THE EXISTENCE OF CERTAIN TYPES OF MANIFOLDS

BY

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Introduction. The first part of the paper gives a class of polyhedral 3-manifolds in the 4-sphere S^4 and with homology groups $H_1(M) = H_2(M) = 0$, but with $\pi_1(M) \neq 0$. Thus some "Poincare" spaces are imbeddable in S^4. Also, necessary and sufficient conditions are given for a group to be $H_1(M)$ for some 3-manifold M in S^4. In Part Two a negative answer is found for the following question: Since the metric cases of 2-gcms reduce to the classical types [12, IX], and the 3-gcms generally do not, can one state that a sphere-like 3-gcm in S^4 must be a classical manifold? We construct 3-gcms in S^4 which have the same homology and homotopy character as the 3-sphere (globally and locally), but which are not locally euclidean. In part three of the paper a negative answer is found for a question proposed by Griffiths [6] by showing the existence of a 3-dimensional "homotopy manifold" which is not locally euclidean. This is a space previously obtained by R. H. Bing by identifying the points on a certain set of tame arcs in S^3.

Part I. Let P be a connected 2-polyhedron which is contained in S^4 as a subcomplex of some simplicial decomposition Σ of S^4. Before anything else is done we make a barycentric subdivision of Σ. This is to make sure that the resulting decomposition of P will be complete in the sense that if all of the vertices of a simplex of S^4 lie in P, then the simplex itself is in P.

Now every simplex of $\mathrm{St}(P) - P$ is the join of a simplex of P with a simplex in $S^4 - \mathrm{St}(P)$. Hence each $x \in (\mathrm{St}(P) - P)$ may be assigned a number $t(x)$ between 0 and 1, its "distance from P". Newman [8] has shown that the set $M = [x \mid t(x) = 1/2]$ is a manifold. The set M is clearly 3-dimensional, polyhedral and orientable. If we let $X = [x \mid 0 \leq t(x) < 1/2]$, then $\overline{X}$, X and P all have the same homotopy type. The homology groups of M and P are related in the following manner.

PROPOSITION I.1. *If* $H_2(P, \mathfrak{p}) = 0$, *where* $\mathfrak{p}$ *is the additive group of reals modulo 1, then* $H_1(M, Z) = H_1(P, Z)$, *where* Z *is the additive group of integers.*

Proof. Let $Y = S^4 - \overline{X}$. By the Mayer-Vietoris theorem,

$$(1) \qquad H_1(M, Z) \cong H_1(\overline{X}, Z) + H_1(\overline{Y}, Z).$$

If $H_2(P, \mathfrak{p}) = 0$, then $H_2(X, \mathfrak{p}) = 0$, and $H_1(\overline{Y}, Z) = 0$ since $H_2(X, \mathfrak{p})$ and $H_1(\overline{Y}, Z)$ are dually paired. Hence (1) gives

$$H_1(M, Z) \cong H_1(\overline{X}, Z) \cong H_1(P, Z).$$

In the remainder of the paper, $H_1(M)$ will always be taken to mean $H_1(M, Z)$.

The natural contraction of $\overline{X}$ onto P induces a map $\phi: M \to P$, and this induces a homomorphism on the fundamental group of M.

PROPOSITION I.2. *The homomorphism $\phi_\#: \pi_1(M) \to \pi_1(P)$ induced by ϕ is onto.*

Proof. We apply the type of argument given in [8]. If $s: I \to P$ is a closed edge path in P, we choose for each edge σ', an edge $\bar{\sigma}'$ in M such that $\phi(\bar{\sigma}') = \sigma'$. Next the edges are to be connected so that we get an edge path $\bar{s}$ in M such that $\phi(\bar{s}) = s$. This will prove the lemma, and it can clearly be done if we show that for each vertex v in P, $\phi^{-1}(v)$ is connected.

