat{xz} + i\widehat{yz})\frac{\partial \bar{z}}{\partial s} = 0 \text{ at the boundary of a cross-section,} \tag{13}$$

where $\bar{z}=x-iy$ is the complex conjugate of z. Here and elsewhere a bar over a symbol denotes the complex conjugate. It is condition (13) that will furnish the solution of the problem. Up to this point the investigation is general, and applies equally well to elastic and plastic deformation.

2. **The stress component $\widehat{zz}$.** We now particularize the general investigation of §1 by assuming that

(i) the material is undergoing infinitesimal elastic deformation;

(ii) the material is isotropic.

For such a state of affairs, in the assumed absence of body force, we have Beltrami's equations [4]

$$(1 + \eta)\nabla^2\widehat{pq} + \partial^2(\widehat{xx} + \widehat{yy} + \widehat{zz})/\partial p\partial q = 0,$$

where η is Poisson's ratio, ∇^2 is Laplace's operator $\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial R^2$, and $\widehat{pq}$ is any one of the six stress components.

In our case of antiplane stress given by (10), Beltrami's equations yield

$$\frac{\partial^2\widehat{zz}}{\partial x^2} = 0, \qquad \frac{\partial^2\widehat{zz}}{\partial x\partial y} = 0, \qquad \frac{\partial^2\widehat{zz}}{\partial y^2} = 0,$$

and therefore $\widehat{zz}$ is a linear function x and y. Therefore from (12) we can write

$$\widehat{zz} = E(R - L)(\alpha_x x + \alpha_y y + \gamma), \tag{14}$$

where

$$E = 2\mu(1 + \eta) \tag{15}$$

is Young's modulus and α_x, α_y, γ are constants. Therefore from (5) we must have

$$\int (\alpha_x x + \alpha_y y + \gamma) dS = 0, \text{ taken over a cross-section}$$

and so

$$\alpha_x x_G + \alpha_y y_G + \gamma = 0, \tag{16}$$

where (x_G, y_G) are the coordinates of the centroid of the section. Therefore from (14) and (16)

$$\widehat{zz} = E(R - L)\left[\alpha_x(x - x_G) + \alpha_y(y - y_G)\right]. \tag{17}$$

To determine α_x, α_y we apply (6) to give

$$\alpha_x \int x(x - x_G)dS + \alpha_y \int x(y - y_G)dS = \frac{W_x}{E},$$

$$\alpha_x \int y(x - x_G)dS + \alpha_y \int y(y - y_G)dS = \frac{W_y}{E}.$$

Now since (x_G, y_G) is the centroid

$$\int (x - x_G)dS = \int (y - y_G)dS = 0$$

and therefore

$$\int x(x - x_G)dS = \int (x - x_G)^2 dS$$

with two similar results. Thus if we write

$$A = \int (y - y_G)^2 dS, \quad B = \int (x - x_G)^2 dS, \quad H = \int (x - x_G)(y - y_G)dS, \tag{18}$$

so that A, B, H are the second moments and products of area with respect to axes through the centroid of the section parallel to the x and y axes, we have

$$\alpha_x B + \alpha_y H = W_x/E, \qquad \alpha_x H + \alpha_y A = W_y/E$$

and therefore

$$\alpha_x = \frac{AW_x - HW_y}{E(AB - H^2)}, \qquad \alpha_y = \frac{BW_y - HW_x}{E(AB - H^2)}. \tag{19}$$

Thus $\widehat{zz}$ is completely determined by (17), (18), (19). We now introduce two useful parameters

$$\beta = \alpha_x + i\alpha_y, \qquad P = W_x + iW_y. \tag{20}$$

Then (19) gives

(21) $$\beta = \{(A + B)P - (B - A + 2iH)\bar{P}\}/2E(AB - H^2).$$

Write

(22) $$\xi = x_G + iy_G.$$

Then from (18) we find, after a simple reduction,

(23) $$A + B = \int (z - \xi)(\bar{z} - \bar{\xi})dS = -\frac{1}{4} i \int_C (\bar{z} - \bar{\xi})^2(z - \xi)dz,$$

(24) $$B - A + 2iH = \int (z - \xi)^2 dS = -\frac{1}{2} i \int_C (z - \xi)^2(\bar{z} - \bar{\xi})dz,$$

where C denotes the contour of a cross-section. These results follow from the Stokes' complex theorem [5]. Moreover on C, $\bar{z}$ is a function of z and therefore $A+B$, $B-A+2iH$ can be expressed as ordinary contour integrals to which Cauchy's residue theorem may be applied. This method probably gives the simplest analytic process for finding plane surface integrals.

The foregoing results show that all formulae are greatly simplified by taking the origin at the centroid of the section, for then $\xi=0$. The centroid is, however, not always the origin most convenient analytically.

