

# ON THE ZEROS OF THE DERIVATIVES OF SOME ENTIRE FUNCTIONS<sup>(1)</sup>

BY

ROBERT M. McLEOD

**1. Introduction.** 1.1. *Objective.* Let  $f$  be an entire function. Let  $\mathcal{L}_f$  be the set of points  $z$  such that to each disk  $D$  centered at  $z$  there corresponds a sequence of integers  $\{n_k\}$  and a sequence of points  $\{z_k\}$ ,  $z_k \in D$ , such that  $f^{(n_k)}(z_k) = 0$ . In [6] Pólya determined  $\mathcal{L}_f$  for functions  $f$  of finite order at least 2 having only a finite set of zeros. The object of this paper is to extend the results by relaxing the restriction on the set of zeros of  $f$ .

1.2. *Notations.* Throughout the paper the following notations will be used:  $\mathbf{C}$  is the complex plane.

$$D(z, r) = \{u: |u - z| < r\}; \quad C(z, r) = \{u: |u - z| = r\}.$$

If  $c = |c|e^{i\gamma}$ ,  $0 \leq \gamma < 2\pi$ ,  $|c| > 0$ , then

$$(1.1) \quad \alpha_k(c) = \exp[-\gamma i/q + 2\pi k i/q], \quad k = 0, 1, \dots, q-1,$$

and  $\beta_k(c) = |c|^{-1/q} \alpha_k(c)$ . Also,

$$A_k(c, \rho) = \{u: |u| > 0, |\arg[\alpha_k(c)/u]| < \rho\}$$

and  $E(c, \rho) = \bigcup_{k=0}^{q-1} A_k(c, \rho)$ . If  $w \in \mathbf{C}$ ,  $F(z) = w$ , then  $F^*(z) = w^*$ , where  $w^*$  is the complex conjugate of  $w$ . If  $F$  is a bounded, real-valued function,

$$M(F, z, r) = \sup_{u \in D(z, r)} F(u).$$

1.3. *Results.* Suppose  $f(z) = \phi(z) \exp(cz^q + dz^{q-1})$ ,  $q \geq 2$ , where

$$M(\log |\phi|, 0, r) = o(r^{q+1}).$$

Let  $\mathcal{R}_f$  be the set made up of the  $q$  rays emanating from the point  $-d/(qc)$  and passing through  $-d/(qc) + \alpha_k(c)e^{i\pi/q}$ ,  $k=0, 1, \dots, q-1$ . Pólya proved that  $\mathcal{L}_f = \mathcal{R}_f$  if  $f$  has a finite set of zeros. By application of his result to functions approximating to an  $f$  having an infinite set of zeros, it soon becomes clear that only in the directions  $\alpha_k(c)$  do the zeros of  $f$  influence  $\mathcal{L}_f$ . For functions  $f$  of the classes  $\mathcal{F}$  and  $\mathcal{G}$  defined below it is again true that

Presented to the Society, September 2, 1955; received by the editors February 6, 1956 and, in revised form, June 22, 1957.

<sup>(1)</sup> This paper is based on a thesis presented to the faculty of the Rice Institute. The author expresses sincere thanks to Professor G. R. MacLane for his skillful direction of the preparation of the thesis. This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under contract No. AF 18(600)-1341. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$\mathfrak{L}_f = \mathfrak{R}_f$ . The theorem to this effect is given in §4.

DEFINITION 1.  $f \in \mathfrak{F}$  if and only if  $f$  is entire and there exist  $c$  and  $d$  in  $\mathbf{C}$ ,  $c \neq 0$ , an integer  $q$ ,  $q \geq 2$ , and  $\rho > 0$  such that

$$(1.2) \quad \max_{|z|=r} \log |f(z)e^{-cz^q}| = o(r^q) \quad \text{as } r \rightarrow \infty,$$

$$(1.3) \quad \log |f(z)e^{-cz^q-dz^{q-1}}| = o(|z|^{q-1}), \quad z \in E(c, \rho).$$

Note that (1.3) requires  $f$  to have a finite set of zeros in  $E(c, \rho)$ .

DEFINITION 2.  $f \in \mathfrak{G}$  if and only if  $f$  is entire and there exist  $c$  and  $d$  in  $\mathbf{C}$ ,  $c \neq 0$ , and  $\rho > 0$  such that

$$(1.4) \quad \max_{|z|=r} \log |f(z)e^{-cz^2}| = o(r^2) \quad \text{as } r \rightarrow \infty,$$

$$(1.5) \quad \log |f(z)e^{-cz^2}| = o(|z|^2) \quad \text{as } z \rightarrow \infty, z \in E(c, \rho),$$

$$(1.6) \quad \log |\phi(z)/\phi(w_z)| = o(|z|), \quad \text{as } z \rightarrow \infty, z \in E(c, \rho),$$

where  $\phi(z) = f(z) \exp[-cz^2 - dz]$  and  $w_z$  is the reflection of  $z$  in the line through the origin and the points  $\alpha_k(c)e^{i\pi/2}$ ,  $k=0, 1$ .

The classes  $\mathfrak{F}$  and  $\mathfrak{G}$  intersect.  $\mathfrak{F}$  includes the class treated by Pólya.

In §6 there is given a class with yet weaker conditions on the zeros of  $f$  for which  $\mathfrak{R}_f$  may be a proper subset of  $\mathfrak{L}_f$ .

1.4. *Methods.* The proof here is fundamentally the same as that used by Pólya. But the details are somewhat simpler. The method of proof is: (1) find the asymptotic behavior of  $f^{(n)}(z)$  in certain sectors using a modification of a generalization of Stirling's formula due to Hayman [2, p. 69], (2) apply a theorem of Ganelius [1, p. 33] which gives an estimate from below on the number of zeros in certain neighborhoods.

