DERIVATIVE MANIFOLDS AND TAYLOR SERIES IN THE MEAN(1)

BY

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1. **Introduction.** Let $f(x)\in C^\infty$ $(-\infty, \infty)$ and let there be some $p\geqq 1$ such that $f^{(n)}(x)\in L^p(-\infty, \infty)$ $(n\geqq 0)$. It has been shown [1, p. 38] that the closed linear manifold $D[f]$ of L^p spanned by $f(x), f'(x), f''(x), \cdots$ is contained in the closed linear manifold $T[f]$ spanned by the translates $f(x+h)$. For $p=2$ there is the stronger result [2, p. 130](2)

$$\underset{h\to 0}{\text{l.i.m.}}\ \frac{f^{(n)}(x+h)-f^{(n)}(x)}{h} = f^{(n+1)}(x).$$

It is thus of interest to determine when $D[f]=T[f]$. This question has been studied by Mandelbrojt [1, pp. 39–40], who has found it necessary to consider classes of functions designated $L^p\{M_n\}$, where $\{M_n\}$ is a sequence of positive reals and $L^p\{M_n\}$ consists of all C^∞ $(-\infty, \infty)$ functions such that $\|f^{(n)}(x)\|_p \leqq M_n$ $(n\geqq 0)$. His principal result states that $D[f]=T[f]$ for all $f(x)\in L^p\{M_n\}$ if and only if the class $C\{M_n\}$ is quasi-analytic, $C\{M_n\}$ being the class of all $f(x)\in C$ $(-\infty, \infty)$ such that $|f^{(n)}(x)|\leqq k^n M_n$ $(-\infty\leqq x\leqq\infty)$ $(n\geqq 0)$ for some positive k. If $C\{M_n\}$ is not quasi-analytic, then there is at least one $f(x)\in L^p\{M_n\}$ for which $D[f]\neq T[f]$. Hence, if the derivatives of $f(x)$ are too large in norm, Mandelbrojt's theorem cannot be used to decide whether $D[f]=T[f]$.

For $p=2$ we shall establish the following necessary and sufficient condition on $f(x)$ for $D[f]=T[f]$:

THEOREM 1.1. *Let $f(x)$ be a $C^\infty(-\infty, \infty)$ function such that $f^{(n)}(x)\in L^2(-\infty, \infty)$ $(n\geqq 0)$ and let $F(x)$ be the Fourier transform of $f(x)$. Then $D[f]=T[f]$ if and only if the distribution function $\psi(t)=\int_{-\infty}^{t}|F(x)|^2dx$ is the solution of a determined Hamburger moment problem.*

Thus in L^2 it is not necessary to consider classes of functions. Furthermore, it will be shown that if $D[f]=T[f]$ for an L^2 function, then $D[f^{(n)}]=T[f]$ $(n\geqq 0)$, which in turn implies an interesting result in the theory of orthogonal polynomials.

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(1) This paper is the author's dissertation, done under the supervision of Professor I. J. Schoenberg and submitted to the Faculty of the Graduate School of the University of Pennsylvania in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

(2) As a corollary of our Theorem 3.1, this result is actually valid for general p.

We shall also consider approximating translates by Taylor series; that is, the possibility of representing $f(x+h)$ by

$$\operatorname*{l.i.m.}_{n\to\infty} \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(\nu)}(x).$$

It will be shown that the class of L^p functions admitting such a representation is the generalization to L^p of an analytic function class in L^2 studied by Paley and Wiener [3, pp. 3–13] and that the "Taylor series in the mean" can be used to prove a theorem of Paley and Wiener concerning the Fourier transforms of functions in the Paley-Wiener class.

2. **Derivative manifolds in L^2 and the Hamburger moment problem.** In this chapter we shall restrict ourselves to L^2. Our principal aim is to prove Theorem 1.1, thus showing the intimate connection between the Hamburger moment problem and the derivative manifold in L^2.