Considering the decomposition of S^4 utilizing the vertices of M as well as all vertices in the previous decomposition, we have that $\phi^{-1}(v) = \beta(\mathrm{St}(v)) - \mathrm{St}(P)$, $\beta(\mathrm{St}(v))$ denoting the boundary of the star of v. We show that P does not separate $\beta(\mathrm{St}(v))$ since $\beta(\mathrm{St}(v))$ is a 3-sphere and $\beta(\mathrm{St}(v)) \cap P$ is at most 1-dimensional. If σ^2 were a 2-simplex in $\beta(\mathrm{St}(v)) \cap P$ then the join $\sigma^2 \vee v$ is a 3-simplex whose vertices lie in P. Since P is complete $\sigma^2 \vee v$ is in P, contradicting the 2-dimensionality of P. Since P does not separate $\beta(\mathrm{St}(v))$, neither does $\mathrm{St}(P)$ and $\phi^{-1}(v) = \beta(\mathrm{St}(v)) - \mathrm{St}(P)$ is connected. This proves the proposition.

We now construct some 2-polyhedra in S^4 to which we can apply the two propositions of this part.

(1) *A disk in S^4 whose boundary is n times a simple closed curve.*

This is the obvious generalization of a cross-cap. Divide the half-circular disk shown into $2n$ equal sectors.

It is clear that the curved boundaries of sectors $0\, p_{2k-1} p_{2k} p_{2k+1}\ (k = 1, \cdots, n)$ can be identified in E^3 without any other identifications, that is, without any self intersections except at the curved boundary, so that each arc $p_{2k-1} p_{2k} p_{2k+1}$ becomes a simple closed curve, and a preassigned orientation is preserved along $p_1 p_2 \cdots p_{2n+1}$. Now consider E^3 in E^4 and choose a point $q \in E^4 - E^3$. We add to the polyhedron the join of q with the simple closed curve $p_1 0 p_{2n+1}$. The resulting polyhedron is of the desired type.

APPLICATION. Given any finitely-generated abelian group G we take a basis $g_1, g_2, \cdots, g_n$. Then the relations are all of the form $k_i g_i = 0$ (with the convention that $k_i = \infty$ means that no multiple of g_i is the identity). We use coordinates x, y, z, w in E^4. In the xy plane we take n simple closed curves with a single common point. Call these $c_1, \cdots, c_n$. For each i we attach a disk in E^4 whose boundary is k_i times c_i. For $k_i = \infty$ no disk is attached. The polyhedron P so constructed is connected, 2-dimensional, and $H_1(P) = G$.
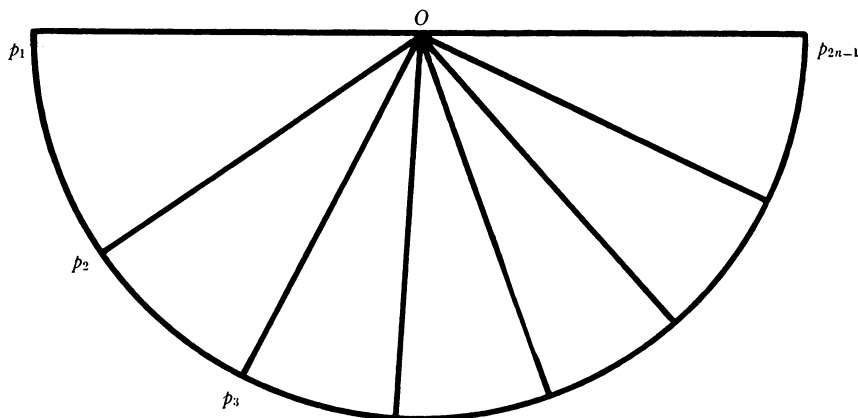
FIGURE 1

THEOREM I.1.([2]) *Let G be a finitely-generated abelian group. Then G is isomorphic with $H_1(M, Z)$ for some orientable polyhedral 3-manifold M in S^4 if and only if the torsion subgroup of G is the direct sum of two isomorphic groups.*

