ON LEBESGUE AREA. II(1)

BY

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1. **Introduction.** The theory of k dimensional area is the study of certain geometric properties of continuous maps

$$f: X \to E_n$$

where X is a k dimensional finitely triangulable space and E_n is a Euclidean space of dimension $n \geqq k$. In particular, the theory of Lebesgue area concerns the problem of approximating f, uniformly and in area, by maps of an elementary character. Except for the relatively simple case in which f satisfies a Lipschitz condition, and except for the special situation considered in [F11], it has always been assumed in previous work that X can be embedded in a k dimensional manifold. Our present purpose is to do without this assumption on X.

One of our new tools is the norm of a cohomology class, introduced in §2. For $k=2$, an important role is also played by dimension theoretic considerations and by some new information (Corollary 5.2) about the local structure of monotone images of finitely triangulable spaces. Another new topological result (Theorem 4.2) relates k dimensional stability of a map to $k-1$ dimensional stability of its slices.

The particular problem studied here is the validity of the inequality

$$L_k(f) \leqq \sum_{\xi \in \Omega_n^k} L_k(P_n^\xi \circ f),$$

where L_k is k dimensional Lebesgue area, and Ω_n^k consists of all strictly increasing sequences

$$\xi: \{1, 2, \cdots, k\} \to \{1, 2, \cdots, n\}$$

to each of which corresponds the orthogonal projection P_n^ξ of E_n onto E_k mapping $(x_1, x_2, \cdots, x_n)$ onto $(x_{\xi_1}, x_{\xi_2}, \cdots, x_{\xi_k})$. This inequality, which has been basic in the development of area theory, was proved in [CE2] for the case in which X is a 2-cell and $n=3$; then in [F10] for the case(2) in

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(2) For this case another proof was announced by L. Cesari in an abstract, Bull. Amer. Math. Soc. vol. 59 (1953) p. 534 but has not yet been published. Cesari's recent book *Surface area*, Annals of Mathematics Study 35, Princeton, 1956, deals only with $n=3$.

which $X \subset E_2$ as well as for the case in which X is contained in a k dimensional manifold and $f(X)$ has $k+1$ dimensional Hausdorff measure 0; finally in [S] for the case in which X is contained in a 2 dimensional manifold. *Here we show that the inequality holds* (*Theorem* 3.16) *in case* $f(X)$ *has* $k+1$ *dimensional Hausdorff measure* 0, *and* (*Theorem* 5.7) *in case* $k=2$ (*without any restriction on* X *or* f). We also find (Theorem 3.17) that if $n=k$ then $L_k(f)$ equals the integral, with respect to k dimensional Lebesgue measure, of a new multiplicity function defined in terms of norms of cohomology classes.

Since this paper is a continuation of [F10], we readopt the terminology used there, assuming that the reader has a copy of [F10] at hand.

2. The norm of a cohomology class.

Here we introduce the norm $|\lambda|$, a nonnegative integer or ∞, associated with any integral Čech relative cohomology class λ. We shall use this concept in §3 to define the multiplicity function M.

2.1. DEFINITION. For each k dimensional integral Čech cohomology class

$$\lambda \in H^k(X, A)$$

of a compact (Hausdorff) pair (X, A) we define

$$|\lambda|$$

as the infimum (possibly ∞) of all non-negative integers m with the following property:

$X-A$ contains m disjoint k-cells $Q_1, Q_2, \cdots, Q_m$ with boundaries $B_1, B_2, \cdots, B_m$ such that

$$R = X - \bigcup_{i=1}^{m} (Q_i - B_i)$$

is closed in X, and there exists a cohomology class

$$\mu \in H^k(X, R)$$

such that the homomorphism

$$H^k(X, R) \to H^k(X, A) \text{ induced by } (X, A) \subset (X, R)$$

maps μ onto λ, and each of the homomorphisms

$$H^k(X, R) \to H^k(Q_i, B_i) \text{ induced by } (Q_i, B_i) \subset (X, R)$$

maps μ onto a generator of $H^k(Q_i, B_i)$.

2.2. LEMMA. *If $\lambda \in H^k(X, A)$ with $|\lambda| \leq m < \infty$ and if $Q_1, Q_2, \cdots, Q_m$ are k-cells with boundaries $B_1, B_2, \cdots, B_m$ satisfying the conditions of Definition* 2.1, *then these conditions hold also for any relative k-cells $(Q_1', B_1'), \cdots, (Q_m', B_m')$ such that $Q_i' \subset Q_i$ for $i=1, 2, \cdots, m$.*

Proof. It is easy to verify that the corresponding set R' is closed in X, to represent all the homomorphisms induced by the relevant inclusion maps in a commutative diagram, and to complete the argument using the excision axiom.

2.3. THEOREM. *Suppose* $\lambda, \lambda' \in H^k(X, A)$. *Then*

$$|\lambda| = 0 \text{ if and only if } \lambda = 0,$$
$$|-\lambda| = |\lambda|,$$
$$|\lambda + \lambda'| \leqq |\lambda| + |\lambda'|.$$

Proof. The first two statements are trivial. In checking the third, one may use Lemma 2.2 to make the k-cells for λ disjoint from the k-cells for λ'.

2.4. THEOREM. *If* $f: (X', A') \to (X, A)$ *is a relative homeomorphism, then*

$$|f^*(\lambda)| = |\lambda| \text{ whenever } \lambda \in H^k(X, A).$$

2.5. THEOREM. *If* $(X, A) \supset (X', A')$ *are compact pairs and if the induced homomorphism*

$$H^k(X, A) \to H^k(X', A')$$

maps λ *onto* λ', *then* $|\lambda| \geqq |\lambda'|$.

If furthermore $X - A = X' - A'$, *then* $|\lambda| = |\lambda'|$.

Proof. The second statement follows from Theorem 2.4. In order to check the first statement one may use Lemma 2.2 to arrange it so that each k-cell for λ is either contained in X' or disjoint from X'.

2.6. THEOREM. *Suppose that* (X, A) *is a compact pair and* X *is the union of disjoint closed subsets* X_1 *and* X_2. *If* $\lambda \in H^k(X, A)$ *and* λ *is mapped onto* λ_i *by the homomorphism*

$$H^k(X, A) \to H^k(X_i, X_i \cap A)$$

induced by $(X_i, X_i \cap A) \subset (X, A)$ *for* $i = 1, 2$, *then*

$$|\lambda| = |\lambda_1| + |\lambda_2|.$$

2.7. THEOREM. *Suppose* (X, A) *is a* k *dimensional compact pair, with* $k \geqq 2$, S *is a* k*-cell with boundary* T, *and*

$$f: (X, A) \to (S, T)$$

is a continuous map. Let η *be a generator of* $H^k(S, T)$, *suppose*

$$|f^*(\eta)| = m < \infty$$

and let $(Q_1, B_1), \cdots, (Q_m, B_m), R, \mu$ *be as in Definition* 2.1 *with* $\lambda = f^*(\eta)$. *For* $i = 1, 2, \cdots, m$ *let*

$$g_i: (Q_i, B_i) \to (S, T)$$

be a continuous map such that $g_i^*(\eta)$ is the image of μ under the homomorphism induced by $(Q_i, B_i) \subset (X, R)$.

Under these conditions there exists a continuous map

$$h: (X, R) \to (S, T)$$

such that $h|A = f|A$ and $h|Q_i = g_i$ for $i = 1, 2, \cdots, m$.