1.5. *Related work.* Wyman and Moser have developed, in [4; 5], asymptotic series expansions for  $f^{(n)}(z)$  where  $f$  is the exponential of a polynomial. Theorem 1 of this paper gives only the first term of the asymptotic series.

Results similar to that of Ganelius have been given by Kay in [3].

General surveys of the study of the zeros of the sequence of derivatives are available in [7; 10].

## 2. A generalization of Stirling's formula.

2.1. *Definitions.* For convenience let  $D(0, \infty) = \mathbf{C}$ . Suppose  $f$  is holomorphic in  $D(0, R)$ ,  $0 < R \leq \infty$ . Associate with  $f$  the functions  $a_f$  and  $b_f$  defined by

$$(2.1) \quad a_f(z) = zf'(z)/f(z),$$

$$(2.2) \quad b_f(z) = za_f'(z).$$

DEFINITION 3. The class  $\mathbf{Z}_R$  consists of those functions  $f$ , holomorphic in  $D(0, R)$ , with the following properties:

(a) *There exist numbers  $K_f$  and  $R_f$ ,  $0 < R_f < R$ , and for each  $r \geq R_f$  a nonvoid set  $I_f(r)$  such that  $z \in I_f(r)$  implies  $|z| = r$  and*

$$(2.3) \quad [\operatorname{Im} a_f(z)]^2 / [\operatorname{Re} b_f(z)] \leq K_f.$$

(b) *There exists a real-valued function  $\delta_f$  defined on the interval  $(R_f, R)$  such that  $0 < \delta_f(r) < \pi$  and*

$$(2.4) \quad f(ze^{it}) = [1 + o(1)]f(z) \exp [it a_f(z) - (t^2/2)b_f(z)]$$

*as  $r \rightarrow R$ ,  $z \in I_f(r)$ , uniformly for  $|t| \leq \delta_f(r)$ , while*

$$(2.5) \quad f(ze^{it}) = o[f(z)[b_f(z)]^{-1/2}]$$

*as  $r \rightarrow R$ ,  $z \in I_f(r)$ , uniformly for  $\delta_f(r) \leq |t| \leq \pi$ .*

(c) *There exists a number  $M_f$  such that*

$$(2.6) \quad |b_f(z)/\operatorname{Re} b_f(z)| \leq M_f, \quad z \in I_f(r), R_f \leq r < R.$$

*Furthermore,*

$$(2.7) \quad |b_f(z)| \rightarrow \infty \quad \text{as } r \rightarrow R, z \in I_f(r).$$

2.2. THEOREM 1. *Let  $f \in \mathbb{Z}_R$ . Then, as  $r \rightarrow R$ ,*

$$(2.8) \quad \frac{f^{(n)}(0)}{n!} z^n = \frac{f(z)}{[2\pi b_f(z)]^{1/2}} \left\{ \exp \left[ -\frac{(a_f(z) - n)^2}{2b_f(z)} \right] + o(1) \right\},$$

*if  $z \in I_f(r)$ , uniformly for all integers  $n$ .*

*Moreover, if  $\mathbb{W}$  is a subclass of  $\mathbb{Z}_R$  such that there is a number  $R_0$  satisfying  $R_f \leq R_0$ ,  $f \in \mathbb{W}$ , and such that (2.3), (2.4), (2.5), (2.6), and (2.7) hold uniformly for all  $f$  in  $\mathbb{W}$ , then (2.8) holds uniformly for all  $f$  in  $\mathbb{W}$ .*

This theorem generalizes that of Hayman [2, p. 69]. The difference is primarily that Hayman requires that  $f(z)$  be real if  $z$  is real and that  $I_f(r) = \{r\}$ .

The following lemma anticipates our needs in proving Theorem 1.

2.3. LEMMA 1. *Let  $\delta$  be a positive number. Let  $a$  and  $b$  be complex numbers with  $\operatorname{Re} b > 0$ . Let  $\omega$  be a continuous, complex-valued function on the interval  $[-\delta, \delta]$ . Let  $\Omega = \max |\omega(t)|$ . Set  $a = a_1 + ia_2$ ,  $b = b_1 + ib_2$ , and  $b_1 = (B_1)^2$ ,  $B_1 > 0$ . Then*

$$(2.9) \quad \left| \int_{-\delta}^{\delta} \omega(t) e^{at - bt^2} dt \right| \leq \Omega \pi^{1/2} B_1^{-1} e^{a_1^2/(4b_1)}.$$

*Furthermore, if  $B_1 \delta - a_1(2B_1)^{-1} > 0$ , then*

$$(2.10) \quad \left| \int_{\delta}^{\infty} e^{at - bt^2} dt \right| \leq B_1^{-1} e^{-B_1^2 \delta^2 + a_1 \delta}.$$

**Proof.** By obvious steps,

$$\begin{aligned} \left| \int_{-\delta}^{\delta} \omega(t) e^{at-bt^2} dt \right| &\leq \Omega \int_{-\delta}^{\delta} e^{a_1 t - b_1 t^2} dt \\ &= \Omega e^{a_1^2 (4B_1^2)^{-1}} \int_{-\delta}^{\delta} e^{-(B_1 t - a_1/2B_1)^2} dt. \end{aligned}$$

The dominant becomes larger if  $\int_{-\infty}^{\infty}$  replaces  $\int_{-\delta}^{\delta}$ . But

$$\int_{-\infty}^{\infty} \exp [-(B_1 t - a_1/2B_1)^2] dt = (1/B_1) \int_{-\infty}^{\infty} \exp (-x^2) dx = \pi^{1/2} (1/B_1).$$

Thus we have (2.9).

In like manner,

$$\left| \int_{\delta}^{\infty} e^{at-bt^2} dt \right| \leq B_1^{-1} e^{a_1^2 (4B_1^2)^{-1}} \int_p^{\infty} e^{-x^2} dx$$

where  $p = B_1 \delta - a_1 (2B_1)^{-1}$ . But, as one shows easily by a change of variable,  $p > 0$  implies  $\int_p^{\infty} \exp (-x^2) dx < \exp (-p^2)$ . Thus we have (2.10).