From (14), (22), (16) we have

(25) $$\widehat{zz} = \frac{1}{2} E(R - L)(\beta\bar{z} + \bar{\beta}z + 2\gamma),$$

(26) $$2\gamma = -\beta\bar{\xi} - \bar{\beta}\xi.$$

If we observe that

$$4(AB - H^2) = (A + B)^2 - (B - A + 2iH)(B - A - 2iH),$$

the value of β is at once determined if we calculate the integrals (23) and (24).

3. **The shears.** We proceed to find the shears $\widehat{xz}$, $\widehat{yz}$. Substitute the expression (14) for $\widehat{zz}$ in the third stress equilibrium Equation (11). This equation can then be written in the form

$$\frac{\partial}{\partial x}\left\{\widehat{xz} + \frac{1}{2} E(\alpha_x x^2 + \gamma x)\right\} + \frac{\partial}{\partial y}\left\{\widehat{yz} + \frac{1}{2} E(\alpha_y y^2 + \gamma y)\right\} = 0.$$

Therefore there exists a real-valued stress function, which we denote by $\mu F_1(x, y)$ or μF_1, such that

(27) $$\widehat{xz} = \mu\partial F_1/\partial y - \frac{1}{2} E(\alpha_x x^2 + \gamma x),$$

(28) $$\widehat{yz} = -\mu\partial F_1/\partial x - \frac{1}{2} E(\alpha_y y^2 + \gamma y).$$

Therefore since $E=2\mu(1+\eta)$, we have in terms of $z=x+iy$

$$(29)\quad \widehat{xz} - i\widehat{yz} = 2i\mu\partial F_1/\partial z - \frac{1}{4}\mu(1+\eta)\{\alpha_x(z+\bar{z})^2 + i\alpha_y(z-\bar{z})^2 + 4\gamma\bar{z}\}.$$

In terms of the parameter β of (20) we reduce this to

$$(30)\quad \widehat{xz} - i\widehat{yz} = 2i\mu\partial F_1/\partial z - \frac{1}{4}\mu(1+\eta)\{\beta(z^2+\bar{z}^2 - 2\xi\bar{z}) + 2\bar{\beta}(z\bar{z} - \xi\bar{z})\}.$$

Again from Beltrami's equations (§2) we have

$$(31)\qquad \frac{\partial^2\widehat{zz}}{\partial x\partial R} + (1+\eta)\nabla_1^2\widehat{xz} = 0, \qquad \frac{\partial^2\widehat{zz}}{\partial y\partial R} + (1+\eta)\nabla_1^2\widehat{yz} = 0,$$

where, since $\widehat{xz}$ and $\widehat{yz}$ are independent of R,

$$(32)\qquad \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\partial\bar{z}}\cdot$$

Therefore using (14) once more

$$(33)\qquad \nabla_1^2(\widehat{xz} - i\widehat{yz}) = -2\mu\alpha_x + 2\mu i\alpha_y = -2\mu\bar{\beta}.$$

Using (32) to eliminate $\widehat{xz}-i\widehat{yz}$ from (30) and (33) we get

$$(34)\qquad 4\frac{\partial^3 F_1}{\partial z^2\partial\bar{z}} = -i\eta\bar{\beta}.$$

Integration with respect to z then gives

$$\nabla_1^2 F_1 = \frac{4\partial^2 F_1}{\partial z\partial\bar{z}} = -i\eta\bar{\beta}z + i\eta\beta\bar{z} - 2\tau$$

where 2τ is an arbitrary real constant and the right-hand side has been adjusted to be real, since $\nabla_1^2 F_1$ is real, by adding the conjugate complex term $i\eta\beta\bar{z}$.

Integration with respect to $\bar{z}$ leads to

$$4\partial F_1/\partial z = 2F_2(z) - 2\tau\bar{z} + \frac{1}{2}i\eta\beta\bar{z}^2 - i\eta\bar{\beta}z\bar{z}$$

where $2F_2(z)$ is an arbitrary function of z. Substitution for $\partial F_1/\partial z$ in (30) then gives

$$(35)\qquad \begin{aligned}\widehat{xz} - i\widehat{yz} = {} & i\mu F_2(z) - i\mu\tau\bar{z} - \frac{1}{4}\mu\eta\beta\bar{z}^2 + \frac{1}{2}\mu\eta\bar{\beta}z\bar{z} \\ & - \frac{1}{4}\mu(1+\eta)\{\beta(z^2+\bar{z}^2 - 2\xi\bar{z}) + 2\bar{\beta}(z\bar{z} - \xi\bar{z})\}.\end{aligned}$$

Since $iF_2(z)$ is an arbitrary function we may replace it by $iF_2(z)=\Phi(z)+(1+\eta)\beta z^2/4$, and then (35) becomes

$$\widehat{xz} - i\widehat{yz} = \mu\{\Phi(z) - p\bar{z} - q\bar{z}^2 - rz\bar{z}\}, \tag{36}$$

where

$$p = i\tau - \frac{1}{2}(1+\eta)(\beta\bar{\xi} - \bar{\beta}\xi), \tag{37}$$

$$q = \frac{1}{4}(1+2\eta)\beta, \qquad r = \frac{1}{2}\bar{\beta}. \tag{38}$$

We call $\Phi(z)$ the *flexure function.* In so far as the stress is finite and continuous over the whole of the cross-section, the function $\Phi(z)$ is holomorphic over the whole cross-section. The solution of the flexure problem is therefore reduced to the determination of the flexure function $\Phi(z)$.