Proof. Apply Lemma 6.21 [F10] with

$$P = X, \qquad Q = \bigcup_{i=1}^{m} Q_i, \qquad B = \bigcup_{i=1}^{m} B_i, \qquad g = \bigcup_{i=1}^{m} g_i.$$

Since $g^*(\eta) = q(\mu)$, then $f^* = p \circ q^{-1} \circ g^*$.

2.8. Remark. In case (X, A) is a finitely triangulable k dimensional pair and $\lambda \in H^k(X, A)$, then $|\lambda|$ may be computed as follows:

Suppose K is any finite simplicial complex whose space is homeomorphic to X, such that A corresponds to a subcomplex L of K, and choose a definite orientation for the k dimensional simplexes of K (for instance by ordering the vertices of K). Then λ is represented by a cohomology class consisting of cocycles u of (K, L). With each such u associate the sum

$$\sum_{\sigma} |u(\sigma)|$$

extended over the set of all k dimensional simplexes in K. The minimum of these sums equals $|\lambda|$.

In this paper we make no use of this fact, which may be verified readily with the help of Lemma 2.2 and Theorem 2.6.

3. **General theorems on multiplicity and area.** We shall now study the multiplicity $M(f, y)$ with which a continuous map f of a compact space into E_k assumes the value y, and establish the basic theorems connecting this multiplicity function with the theory of Lebesgue area.

3.1. Definition. *Suppose f is a continuous map of a compact space X into Euclidean k-space E_k and U is a nonempty connected open subset of E_k with compact closure.* Choosing any compact set Y such that

$$f(X) \cup U \subset Y \subset E_k$$

we consider the induced homomorphism

$$f^*: H^k(Y, Y - U) \to H^k(X, f^{-1}(Y - U))$$

and define

$$D(f, U) = |f^*(\eta)|$$

where η is a generator of $H^k(Y, Y - U)$.

(This notation is permissible because changes in Y and η will not affect $|f^*(\eta)|$.)

3.2. Definition. For each continuous map f of a compact space X into E_k and for each $y \in E_k$ we define

$$M(f, y)$$

as the *supremum of* $D(f, U)$ *for all connected open subsets* U *of* E_k *with compact closure and such that* $y \in U$.

3.3. Lemma. *Suppose* f *and* g *are continuous maps of a compact space* X *into* E_k,

U *and* U' *are nonempty open connected subsets of* E_k,

Y *is a compact subset of* E_k *for which* $U' \subset U \subset Y$ *and* $g(X) \cup f(X) \subset Y$.

If

$$f^{-1}(Y - U) \subset g^{-1}(Y - U')$$

and if the homomorphisms

$$H^k(Y, Y - U') \rightarrow H^k(X, f^{-1}(Y - U))$$

induced by f *and* g *are identical, then*

$$D(g, U') \geqq D(f, U).$$

Proof. Consider the diagram

$$\begin{array}{ccc} H^k(Y, Y - U') & \xrightarrow{g^*} & H^k(X, g^{-1}(Y - U')) \\ \downarrow & & \downarrow \\ H^k(Y, Y - U) & \xrightarrow{f^*} & H^k(X, f^{-1}(Y - U)) \end{array}$$

where the vertical homomorphisms are induced by inclusions. The last assumption of the hypothesis means precisely that the preceding diagram is commutative. Since the left homomorphism maps a generator η' onto a generator η, the right one maps $g^*(\eta')$ onto $f^*(\eta)$. It follows from Theorem 2.5 that

$$D(g, U') = |g^*(\eta')| \geqq |f^*(\eta)| = D(f, U).$$

3.4. Theorem. *If* f *and* U *are as in Definition* 3.1 *and if* U' *is a nonempty open connected subset of* U, *then*

$$D(f, U') \geqq D(f, U).$$

Proof. Apply Lemma 3.3 with $g = f$.

3.5. Lemma. *If* f *is a univalent continuous map of a compact space* X *into* E_k *and* $y \in \text{Interior } f(X)$, *then* $M(f, y) = 1$.

3.6. Theorem. *If* f *is a continuous map of a compact space* X *into* E_k, X *is the union of the disjoint closed sets* X_1 *and* X_2, *and* $y \in E_k$, *then*

$$M(f, y) = M(f|X_1, y) + M(f|X_2, y).$$

3.7. THEOREM. *If f is a continuous map of a compact space X into E_k, $y \in E_k$, and X' is a closed subset of X, then*

$$M(f, y) \geqq M(f|X', y).$$

If furthermore $f(x) \neq y$ for $x \in$ Closure $(X - X')$, *then*

$$M(f, y) = M(f|X', y).$$

Proof. Suppose U and Y are as in Definition 3.1, with $y \in U$. Applying the first part of Theorem 2.5 with $A = f^{-1}(Y - U)$ and $A' = A \cap X'$ we find that

$$D(f, U) \geqq D(f|X', U).$$

In case $y \notin f$(Closure $(X - X')$) we may, by virtue of Theorem 3.4, assume that U does not meet $f(X - X')$. Then $X - A = X' - A'$, and the second part of Theorem 2.5 implies that

$$D(f, U) = D(f|X', U).$$

3.8. COROLLARY. *If f and g are continuous maps of a compact space X into E_k, $y \in E_k$, and if there exists a closed set $C \subset X$ such that*

$$g(x) \neq y \text{ for } x \in C,$$
$$g(x) = f(x) \textit{ for } x \in (X - C),$$

then

$$M(f, y) \geqq M(g, y).$$

3.9. THEOREM. *If f and g are continuous maps of a compact space X into E_k, $y \in E_k$, and if there exists a homotopy*

$$h: X \times I \to E_k, \quad \textit{where } I = \{t \mid 0 \leqq t \leqq 1\},$$

such that

$$h(x, 0) = f(x) \quad \textit{and} \quad h(x, 1) = g(x) \textit{ for } x \in X,$$
$$\{x \mid h(x, t) = y\} = \{x \mid f(x) = y\} \textit{ for } 0 \leqq t \leqq 1,$$

then

$$M(f, y) = M(g, y).$$

Proof. Suppose U is as in Definition 3.2 and Y is a compact subset of E_k such that $h(X \times I) \cup U \subset Y$. Since $y \notin h(f^{-1}(Y - U) \times I)$, there exists a connected open set U' such that

$$y \in U' \subset U - h(f^{-1}(Y - U) \times I).$$

Accordingly, f and g are joined by the homotopy

$$h: (X \times I, f^{-1}(Y - U) \times I) \to (Y, Y - U'),$$

and Lemma 3.3 implies that $D(g, U') \geqq D(f, U)$.

Consequently $M(g, y) \geqq M(f, y)$. The opposite inequality follows by symmetry.

3.10. THEOREM. *For each compact space X, the function M is lower semicontinuous on the cartesian product of the function space $(E_k)^X$, with its uniform topology, and E_k.*

Proof. Suppose f, y and U are as in Definition 3.2. Choose $\epsilon > 0$ so that

$$|z - y| \geqq 2\epsilon \text{ for } z \in E_k - U$$

and choose a compact set Y such that $U \subset Y \subset E_k$ and

$$|z - f(x)| \geqq \epsilon \text{ for } z \in E_k - Y, x \in X.$$

Furthermore let

$$U' = E_k \cap \{z \mid |z - y| < \epsilon\} \quad \text{and} \quad I = \{t \mid 0 \leqq t \leqq 1\}.$$

Now, if g is any continuous map of X into E_k such that

$$|g(x) - f(x)| < \epsilon \text{ for } x \in X,$$

then f may be deformed into g by the homotopy

$$h: (X \times I, f^{-1}(Y - U) \times I) \to (Y, Y - U'),$$
$$h(x, t) = (1 - t)f(x) + tg(x) \text{ for } x \in X, t \in I.$$

It follows from Lemma 3.3 that $D(g, U') \geqq D(f, U)$.