**2.4. Proof of Theorem 1.** For convenience, we omit the subscript  $f$ .

From Cauchy's formula,

$$\frac{f^{(n)}(0)}{n!} z^n = \frac{1}{2\pi} \int_{-\delta(r)}^{2\pi - \delta(r)} f(ze^{it}) e^{-nit} dt.$$

With the same integrand, set  $I_1 = (1/2\pi) \int_{-\delta(r)}^{\delta(r)}$  and  $I_2 = (1/2\pi) \int_{\delta(r)}^{2\pi - \delta(r)}$ . From (2.5),  $I_2 = f(z) [2\pi b(z)]^{-1/2} o(1)$  as  $r \rightarrow R$ , if  $z \in I(r)$ , uniformly with respect to  $n$ .

From (2.4),

$$I_1 = \frac{f(z)}{2\pi} \int_{-\delta(r)}^{\delta(r)} [1 + \omega(t)] \exp [i(a(z) - n)t - b(z)t^2/2] dt$$

where  $\omega(t) = o(1)$ . Call the exponential in the integrand  $E(t)$ . Set

$$I_1 = \frac{f(z)}{2\pi} \left\{ \int_{-\infty}^{\infty} E(t) dt + \int_{-\delta(r)}^{\delta(r)} \omega(t) E(t) dt - \left[ \int_{-\infty}^{-\delta(r)} + \int_{\delta(r)}^{\infty} \right] E(t) dt \right\}.$$

To apply Lemma 1 to the various terms we need some deductions from (2.4) and (2.5).

Taking in turn  $t = \delta(r)$ ,  $t = -\delta(r)$  in (2.4) and (2.5), one infers that

$$\exp \left[ \delta(r) \operatorname{Im} a(z) - \frac{1}{2} \delta^2(r) \operatorname{Re} b(z) \right] = o([b(z)]^{-1/2})$$

and

$$\exp \left[ -\delta(r) \operatorname{Im} a(z) - \frac{1}{2} \delta^2(r) \operatorname{Re} b(z) \right] = o([b(z)]^{-1/2}).$$

Out of these relations and (2.7) it follows that

$$\frac{1}{2} \delta^2(r) \operatorname{Re} b(z) - \delta(r) |\operatorname{Im} a(z)| \rightarrow \infty \text{ as } r \rightarrow R.$$

In particular,  $\operatorname{Re} b(z)$  is positive for  $r$  sufficiently large.

From these facts we may verify the hypotheses of Lemma 1 for the last three integrals in the expression for  $I_1$ . Therefore, in view of (2.3),

$$I_1 = \frac{f(z)}{2\pi} \left\{ \int_{-\infty}^{\infty} E(t) dt + o([\operatorname{Re} b(z)]^{-1/2}) \right\}$$

uniformly for all integers  $n$ . But now

$$\int_{-\infty}^{\infty} E(t) dt = e^{-c^2} [b(z)/2]^{-1/2} \int_L e^{-y^2} dy$$

where  $c = [a(z) - n][2b(z)]^{-1/2}$  and  $L$  is the parametrized path given by the function  $t \rightarrow t[b(z)/2]^{1/2} - ic$ ,  $t$  real. (The square root  $[b(z)/2]^{1/2}$  is the one having positive real part.) A simple application of Cauchy's theorem gives  $\int_L \exp(-y^2) dy = \int_{-\infty}^{\infty} \exp(-x^2) dx$ .

Finally, in view of (2.6),

$$I_1 = \frac{f(z)}{[2\pi b(z)]^{1/2}} \left\{ \exp \left[ -\frac{(a(z) - n)^2}{2b(z)} \right] + o(1) \right\}.$$

On combining this with the earlier estimate on  $I_2$  we have (2.8).

The statement regarding uniformity over a class  $\mathfrak{W}$  is easy to check.

### 3. The classes $\mathfrak{F}$ and $\mathfrak{G}$ .

3.1. *General lemmas.* The first of the following lemmas has been given by Hayman, [2, p. 78], in very slightly different form.

LEMMA 2. *Let  $f$  be a function which is holomorphic and has no zeros in the disk  $D(w, \rho|w|)$ ,  $0 < \rho \leq 1$ . Let  $a_f$  and  $b_f$  be defined by (2.1) and (2.2). If*

$$(3.1) \quad |b_f(z)| < C |b_f(w)|, \quad z \in D(w, \rho|w|),$$

then

$$(3.2) \quad \log f(we^{it}) = \log f(w) + ita_f(w) - t^2 b_f(w)/2 + \eta(w, t)$$

where  $|\eta(w, t)| < C |b_f(w) t^3| / \rho$  for  $|t| \leq \rho/2$ .

LEMMA 3. *Let  $f$  be a function which is holomorphic and has no zeros in  $D(w, \rho|w|)$ ,  $w \neq 0$ . Let  $g_k(z) = (d^k/dz^k) \log f(z)$ ,  $k \geq 1$ ,  $g_0(z) = \log f(z)$ , and  $h(z) = \log [f(z)/f(w)]$ . If  $0 < \tau < \sigma < \rho$ , there is a positive number  $A$ , depending on  $\sigma$  and  $\tau$ , such that*

$$(3.3) \quad |g_k(z)| \leq k! A^k |w|^{-k} M(\operatorname{Re} h, w, \sigma |w|)$$

for  $k \geq 1$  and  $z$  in  $D(w, \tau |w|)$ .