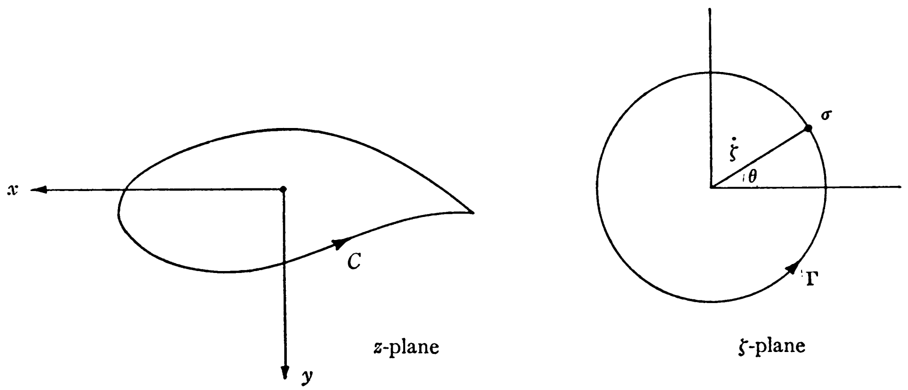

FIG. 3

4. **Solution of the flexure problem.** From (13) and (36) the boundary condition is

$$\Phi(z)\frac{\partial z}{\partial s} - \bar{\Phi}(\bar{z})\frac{\partial \bar{z}}{\partial s} = (p + q\bar{z} + rz)\bar{z}\frac{\partial z}{\partial s} - (\bar{p} + \bar{q}z + \bar{r}\bar{z})z\frac{\partial \bar{z}}{\partial s} \tag{39}$$

at the boundary C of a cross-section of the beam. We now show that this condition suffices to determine uniquely the holomorphic function $\Phi(z)$ whenever the cross-section can be mapped conformally on the unit circle.

Let the cross-section C and its interior be mapped on the circumference Γ and the interior of the unit circle $|\zeta| \leq 1$ in a ζ-plane by the mapping function

$$z = m(\zeta). \tag{40}$$

Write

$$\Phi(z) = \Phi[m(\zeta)] = \psi(\zeta). \tag{41}$$

Let $\sigma = e^{i\theta}$ denote a point on the circumference Γ. Then $\bar{\sigma} = e^{-i\theta} = 1/\sigma$ and therefore for points z on the contour of the section

$$z = m(\sigma), \qquad \bar{z} = \bar{m}(\bar{\sigma}) = \bar{m}(1/\sigma) \tag{42}$$

so that

$$\frac{\partial z}{\partial s} = m'(\sigma)\frac{d\sigma}{ds}, \qquad \frac{\partial \bar{z}}{\partial s} = -\frac{1}{\sigma^2}\bar{m}'\left(\frac{1}{\sigma}\right)\frac{d\sigma}{ds},$$

and therefore the boundary condition (39) becomes

$$\psi(\sigma)m'(\sigma) + \frac{1}{\sigma^2}\bar{\psi}(1/\sigma)\bar{m}'(1/\sigma) = H(\sigma), \tag{43}$$

where

$$\begin{aligned} H(\sigma) = {} & [p + q\bar{m}(1/\sigma) + rm(\sigma)]\bar{m}(1/\sigma)m'(\sigma) \\ & + \frac{1}{\sigma^2}[\bar{p} + \bar{q}m(\sigma) + \bar{r}\bar{m}(1/\sigma)]m(\sigma)\bar{m}'(1/\sigma). \end{aligned} \tag{44}$$

Multiply (43) by $d\sigma/[2\pi i(\sigma - \zeta)]$ and integrate round Γ. We then get

$$\frac{1}{2\pi i}\int_\Gamma \frac{\psi(\sigma)m'(\sigma)d\sigma}{\sigma - \zeta} + \frac{1}{2\pi i}\int_\Gamma \frac{\bar{\psi}(1/\sigma)\bar{m}'(1/\sigma)d\sigma}{\sigma^2(\sigma - \zeta)} = \frac{1}{2\pi i}\int_\Gamma \frac{H(\sigma)d\sigma}{\sigma - \zeta}.$$

If ζ is inside Γ, $\psi(\zeta)$ and $m'(\zeta)$ are necessarily holomorphic, while the point $1/\zeta$ is correspondingly outside Γ and therefore $\bar{\psi}(1/\zeta)$ and $\bar{m}'(1/\zeta)$ are holomorphic when $1/\zeta$ is outside and therefore ζ is inside Γ. Therefore by Cauchy's formula [6] the first integral becomes $\psi(\zeta)m'(\zeta)$, and the second integral vanishes by Cauchy's theorem. Therefore

$$\psi(\zeta)m'(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{H(\sigma)d\sigma}{\sigma - \zeta}. \tag{45}$$

This determines $\psi(\zeta)$ and therefore from (41) $\Phi(z)$ and consequently from (36), $\widehat{xz} - i\widehat{yz}$. Thus (45) constitutes the solution of the flexure problem.