To characterize $D[f]$ and $T[f]$ we work with the Fourier transform

$$F(x) = \operatorname*{l.i.m.}_{n\to\infty} (2\pi)^{-1/2} \int_{-n}^{n} f(t)e^{-ixt}dt$$

of $f(x)$. Let A be the (measurable) set of all x for which $F(x)\neq 0$. Then by a theorem of Bochner and Chandrasekharan [2, p. 149], $T[f]$ consists of all $g(x)\in L^2(-\infty, \infty)$ whose Fourier transforms vanish almost everywhere on the complement of A. (As a special case of the theorem, $T[f]=L^2$ if and only if the complement of A has measure zero. This result is due to Wiener [4, p. 100].)

Since the Fourier transform of $f^{(n)}(x)$ is equal to $(ix)^n F(x)$, $D[f]$ consists of all L^2 functions whose Fourier transforms belong to the closed linear manifold spanned by $F(x)$, $xF(x)$, $x^2F(x)$, $\cdots$. It therefore follows that $D[f]=T[f]$ if and only if $F(x)$, $xF(x)$, $x^2F(x)$, $\cdots$ span $L^2(A)$.

We next introduce the distribution function

$$\psi(t) = \int_{-\infty}^{t} |F(x)|^2 dx.$$

Since

$$\int_{-\infty}^{\infty} t^{2n} d\psi(t) = \int_{-\infty}^{\infty} |x^n F(x)|^2 dx < \infty, \tag{2.1}$$

$\psi(t)$ has moments of all orders. Conversely, if $\psi(t)$ is an absolutely continuous distribution having moments of all orders and $F(t)$ is any measurable function such that $|F(t)|^2=\psi'(t)$ almost everywhere, then

$$f(x) = \operatorname*{l.i.m.}_{n\to\infty} \frac{1}{(2\pi)^{1/2}} \int_{-n}^{n} F(t)e^{itx}dt \tag{2.2}$$

is equivalent to a $C^\infty(-\infty, \infty)$ function all of whose derivatives belong to L^2.

Consider now the Hilbert space L^2_ψ consisting of all ψ-measurable $g(t)$ such that

$$\|g(t)\|_\psi^2 = \int_{-\infty}^{\infty} |g(t)|^2 d\psi(t) < \infty.$$

It follows from (2.1) that

$$\|t^n\|_\psi = \|x^n F(x)\|. \tag{2.3}$$

We are thus led to consider the following isometry of $L^2(A)$ onto L^2_ψ

$$g(x)(\epsilon L^2(A)) \leftrightarrow h(t)(\epsilon L^2_\psi),$$

$$h(t) = g(t)/F(t), \tag{2.4}$$

$$g(x) = F(x)h(x). \tag{2.5}$$

(Note that the points where $h(t)$ might be undefined by (2.4) are of ψ-measure zero.) Hence, $D[f]=T[f]$ if and only if $1, t, t^2, \cdots$ are complete in L^2_ψ.

The question of completeness of polynomials in L^2_ψ is completely resolved by the following theorem of M. Riesz [5; 6, p. 62]:

THEOREM 2.1. *Let $\psi(t)$ be a distribution function having moments of all orders. Then* $1, t, t^2, \cdots$ *span L^2_ψ if and only if one of the following condtions holds*:

(1) *$\psi(t)$ is the solution of a determined Hamburger moment problem.*

(2) *The Hamburger moment problem of which $\psi(t)$ is a solution is indeterminate, but $\psi(t)$ is one of R. Nevanlinna's* [6, *p.* 60; 7] *extremal solutions.*

We need not consider here the definition of "extremal solution"; for our purpose we need only remark that such distribution functions are known [6, p. 60; 7] to have discontinuous spectra. Since our $\psi(t)$ is absolutely continuous, the validity of Theorem 1.1 follows.

Theorem 1.1 allows us to construct functions for which $D[f]\neq T[f]$. If $\psi(t)$ is an absolutely continuous solution of an indeterminate Hamburger moment problem(3) and $F(t)$ is a measurable function such that $|F(t)|^2 =\psi'(t)$ almost everywhere, then such an $f(x)$ may be obtained from $F(t)$ via (2.2). An example of such a $\psi'(t)$ is $\exp(-|t|^{1/2})$ [6, p. 22]. Hence for

$$f(x) = \int_0^\infty \exp(-t^{1/2}/2) \cos xt\,dt$$

we have $D[f]\neq T[f]$.