Consequently, if $z \in U'$, then

$$M(g, z) \geqq D(g, U') \geqq D(f, U).$$

3.11. THEOREM. *If f is a continuous map of a compact space X into E_k, $k \geqq 2$, $y \in E_k$, $M(f, y) = m < \infty$ and $\epsilon > 0$, then there is a continuous map u of X into E_k such that*

$$N(u, X, y) = m \quad \text{and} \quad |u(x) - f(x)| < \epsilon \text{ for } x \in X.$$

Accordingly[3], $M(f, y) \geqq S(f, y)$.

Proof. Choose a k-cell S contained in E_k with boundary T such that

$$\operatorname{diam} S < \epsilon, \qquad y \in S - T \quad \text{and} \quad D(f, S - T) = m.$$

Then apply Theorem 2.7 with X, A, f replaced by $f^{-1}(S)$, $f^{-1}(T)$, $f|f^{-1}(S)$ respectively, letting $g_1, g_2, \cdots, g_m$ be suitable homeomorphisms. Finally extend h to u by defining $u(x) = f(x)$ whenever $f(x) \notin S$.

3.12. THEOREM. *If f is a quasi-linear function on a k dimensional space X into E_k, then*

(3) See [F10, Definition 3.10].

$$M(f, y) = N(f, X, y) \text{ for } \mathfrak{L}_k \text{ almost all } y \in E_k.$$

Proof. This is an obvious consequence of Theorems 3.7, 3.6 and Lemma 3.5.

3.13. THEOREM. *If X is a k dimensional finitely triangulable space and f is a continuous map of X into E_k, then*

$$\int_{E_k} M(f, y) d\mathfrak{L}_k y \leqq L_k(f).$$

Proof. Suppose $g_1, g_2, \cdots$ are quasi-linear maps of X into E_k which converge uniformly to f and for which

$$L_k(f) = \lim_{n\to\infty} \int_{E_k} N(g_n, X, y) d\mathfrak{L}_k y.$$

Using Theorem 3.12, Fatou's Lemma and Theorem 3.10, we see that

$$\begin{aligned} L_k(f) &= \lim_{n\to\infty} \int_{E_k} M(g_n, y) d\mathfrak{L}_k y \\ &\geqq \int_{E_k} \liminf_{n\to\infty} M(g_n, y) d\mathfrak{L}_k y \geqq \int_{E_k} M(f, y) d\mathfrak{L}_k y. \end{aligned}$$

3.14. THEOREM. *Suppose*:

(1) *X is a k dimensional finitely triangulable space, with $k \geqq 2$.*

(2) *v is a continuous function on X to E_n, with $n \geqq k$.*

(3) *$v_1, v_2, \cdots, v_n$ are the real valued functions on X such that*

$$v(x) = (v_1(x), v_2(x), \cdots, v_n(x)) \text{ for } x \in X.$$

(4) *$a \in E_n$, $\epsilon > 0$ and $N_i = \{a_i + \epsilon m \mid m$ is an even integer$\}$ for $i = 1, 2, \cdots, n$.*

(5) *If $x \in X$, then the set*

$$\{i \mid v_i(x) \in N_i\}$$

has at most k elements.

(6) *Δ is the set of all k termed sequences of even integers.*

Under these conditions there exists a quasi-linear function u on X to E_n such that

$$|u(x) - v(x)| < \epsilon 2 n^{1/2} \text{ for } x \in X$$

and

$$\int_{E_n} N(u, X, y) d\mathcal{H}_n^k y \leqq \sum_{(\xi,\delta)\in\Omega_n^k\times\Delta} 2^k \epsilon^k M[P_n^\xi \circ v, P_n^\xi(a) + \epsilon\delta].$$

Proof. (A modification of the proof of Theorem 7.6 [F10].) In view of an obvious reduction by homothetic transformations, we shall only consider the special case in which a is the origin of E_n and $\epsilon = 1$. Under these conditions N_i is the set of all even integers for $i = 1, 2, \cdots, n$. We may also assume that

$$\sum_{(\xi,\delta)\in\Omega_n^k\times\Delta} M(P_n^\xi \circ v, \delta) < \infty.$$

We readopt the notation of Remark 7.4 [F10]. Thus we can express the assumption (5) more simply by stating that v maps X into $E_n - C''_{n-k-1}$. Using the deformation

$$\rho_{k+1}: (E_n - C''_{n-k-1}) \times \{t \mid 0 \leqq t \leqq n - k\} \to E_n - C''_{n-k-1}$$

we define the function w on X to C'_k by the formula

$$w(x) = \rho_{k+1}[v(x), n - k] \text{ for } x \in X.$$

If $\xi \in \Omega_n^k$ and $\delta \in \Delta$, then

$$M(P_n^\xi \circ w, \delta) = M(P_n^\xi \circ v, \delta).$$

In fact the functions $P_n^\xi \circ v$ and $P_n^\xi \circ w$ are joined by the homotopy which has the value

$$(P_n^\xi \circ \rho_{k+1})[v(x), t]$$

for $(x, t) \in X \times \{t \mid 0 \leqq t \leqq n-k\}$; inasmuch as the last statement in Remark 7.4 [F10] implies that

$$\{x \mid (P_n^\xi \circ \rho_{k+1})[v(x), t] = \delta\} = \{x \mid (P_n^\xi \circ v)(x) = \delta\}$$

for $0 \leqq t \leqq n-k$, our assertion follows from Theorem 3.9.

Next consider the family

$$F = \{w^{-1}(S) \mid S \text{ is a } k\text{-cell in } C' \text{ and } w^{-1}(S) \neq 0\}.$$

For each $V \in F$ let (S_V, T_V) be the k-cell of C' and its boundary such that $w^{-1}(S_V) = V$, and let

$$m(V) = |(w \mid V)^*(\eta)|$$

where η is a generator of $H^k(S_V, T_V)$ and

$$(w \mid V)^*: H^k(S_V, T_V) \to H^k(V, w^{-1}(T_V)).$$

Defining

$$U_\delta = E_k \cap \{x \mid |x_i - \delta_i| < 1 \text{ for } i = 1, \cdots, k\} \text{ for } \delta \in \Delta,$$

$$G(\xi, \delta) = F \cap \{V \mid P_n^\xi(S_V - T_V) = U_\delta\} \text{ for } \xi \in \Omega_n^k \text{ and } \delta \in \Delta,$$

we readily see from the construction of C' that the members of each family

$G(\xi, \delta)$ are disjoint, that no two such families have a member in common, and that

$$F = \bigcup_{(\xi,\delta)\in\Omega_n^k\times\Delta} G(\xi, \delta).$$

Furthermore

$$m(V) = D(P_n^\xi \circ w \mid V, U_\delta) \text{ for } V \in G(\xi, \delta),$$

and we infer with the help of Theorems 3.6 and 3.7 that

$$\begin{aligned}\sum_{V\in F} m(V) &= \sum_{(\xi,\delta)\in\Omega_n^k\times\Delta} \sum_{V\in G(\xi,\delta)} D(P_n^\xi \circ w \mid V, U_\delta)\\ &\leqq \sum_{(\xi,\delta)\in\Omega_n^k\times\Delta} \sum_{V\in G(\xi,\delta)} M(P_n^\xi \circ w \mid V, \delta)\\ &\leqq \sum_{(\xi,\delta)\in\Omega_n^k\times\Delta} M(P_n^\xi \circ w, \delta) < \infty.\end{aligned}$$