NOTES.

(i) Equation (45) determines $\psi(\zeta)$ uniquely.

(ii) We can use (40) and (45) together with (36) to express the shears in terms of ζ as a parameter, related to z by (40). This fact is useful because elimination of ζ may not always be convenient or even feasible.

(iii) Whenever the mapping function $m(\zeta)$ is rational, $H(\sigma)$ given by (44) is also a rational function, and therefore $\psi(\zeta)$ can be expressed in terms of elementary functions.

(iv) If we can write $H(\sigma)$ in the form $H(\sigma)=H_1(\sigma)+H_2(1/\sigma)$, where all and only the negative powers of σ occur in $H_2(\sigma)$, then (45) gives $\psi(\zeta)m'(\zeta)=H_1(\zeta)$. In other words in this case we can write down the solution without calculation. This will always happen, in particular, when the mapping function $m(\zeta)$ is rational.

(v) Looking at (37), (38), (44) we see that the solution is obtained as a function of ξ, β, τ, wherein ξ depends only on the geometry of the cross-section, β includes the whole effect of the loading and τ will be interpreted in §6 as a twist.

(vi) It will be shown in §5 that the displacement is also determined to within a rigid body movement from $\Phi(z)$.

(vii) If the distribution applied to the face $R=L$ is such that in equations (1), $W_x=W_y=0$, $M\neq 0$, the action on the face reduces to a couple of moment M and we have the torsion problem. In this case (21) shows that $\beta=0$ and therefore from (37) and (38) $p=i\tau$, $q=r=0$. Thus the solution of the torsion problem is given by

$$\widehat{xz} - i\widehat{yz} = \mu[\Phi(z) - i\tau\bar{z}],$$

and the function $H(\sigma)$ in (45) simplifies to

$$H(\sigma) = i\tau\{\bar{m}(1/\sigma)m'(\sigma) - m(\sigma)\bar{m}'(1/\sigma)/\sigma^2\}.$$

5. **The displacement.** If (u, v, w) are the components of the displacement, Hooke's law leads to six equations of which two are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

which show that u, v are conjugate functions. If then we write $V=u+iv$, $\overline{V}=u-iv$, V is independent of $\bar{z}$ and $\overline{V}$ is independent of z, but both depend on R. The remaining equations of Hooke's law, then reduce to

$$\frac{\partial V}{\partial z} + \frac{\partial \overline{V}}{\partial \bar{z}} = -\eta(R - L)(\beta\bar{z} + \bar{\beta}z + 2\gamma),$$

$$\frac{\partial \overline{V}}{\partial R} + 2\frac{\partial w}{\partial z} = \Phi(z) - p\bar{z} - q\bar{z}^2 - rz\bar{z},$$

$$\frac{\partial w}{\partial R} = \frac{1}{2}(R - L)(\beta\bar{z} + \bar{\beta}z + 2\gamma).$$

We then get

$$\frac{\partial \overline{V}}{\partial \bar{z}} + \eta(R - L)(\beta\bar{z} + \gamma) = -\frac{\partial V}{\partial z} - \eta(R - L)(\bar{\beta}z + \gamma) = if_1(R)$$

since one side is independent of z, and the other of $\bar{z}$, and they are conjugate complex. Here $f_1(R)$ is a real valued function of R. Forming and eliminating $\partial^2\overline{V}/\partial R\partial\bar{z}$ we find that

$$2\frac{\partial^2 w}{\partial z\partial\bar{z}} = -rz - \bar{r}\bar{z} - i\tau - \gamma - if_1'(R).$$

Expressing that the right-hand side of this equation is real since the left-hand side is, we find that $f_1'(R) = -\tau$. After some reduction we finally obtain

$$u - iv = \overline{V} = -\frac{1}{6}\bar{\beta}(R-L)^3 - (R-L)\left\{\eta\left(\frac{1}{2}\beta\bar{z}^2 + \gamma\bar{z}\right) + i\tau\bar{z}\right\}, \tag{46}$$

$$\begin{aligned} 2w &= \phi(z) + \bar{\phi}(\bar{z}) - \frac{1}{2}rz^2\bar{z} - \frac{1}{2}\bar{r}z\bar{z}^2 - \gamma z\bar{z} \\ &\quad + \frac{1}{2}(R-L)^2\{\bar{\beta}z + \beta\bar{z} + 2\gamma\}, \end{aligned} \tag{47}$$

wherein a rigid body movement has been omitted and $\phi(z)$ denotes the indefinite integral of $\Phi(z)$.