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(3) It is of interest to note that Hamburger [6, p. 61; 9] has shown that every indeterminate Hamburger moment problem has absolutely continuous solutions.

Theorem 1.1 may also be used to show that if $D[f]=T[f]$, then derivatives of all orders are not necessary to approximate translates. In fact, we have:

THEOREM 2.2. *Let* $f(x)\in C^\infty(-\infty,\infty)$ *and let* $f^{(n)}(x)\in L^2(-\infty,\infty)$ $(n\geqq 0)$. *Then if* $D[f]=T[f]$, $D[f^{(n)}]=T[f]$ $(n\geqq 0)$. *In other words, for each non-negative integer* n *and each real* h, $f(x+h)$ *is the limit in norm of finite linear combinations of the form*

$$C_n f^{(n)}(x) + \cdots + C_{n+N} f^{(n+N)}(x).$$

To prove Theorem 2.2, we first show that $D[f]=T[f]$ implies that $f(x)\in D[f']$. Taking Fourier transforms and using the isometry (2.4) of $L^2(A)$ onto L^2_ψ, this is seen to be equivalent to showing that 1 belongs to the closed linear manifold of L^2 spanned by $t, t^2, \cdots$. To show this, we evaluate

$$\rho_n = \min_{C_1,\cdots,C_n} \|1 - C_1 t - \cdots - C_n t^n\|_\psi^2.$$

If $\omega_n(t)$ is the orthonormal polynomial of degree n (i.e., $\omega_n(t)$ is of degree n and $\int_{-\infty}^{\infty}\omega_m(t)\omega_n(t)d\psi(t)=\delta_{mn}$) and if

$$K_n(t,0) = \sum_{k=0}^{n} \omega_k(0)\omega_k(t),$$

then for any polynomial $P(t)$ of degree not exceeding n

$$P_n(0) = \int_{-\infty}^{\infty} P_n(t) K_n(t,0) d\psi(t).$$

Applying the Schwarz inequality,

$$|P_n(0)|^2 \leqq \|P_n(t)\|_\psi^2 \sum_{k=0}^{n} \omega_k^2(0).$$

Hence, it is seen that $\rho_n = 1/\sum_{k=0}^{n}\omega_k^2(0)$, the minimum being attained when the C_k are the coefficients of

$$\frac{K_n(t,0)}{K_n(0,0)} - 1.$$

But in the theory of the Hamburger moment problem [6, p. 44], it is shown that $1/\sum_{k=0}^{\infty}\omega_k^2(0)$ is the minimum mass at $t=0$ for any distribution having the same moments as $\psi(t)$. Hence, since $\psi(t)$ is absolutely continuous and determined by its moments, $\lim_{n\to\infty}\rho_n=0$.

Thus $D[f']=T[f]$. But $T[f]=T[f']$, since $\{x\mid F(x)\neq 0\}$ and $\{x\mid xF(x)\neq 0\}$ differ by at most a set of measure zero. Hence, we have

$$D[f] = T[f] = T[f'] = D[f''] = \cdots = D[f^{(n)}] = \cdots.$$

Theorem 2.2 is considerably stronger than the following theorem of Mandelbrojt [1, pp. 39–40], which only states that the first derivative is not needed for approximating translates provided $f(x)$ belongs to a class of functions $\{g(x)\}$ for each of which $D[g]=T[g]$:

THEOREM 2.3. *Let $f(x)\in L^p\{M_n\}$ and let $C\{M_n\}$ be quasi-analytic. Then for each real h, $f(x+h)$ is the limit in norm of finite linear combinations of the form*

$$f(x) + C_2f''(x) + \cdots + C_Nf^{(N)}(x).$$

It is interesting to note the following immediate consequence of Theorem 2.2:

THEOREM 2.4. *Let $F(x)$ be a measurable function which is positive almost everywhere on the interval (a, b), where a may equal $-\infty$ and b may equal $+\infty$ and let*

$$\int_a^b x^{2n}F(x)dx < \infty \qquad (n \geqq 0).$$

Then if $(F(x))^{1/2}$, $x(F(x))^{1/2}$, $x^2(F(x))^{1/2}, \cdots$ are complete in $L^2(a, b)$, it follows that for each positive integer n, $x^n(F(x))^{1/2}$, $x^{n+1}(F(x))^{1/2}$, $x^{n+2}(F(x))^{1/2}$, $\cdots$ are also complete in $L^2(a, b)$.

Theorem 2.4 is believed to be new. It would be interesting to see a proof which does not depend on Theorem 2.1.

3. **Taylor series in the mean.** We shall now consider the use of Taylor series to approximate translates; i.e., given $f(x)\in C^\infty(-\infty, \infty)$ such that $f^{(n)}(x)\in L^p(-\infty, \infty)$ $(n\geqq 0)$, we wish to represent $f(x+h)$ by

$$\text{(3.1)} \qquad (\text{l.i.m.})_p \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(\nu)}(x).$$

The expression (3.1) will be called the "L^p-Taylor series" of $f(x)$ (h being restricted to real values for the present).

To study the convergence of L^p-Taylor series, we shall first establish the following analog of Taylor's theorem:

THEOREM 3.1. *Let $f(x)\in C^n(-\infty, \infty)$ and let $f^{(\nu)}(x)\in L^p(-\infty, \infty)$ $(0\leqq\nu\leqq n)$. Then the remainder*

$$r_{n,h}(x) = f(x+h) - \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(\nu)}(x)$$

admits the following estimates:

$$\text{(3.2)} \qquad \|r_{n,h}(x)\|_p \leqq \frac{|h|^n}{n!} \|f^{(n)}(x)\|_p,$$

$$(3.3)\quad |r_{n,h}(x)| \leqq \begin{cases} \dfrac{|h|^{n-1}}{(n-1)!}\|f^{(n)}(x)\|_p[|h|/(1+q(n-1))]^{1/q} & \text{if } p>1, \\ \dfrac{|h|^{n-1}}{(n-1)!}\|f^{(n)}(x)\|_p & \text{if } p=1, \end{cases}$$

where $q=p/(p-1)$.

The proof of Theorem 3.1 depends on the well-known formula

$$(3.4)\qquad r_{n,h}(x) = \int_0^h \frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(x+t)dt.$$

Applying the generalized Minkowski inequality [9, Theorem 202] to (3.4):

$$\|r_{n,h}(x)\|_p \leqq \left| \int_0^h \frac{|h-t|^{n-1}}{(n-1)!}\|f^{(n)}(x+t)\|_p dt \right|$$

which immediately yields (3.2).

To prove (3.3) for $p=1$, we note that

$$|r_{n,h}(x)| \leqq \frac{|h|^{n-1}}{(n-1)!}\left| \int_0^h |f^{(n)}(x+t)|\, dt \right| \leqq \frac{|h|^{n-1}}{(n-1)!}\int_{-\infty}^{\infty} |f^{(n)}(x+t)|\, dt.$$

For $p>1$, we note that

$$\frac{(h-t)^{n-1}}{(n-1)!} \in L^q(0, h)$$

and apply Hölder's inequality.

The right hand side of (3.2) is equal to the norm of the first neglected term of the L^p-Taylor series, a quantity which must tend to zero in order that the series converge. Hence, for the convergence of the L^p-Taylor series, it is both necessary and sufficient that the nth term converge strongly to zero. Furthermore, convergent L^p-Taylor series always converge to the proper limit $f(x+h)$, in contrast to the behavior of ordinary Taylor series.