We may now assume that X is the space of a finite Euclidean simplicial complex K' with the following property: To each $V \in F$ corresponds a family $H(V)$ consisting of $m(V)$ cubes contained in $V - w^{-1}(T_V)$, whose first barycentric subdivisions are subcomplexes of K', and which are fitting with respect to Definition 2.1; more precisely, supposing that

$$H(V) = \{Q_i \mid i = 1, 2, \cdots, m(V)\},$$

we require that the conditions stated in that definition shall hold with (X, A) replaced by $(V, V \cap w^{-1}(T_V))$, with $\lambda = (w \mid V)^*(\eta)$, where η is a generator of $H^k(S_V, T_V)$, and with a suitable μ. For $i = 1, 2, \cdots, m(V)$ we choose a properly oriented barycentric homeomorphism g_i of Q_i onto S_V, and apply Theorem 2.7 with $f = w \mid V$ to obtain a continuous map

$$h_V: V \to S_V$$

such that

$$h_V(x) = w(x) \quad \text{whenever} \quad x \in V \cap w^{-1}(C'_{k-1}),$$

$$h_V\Big(V - \bigcup_{Q\in H(V)} Q\Big) \subset C'_{k-1},$$

and h_V maps each cube of $H(V)$ homeomorphically and barycentrically onto S_V.

If V and V' are distinct elements of F, then

$$V \cap V' \subset w^{-1}(C'_{k-1}), \qquad h_V(x) = h_{V'}(x) \text{ for } x \in V \cap V'.$$

Therefore

$$g = \bigcup_{V\in F} h_V$$

is a continuous map of X into E_n which maps each element of the family

$$H = \bigcup_{V\in F} H(V)$$

homeomorphically and barycentrically onto some k dimensional cube in C', and for which

$$g\left(X - \bigcup_{Q\in H} Q\right) \subset C'_{k-1}.$$

Let H' be the set of all those k dimensional simplices which belong to the first barycentric subdivision of some element of H, and let K be the first barycentric subdivision of a finite subcomplex of C' containing the range of g. Then $H'\subset K'$, g maps each element of H' barycentrically onto some element of K, and g maps $K'-H'$ into the $k-1$ dimensional skeleton of K.

According to Lemma 6.20 [F10] there exists a function u on X to E_n, a finite simplicial subdivision K'' of K', and a function q on H' to K'' such that u maps each element of K'' barycentrically onto some element of K,

$$u[g^{-1}(W)] \subset W \text{ for } W \in K,$$
$$q(S) \subset S \quad\text{and}\quad u[q(S)] = g(S) \text{ for } S \in H',$$

and u maps $K''-q(H')$ into the $k-1$ dimensional skeleton of K.

If $x\in X$ and $v(x)\in Y\in C'$, then also $w(x)\in Y$, $g(x)\in Y$, and $u(x)\in Y$, hence

$$\left| u(x) - v(x) \right| \leqq \operatorname{diam}(Y) \leqq 2n^{1/2}.$$

Finally we compute

$$\begin{aligned}\int_{E_n} N(u, X, y)d\mathcal{H}_n^k y &= \sum_{Y\in H'} \mathcal{H}_n^k(u[q(Y)])\\ &= \sum_{Y\in H'} \mathcal{H}_n^k(g(Y)) = \sum_{Q\in H} \mathcal{H}_n^k(g(Q)) = \sum_{Q\in H} 2^k\\ &= \sum_{V\in F}\sum_{Q\in H(V)} 2^k = \sum_{V\in F} 2^k m(V)\\ &\leqq \sum_{(\xi,\delta)\in\Omega_n^k\times\Delta} 2^k M(P_n^\xi \circ v, \delta).\end{aligned}$$

3.15. Lemma. *If X is a k dimensional finitely triangulable space, f is a continuous function on X to E_n with $n\geqq k$, Δ is the set of all k termed sequences of even integers, $\epsilon>0$, and R is the set of all those points $a\in E_n$ for which*

$$\sum_{(\xi,\delta)\in\Omega_n^k\times\Delta} 2^k\epsilon^k M[P_n^\xi\circ f, P_n^\xi(a) + \epsilon\delta] \leqq \sum_{\xi\in\Omega_n^k}\int_{E_k} M(P_n^\xi\circ f, z)d\mathfrak{L}_k z,$$

then $\mathfrak{L}_n(R)>0$.

Proof. Theorem 3.10 assures us that the proof of Lemma 7.7 [F10] remains valid.

3.16. THEOREM. *If X is a k dimensional finitely triangulable space, with $k \geqq 2$, f is a continuous map of X into E_n, with $n \geqq k$, and*

$$\mathcal{H}_n^{k+1}(f(X)) = 0,$$

then

$$L_k(f) \leqq \sum_{\xi \in \Omega_n^k} \int_{E_k} M(P_n^\xi \circ f, z) d\mathcal{L}_k z.$$

Proof. Use the proof of Theorem 7.8 [F10], substituting Lemma 3.15 and Theorem 3.10 for Lemma 7.7 and Theorem 7.6 of [F10].

3.17. COROLLARY. *If X is a k dimensional finitely triangulable space, $k \geqq 2$, and f is a continuous function on X to E_k, then*

$$L_k(f) = \int_{E_k} M(f, z) d\mathcal{L}_k z.$$

Proof. Combine Theorems 3.16 and 3.13.

3.18. REMARK. *In the preceding corollary the algebraic multiplicity function M cannot be replaced by the stable multiplicity function S, even in case $k=2$ and f is a light mapping.* This shown by the following example:

Suppose m is a positive integer and $0 < \epsilon < 4$.

We let X be the union of the $m+1$ rectangles

$$X_i = E_3 \cap \{x \mid |x_1| \leqq 1, |x_2| \leqq 1, x_3 = ix_1\}$$

corresponding to $i = 0, 1, \cdots, m$ and define

$$g: X \to E_2, \; g(x) = (x_1, x_2) \text{ for } x \in X.$$

Considering those points $z \in E_2$ which are stable values of g, we have

$$S(g, z) = m + 1 \quad \text{whenever} \quad 0 < |z_1| < 1 \text{ and } |z_2| < 1,$$
$$S(g, z) = 1 \quad \text{whenever} \quad z_1 = 0 \text{ and } |z_2| < 1.$$

Next we choose a homeomorphism h of the square $g(X)$ onto itself such that

$$\mathcal{L}_2[h(E_2 \cap \{z \mid z_1 = 0, |z_2| \leqq 1\})] = 4 - \epsilon,$$

and let $f = h \circ g$. Evidently

$$\int_{E_2} S(f, z) d\mathcal{L}_2 z = \epsilon \cdot (m + 1) + (4 - \epsilon) \cdot 1 = \epsilon m + 4.$$

To compute $L_2(f)$, we observe that X is not locally Euclidean at any point of the line segment

$$A = E_3 \cap \{x \mid x_1 = 0, \ |x_2| \leqq 1, x_3 = 0\},$$

hence A is contained in the 1-skeleton of every triangulation of X; since any two of the rectangles X_i intersect only along A, it follows that

$$L_2(\phi) = \sum_{i=0}^{m} L_2(\phi \mid X_i)$$

for every quasilinear map ϕ of X into E_2. Accordingly

$$L_2(f) \geqq \sum_{i=0}^{m} L_2(f \mid X_i) \geqq 4 \cdot (m + 1)$$

because f maps each rectangle X_i homeomorphically onto $g(X)$. Furthermore

$$M(f, z) = m + 1 \text{ for } z \in f(X - A)$$

and $f(X-A)$ is dense in $f(X)$, hence

$$M(f, z) \leqq m + 1 \text{ for } z \in E_2,$$

and we conclude that

$$L_2(f) = 4m + 4.$$

4. A theorem on stable values. Here we connect the stable values of a continuous map f of a finitely triangulable k dimensional space X into E_k with the stable values of the sections of f corresponding to a fibering of E_k into parallel $k-1$ dimensional planes. Such a connection was known previously [F8, Theorem 8.10] only in case $k=2$ and $X \subset E_2$, and the proof depended on very special properties of E_2.