6. **The center of flexure.** The rotation is

$$\Omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{1}{2}i\left(\frac{\partial V}{\partial z} - \frac{\partial\overline{V}}{\partial\bar{z}}\right) = \frac{1}{2}(R-L)[2\tau - i\eta(\beta\bar{z} - \bar{\beta}z)].$$

Thus in the torsion case, i.e. when $\beta = 0$, the rotation is $(R-L)\tau$, which interprets τ as a twist per unit length.

Moreover when $\beta \neq 0$, that is to say in the general case, flexure is usually accompanied by torsion. The rate of change of Ω with R is

$$\omega = \partial\Omega/\partial R = \tau - \frac{1}{2}i\eta(\beta\bar{z} - \bar{\beta}z), \tag{48}$$

which is called the *local twist*.

The mean value of the local twist over the area of the cross-section is

$$\omega_m = \tau - \frac{1}{2}i\eta(\beta\bar{\xi} - \bar{\beta}\xi)$$

which is the same as the value of ω at the centroid of the section.

DEFINITION. The *center of flexure* is the position of the load point when the local twist at the centroid is zero.

This occurs when

$$\tau = \frac{1}{2}i\eta(\beta\bar{\xi} - \bar{\beta}\xi). \tag{49}$$

The corresponding value of p from (37) is

$$p_0 = -\frac{1}{2}\beta\bar{\xi}(1+2\eta) - \frac{1}{2}\bar{\beta}\xi. \tag{50}$$

In particular when the origin is at the centroid, $p_0=0$.

7. **The moment.** Let M be the moment of the applied load with respect to the point O_L. We then have, cf. (1),

$$\frac{M+iM'}{\mu} = \frac{1}{\mu}\int iz(\widehat{xz} - i\widehat{yz})dS = i\int z\Phi(z)dS - i\int pz\bar{z}dS$$
$$- i\int qz\bar{z}^2dS - i\int rz^2\bar{z}dS,$$

where the integrals are taken over the area of the section $R=L$ and M' is a real number whose value does not concern us. We write

$$I = \int z\bar{z}dS = -\frac{1}{4}i\int_C z\bar{z}^2dz, \tag{51}$$

$$J = 6i\int z\bar{z}^2dS = \int_C z\bar{z}^3dz, \tag{52}$$

$$K = 2i\int z\Phi(z)dS = \int_C z\bar{z}\Phi(z)dz, \tag{53}$$

where we have applied Stokes' complex theorem to the surface integrals, which we note do not depend on the loading. We also note that I is the polar second moment of the area of the cross-section with respect to the origin O_L. Thus

$$M + iM' = \mu\left\{\frac{1}{2}K - ipI - \frac{1}{6}qJ + \frac{1}{6}r\bar{J}\right\}$$

and therefore writing the conjugate and using (37), (38) we have

$$2M/\mu = \frac{1}{2}(K+\bar{K}) + 2\tau I + \frac{1}{24}(1-2\eta)(\beta J + \bar{\beta}\bar{J}). \tag{54}$$

If z_0 is the load point, (3) gives

$$2M = iz_0\bar{P} - i\bar{z}_0P. \tag{55}$$

By equating the values of M given by (54) and (55) we find the value of τ_0 for a given load point, or alternatively the load point corresponding to a given value of τ.

We also note from (55) that

$$\bar{z}_0 = 2i\partial M/\partial P. \tag{56}$$

To find the center of flexure z_c we insert in (56) the value of τ given by (49) namely $\tau = 2^{-1}i\eta(\beta\bar{\xi} - \bar{\beta}\xi)$, and so find z_c from (56) in the form

$$\bar{z}_c = 2i\partial M_0/\partial P, \tag{57}$$

where M_0 is the moment about the origin when the local twist at the centroid is zero.

In performing the calculations of (56) or (57) we note that M or M_0 involve the load P only through β and $\bar{\beta}$ as given by (21); also that M is linear in β and $\bar{\beta}$ and therefore in P.

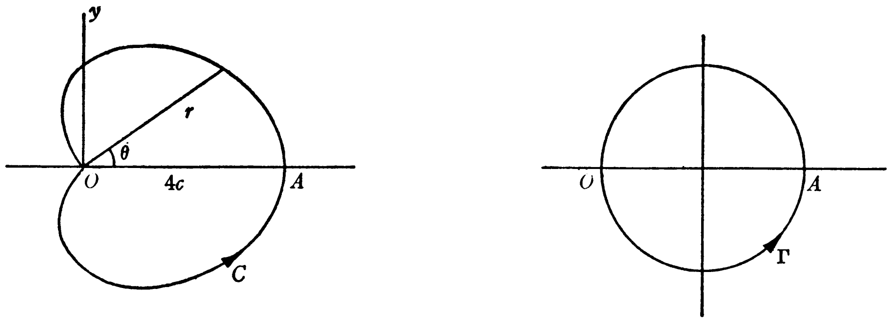

FIG. 4

8. **We take some illustrations.**

(i) *Cross-section a circle of radius a.* This is mapped on the unit circle by $z = a\zeta = m(\zeta)$ and so from (44)