We are now able to compute the radius of convergence of the L^p-Taylor series. Using the well-known root criterion of Cauchy, it is seen that for the series to converge, the relation

$$|h| \limsup_{n\to\infty} \left[\frac{\|f^{(n)}(x)\|_p}{n!}\right]^{1/n} \leqq 1$$

must hold. Hence, the following theorem is true:

THEOREM 3.2. *Let* $f(x)\in C^\infty(-\infty, \infty)$ *and let* $f^{(n)}(x)\in L^p(-\infty, \infty)$ $(n\geqq 0)$. *Then for*

$$-\rho < |h| < \rho = \left(\limsup\left[\frac{\|f^{(n)}(x)\|_p}{n!}\right]^{1/n}\right)^{-1},$$

$$f^{(k)}(x+h) = (\lim_{n\to\infty})_p \sum_{\nu=0}^{n-1} \frac{h^\nu}{\nu!} f^{(k+\nu)}(x) \qquad (k \geqq 0).$$

For $|h| > \rho$, the L^p-Taylor series for $f(x)$ and its derivatives do not converge.

Thus if $\rho > 0$, $f^{(n)}(x+h) \in D[f]$ for all n and for $|h| < \rho$. We shall now show that the restriction on h is not needed; i.e., the condition $\rho > 0$ implies that $f^{(n)}(x+h) \in D[f]$ for all h and for all n. This follows from the result of Mandelbrojt [1, pp. 39–40] cited in Chapter 1, since the class $C\{\|f^{(n)}(x)\|_p\}$ is regular in the strip $-\rho < \operatorname{Im} z < \rho$. It is also possible to give a proof based on Theorem 3.2. Consider any h_0 smaller than ρ in absolute value. Since L^p norms are invariant under translation, the L^p-Taylor series of $f(x+h_0)$ also has ρ as its radius of convergence. Hence by a finite number of "continuations in the mean," every translate of $f^{(n)}(x)$ can be shown to be in $D[f]$.

It might be thought possible to obtain a larger radius of convergence for the L^p-Taylor series by considering weak L^p convergence instead of strong L^p convergence. This is not so. If the L^p-Taylor series converges weakly, then its partial sums, and hence its terms, must be bounded in norm. But for $|h| > \rho$ the terms are not bounded in norm. Hence the weak and strong radii are equal.

We shall now determine what functions may be represented by L^p-Taylor series. From the estimate (3.3), it is seen that if $\rho > 0$, then the ordinary Taylor series for $f(x)$ about each real x_0 converges to $f(x)$ for $x_0 - \rho < x < x_0 + \rho$. Hence $f(x)$ may be extended to an analytic function $f(z)$ which is regular in the strip S_ρ: $-\rho < \operatorname{Im} z < \rho$.

Hence we may extend the definition of L^p-Taylor series to complex h. For any h such that $|h| < \rho$, the series

$$\sum_{n=0}^{\infty} \frac{|h|^n}{n!} \|f^{(n)}(x)\|_p$$

converges, so that the limit (3.1) exists for such h and is necessarily equal to $f(x+h)$. More generally, if $h = \xi + i\eta \in S_\rho$

$$f(x+h) = (\text{l.i.m.})_p \sum_{\nu=0}^{n-1} \frac{(i\eta)^\nu}{\nu!} f^{(\nu)}(x+\xi)$$

with $n \to \infty$ under the l.i.m., and

$$\|f(x+h)\|_p \leqq \sum_{n=0}^{n} \frac{|\eta|^n}{n!} \|f^{(n)}(x)\|_p.$$

Thus we have:

THEOREM 3.3. *If the L^p-Taylor series of $f(x)$ has radius of convergence $\rho>0$, then $f(x)$ may be extended to an analytic function $f(z)$ which is regular in the strip S_ρ. Furthermore, for each positive σ which is less than ρ,* $\sup_{h\in S_\sigma}\|f(x+h)\|_p<\infty$.

We thus find it convenient to make the following definitions:

DEFINITION 3.1. The function class $W_p(\sigma)$ consists of all analytic functions $f(z)$ which are regular in S_σ and are such that $f(x+h)\in L^p(-\infty,\infty)$ for each h in S_σ, $\|f(x+h)\|_p$ being bounded in each S_λ for which $\lambda<\sigma$.