4.1. LEMMA. *Suppose* (A, B) *is a compact pair and*

$$u: A \to E_1, \qquad v: A \to E_{k-1}, \qquad f: A \to E_k = E_1 \times E_{k-1}$$

are continuous maps such that

$$f(x) = (u(x), v(x)) \text{ for } x \in A.$$

Furthermore suppose Q *and* W *are* $k-1$ *dimensional cells in* E_{k-1},

$$W \subset \operatorname{Int} Q, \qquad v(A, B) \subset (Q, Q - \operatorname{Int} W),$$

and G *is a subgroup of* $H^{k-1}(A, B)$.

Let Γ *be the class of all closed subsets* C *of* A *such that* u *is constant on* C *and such that the homomorphisms*

$$H^{k-1}(Q, Q - \operatorname{Int} W) \xrightarrow{v^*} H^{k-1}(A, B) \xrightarrow{i^*} H^{k-1}(C, C \cap B),$$

induced by v and $C \subset A$, satisfy the conditions

$$i^* \circ v^* \neq 0 \quad \textit{and} \quad \text{kernel } i^* = G.$$

Then the following statement holds:

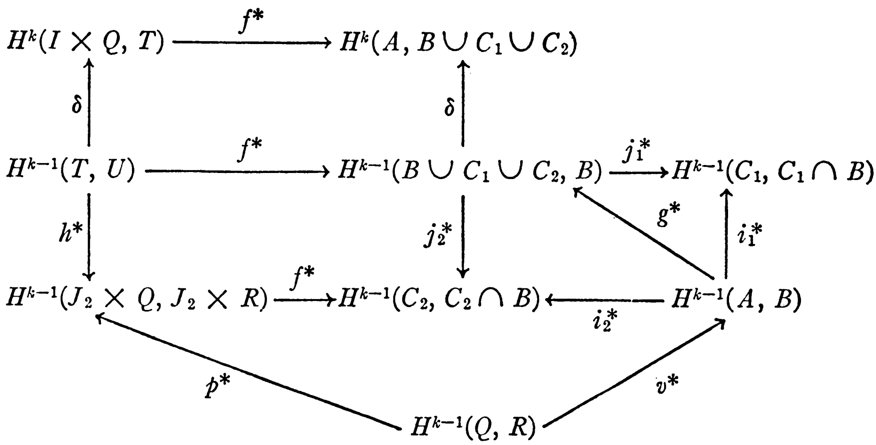
If $\inf u(\Gamma) < s < \sup u(\Gamma)$ *and* $t \in \operatorname{Int} W$, *then* (s, t) *is a stable value of* f.

Proof. Suppose $C_1 \in \Gamma$, $C_2 \in \Gamma$ and $u(C_1) < u(C_2)$.

We choose a compact interval I such that $u(A) \subset I$, let

$$J_1 = I \cap \{s \mid s \leqq u(C_1)\}, \qquad J_2 = I \cap \{s \mid s \geqq u(C_2)\}, \qquad R = Q - \operatorname{Int} W,$$
$$T = (I \times R) \cup (J_1 \times Q) \cup (J_2 \times Q), \qquad U = (I \times R) \cup (J_1 \times Q),$$

and consider the commutative diagram:



Here p is the projection mapping $(s, t) \in J_2 \times Q$ onto $t \in Q$, while g, h, i_1, i_2, j_1, j_2 are inclusion maps.

Since $i_1^* \circ v^* \neq 0$, we may choose

$$\eta \in H^{k-1}(Q, R) \text{ with } (i_1^* \circ v^*)(\eta) \neq 0.$$

Observing that h^* is an isomorphism we further define

$$\zeta = (h^{*-1} \circ p^*)(\eta),$$

and we shall prove that

$$(f^* \circ \delta)(\zeta) \neq 0.$$

In fact, if $(\delta \circ f^*)(\zeta) = (f^* \circ \delta)(\zeta) = 0$, then the exactness of the cohomology sequence of the triple $(A, B \cup C_1 \cup C_2, B)$ implies the existence of

$$\alpha \in H^{k-1}(A, B) \text{ with } g^*(\alpha) = f^*(\zeta),$$

whence

$$\begin{aligned} i_2^*(\alpha) &= (j_2^* \circ g^*)(\alpha) = (j_2^* \circ f^*)(\zeta) = (f^* \circ h^*)(\zeta) \\ &= (f^* \circ p^*)(\eta) = i_2^*[v^*(\eta)], \end{aligned}$$

$$i_2^*[\alpha - v^*(\eta)] = 0, \qquad \alpha - v^*(\eta) \in G, \qquad i_1^*[\alpha - v^*(\eta)] = 0,$$

$$i_1^*(\alpha) = (i_1^* \circ v^*)(\eta) \neq 0$$

but also

$$i_1^*(\alpha) = (j_1^* \circ g^*)(\alpha) = (j_1^* \circ f^*)(\zeta) = 0$$

because $f \circ j_1$ maps C_1 into U.

We conclude that the homomorphism

$$f^*: H^k(I \times Q, T) \to H^k(A, B \cup C_1 \cup C_2)$$

is nontrivial, and hence that every point of the set

$$(I \times Q) - T = \{s \mid u(C_1) < s < u(C_2)\} \times \text{Interior } W$$

is a stable value of f.

4.2. Theorem. *If X is a k dimensional finitely triangulable space and*

$$u: X \to E_1, \qquad v: X \to E_{k-1}, \qquad f: X \to E_k = E_1 \times E_{k-1}$$

are continuous maps such that

$$f(x) = (u(x), v(x)) \text{ for } x \in X,$$

then there is a countable set $D \subset E_1$ such that

$$S[f, (s, t)] \geqq S[v \mid u^{-1}(\{s\}), t] \text{ for } (s, t) \in (E_1 - D) \times E_{k-1}.$$

Proof. Let Q be a $k-1$ cell in E_{k-1} such that

$$v(X) \subset \text{Interior } Q,$$

and let H be a countable family of $k-1$ cells contained in the interior of Q and forming a base at each point of $v(X)$.

Choose a countable family F of compact pairs contained in X such that $H^{k-1}(A, B)$ is finitely generated whenever $(A, B) \in F$, and such that every compact pair in X is the intersection of a subfamily of F. (For instance, F may consist of all simplicial pairs which are subcomplexes of the successive barycentric subdivisions of a particular triangulation of X.)

Let Δ be the set of all quadruples

$$\delta = (W, A, B, G)$$

such that $W \in H$, $(A, B) \in F$, $v(B) \subset Q - W$, and G is a subgroup of $H^{k-1}(A, B)$.

Clearly Δ is countable.

For each $\delta \in \Delta$, let $\Gamma(\delta)$ be the class of all closed subsets C of A such that u is constant on C and such that the homomorphisms

$$H^{k-1}(Q, Q - \text{Int } W) \xrightarrow{(v \mid A)^*} H^{k-1}(A, B) \xrightarrow{i^*} H^{k-1}(C, C \cap B),$$

induced by $v|A$ and by $C \subset A$, satisfy the conditions

$$i^* \circ (v \mid A)^* \neq 0 \quad \text{and} \quad \text{kernel } i^* = G.$$

We define

$$D = \{s \mid \dim u^{-1}(\{s\}) = k\} \cup \bigcup_{\delta \in \Delta} \{\inf u[\Gamma(\delta)], \sup u[\Gamma(\delta)]\},$$

and observe that D is countable, because every k dimensional subset of X has interior points.