Proof. Suppose M is such a manifold in S^4. Let X and Y be the complementary domains. Then by the Mayer-Vietoris theorem

$$H_1(M, Z) \approx H_1(\overline{X}, Z) + H_1(\overline{Y}, Z).$$

By the Alexander duality theorem the torsion subgroups are such that

$$T_1(\overline{X}, Z) \approx T_1(Y, Z) \approx T_1(\overline{Y}, Z)$$

so that

$$T_1(M, Z) \approx T_1(\overline{X}, Z) + T_1(\overline{X}, Z).$$

Conversely, suppose $G = F + T + T$ where $T + T$ is the torsion subgroup of G. We use the technique described above to construct a polyhedron P having $H_1(P, Z) = F + T$. Then for the corresponding manifold M constructed as in the introductory paragraphs of Part I, we have

$$H_1(M, Z) \approx H_1(P, Z) + H_1(Y, Z).$$

Furthermore, $H_1(Y, Z)$ is a torsion group as we see by applying the Alexander duality to $\overline{X}$ and Y using a field of characteristic zero as coefficients. Hence $H_1(Y, Z) = T_1(Y, Z)$ and, again using the Alexander duality,

$$T_1(Y, Z) \approx T_1(\overline{X}, Z) \approx T_1(P, Z) \approx T.$$

Hence

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([2]) We are indebted to the referee for pointing out this theorem which replaces an incorrect version.

$$H_1(M, Z) \approx F + T + T = G.$$

(2) *A polyhedron P in S^4 having an arbitrary (but finitely-presentable) group as $\pi_1(P)$.*

Suppose the given group π has the presentation $(g_1, \cdots, g_k | r_1, \cdots, r_q)$. We take simple closed curves $c_1, \cdots, c_k$ in the xy plane having the origin as the only intersection. Each relator r_j is a word in the g_i's and we let a_j denote the number of letters in the word r_j. For each r_j we take a_j sectors $s_1, \cdots, s_{a_j}$ and identify the curved boundary of each s_t with the simple closed curve corresponding to the tth letter in r_j. Next, if the tth and the $(t+1)$st letter are the same we identify the last radial edge of s_t with the first radial edge of s_{t+1}. The points of the sectors at the centers of the corresponding circles are identified. This can all be accomplished in E^3 without any anwanted intersections. Each time the tth and the $(t+1)$st letters are different we are left with an "eye", and the same is true at the end of each word. Each eye is to be closed by taking the join of it with a point outside of E^3 in E^4. These joins can all be made disjoint (except at the common vertex of the s_t's and the origin). By the construction $\pi_1(P) = \pi$.

THEOREM I.2. *There exist 3-manifolds M in E^4 with $H_1(M) = 0$ but with $\pi_1(M) \neq 0$.*

Proof. Let π be a finitely-presentable perfect group. We construct P and M as above with $\pi_1(P) = \pi$. Since $\pi_1(M)$ maps onto $\pi_1(P)$, it follows that $\pi_1(M) \neq 0$. However, since π is perfect $H_1(P) = 0$. Hence to complete the proof, we must simply exhibit a perfect group which will make the corresponding P such that $H_2(P, \mathfrak{p}) = 0$; then by Proposition I.1, $H_1(M) = 0$. We take generators a, b and relations $a^{-3}(ab)^2 = 1$, $b^{-5}(ab)^2 = 1$, and a routine calculation gives $H_2(P, Z) = 0$. Now $H_2(P, \mathfrak{p}) = H_2(P, Z) \otimes \mathfrak{p} + \mathrm{Tor}\,(H_1(P, Z), \mathfrak{p})$ by the universal coefficient theorem. Since $H_1(P, Z) = 0$, we get the desired result.

Part II. It is known [13] that if M is an orientable n-gcm and $f(M) = M'$ is an $(n-1)$-monotone mapping of M onto a Hausdorff space M', then M' is an orientable n-gcm of the same homology type as M. Consequently one may hope to obtain n-gcms having curious properties by the simple device of identification of points in acyclic subsets of ordinary manifolds.

One such class of identifications is that wherein one identifies all points on a "wild" arc in ordinary 3-space.

Let C be an arc in E^3. If C is shrunk to a point the resulting space may not be E^3. However, if C is put in E^4 by putting E^3 in E^4 and then C is shrunk to a point, the resulting space is E^4. To show this we use the following lemma due to Klee [7, Lemma 3.3].