$$H(\sigma) = a^3(r + \bar{q}) + (p + \bar{p})a^2/\sigma + (q + \bar{r})a^3/\sigma^2$$

and therefore from 4(iv)

$$\Phi(z)\cdot a = \psi(\zeta)\cdot a = a^3(r + \bar{q}) = a^3\bar{\beta}(3 + 2\eta)/4$$

from (38). Now for the circle, $A = B = \pi a^4/4$, $H = 0$ and therefore from (21) $\beta = 2P/\{\mu\pi a^4(1+\eta)\}$. Also $\xi = 0$ and so, from (37), $p = i\tau$. Thus we get from (36)

$$\widehat{xz} - i\widehat{yz} = \{[(3 + 2\eta)a^2 - 2z\bar{z}]\bar{P} - (1 + 2\eta)\bar{z}^2P\}/\{2\pi a^4(1 + \eta)\} - \mu i\tau\bar{z}.$$

For instance if $\tau = 0$ and $P = W$, i.e. if $W_x = W$, $W_y = 0$, we get

$$\widehat{xz} = (W/2\pi a^4(1 + \eta))[(3 + 2\eta)a^2 - 3x^2 - y^2 - 2\eta(x^2 - y^2)],$$

$$\widehat{yz} = -(W(1 + 2\eta)/\pi a^4(1 + \eta))xy.$$

(ii) *Cross-section a cardioid.*

We take the cardioid $r=2c(1+\cos\theta)$. This is mapped on the unit circle by

$$z = m(\zeta) = c(1+\zeta)^2.$$

Here (49) gives

$$\frac{H(\sigma)}{2c^2} = \frac{p(1+\sigma)^3}{\sigma^2} + qc\frac{(1+\sigma)^5}{\sigma^4} + rc\frac{(1+\sigma)^5}{\sigma^2} + \bar{p}\frac{(1+\sigma)^3}{\sigma^3}$$
$$+ \bar{q}c\frac{(1+\sigma)^5}{\sigma^3} + \bar{r}c\frac{(1+\sigma)^5}{\sigma^5}.$$

Picking out $H_1(\sigma)$ we get from 4 (iv)

$$\frac{\Phi(z)(1+\zeta)}{c} = p(\zeta+3) + qc(\zeta+5) + rc(\zeta^3+5\zeta^2+10\zeta+10)$$
$$+ \bar{p} + \bar{q}c(\zeta^2+5\zeta+10) + \bar{r}c,$$

and so eliminating by means of $\zeta=(z/c)^{1/2}-1$ we get

$$\Phi(z) = rcz + c^2(2r+\bar{q})(z/c)^{1/2} + c^2(q+3\bar{q}+3r)$$
$$+ c(2p+\bar{p}+4qc+6\bar{q}c+4rc+\bar{r}c)(c/z)^{1/2}.$$

We note that this gives infinite stress at the cusp $z=0$. This is to be expected at the reentrant point and physically leads to plastic yielding.

To find ξ we have for the area

$$S = \int_S dS, \quad \text{and} \quad \xi S = \int_S z dS.$$

The Stokes' complex theorem gives

$$S = -\frac{1}{2}i\int_C \bar{z}dz = -ic^2\int_\Gamma \frac{(1+\sigma)^3 d\sigma}{\sigma^2},$$
$$\xi S = -\frac{1}{2}i\int_C z\bar{z}dz = -ic^3\int_\Gamma \frac{(1+\sigma)^5 d\sigma}{\sigma^2}.$$

Thus by the Residue Theorem $S=6\pi c^2$, $\xi S=10\pi c^3$, $\xi=5c/3$. To find the second moments of area we use (23) and (24) which give in similar fashion

$$A+B = 55\pi c^4/3, \qquad B-A+2iH = -8\pi c^4/3,$$

whence we get

$$\beta = \frac{55P+8\overline{P}}{21\times 47(1+\eta)\pi\mu c^4}.$$

To calculate the moment M about the origin we need the constants I, J, K of §7. We have

$$I = A + B + \xi^2 S = 35\pi c^4,$$

$$J = \int_C z\bar{z}^3 dz = 2\pi 2c^5 \times \text{residue of } (1+\sigma)^9/\sigma^6 = 504\pi i c^5,$$

$$K = \int_C z\bar{z}\Phi(z)dz = 4\pi i c^4\{4\bar{r}c + 50rc + 45\bar{q}c + 21qc + 13p + 4\bar{p}\}.$$

Combining these (54) gives for the moment

$$2M/(\mu\pi c^4) = 34\tau - (37 + 66\eta)(\beta - \bar{\beta})ic.$$

To find the center of flexure we use (49) to give $\tau = 5c\eta i(\beta - \bar{\beta})/6$ leading to

$$2M_0/(\mu\pi c^4) = -i\frac{c}{3}(\beta - \bar{\beta})(111 + 113\eta),$$

when using the value of β just found and (57) we get

$$\bar{z}_c = c(111 + 113\eta)/\{63(1 + \eta)\}$$

which is real and therefore the center of flexure is on the axis at the point $x_c = \bar{z}_c$, $y_c = 0$. This agrees with the calculation of Sokolnikoff [7] by a different method.