Now suppose $s \in E_1 - D$, $t \in E_{k-1}$, m is a positive integer, and

$$S[v \mid u^{-1}(\{s\}), t] \geqq m.$$

Let $C_1, \cdots, C_m$ be disjoint closed subsets of $u^{-1}(\{s\})$ such that t is a stable value of each of the maps $v|C_1, \cdots, v|C_m$. Since $t \in v(X)$ and

$$\dim C_j \leqq \dim u^{-1}(\{s\}) \leqq k - 1$$

for $j=1, \cdots, m$, we can choose $W \in H$ and closed subsets T_j of C_j such that

$$t \in \text{Int } W, \qquad v(T_j) \subset Q - W$$

and such that all the homomorphisms

$$(v \mid C_j)^*: H^{k-1}(Q, Q - \text{Int } W) \to H^{k-1}(C_j, T_j)$$

are nontrivial.

Next we choose $(A_j, B_j) \in F$ so that

$$(C_j, T_j) \subset (A_j, B_j) \quad \text{and} \quad v(B_j) \subset Q - W$$

for $j=1, \cdots, m$, and so that $A_1, \cdots, A_m$ are disjoint. From the commutative diagram

$$\begin{array}{ccc} H^{k-1}(Q, Q - \text{Int } W) & \xrightarrow{(v \mid C_j)^*} & H^{k-1}(C_j, T_j) \\ (v \mid A_j)^* \downarrow & & \uparrow h_j^* \\ H^{k-1}(A_j, B_j) & \xrightarrow[i_j^*]{} & H^{k-1}(C_j, C_j \cap B_j), \end{array}$$

where i_j and h_j are inclusion maps, we see that

$$i_j^* \circ (v \mid A_j)^* \neq 0.$$

Finally we define

$$G_j = \text{kernel } i_j^*, \qquad \delta_j = (W, A_j, B_j, G_j)$$

whence

$$\delta_j \in \Delta \quad \text{and} \quad C_j \in \Gamma(\delta_j),$$

and infer from Lemma 4.1 that (s, t) is a stable value of $f|A_j$ for $j=1, \cdots, m$. Accordingly

$$S[f, (s, t)] \geqq m.$$

4.3 Remark. *In the preceding theorem some local smoothness of X is essential.* For example let Y be any nondense perfect subset of E_1, let Z be a $k-1$ dimensional cube in E_{k-1}, and let $X = Y \times Z$, with $u(s, t) = s$, $v(s, t) = t$, $(s, t) = (s, t)$ for $s \in Y$, $t \in Z$. Then f is the inclusion map of the $k-1$ dimensional compact space X into E_k, hence f has no stable value, even though t is a stable value of $v|u^{-1}(\{s\})$ whenever $s \in Y$ and t is an interior point of Z.

On the other hand *the opposite inequality*

$$S[f, (s, t)] \leqq S[v \mid u^{-1}(\{s\}), t]$$

holds for any metric space X and for all $(s, t) \in E_1 \times E_{k-1}$. In fact suppose A is a compact subset of X and t is not a stable value of $v|[A \cap u^{-1}(\{s\})]$. Approximating this map by a map w: $A \cap u^{-1}(\{s\}) \to E_{k-1}$ which does not assume the value t, suitably extending w over A into E_{k-1}, and combining this extension with $u|A$, one obtains a map of A into E_k which approximates f and does not assume the value (s, t)

4.4 Remark. *The hypotheses* (1) *and* (2) *in Theorem* 4.8 *of* [F10] *may be replaced by the hypothesis that X is a 2 dimensional finitely triangulable space.*

In fact our previous argument applies to the new situation, provided the reference to Theorem 8.10 of [F8] is replaced by a reference to Theorem 4.2 of the present paper.

It follows that *the conclusions of Corollary* 4.9 *of* [F10] *hold whenever f is a continuous map of a 2 dimensional finitely triangulable space X into E_n such that*

$$\int_{E_2} S(P_n^{\xi} \circ f, z)\, d\mathfrak{L}_2 z < \infty \quad \text{for } \xi \in \Omega_n^2.$$

5. **On 2 dimensional Lebesgue area.** This section is concerned with the theory of Lebesgue area for continuous maps of a 2 dimensional finitely triangulable space into n-space.

5.1 Theorem. *Suppose X is a locally connected compact metric space such that $H^1(X)$ is finitely generated. If $p \in X$, then either p is a local separating point of X, or X is connectedly separated*(4) *at p.*

Proof. (5) Suppose $p \in X$ and p is not a local separating point of X.

---

(4) See [F10, Definition 5.1].

(5) *Added in proof.* Parts of our argument are similar to methods of R. L. Plunkett, *Some implications of semi-1-connectedness*, Proc. Amer. Math. Soc. vol. 5 (1954) p. 665. This paper was kindly brought to our attention by E. E. Floyd.

Since $H^1(\{p\})$ is trivial and $H^1(X)$ is finitely generated, there exists a region W such that $p \in W$ and such that the homomorphism

$$i^*: H^1(X) \to H^1(\text{Clos } W),$$

induced by Clos $W \subset X$, is trivial.

Let Z be any compact neighborhood of p such that $Z \subset W$.

The boundary of Z can be covered by finitely many regions whose closures are compact subsets of $W - \{p\}$, and any two such regions can be joined by an arc in $W - \{p\}$. Hence there exists a continuum C such that

$$\text{Bdry } Z \subset C \subset W - \{p\}.$$

Now suppose U is any open neighborhood of p for which

$$\text{Clos } U \subset Z - C.$$

Then Bdry U separates p from C in W. Choose $q \in C$, select a closed subset Q of Bdry U such that Q cuts W irreducibly between p and q, and let V be the component of p in $X - Q$.

We observe that

$$V \subset Z - C$$

because otherwise there would exist an arc $A \subset Z \cap V$ such that $p \in A$ and $A \cap C \neq 0$, and $A \cup C$ would be a connected set joining p and q in $W - Q$.

Accordingly Bdry $V \subset Q$ and $q \notin V$, Bdry V separates p and q in W, hence Bdry $V = Q$.

We shall complete the proof by showing that Q is connected.

Assuming that Q is not connected, we secure disjoint open sets R and S such that

$$Q \subset R \cup S \subset W$$

and neither R nor S separates p and q in W. From the Mayer-Vietoris cohomology sequences of the two triads

$$(\text{Clos } W; \text{Clos } W - R, \text{Clos } W - S), \qquad (X; X - R, X - S)$$

we obtain the commutative diagram:

$$\begin{array}{ccc}
H^0(\text{Clos } W - R) \oplus H^0(\text{Clos } W - S) & & \\
\downarrow h_1^* - h_2^* & & \\
H^0[\text{Clos } W - (R \cup S)] & \xleftarrow{j^*} & H^0[X - (R \cup S)] \\
\downarrow \Delta & & \downarrow \Delta \\
H^1(\text{Clos } W) & \xleftarrow{i^*} & H^1(X)
\end{array}$$

Since $R \cup S$ separates p and q in X, there exists

$$\alpha \in H^0[X - (R \cup S)] \text{ with } \alpha(p) \neq \alpha(q).$$

Moreover $i^*=0$, hence $\Delta \circ j^* = i^* \circ \Delta = 0$ and there exist

$$\beta \in H^0(\text{Clos } W - R), \qquad \gamma \in H^0(\text{Clos } W - S)$$

such that

$$j^*(\alpha) = h_1^*(\beta) - h_2^*(\gamma).$$

From the fact that neither R nor S separates Clos W between p and q it follows that

$$\beta(p) = \beta(q), \qquad \gamma(p) = \gamma(q).$$

Accordingly

$$\alpha(p) = \beta(p) - \gamma(p) = \beta(q) - \gamma(q) = \alpha(q),$$

which is incompatible with the choice of α.