Furthermore, there are numbers  $B$  and  $\omega$  such that for  $k \geq 0$  and  $z$  in  $D(0, \omega |w|)$

$$(3.4) \quad g_k(w+z) = g_k(w) + \eta_k(w, z)$$

where

$$|\eta_k(w, z)| < (k+1)! |z| |B/w|^{k+1} M(\operatorname{Re} h, w, \sigma |w|).$$

**Proof.** Set  $\epsilon = (\tau + \sigma)/2$ . Since the functions  $g_k$ ,  $k \geq 1$ , are also derivatives of  $h$ , Cauchy's integral yields

$$|g_k(z)| \leq \frac{k! \epsilon}{(\epsilon - \tau)^{k+1}} |w|^{-k} M(|h|, w, \epsilon |w|)$$

if  $z \in D(w, \tau |w|)$ . The Borel-Carathéodory inequality,

$$\max_{|z|=r} |\phi(z)| \leq \frac{2r}{R-r} \max_{|z|=R} \operatorname{Re} \phi(z) + \frac{R+r}{R-r} |\phi(0)|, \quad 0 < r < R,$$

(see [8, p. 174], e.g.) gives

$$M(|h|, w, \epsilon |w|) \leq \frac{2\epsilon}{\sigma - \epsilon} M(\operatorname{Re} h, w, \sigma |w|).$$

The conclusion (3.3) is now only a matter of naming an appropriate number  $A$ .

The Taylor expansion

$$g_k(w+z) = g_k(w) + \sum_{p=1}^{\infty} g_{k+p}(w) z^p / p!$$

yields  $\eta_k(w, z)$  in an obvious way. From (3.3)

$$\left| g_{k+p}(w) \frac{z^p}{p!} \right| \leq \left| \frac{A}{w} \right|^k M(\operatorname{Re} h, w, \sigma |w|) \frac{(k+p)!}{p!} \left| \frac{Az}{w} \right|^p.$$

Since  $\sum_{p=1}^{\infty} x^p (k+p)! / p! = k! [(1-x)^{-k-1} - 1]$  when  $|x| < 1$ , it is an easy matter to name suitable numbers  $\omega$  and  $B$ .

**3.2. Lemmas for entire functions.** We shall deal with entire functions  $f$  satisfying one or more of the following:

$$(3.5) \quad \max_{|z|=r} \log |f(z)| = o(r^q) \quad \text{as } r \rightarrow \infty;$$

in a certain unbounded region  $G$ ,

$$(3.6) \quad \log |f(z)| = o(|z|^q) \quad \text{as } z \rightarrow \infty;$$

$$(3.7) \quad \log |f(z)/f(-z^*)| = o(|z|) \quad \text{as } z \rightarrow \infty \text{ in } G.$$

Note that (3.6) implies a finite set of zeros of  $f$  in  $G$ .

LEMMA 4. Let  $f$  satisfy (3.5), (3.6), and (3.7) with  $q=2$  and  $G=E(1, \rho)$  for some  $\rho>0$ . Then, for  $\sigma<\rho$ ,

$$\log \left| \frac{f(w+z)}{f(-w^*+u)} \right| = o(|w|) \quad \text{as } w \rightarrow \infty \text{ in } E(1, \sigma),$$

uniformly for  $z$  and  $u$  in  $D(0, R)$ .

**Proof.** In the notation of the previous lemma, consider

$$\operatorname{Re}[g_0(w+z) - g_0(-w^*+u)].$$

Use (3.4). The dominants for  $\eta_0(w, z)$  and  $\eta_0(-w^*, u)$  in combination with (3.5), (3.6), and (3.7) yield the desired conclusion.

LEMMA 5. Under the hypotheses of Lemma 4,

$$a_f^*(-w^*) - a_f(w) = o(|w|)$$

as  $w \rightarrow \infty$  in  $E(1, \sigma)$ ,  $\sigma < \rho$ .

**Proof.** If  $\phi$  is a function holomorphic in an open set  $E$ , then  $\phi^*: z \rightarrow -\phi^*(-z^*)$  is holomorphic in the set  $E^*$  which is the reflection of  $E$  across the imaginary axis. Furthermore,  $\phi^{**} = -\phi^*$ . Set  $\phi = \log f$ . There is a number  $M$  such that both  $\phi$  and  $\phi^*$  are holomorphic in  $E(1, \rho) \cap \{z: |z| > M\}$ . Let  $w \in E(1, \sigma)$ ,  $\sigma < \rho$ , and set  $\psi(z) = \phi^*(z) + \phi(z) - \phi^*(w) - \phi(w)$ . There is a number  $\tau$  such that  $D(w, 2\tau|w|) \subset E(1, \rho)$ . Then

$$|\psi'(w)| \leq (\tau|w|)^{-1} M(|\psi|, w, \tau|w|)$$

for large  $|w|$ . Using the Borel-Carathéodory inequality we get the further inequality

$$|\psi'(w)| \leq \frac{2}{\tau|w|} M(\operatorname{Re} \psi, w, 2\tau|w|).$$

But  $\operatorname{Re} \psi(z) = \log |f(z)/f(-z^*)| - \log |f(w)/f(-w^*)|$ . Therefore (3.7) implies that  $\psi'(w) = o(1)$  as  $w \rightarrow \infty$  in  $E(1, \sigma)$ .

Now  $a_f^*(-w^*) - a_f(w) = -w\psi'(w)$ . The conclusion is immediate.

LEMMA 6. Let  $f$  be an entire function such that  $f(z) = \phi(z) \exp(cz^q)$  where  $\phi$  satisfies (3.5) and (3.6) with  $G=E(c, \rho)$  for some  $\rho$  such that  $0 < \rho < \pi/q$ .

A. Let  $f_u$  be the function  $z \rightarrow f(z+u)$ . Given  $R>0$  there exists  $x_R$  such that to each  $x$ ,  $x \geq x_R$ , there corresponds a set of  $q$  functions  $\psi_{zk}$ , defined on  $D(0, R)$  and holomorphic there, with the property that  $a_{f_u}(\psi_{zk}(u)) = x$ ,  $u \in D(0, R)$ .