(iii) *Cross-section one loop of Bernoulli's lemniscate.*

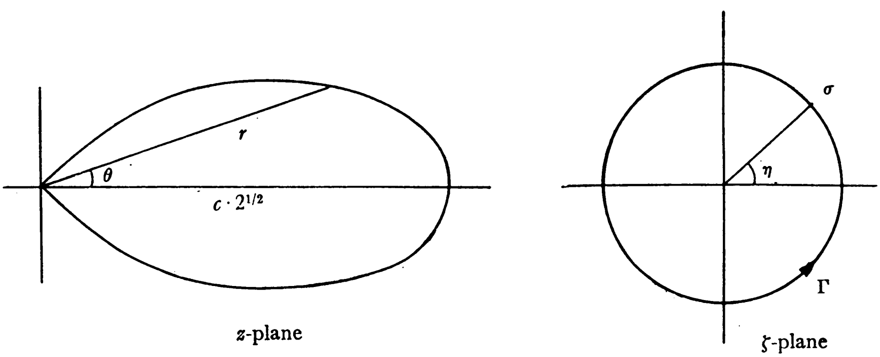

FIG. 5

One loop of Bernoulli's lemniscate

$$r^2 = 2c^2 \cos 2\theta$$

is mapped on the unit circumference Γ by

$$z = c(1 + \sigma)^{1/2}, \qquad \sigma = e^{i\eta}.$$

Thus $m(\sigma) = c(1+\sigma)^{1/2}$ is not a rational function so that the simple method of 4(iv) is not available. We have in fact

$$H(\sigma) = \frac{1}{2}pc^2\sigma^{-1/2} + \frac{1}{2}qc^3(1+\sigma)^{1/2}\sigma^{-1} + \frac{1}{2}rc^3(1+\sigma)^{1/2}\sigma^{-1/2}$$
$$+ \frac{1}{2}\bar{p}c^2\sigma^{-3/2} + \frac{1}{2}\bar{q}c^3(1+\sigma)^{1/2}\sigma^{-3/2} + \frac{1}{2}\bar{r}c^3(1+\sigma)^{1/2}\sigma^{-2}.$$

Using the identities

$$\frac{1}{\sigma(\sigma-\zeta)} = \frac{1}{\zeta}\left(\frac{1}{\sigma-\zeta} - \frac{1}{\sigma}\right),$$
$$\frac{1}{\sigma^2(\sigma-\zeta)} = \frac{1}{\zeta^2}\left(\frac{1}{\sigma-\zeta} - \frac{1}{\sigma}\right) - \frac{1}{\zeta}\frac{1}{\sigma^2}$$

the problem reduces effectively to calculating

$$\frac{1}{2\pi i}\int_\Gamma \frac{(1+\sigma)^{1/2}d\sigma}{\sigma-\zeta}, \qquad \frac{1}{2\pi i}\int_\Gamma \frac{\sigma^{1/2}d\sigma}{\sigma-\zeta}, \qquad \frac{1}{2\pi i}\int_\Gamma \frac{(1+\sigma)^{1/2}\sigma^{1/2}d\sigma}{\sigma-\zeta},$$

their values when $\zeta=0$, and their first derivatives with respect to ζ, when $\zeta=0$. Denote the integrals by $f_1(\zeta)$, $f_2(\zeta)$, $f_3(\zeta)$ respectively. We have at once $f_1(\zeta)=(1+\zeta)^{1/2}$ since the function $(1+\zeta)^{1/2}$ has no singularities inside the circle. The other two integrands present branch points at $\sigma=0$.