5.2 COROLLARY. *Suppose X is a monotone image of a finitely triangulable space. If $p \in X$, then either p is a local separating point of X, or X is connectedly separated at p.*

Proof. Monotone maps induce monomorphisms of H^1, according to [K, §51, XI, 4]. Thus $H^1(X)$ is finitely generated.

5.3 LEMMA. *Every compact, locally connected metric space X contains a countable set T with the following property:*

If $p \in X - T$ and p is a local separating point of X, then p has arbitrarily small neighborhoods whose boundary consists of two elements of T.

Proof. Let A be the set of all local separating points of X, and let B be the set of all points of order 2 in X. Then $A - B$ is countable [K, §46, V, 9].

Let F be the family of all those open subsets of X whose boundary consists of two points. Then F is a basis at each point of B. Hence F has a countable subfamily G such that G is a basis at each point of B. Let

$$T = (A - B) \cup \bigcup_{W \in G} \text{Bdry } W.$$

5.4 LEMMA. *If C is the union of finitely many $n-2$ dimensional planes in E_n, $\epsilon > 0$, X is a metric space, H is a finite family of open subsets of X with finite boundaries, and*

$$h: \left(X, \bigcup_{W \in H} \text{Bdry } W\right) \to (E_n, E_n - C)$$

is a continuous map such that diam $[h(W)] < \epsilon$ *for $W \in H$, then there exists a continuous map*

$$\psi: \left(X, \bigcup_{W \in H} \text{Clos } W\right) \to (E_n, E_n - C)$$

such that

$$|\psi(x) - h(x)| < \epsilon \text{ for } x \in X,$$
$$\psi(x) = h(x) \text{ for } x \in X - \bigcup_{W \in H} W.$$

Proof. Ordering H and subtracting from each member of H the union of the closures of the preceding members, one may reduce the problem to the case in which H is disjointed, which in turn follows trivially from the case in which H has only one element.

Suppose therefore that $H = \{W\}$, and let U be a convex open subset of E_n such that $h(W) \subset U$ and diam $(U) < \epsilon$. The map

$$h | \text{Bdry } W: \text{Bdry } W \to E_n - C$$

is inessential, because Bdry W is finite and $U - C$ is arcwise connected, and hence there exists a continuous map

$$f: \text{Clos } W \to (U - C) \text{ with } f | \text{Bdry } W = h | \text{Bdry } W.$$

If $x \in W$, then $f(x) \in U$ and $h(x) \in U$, hence $|f(x) - h(x)| < \epsilon$. Accordingly a suitable function ψ is given by the formulae

$$\psi(x) = f(x) \text{ for } x \in W, \qquad \psi(x) = h(x) \text{ for } x \in X - W.$$

5.5 Remark. We observe that Theorem 5.3 of [F10] may be generalized as follows: In place of (1), assume that X is a metric space, A is a compact subset of X, and X is connectedly separated at each point of A. In the last sentence of the conclusion, replace the phrase "there is no point $x \in X$ such that" by the phrase "there is no point $x \in A$ such that." In the proof, just replace C^s by $C^s \cap A$.

5.6 Theorem. *Suppose*:

(1) *X is a compact metric space. If $p \in X$, then either p is a local separating point of X, or X is connectedly separated at p.*

(2) *$T \subset X$. If $p \in X - T$ and p is a local separating point of X, then p has arbitrarily small neighborhoods whose boundary consists of two points of T.*

(3) *$g_1, \cdots, g_n$ are real valued continuous functions on X.*

(4) *$F_1, \cdots, F_n$ are finite sets of real numbers.*

(5) *$T \cap g_i^{-1}(F_i) = 0$ for $i = 1, 2, \cdots, n$.*

(6) *If i, j, k are three distinct positive integers less than or equal to n, and $s \in F_i$, $t \in F_j$, $u \in F_k$, then $g_i^{-1}(\{s\})$ has dimension 0 at each point of*

$$g_i^{-1}(\{s\}) \cap g_j^{-1}(\{t\}) \cap g_k^{-1}(\{u\}).$$

(7) $\epsilon > 0$.

Under these conditions there exist closed subsets $D_{i,j}$ of X and real valued continuous functions ψ_i on X with the following properties:

If i is a positive integer less than or equal to n, then

$$|\psi_i(x) - g_i(x)| < \epsilon \textit{ for } x \in X.$$

If i and j are two distinct integers less than or equal to n, then

$$\psi_i^{-1}(F_i) \cap \psi_j^{-1}(F_j) \subset X - D_{i,j},$$
$$\psi_j(x) = g_i(x) \quad \textit{and} \quad \psi_j(x) = g_j(x) \textit{ for } x \in X - D_{i,j}.$$

If i, j, k are three distinct integers less than or equal to n, then

$$\psi_i^{-1}(F_i) \cap \psi_j^{-1}(F_j) \cap \psi_k^{-1}(F_k) = 0.$$

Proof. Let

$$g: X \to E_n, \qquad g(x) = (g_1(x), \cdots, g_n(x)) \text{ for } x \in X,$$

let G be the family of all those open subsets W of X such that diam $[g(W)] < \epsilon/4$ and the boundary of W consists of two points of T, and consider the compact set

$$A = \bigcup_{i=1}^{n} g_i^{-1}(F_i) - \bigcup_{W \in G} W.$$

If $p \in A$, then $p \notin T$, by (5); hence X is connectedly separated at p, by (2) and (1).

It follows from Theorem 5.3 of [F10], as modified in Remark 5.5, that there exist closed subsets $D_1, \cdots, D_n$ of X and real valued continuous functions $h_1, \cdots, h_n$ on X with the following two properties:

If i is a positive integer less than or equal to n, then

$$h_i(x) = g_i(x) \text{ for } x \in X - D_i, \qquad h_i(x) \notin F_i \text{ for } x \in D_i,$$
$$|h_i(x) - g_i(x)| < \epsilon/(8n) \text{ for } x \in X.$$

If i, j, k are three distinct positive integers less than or equal to n, then

$$A \cap \overset{-1}{h_i}(F_i) \cap \overset{-1}{h_j}(F_j) \cap \overset{-1}{h_k}(F_k) = 0.$$

Next let

$$h: X \to E_n, \qquad h(x) = (h_1(x), \cdots, h_n(x)) \text{ for } x \in X,$$
$$B = \bigcup_{1 \leq i < j < k \leq n} E_n \cap \{z \mid z_i \in F_i, z_j \in F_j, z_k \in F_k\},$$
$$C = \bigcup_{1 \leq i < j \leq n} E_n \cap \{z \mid z_i \in F_i, z_j \in F_j\},$$

and observe that

$$h^{-1}(B) \subset g^{-1}(B), \qquad h^{-1}(B) \cap A = 0, \qquad h^{-1}(B) \subset \bigcup_{W \in G} W.$$

Since $h^{-1}(B)$ is compact, G has a finite subfamily H such that

$$h^{-1}(B) \subset \bigcup_{W \in H} W.$$

If $W \in H$, then $h(\text{Bdry } W) \subset E_n - C$. In fact, if $p \in \text{Bdry } W$, then $p \in T$, hence $g_i(p) \notin F_i$, $h_i(p) \notin F_i$ for $i = 1, \cdots, n$.