B. To each  $u$  and  $k$  there corresponds a function  $\zeta_{uk}$  defined on a half-line  $[m, \infty)$  such that  $\operatorname{Im} a_{f_u}(\zeta_{uk}(r)) = 0$ ,  $r \geq m$ , and  $\zeta_{uk}(r) = r\alpha_k(c) \exp[i\theta_{uk}(r)]$ ,  $\theta_{uk}(r)$  real,  $\theta_{uk}(r) \rightarrow 0$  as  $r \rightarrow \infty$ . Moreover  $\psi_{zk}(u) = \zeta_{uk}(|\psi_{zk}(u)|)$ .

C. If  $c > 0$ ,  $q = 2$ , and  $\phi$  also satisfies (3.7), then  $\psi_{z0}(u) + \psi_{z1}^*(u) = o(1)$  as  $x \rightarrow \infty$ .

D. If  $\phi(z) = \phi_1(z)\phi_2(z) \exp(dz^{q-1})$  where  $\log|\phi_2(z)| = o(|z|^{q-1})$  as  $z \rightarrow \infty$  in  $E(c, \rho)$ , then

$$(3.8) \quad \psi_{zk}(u) = t_{zk} - (qu + d/c)(q-1)q^{-2} + o(1) \quad \text{as } x \rightarrow \infty,$$

uniformly in  $D(0, R)$ , where  $t_{zk}$  is the number  $\psi_{zk}(0)$  got by taking  $f(z) = \phi_1(z) \exp(cz^q)$  in part A. In particular, if  $f(z) = \exp(cz^q)$ , then  $t_{zk} = \beta_k(c)(x/q)^{1/q}$ . In general,  $t_{zk} = \beta_k(c)(x/q)^{1/q}(1 + o(1))$ .

**Proof.** It suffices to take  $c = 1$  since the general case is easily derived from this special case. We abbreviate  $A_k(1, \rho)$  to  $A_k(\rho)$  and  $\alpha_k(1)$  to  $\alpha_k$ . We regard  $k$  as fixed throughout the proof.

Let  $\phi_u$  be the function  $z \rightarrow \phi(z+u)$ . Let  $R$  and  $\sigma$  be positive numbers with  $\sigma < \rho$ . Then  $\phi_u$  satisfies (3.5) and (3.6) with  $G = A_k(\sigma)$ , uniformly for  $u$  in  $D(0, R)$ . With the aid of (3.3) we find that

$$a_{f_u}(z) = qz^q + o(z^{q-1}), \quad b_{f_u}(z) = q^2z^q + o(z^{q-1})$$

as  $z \rightarrow \infty$  in  $A_k(\sigma)$ , uniformly for  $u$  in  $D(0, R)$ .

Let  $w$  be a number in  $A_k(\sigma)$  such that  $w^q > 0$ . Then by quite simple considerations one may show that there is a positive number  $t_\epsilon$  such that  $|z^q - w^q| \geq t_\epsilon |z|^q$  for all  $z$  inside  $A_k(\sigma)$  but outside  $D(w, \epsilon|w|)$ .

Let  $S_k(s, \sigma) = A_k(\sigma) \cap \{z: |z| > s\}$ . Fix  $\epsilon$  for the moment and choose  $s$  so that

$$|a_{f_u}(z) - qz^q| \leq |z|^q t_\epsilon / 2$$

for all  $z$  in  $S_k(s, \sigma)$  and all  $u$  in  $D(0, R)$ . Let  $x$  be positive and set  $w = \alpha_k(x/q)^{1/q}$ . Evidently there is a number  $x_R$  such that  $x \geq x_R$  implies  $|a_{f_u}(z) - x| \geq qt_\epsilon |z|^q / 2$  for all  $z$  inside  $S_k(s, \sigma)$  but outside  $D(w, \epsilon|w|)$  and  $|a_{f_u}(z) - qz^q| < |qz^q - x|$  for  $z$  on  $C(w, \epsilon|w|)$ . The disk  $D(w, \epsilon|w|)$  is the only part of  $S_k(s, \sigma)$  in which a root of  $a_{f_u}(z) = x$  can lie. Rouché's theorem guarantees exactly one such root there. Denote it by  $\psi_{zk}(u)$ . The function  $\psi_{zk}: u \rightarrow \psi_{zk}(u)$  is defined at least on  $D(0, R)$  and is holomorphic. This last property may be deduced from the standard implicit function existence theorem.

The preceding argument is valid no matter how small  $\epsilon$  may be. Thus

$$\psi_{zk}(u) = \alpha_k(x/q)^{1/q}(1 + o(1))$$

as  $x \rightarrow \infty$ , uniformly for  $u$  in  $D(0, R)$ .

The problem in part B is to study the equation  $\operatorname{Im} a_{f_u}(z) = 0$ . We know already that the numbers  $\psi_{zk}(u)$  satisfy it. Since the function  $x \rightarrow \psi_{zk}(u)$  is continuous, the image of  $(x_R, \infty)$  is connected; thus it meets the circle



$C(0, r)$  if  $r$  is large enough. Actually the intersection consists of only one point. One may show this by applying elementary calculus to the function  $\theta \rightarrow \operatorname{Im} a_{f_u}(r e^{i\theta})$ ,  $\theta$  real. The conclusions of part B are now immediate from the facts given in this paragraph and the one before it.

The first step in proving C and D is to show that there are numbers  $\sigma$  and  $r$  such that

$$(3.9) \quad |a_{f_u}(w+z) - a_{f_u}(w)| \geq |zw^{q-1}|$$

provided  $w \in S_k(r, \sigma)$  and  $z \in D(0, \sigma|w|)$ .