DEFINITION 3.2. The function class $T_p(\sigma)$ consists of all $f(x)\in C^\infty(-\infty,\infty)$ such that $f^{(n)}(x)\in L^p(-\infty,\infty)$ $(n\geqq 0)$ whose L^p-Taylor series have radius of convergence at least equal to σ.

According to Theorem 3.3, $T_p(\sigma)\subset W_p(\sigma)$. We shall show that $T_p(\sigma)=W_p(\sigma)$. To prove this we first establish the following generalization of a theorem of Paley and Wiener [3, p. 5].

THEOREM 3.4. *Let $f(z)$ be regular in $\bar{S}_\lambda$, the closure of S_λ. Furthermore, let*

$$\sup_{h\in\bar{S}_\lambda}\|f(x+h)\|_p=M<\infty.$$

Then for each $\zeta\in S_\lambda$

$$f^{(n)}(\zeta)=\frac{n!}{2\pi i}\left[\int_{-\infty}^{\infty}\frac{f(x-i\lambda)}{(x-i\lambda-\zeta)^{n+1}}dx-\int_{-\infty}^{\infty}\frac{f(x+i\lambda)}{(x+i\lambda-\zeta)^{n+1}}dx\right] \quad (n\geqq 0). \tag{3.5}$$

To prove Theorem 3.4, we use Cauchy's integral formula, observing that for sufficiently large A

$$\begin{aligned}f^{(n)}(\zeta)&=\frac{n!}{2\pi i}\left[\int_{-A}^{A}\frac{f(x-i\lambda)}{(x-i\lambda-\zeta)^{n+1}}dx-\int_{-A}^{A}\frac{f(x+i\lambda)}{(x+i\lambda-\zeta)^{n+1}}dx\right.\\&\qquad\left.+i\int_{-\lambda}^{\lambda}\frac{f(A+iy)}{(A+iy-\zeta)^{n+1}}dy-i\int_{-\lambda}^{\lambda}\frac{f(-A+iy)}{(-A+iy-\zeta)^{n+1}}dy\right]\\&=\frac{n!}{2\pi i}\left[I_1(A)-I_2(A)+iI_3(A)-iI_4(A)\right].\end{aligned}$$

Thus for sufficiently large B

$$f^{(n)}(\zeta)=\frac{n!}{2\pi i}\int_B^{B+1}\left[I_1(A)-I_2(A)+iI_3(A)-iI_4(A)\right]dA.$$

Now

$$\int_B^{B+1}I_3(A)dA=\int_{-\lambda}^{\lambda}dy\int_B^{B+1}\frac{f(A+iy)}{(A+iy-\zeta)^{n+1}}dA.$$

If $p>1$, we apply Hölder's inequality to obtain

$$\left|\int_B^{B+1} I_3(A)dA\right|$$
$$\leqq \int_{-\lambda}^{\lambda} dy \left(\int_B^{B+1} |f(A+iy)|^p dA\right)^{1/p} \left(\int_B^{B+1} \frac{dA}{|A+iy-\zeta|^{q(n+1)}}\right)^{1/q}$$
$$\leqq \frac{2\lambda M}{|B-\operatorname{Re}\zeta|^{n+1}}.$$

For $p=1$, the same inequality is even more easily established. Hence $\int_B^{B+1} I_3(A)dA \to 0$ as $B\to\infty$. Similarly, $\int_B^{B+1} I_4(A)dA\to 0$. Clearly $\int_B^{B+1} I_1(A)dA \to I_1(\infty)$, $\int_B^{B+1} I_2(A)dA \to I_2(\infty)$, and the theorem is proven.