LEMMA. *If the linear space L is represented as a sum $L = L_1 + L_2$, C is a closed subset of L_2, and h is a homeomorphism of C into L_1, then there exists an*

isotopy of L onto itself, the initial stage of the isotopy being the identity and the final stage being h on C.

Let $E^4 = E^3 + E^1$ and h be a homeomorphism of C (in E^3) onto the unit interval in E^1. Let $f: E^4 \rightarrow E^4$ be the final stage of the isotopy furnished by the lemma, and let g_1 and g_2 be the quotient maps of E^4 obtained by shrinking C and $h(C)$ to points. Now $g_2(E^4)$ is homeomorphic with E^4 since $h(C)$ is a linear interval. The map $g_2 f g_1^{-1}$ is a homeomorphism of $g_1(E^4)$ to $g_2(E^4)$ and our assertion is proved. This means that an arc which is wild in E^3 will have to be tame in E^4.

In this part of the paper we use this result to obtain some non-euclidean generalized manifolds in the 4-sphere. In particular we shrink some of the Artin-Fox wild arcs [1].

(1) *A 3-gcm M in S^4 which has the homology and homotopy of S^3, is locally euclidean except at a point p, and $M - p$ is not simply connected.*

This is obtained by shrinking the arc of example 1.1 of [1]. The fact that M has the homotopy type of S^3 follows from a theorem of J. H. C. Whitehead [10, p. 1126]. The map $g_2 f g_1^{-1}$ imbeds M in S^4, and the other conclusions are clear from the properties of the arc used.

It should be noted that not only does the resulting 3-gcm M have the properties stated, but if U is any neighborhood of p in M, then $U - p$ contains a singular 1-sphere that is not contractible in $M - p$; i.e., $M - p$ is not $1 - LC$ at p [4, p. 614].

(2) Example 1.2 of [1] gives an M which is homeomorphic with S^3, because the complement of the arc is an open 3-cell. This shows, incidentally, that *an arc may be wild, yet its image in the quotient space obtained by identification of its points may not be wild.*

(3) *A 3-gcm M in S^4 which has the homotopy type of S^3 and contains a point p such that $M - p$ is locally euclidean and simply connected, but such that $M - p$ is not $1 - LC$ at p.*

This is obtained by using example 1.3 of [1].

(4) If example 1.4 of [1] is used, the result is S^3 just as in the case of example 1.2. To see this we must show that the complement of the arc $H^\natural$ is an open 3-cell. Let $A_1 = II^\#$, $A_2 = H^b$ and note that $A_1 \cap A_2 = q$. Let T_i ($i = 1, 2$) be a tubular neighborhood of $A_i - q$ (closely approximating this set), such that $\overline{T}_1 \cap \overline{T}_2 = q$. There is a homeomorphism h_i of $\overline{T}_i$ onto the 3-cell E ($x^2 + y^2 + z^2 \leq 1$) which carries q into $(1, 0, 0)$ and A_i onto the interval I ($0 \leq x \leq 1$, $y = 0 = z$). Also, there exists a mapping $g(E) = E$ which leaves points on the boundary of E fixed, is a homeomorphism on $E - I$, and carries I into $(1, 0, 0)$ (compare with [1, 984]).

The mapping f which is (1) the identity on $S^3 - (T_1 \cup T_2)$, (2) $h_i^{-1} g h_i$ on T_i, $i = 1, 2$, is a homeomorphism on $S^3 - H^\natural$ and carries $H^\natural$ into q. Hence $S^3 - H^\natural$ is an open 3-cell.

Part III. The examples of non-euclidean generalized manifolds in Part II

can all be distinguished from classical manifolds by some local homotopy properties. In this part we show that a certain example used by Bing [3] is a non-euclidean 3-dimensional manifold which has nicer local homotopy properties([3]). In particular, it is a homotopy manifold as defined by Griffiths [6].