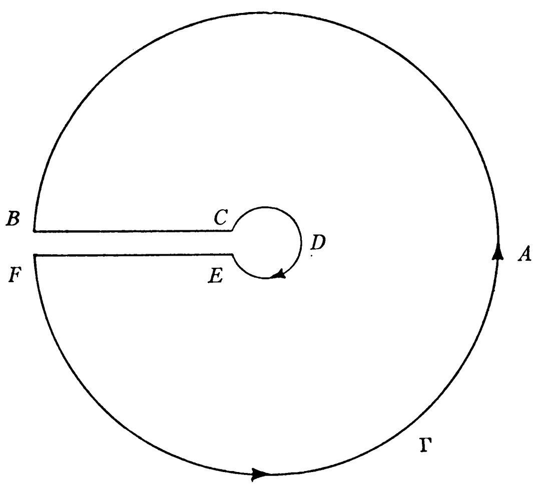

FIG. 6

To illustrate the process of calculating $f_2(\zeta)$, $f_3(\zeta)$ we shall calculate $f_3(\zeta)$, using the contour shown in Fig. 6, namely the contour $ABCDEFA$, where CDE is a circle concentric with Γ, of small radius r, and BC, EF are lines parallel and near to the negative real axis. Within this contour the integrand is holomorphic and one-valued so that by Cauchy's theorem the integral is zero. If we denote by L the limit of the path $BCDEF$ when $r \to 0$, we have

$$\int_\Gamma \frac{(1+\sigma)^{1/2}\sigma^{1/2}d\sigma}{\sigma - \zeta} + \int_L \frac{(1+\sigma)^{1/2}\sigma^{1/2}d\sigma}{\sigma - \zeta} = 0$$

that is to say

$$2\pi i f_3(\zeta) = -\int_L \frac{(1+\sigma)^{1/2}\sigma^{1/2}d\sigma}{\sigma - \zeta} = 2\,i\int_0^1 \frac{x^{1/2}(1-x)^{1/2}dx}{x+\zeta}$$

where x is a real variable. This integral can be evaluated by the substitution $x = 1/(1+t^2)$, whence we find $f_3(\zeta) = (1+2\zeta)/2$.

The same contour and the substitution $x = t^2$ for the integral on L gives

$$f_2(\zeta) = \frac{1}{2\pi}\left\{1 + \zeta^{1/2}\tan^{-1}\zeta^{1/2}\right\}.$$

Thus from (41) and (45)

$$\begin{aligned}\frac{1}{2}c\Phi(z)(1+\zeta)^{-1/2} &= (pc^2/\pi)\zeta^{-1/2}\tan^{-1}\zeta^{1/2} \\ &+ (\bar{p}c^2/\pi)\left\{-\zeta^{-1} + \zeta^{-3/2}\tan^{-1}\zeta^{1/2}\right\} + \frac{1}{2}qc^3\left\{-1 + (1+\zeta)^{1/2}\right\}\zeta^{-1} \\ &+ \frac{1}{2}rc^3 + \frac{1}{2}\bar{r}c^3\left\{-1 - \frac{1}{2}\zeta + (1+\zeta)^{1/2}\right\}\zeta^{-2}.\end{aligned}$$

Since $\tan^{-1}\zeta^{1/2} = \zeta^{1/2} - \zeta^{3/2}/3 + \cdots$, by Gregory's series and $(1+\zeta)^{1/2} = 1 + \zeta/2 + \cdots$, there is no singularity at $\zeta = 0$ or at $\zeta = -1$ which gives the vertex of the loop of the lemniscate. Thus the stress at the vertex is finite and is in fact zero, which is what we should expect at a salient point.

The flexure function $\Phi(z)$ in this case is easily expressed in terms of z since $\zeta = z^2/c^2 - 1$, from the mapping function. The calculation of β presents no difficulty, so that the flexure problem is effectively solved.

9. **Half-sections.**

Suppose we have a section for which $0x$ is an axis of symmetry. Let this section be mapped on the unit circle in a t-plane by

$$z = m(t)$$

in such a way that the axis of symmetry OA of the section maps onto the diameter OA on the real axis in the t-plane. Then the same mapping function maps the half-section OAB in the z-plane on the semicircle OAB in the t-plane. This semicircle can in turn be mapped on the unit circle in the ζ-plane by

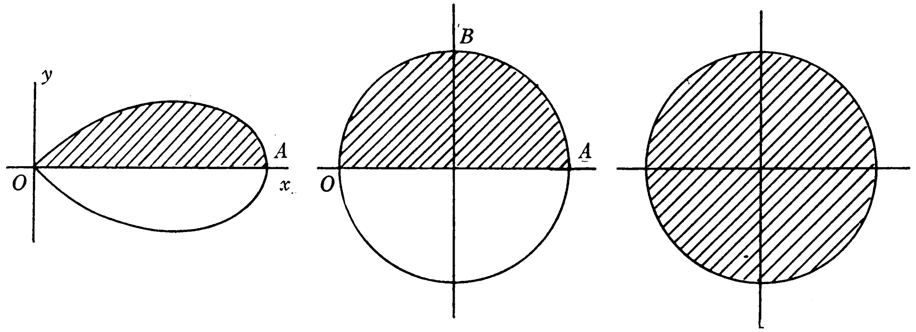

FIG. 7

$$(\zeta - 1)/(\zeta + 1) = -i\{(t - 1)/(t + 1)\}^2,$$

and these two mappings map the half-section in the z-plane on the unit circle in the ζ-plane, so that we can also solve the flexure problem for the half-section.

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BROWN UNIVERSITY,
PROVIDENCE, R. I.