We are now ready to prove:

THEOREM 3.5. *Under the conditions of Theorem* 3.4, $f^{(n)}(x)\in L^p(-\infty,\infty)$ $(n\geqq 0)$ *and*

$$\|f^{(n)}(x)\|_p \leqq \frac{n!M}{\lambda^n} \qquad (n\geqq 0). \tag{3.6}$$

Proof. Applying the generalized Minkowski inequality [9, Theorem 202] to (3.5), we have for $n\geqq 1$

$$\left(\int_{-\infty}^{\infty} |f^{(n)}(t)|^p dt\right)^{1/p} \leqq \frac{n!}{2\pi}\left[\int_{-\infty}^{\infty} \frac{dx}{|x-i\lambda|^{n+1}}\left(\int_{-\infty}^{\infty} |f(x+t-i\lambda)|^p dt\right)^{1/p}\right.$$
$$\left.+\int_{-\infty}^{\infty} \frac{dx}{|x+i\lambda|^{n+1}}\left(\int_{-\infty}^{\infty} |f(x+t+i\lambda)|^p dt\right)^{1/p}\right]$$
$$\leqq \frac{n!M}{2\pi}\left[\int_{-\infty}^{\infty} \frac{dx}{|x-i\lambda|^{n+1}} + \int_{-\infty}^{\infty} \frac{dx}{|x+i\lambda|^{n+1}}\right]$$
$$= \frac{n!M}{\pi\lambda^{n-1}}\int_{-\infty}^{\infty} \frac{dx}{x^2+\lambda^2} = \frac{n!M}{\lambda^n}.$$

Now we shall prove:

THEOREM 3.6. *The classes* $W_p(\sigma)$ *and* $T_p(\sigma)$ *are identical. Given* $f(z)\in W_p(\sigma)$ *the radius of convergence* ρ *of the* L^p-*Taylor series of* $f(x)$ *is given by*

$$\rho = \sup\{\sigma \mid f(z)\in W_p(\sigma)\}. \tag{3.7}$$

Proof. By Theorem 3.3, $T_p(\sigma)\subset W_p(\sigma)$. Let $f(z)\in W_p(\sigma)$ and let $|h|<\sigma$. Choose λ so that $|h|<\lambda<\sigma$. Then by (3.6)

$$\frac{|h|^n}{n!}\|f^{(n)}(x)\|_p \leqq \frac{|h|^n M}{\lambda^n}\to 0 \qquad (n\to\infty).$$

Hence $\rho\geqq\sigma$ and $W_p(\sigma)=T_p(\sigma)$. Since $f(z)\in W_p(\rho)$, (3.7) is clear.

The class $W_2(\sigma)$ was studied by Paley and Wiener [3, pp. 3–13]. Their principal theorem concerning this class may be stated as follows:

THEOREM 3.7. *Let $f(z)\in W_2(\sigma)$ and let $F(x)$ be the Fourier transform of $f(x)$. Then for each $h\in S_\sigma$, $e^{ihx}F(x)$ is the Fourier transform of $f(x+h)$ and is absolutely integrable so that*

$$f(z) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{izt}F(t)dt \qquad (z \in S_\sigma). \tag{3.8}$$

Conversely, if $e^{ihx}F(x)L^2(-\infty, \infty)$ for each $h\in S_\sigma$, then it is absolutely integrable for each such h and $F(x)$ substituted in (3.8) *yields an $f(z)\in W_2(\sigma)$.*

We shall give a new proof of the first part of the theorem. By Theorem 3.6, $f(x+h)$ is represented for all $h=\xi+i\eta\in S_\sigma$ by

$$\operatorname*{l.i.m.}_{n\to\infty} \sum_{\nu=0}^{n-1} \frac{(i\eta)^\nu}{\nu!} f^{(\nu)}(x+\xi). \tag{3.9}$$

Since the Fourier transform of $f^{(n)}(x+\xi)$ is equal to $(ix)^n e^{i\xi x}F(x)$, the Fourier transform of (3.9) is seen to equal

$$\lim_{n\to\infty} \sum_{\nu=0}^{n-1} \frac{(-\eta x)^\nu}{\nu!} e^{i\xi x}F(x) = e^{(-\eta+i\xi)x}F(x) = e^{ihx}F(x).$$

The absolute integrability of $e^{ihx}F(x)$ is easily established by applying the Schwarz inequality