Let $f: E^3 \to X$ be the mapping of E^3 induced by the decomposition of E^3 into points and tame arcs as defined by Bing [3]. As Bing shows, X is not homeomorphic with E^3. Let

$$C = [x \,|\, f^{-1}(x) \text{ is an arc}].$$

Then C is a Cantor set, and consists of those points of X at which X is not locally euclidean. Let $x \in C$ and suppose x has a neighborhood U homeomorphic with E^3. Referring to the manner of construction, we see that there exists in U a neighborhood of x whose boundary is a homeomorph of one of the "figure eight" surfaces closing down on $f^{-1}(x)$, say $A_{ij\ldots n}$. Let $A_{ij\ldots n}$ be the "A" of Bing's construction, with $f^{-1}(U)$ taking the place of his decomposition space; then the argument given in [3, §8] applies to show that U cannot be homeomorphic with E^3.

LEMMA 3.1. $\pi_k(X) = 0$ for all k, and X is LC^∞.

This follows from a theorem([4]) of Smale [9] which implies that all of the homotopy groups of X must vanish, and that X is LC^∞.

To show that X is a homotopy manifold, we need principally to show that if $x \in X$ and U is a neighborhood of x, then there exist neighborhoods V and W of x such that $W \subset V \subset U$, and if $j: (W-x) \to (V-x)$ is the inclusion map, then the induced homomorphism j_* is such that each image $j_*(\pi_k(W-x))$ is isomorphic with $\pi_k(S^2)$. Since this is certainly the case if $x \notin C$, we confine our attention to the case $x \in C$.

Let $\gamma = f^{-1}(x)$, and let $\{T_i\}$ denote the sequence of figure eight neighborhoods that close down on γ ([3]). Let $\{E_i\}$ be a sequence of 3-cell neighborhoods of γ such that (1) $T_i \supset E_i \supset T_{i+1}$, (2) $E_i - \gamma$ is homotopy equivalent to S^2 (recall that γ is a tame arc of E^3). We observe that each $D_i = f(T_i)$ is a neighborhood of x in X.

Let R_i^k $(k = i, i+1, \cdots)$ denote the space obtained from T_i by applying f to $T_i - T_k$ and leaving T_k alone; i.e., all points of an arc of $f^{-1}(C) \cap (T_i - T_k)$ are identified and only such. In an obvious manner, any continuous sphere $\phi(S^r)$ in $D_i - x$ can be considered to be in some $R_i^k - \gamma$. From the definition of the R_i^k, it follows that f induces maps

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([3]) We give an open manifold here, but since it is homeomorphic with E^3 outside of a bounded set, the 1-point compactification will give a closed manifold.

([4]) Smale's theorem states that if X and Y are arcwise connected, locally compact, separable metric spaces, $f: X \to Y$ is proper and onto, X is LC^n, and for each $y + Y$, $f^{-1}(y)$ is LC^{n-1} and $(n-1)$-connected, then Y is LC^n and f induces isomorphisms on the homotopy groups up through dimension $n-1$ (and a homomorphism onto for dimension n).

(1) $$\eta_i^r: (R_i^{i+r} - \gamma) \to (R_i^{i+r+1} - \gamma).$$

Also we have inclusion maps

(2) $$\tau_i^r: (R_i^{i+r} - \gamma) \to (R_{i-1}^{i+r} - \gamma).$$

The injection map $j_*^i: (D_{i+1}-x) \to (D_i-x)$ may be studied through the medium of the η and τ maps, by virtue of the following lemma.
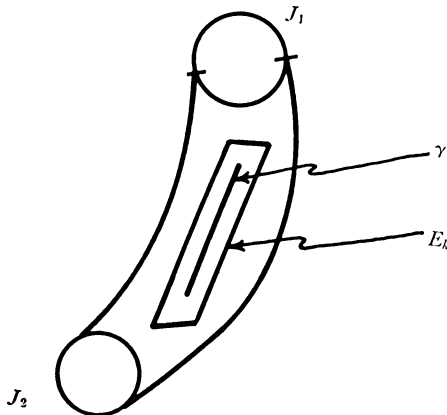

FIGURE 2

LEMMA 3.2. (1) $D_i - x = \lim_r (R_i^{i+r} - \gamma)$,
(2) $j_*^i = \lim_r \tau_i^{r*}$ *for each* i.