There is a number  $r$  such that the right-hand member of

$$a_{f_u}(w+z) - a_{f_u}(w) - \frac{z}{w} b_{f_u}(w) = \sum_{p=2}^{\infty} a_{f_u}^{(p)}(w) z^p / p!$$

has the dominant  $A|z/w|^2|w|^q$  if  $w \in S_k(r, \rho/3)$ ,  $z \in D(0, |w|\rho/3)$ . But  $r$  can be chosen so that, in addition,  $|b_{f_u}(w)| \geq q|w|^q$ . The choice

$$\sigma = \min [\rho/3, q/2A]$$

yields (3.9).

Now we finish C. We can say equally well that (3.9) holds if  $w$  lies in  $E(1, \sigma)$  with  $|w| > r$ . Then, using Lemma 5, we find that

$$|a_{f_u}(-w^* + z) - a_{f_u}^*(w)| \geq |zw|/2$$

provided  $0 < \epsilon \leq |z| \leq \sigma|w|$  and  $|w|$  is large enough. Taking  $w = \psi_{x_0}(u)$ , we conclude, remembering that  $x$  is real, that  $\psi_{x_1}(u)$  lies in  $D(-\psi_{x_0}^*(u), \epsilon)$  since we know from the remark four paragraphs above that it lies in the concentric disk of radius  $\sigma|\psi_{x_0}(u)|$ . The conclusion is now immediate.

To finish D one need only prove that

$$a_{f_u}(t_{zk} - (qu + d/c)(q-1)/q^2) = x + o(t_{zk}^{q-1})$$

and use (3.9). The computation is quite straightforward; we omit the details.

**LEMMA 7.** *Let  $f \in \mathcal{F} \cup \mathcal{G}$ . Let  $\psi_{nk}$  be the function of Lemma 6, part B. Then to each  $R$  there corresponds an  $A$  such that*

$$\left| \frac{f^{(n)}(u)}{n!} \right| \leq \left| \frac{f(u + \psi_{nk}(u))}{[\psi_{nk}(u)]^n} \right| e^{An^{1-1/q}}, \quad u \in D(0, R).$$

**Proof.** From Cauchy's inequality,

$$|f^{(n)}(u)/n!| \leq |\psi_{nk}(u)|^{-n} M(|f|, u, |\psi_{nk}(u)|).$$

The remaining task is to show that  $|f(z)/f(u + \psi_{nk}(u))| \leq \exp(An^{1-1/q})$  if  $z \in C(u, |\psi_{nk}(u)|)$  and  $u \in D(0, R)$ . This will be carried out in the proof of Theorem 2.

3.3. THEOREM 2. Let  $f \in \mathcal{F} \cup \mathcal{G}$ . There are  $q$  sequences  $\{\psi_{nk}\}_{n=N}^{\infty}$ ,  $k=0, 1, \dots, q-1$ , such that  $\psi_{nk}$  is a function holomorphic in a region  $D_{nk}$  and

$$(3.10) \quad \frac{f^{(n)}(u)}{n!} = \frac{f(u + \psi_{nk}(u))}{[\psi_{nk}(u)]^n (2\pi q n)^{1/2}} (1 + o(1)), \quad u \in S_k,$$

uniformly in each compact subset of  $S_k$ , where  $S_k$  is the sector (or half-plane if  $q=2$ ) given by

$$(3.11) \quad S_k = \{u: |v| > 0, |\arg \alpha_k(c) - \arg v| < \pi/q\}$$

where  $v = qu + d/c$ .

If  $B$  is a bounded set, then  $B \subset D_{nk}$  for  $n \geq n_B$ . Finally,  $\psi_{nk}$  is given by (3.8).

3.4. **Proof of Theorem 2; completion of proof of Lemma 7.** Suppose, without loss of generality, that  $c=1$ . Let  $f_u$  be the function  $z \rightarrow f(z+u)$ . The first objective is to show that  $f_u \in \mathcal{Z}_{\infty}$  (see Definition 3) if  $u \in \bigcup_{j=0}^{q-1} S_j$ . However, since the proof of Lemma 7 is also to be completed, certain inequalities will be proved for  $u \in D(0, R)$ .

Throughout the proof  $k$  is regarded as a fixed integer.

From part B of Lemma 6 there is a function  $\zeta$  such that  $\text{Im } a_{f_u}(\zeta(r)) = 0$  and  $\zeta(r) = \alpha_k(1) r e^{i\theta(r)}$  with  $\theta(r)$  real and  $\lim_{r \rightarrow \infty} \theta(r) = 0$ . Let  $I(r) = \{\zeta(r)\}$  and  $\delta(r) = r^{-2q/5}$ . Now (2.3) is satisfied for  $f_u$  with  $K=0$ .

From Lemma 2 we see that (2.4) is satisfied. Moreover, there is a positive number  $\epsilon_1$  such that (2.5) holds for  $\delta(r) \leq |\theta| \leq \epsilon_1$ . Both relations are uniform for  $u$  in  $D(0, R)$ .

Let  $s = r e^{i2\pi k/q}$ .

The first and second derivatives of the function  $\theta \rightarrow -\log |f_u(s e^{i\theta})|$ ,  $\theta$  real, are  $\text{Im } a_{f_u}(s e^{i\theta})$  and  $\text{Re } b_{f_u}(s e^{i\theta})$ , respectively. Therefore  $|f_u(s e^{i\theta})| \leq |f_u(\zeta(r))|$  for  $|\theta| \leq \rho$ . Then in checking (2.5) for  $\epsilon_1 \leq |\theta| \leq \pi$  we may replace  $f_u(\zeta(r))$  by  $f_u(s)$ .

Consider  $|f_u(z)/f_u(s)|$  on that part of  $C(0, r)$  outside  $E(1, \epsilon)$  for any  $\epsilon > 0$ . One sees readily that there is a positive number  $\gamma_1$  such that  $\text{Re } [z^q - s^q] \leq -\gamma_1 r^q$ . It is an easy step to show that to  $\epsilon$  there corresponds  $\gamma > 0$  such that

$$(3.12) \quad \log |f_u(z)/f_u(s)| \leq -\gamma r^q$$

if  $|z| = r$  and  $z \notin E(1, \epsilon)$ , uniformly for  $u$  in  $D(0, R)$ .