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\eta x}\,|\,F(x)\,|\,dx &= \int_{-\infty}^{0} e^{\epsilon x}e^{-(\eta+\epsilon)x}\,|\,F(x)\,|\,dx + \int_{0}^{\infty} e^{-\epsilon x}e^{-(\eta-\epsilon)x}\,|\,F(x)\,|\,dx \\ &\leqq \left(\int_{-\infty}^{0} e^{2\epsilon x}dx\right)^{1/2}\left(\int_{-\infty}^{0} e^{-2(\eta+\epsilon)x}\,|\,F(x)\,|^2dx\right)^{1/2} \\ &\quad + \left(\int_{0}^{\infty} e^{-2\epsilon x}dx\right)^{1/2}\left(\int_{0}^{\infty} e^{-2(\eta-\epsilon)x}\,|\,F(x)\,|^2dx\right)^{1/2}.\end{aligned}$$

The rest of the theorem is easily verified.

Returning to general p, we shall show that two L^p-Taylor series can be multiplied (in a manner analogous to the multiplication of ordinary Taylor series) to give a new L^p-Taylor series whose radius of convergence is at least equal to the smaller of the radii of convergence of the series being multiplied. In other words, if $f_1(x)$ and $f_2(x)$ both belong to $T_p(\sigma)$, then so does $f_1(x)f_2(x)$.

By Theorem 3.6, $f_1(z)$ and $f_2(z)$ both belong to $W_p(\sigma)$. But any $f(z)\in W_p(\sigma)$ is bounded in each S_λ for which $\lambda<\sigma$. This follows immediately from (3.5), by simple estimation when $p=1$ and by Hölder's inequality when $p>1$. Hence if $B=\sup_{z\in S_\lambda}|f_1(z)|$ and $M=\sup_{h\in S_\lambda}\|f_2(x+h)\|_p$ then

$$\|f_1(x+h)f_2(x+h)\|_p \leqq BM \qquad (h\in S_\lambda)$$

and $f_1(z)f_2(z)\in W_p(\sigma)$. Since the L^p-Taylor series of $f_1(x)f_2(x)$ can be formed from the L^p-Taylor series of $f_1(x)$ and $f_2(x)$ by the usual Cauchy method of multiplying power series, our assertion concerning the multiplication of L^p-Taylor series is valid. Two of our results are important enough to be numbered as theorems; viz.,

THEOREM 3.8. *If $f(z)\in W_p(\sigma)$, then $f(z)$ is bounded in each S_λ for which $\lambda<\sigma$.*

THEOREM 3.9. *The class $W_p(\sigma)$ is closed with respect to multiplication.*

As an immediate consequence of Theorem 3.8, we have:

THEOREM 3.10. *If $p_1<p_2$, then $W_{p_1}(\sigma)\subset W_{p_2}(\sigma)$.*

We also have the following more general multiplication theorem:

THEOREM 3.11. *If $f_1(z)\in W_{p_1}(\sigma)$ and $f_2(z)\in W_{p_2}(\sigma)$, then $f_1(z)f_2(z)\in W_p(\sigma)$ for all p such that $p\geqq \min(p_1, p_2)$. If $1/p_1+1/p_2\geqq 1$, then $f_1(z)f_2(z)\in W_p(\sigma)$ for all $p\geqq 1$.*

The first part of the theorem being an obvious consequence of Theorems 3.8 and 3.10, let us consider the case $1/p_1+1/p_2\geqq 1$. In this case $p_2\leqq q_1$, the index conjugate to p_1. Hence $f_2(z)\in W_{q_1}(\sigma)$ and by Hölder's inequality, $f_1(z)f_2(z)\in W_1(\sigma)$.

We close with some examples of functions belonging to various $W_p(\sigma)$. It is readily verified that $e^{-z^2}\in W_p(\sigma)$ for all $p\geqq 1$ and for all positive σ. The function $(z-i\sigma_0)^{-1/p_0}\in W_p(\sigma_0)$ for each $p>p_0$. Its L^p-Taylor series has radius of convergence σ_0 for each such p.

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