Furthermore

$$\left| h(x) - g(x) \right| < \epsilon/8 \text{ for } x \in X,$$
$$\text{diam } [h(W)] < \epsilon/2 \text{ for } W \in H.$$

Accordingly Lemma 5.4 supplies a continuous map ψ of X into E_n such that

$$\left| \psi(x) - h(x) \right| < \epsilon/2 \text{ for } x \in X,$$
$$\psi(x) = h(x) \text{ for } x \in X - \bigcup_{W \in H} W,$$
$$\psi\left(\bigcup_{W \in H} \text{Clos } W \right) \subset E_n - C.$$

It follows that

$$\left| \psi(x) - g(x) \right| < \epsilon \text{ for } x \in X, \qquad \psi^{-1}(B) = 0.$$

In fact, if $\psi(x) \in B$, then $\psi(x) \in C$, x does not belong to a member of H, $\psi(x) = h(x)$, $h(x) \in B$, x does belong to a member of H.

Let $\psi_1, \cdots, \psi_n$ be the real valued functions on X such that

$$\psi(x) = (\psi_1(x), \cdots, \psi_n(x)) \text{ for } x \in X,$$

and let

$$D_{i,j} = D_i \cup D_j \cup \bigcup_{W \in H} \text{Clos } W$$

for any two distinct positive integers i and j less than or equal to n. Obviously

$$\psi_i(x) = h_i(x) = g_i(x) \quad \text{and} \quad \psi_j(x) = h_j(x) = g_j(x) \text{ for } x \in X - D_{i,j}.$$

Furthermore, if $\psi_i(x) \in F_i$ and $\psi_j(x) \in F_j$, then $\psi(x) \in C$, x does not belong to the closure of a member of H, $\psi(x) = h(x)$, $h_i(x) \in F_i$, $h_j(x) \in F_j$, $x \notin D_i$, $x \notin D_j$, hence $x \notin D_{i,j}$.

5.7 THEOREM. *If f is a continuous map of a 2 dimensional finitely triangulable space X into E_n, with $n \geqq 2$, then*

$$L_2(f) \leqq \sum_{\xi \in \Omega_n^2} L_2(P_n^\xi \circ f).$$

Proof. (A refinement of [F10, 7.14, Part 1].) We assume that

$$L_2(P_n^\xi \circ f) < \infty \text{ for } \xi \in \Omega_n^2,$$

and suppose $\epsilon > 0$.

Recalling the light-monotone factorization $f=l_f \circ m_f$ with middle space $\mathfrak{M}_f$, we let $g_1, \cdots, g_n$ be the real valued functions on $\mathfrak{M}_f$ such that

$$l_f(y) = (g_1(y), \cdots, g_n(y)) \text{ for } y \in \mathfrak{M}_f,$$

choose $T \subset \mathfrak{M}_f$ according to Lemma 5.3 (with X replaced by $\mathfrak{M}_f$), and let G be the set of all points $a \in E_n$ such that

$$T \cap g_i^{-1}(\{a_i\}) = 0 \text{ for } i = 1, \cdots, n$$

and such that $g_i^{-1}(\{a_i\})$ has dimension 0 at each point of

$$g_i^{-1}(\{a_i\}) \cap g_j^{-1}(\{a_j\}) \cap g_k^{-1}(\{a_k\})$$

whenever i, j, k are three distinct positive integers less than or equal to n. From the countability of T and Remark 4.4 we see that

$$\mathfrak{L}_n(E_n - G) = 0.$$

Defining Δ and R as in Lemma 3.15 (with $k=2$) and letting Λ be the set of all n termed sequences of even integers, we choose a point $a \in R$ such that

$$a + \epsilon\lambda \in G \quad \text{whenever} \quad \lambda \in \Lambda.$$

Furthermore we let

$$N_i = \{a_i + \epsilon m \mid m \text{ is an even integer}\},$$
$$F_i = N_i \cap \{t \mid |t - g_i(y)| < \epsilon \text{ for some } y \in \mathfrak{M}_f\}$$

for $i=1, \cdots, n$, and apply Theorem 5.6 with X replaced by $\mathfrak{M}_f$ [condition (1) follows from Corollary 5.2] to obtain closed subsets $D_{i,j}$ of $\mathfrak{M}_f$ and real valued continuous functions ψ_i on $\mathfrak{M}_f$ such that

$$|\psi_i(y) - g_i(y)| < \epsilon \text{ for } y \in \mathfrak{M}_f,$$
$$\psi_i^{-1}(F_i) \cap \psi_j^{-1}(F_j) \subset \mathfrak{M}_f - D_{i,j} \text{ for } i \neq j,$$
$$\psi_i(y) = g_i(y) \quad \text{and} \quad \psi_j(y) = g_j(y) \text{ for } y \in \mathfrak{M}_f - D_{i,j}, i \neq j,$$
$$\psi_i^{-1}(F_i) \cap \psi_j^{-1}(F_j) \cap \psi_k^{-1}(F_k) = 0 \text{ for } i \neq j \neq k \neq i.$$

We observe also that in these formulae F_i may be replaced by N_i because $\psi_i(y) \notin N_i - F_i$ for $y \in \mathfrak{M}_f$.

Next we define the functions $v_1, \cdots, v_n$ and v on X by the formulae

$$v_i = \psi_i \circ m_f \text{ for } i = 1, \cdots, n,$$
$$v(x) = (v_1(x), \cdots, v_n(x)) \text{ for } x \in X$$

and note that

$$\{i \mid v_i(x) \in N_i\} \text{ has at most 2 elements},$$
$$|v(x) - f(x)| < \epsilon n^{1/2} \text{ for } x \in X.$$

Furthermore Corollary 3.8 implies that

$$M[P_n^\xi \circ v,\ P_n^\xi(a) + \epsilon\delta] \leqq M[P_n^\xi \circ f,\ P_n^\xi(a) + \epsilon\delta]$$

whenever $\xi \in \Omega_n^2$ and $\delta \in \Delta$, because if $\xi_1 = i$ and $\xi_2 = j$ then

$$\begin{aligned}
P_n^\xi(a) + \epsilon\delta &\in N_i \times N_j,\\
(P_n^\xi \circ v)^{-1}(N_i \times N_j) &= m_f^{-1}[\psi_i^{-1}(N_i) \cap \psi_j^{-1}(N_j)] \subset X - m_f^{-1}(D_{i,j}),\\
(P_n^\xi \circ v)(x) &= ((\psi_i \circ m_f)(x),\ (\psi_j \circ m_f)(x)) = ((g_i \circ m_f)(x),\ (g_j \circ m_f)(x))\\
&= (P_n^\xi \circ f)(x) \text{ for } x \in X - m_f^{-1}(D_{i,j}).
\end{aligned}$$

Finally we choose a quasilinear function u according to Theorem 3.14 and conclude that

$$|u(x) - f(x)| < \epsilon 3 n^{1/2} \text{ for } x \in X$$

with

$$\begin{aligned}
\int_{E_n} N(u, X, z)\, d\mathcal{H}_n^2 z &\leqq \sum_{\xi \in \Omega_n^2} \sum_{\delta \in \Delta} 4\epsilon^2 M[P_n^\xi \circ f,\ P_n^\xi(a) + \epsilon\delta]\\
&\leqq \sum_{\xi \in \Omega_n^2} \int_{E_2} M(P_n^\xi \circ f, z)\, d\mathcal{L}_2 z \leqq \sum_{\xi \in \Omega_n^2} L_2(P_n^\xi \circ f).
\end{aligned}$$

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