The following two inequalities are proved in succeeding paragraphs. Let  $R$  be a positive number and  $H$  a compact subset of  $A_k(1, \pi/q)$ . Then there exists a positive number  $A$  such that

$$(3.13) \quad |f_u(z)/f_u(\zeta(r))| \leq e^{A r^{q-1}}, \quad |z| = r,$$

if  $u \in D(0, R)$  and  $z \in \bigcup_{j \neq k} A_j(1, \epsilon)$ . Also, there are positive numbers  $\gamma$  and  $\epsilon_2$  such that

$$(3.14) \quad |f_u(z)/f_u(\zeta(r))| \leq e^{-\gamma r^{q-1}}$$

if  $qu+d \in H$  and  $z \in \bigcup_{j \neq k} A_j(1, \epsilon_2)$ .

At this point we consider separately the cases  $f \in \mathcal{F}$  and  $f \in \mathcal{G}$ .

Let  $f \in \mathcal{F}$ . Set  $d_u = qu + d$  and  $|d_u| e^{i\tau} = d_u e^{-i2\pi k/q}$ . If  $z = s e^{i\theta}$ , then

$$\operatorname{Re} [d_u(z^{q-1} - s^{q-1})] = |d_u| r^{q-1} [\cos(\tau + (q-1)\theta) - \cos \tau].$$

Set  $\theta = 2\pi j/q + 2\beta/(q-1)$ ,  $1 \leq j \leq q-1$ . Then

$$\cos(\tau + (q-1)\theta) - \cos \tau = -2 \sin(-\pi j/q + \beta + \tau) \sin(-\pi j/q + \beta).$$

Since  $qu+d \in A_k(1, \pi/q)$ ,  $|\tau| < \pi/q$ ; consequently  $-\pi < -\pi j/q + \tau < 0$  and, if  $|\beta|$  is sufficiently small,  $\cos(\tau + (q-1)\theta) - \cos \tau < 0$ . It is now clear that to  $H$  there corresponds an  $\epsilon_2$  and a  $\gamma_1$  such that

$$\operatorname{Re} [d_u(z^{q-1} - s^{q-1})] \leq -\gamma_1 r^{q-1}, \quad |z| = r,$$

if  $z \in \bigcup_{j \neq k} A_j(1, \epsilon_2)$  and  $qu+d \in H$ .

From (1.3),

$$\log |f_u(z)/f_u(s)| \leq \operatorname{Re} [d_u(z^{q-1} - s^{q-1})] + o(r^{q-1}).$$

The last inequality alone makes clear the existence of the number  $A$ . The last two inequalities show the existence of the number  $\gamma$ .

Now let  $f \in \mathcal{G}$ . Set  $d_u = 2u + d$ . In this case the convenient way is to consider  $f_u(z)/f_u(\zeta(r))$ . The set  $E(1, \epsilon)$  consists of the sectors  $A_0(1, \epsilon)$  and  $A_1(1, \epsilon)$ . Put  $z = -\zeta^*(r) e^{i\theta}$ ,  $\theta$  real,  $|\theta| \leq \epsilon$ . Then  $z \in A_l(1, 2\epsilon)$ ,  $l = 1 - k$ , for large  $r$ . Put  $w = -\zeta^*(r)$ . From Lemma 2 one gets

$$\log [f_u(z)/f_u(w)] = ia_{f_u}(w)\theta - \theta^2 b_{f_u}(w)/2 + \eta(w, \theta).$$

There is an  $\epsilon_2$  for which  $|\eta(w, \theta)| \leq \operatorname{Re} b_{f_u}(w)\theta^2/4$  if  $|\theta| \leq \epsilon_2$ . Therefore the real part of the sum of the last two terms is negative. On the other hand, in view of the definition of  $w$  and Lemma 5,  $\operatorname{Re}[ia_{f_u}(w)\theta] = o(r)$ . Thus we assert that

$$\log |f_u(z)/f_u(\zeta(r))| \leq \log |f_u(-\zeta^*(r))/f_u(\zeta(r))| + o(r),$$

uniformly for  $u$  in  $D(0, R)$ . But

$$|f_u(w)/f_u(-w^*)| = |e^{d_u(w+w^*)} \phi_u(w)/\phi_u(-w^*)|.$$

Using Lemma 4 we find that (3.13) and (3.14) hold for  $f \in \mathcal{G}$ .

Lemma 7 follows from (2.5) for  $|\theta| \leq \epsilon_1$ , (3.13), and (3.12) with  $\epsilon = \min(\epsilon_1, \epsilon_2)$  if we choose  $r$  so that  $\zeta(r) = \psi_{nk}(u)$ .

To establish (2.5) on  $\epsilon_1 \leq |\theta| \leq \pi$  it is enough to use (3.14) and (3.12) with  $\epsilon = \min(\epsilon_1, \epsilon_2)$ .

The remaining conditions (2.6) and (2.7) are easy to check.

Taking that  $r$  for which  $\zeta(r)$  becomes  $\psi_{nk}(u)$ , we get (3.10) from (2.8) since

$$[2\pi b_{f_u}(\psi_{nk}(u))]^{1/2} = (2\pi qn)^{1/2}(1 + o(1)).$$

#### 4. The principal theorems.

4.1. *An auxiliary theorem.* The tool to be used in determining  $\mathfrak{L}_f$  is the following generalization of Jentzsch's theorem due to Ganelius [1, p. 33].