Proof. The maps in the following diagram commute.

$$R_i^{i+2} - \gamma \xrightarrow{\;\;\eta_i^2\;\;} R_i^{i+3} - \gamma \xrightarrow{\;\;\eta_i^3\;\;} R_i^{i+4} - \gamma \longrightarrow$$
$$\uparrow \tau_{i+1}^1 \qquad\qquad \uparrow \tau_{i+1}^2 \qquad\qquad \uparrow \tau_{i+1}^3$$
$$R_{i+1}^{i+2} - \gamma \xrightarrow{\;\;\eta_{i+1}^1\;\;} R_{i+1}^{i+3} - \gamma \xrightarrow{\;\;\eta_{i+1}^2\;\;} R_{i+1}^{i+4} - \gamma \longrightarrow$$

The conclusions of the lemma are automatic.

LEMMA 3.3. *Each* η_i^r *is a homotopy equivalence.*

Proof. The spaces $R_i^{i+r} - \gamma$ are LC^∞ by Lemma 3.1. The result now follows from the theorem of Smale used in Lemma 3.1 and a theorem of J. H. C. Whitehead [11] which implies that if a map induces isomorphisms of the homotopy groups, then the spaces have the same homotopy type (for spaces dominated by CW complexes).

LEMMA 3.4. $T_k - \gamma$ *is deformable in* $R_{k-1}^k - \gamma$ *onto* $E_k - \gamma$.
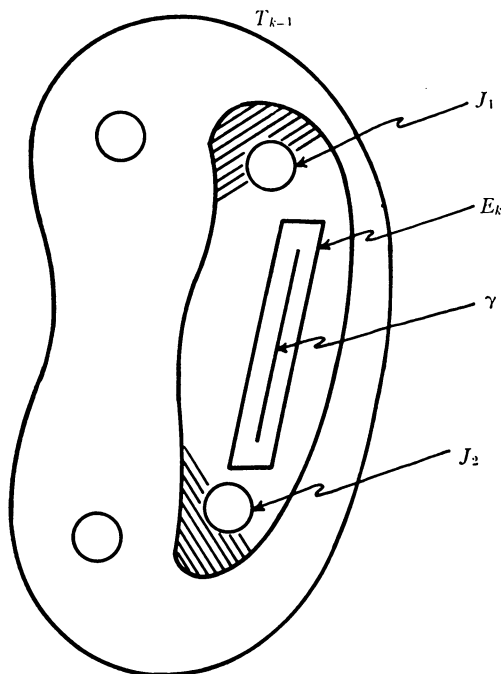
Proof. See Figure 2.

FIGURE 3

J_1 and J_2 are simple closed curves on the holes in T_k as shown. By a contraction of T_k in itself we get a configuration consisting of a 3-cell E_k' and 2 arcs which lie on J_1 and J_2 as shown in Figure 3.

During the contraction J_1 and J_2 remain fixed, as do all points of T_k except those in the shaded area about J_1 and J_2 as shown in Figure 2.

Now R_{k-1}^k is obtained from T_{k-1} by shrinking all arcs $f(x)$ not in T_k to points. Thus points on the boundary of T_k map into themselves, and since the curves J_1 and J_2 are deformable on $T_{k-1} - T_k$ onto the portion on E_k', the same is true in $R_{k-1}^k - \gamma$. Finally, E_k' is now deformable in $T_k - \gamma$ onto E_k.

THEOREM. *The space X is a 3-dimensional homotopy manifold which is not locally euclidean at all points.*

Proof. By definition, X is a 3-dimensional homotopy manifold if (1) dim X = 3, (2) X is LC^3, (3) at each $x \in X$, if U is a neighborhood of x then there exist neighborhoods V and W of x such that $U \supset V \supset W$ and $\pi_r(W-x)$ has image isomorphic to $\pi_r(S^2)$ in $\pi_r(V-x)$ (under the inclusion map).

(1) is clearly satisfied, since X is 3-dimensional at all $x \in X - C$, and for $x \in C$ the neighborhoods of type T_i have 2-dimensional boundaries. No $x \in C$

can have arbitrarily small neighborhoods with boundaries of dimension less than 2 because X satisfies Theorem 3.9 of $[12, \text{p. } 215]$.

(2) was established in Lemma 3.1.

(3) follows from Lemmas 3.3 and 3.4.

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