**THEOREM 3.** *Let  $\{F_n\}_{n=1}^{\infty}$  be a sequence of functions holomorphic in a region containing the closure of the bounded region  $G$ . Let  $z_0$  be a point of  $G$ . Let  $\lambda_n = \sup_{z \in G} \log |F_n(z)|$ . Suppose that  $\lambda_n \rightarrow \infty$  and that there is a positive number  $\delta$  such that  $|F_n(z_0)| \geq \delta$ ,  $n \geq 1$ .*

*Set  $\log^+ x = \max(\log x, 0)$ .*

*Suppose that there is a region  $G_1$  containing  $z_0$  such that*

$$(4.1) \quad \lim_{n \rightarrow \infty} \lambda_n^{-1} \log^+ |F_n(z)| = 0, \quad z \in G_1.$$

*Let  $E$  be the largest such region.*

*Let  $v$  be a boundary point of  $E$  that belongs to  $G$ . Then, to each neighborhood  $V$  of  $v$  there corresponds a subsequence  $\{F_{n_k}\}$  and a positive number  $K$  such that the number of zeros of  $F_{n_k}$  in  $V$  is not less than  $K\lambda_{n_k}$ .*

4.2. *The set  $\mathfrak{L}_f$ .* Recalling from §1.3 the definition of  $\mathfrak{R}_f$ , let us note that  $\mathfrak{R}_f$  is simply the complement of  $\bigcup_{k=0}^{\infty} S_k$ , where  $S_k$  is the sector defined in (3.11).

**THEOREM 4.** *Let  $f \in \mathfrak{F} \cup \mathfrak{G}$ . Let  $v \in \mathfrak{R}_f$  and let  $V$  be a neighborhood of  $v$ . There exist numbers  $N$  and  $K$  such that the number of zeros of  $f^{(n)}$  in  $V$  is not less than  $Kn^{1-1/q}$  if  $n > N$ .*

**Proof.** We may suppose that  $v$  lies on the ray between  $S_k$  and  $S_{k+1}$ . Let  $G$  be the disk with center at  $v$  and unit radius. (We may suppose  $V$  to be a smaller, concentric disk.) Fix  $z_0$  in  $G \cap S_k$ . Using the notation of Theorem 2, set

$$F_n(z) = \frac{f^{(n)}(z) [\psi_{nk}(z)]^n [2\pi q n]^{1/2}}{n! f(z + \psi_{nk}(z))}.$$

The functions  $F_n$  are certainly holomorphic in a region containing the closure of  $G$  for all  $n$  sufficiently large, and the zeros of  $F_n$  in  $V$  are exactly those of  $f^{(n)}$ .

Out of (3.8) and (3.10) we shall establish that

$$\begin{aligned} F_n(z) &= 1 + o(1), & z \in S_k \cap G, \\ \log |F_n(z)| &\geq Mn^{1-1/q}, & z \in S_{k+1} \cap G. \end{aligned}$$

In fact, the first of these is quite clear from the definition of  $F_n$  and (3.10). To prove the second, one must treat two expressions:  $|\psi_{nk}(z)/\psi_{n,k+1}(z)|$  and  $|f(z + \psi_{n,k+1}(z))/f(z + \psi_{nk}(z))|$ . In each instance it is convenient to distinguish the cases  $f \in \mathfrak{F}$  and  $f \in \mathfrak{G}$ .

Set  $\omega = e^{-2\pi i/q}$  and  $\Psi = |\psi_{nk}(z)/\psi_{n,k+1}(z)|$ .

If  $f \in \mathfrak{F}$ ,

$$\Psi = \left| \frac{t_{nk} - m + o(1)}{t_{nk} - m\omega + o(1)} \right|,$$

where  $m = (qz + d/c)(q-1)/q^2$  and  $t_{nk} = \beta_k(c)(n/q)^{1/q}$ . Consequently  $\log \Psi = \operatorname{Re} [m(\omega-1)/t_{nk}] + o(n^{-1/q})$ . With  $z$  in  $S_{k+1}$ ,  $\arg m = \arg t_{nk} + 2\pi/q + \tau$ ,  $|\tau| < \pi/q$ . It follows that  $\log \Psi \geq Kn^{-1/q}$ .

Let  $f \in \mathfrak{G}$  and suppose, without loss of generality, that  $c > 0$ . Then from parts C and D of Lemma 6

$$\Psi = \left| \frac{t_{nk} - m + o(1)}{t_{nk} + m^* + o(1)} \right|$$

and  $\log \Psi = \operatorname{Re} [-(m+m^*)/t_{nk}] + o(n^{-1/2})$ . The conclusion  $\log \Psi \geq Kn^{-1/2}$  may be drawn from the fact that  $\operatorname{Re} m \operatorname{Re} t_{nk} < 0$ .

Now set  $\Phi = |f(z+\psi_{n,k+1}(z))/f(z+\psi_{nk}(z))|$ . Using (1.3) and (3.8) it is not difficult to show that

$$\log \Phi = \operatorname{Re} \left\{ \frac{nm}{q-1} \left[ \frac{1}{t_{n,k+1}} - \frac{1}{t_{nk}} \right] \right\} + o(n^{1-1/q})$$

if  $f \in \mathfrak{F}$ . Calling on Lemma 4, one shows as readily that

$$\log \Phi = -4c \operatorname{Re} m \operatorname{Re} t_{nk} + o(n^{1/2})$$

if  $f \in \mathfrak{G}$ . From each of these it follows that  $\log \Phi \geq Kn^{1-1/q}$ .

Finally,  $\log |F_n(z)| = n \log \Psi + \log \Phi$  if  $z \in S_{k+1} \cap G$ ; consequently  $\log |F_n(z)| \geq Kn^{1-1/q}$ .

The inequality  $K_1 n^{1-1/q} \leq \lambda_n \leq K_2 n^{1-1/q}$  is a consequence of the estimate just obtained and Lemma 7.

The point  $v$  is a boundary point of the maximal region  $E$  for the sequence  $\{F_n\}$  and for each of its subsequences as well. The assertion of Theorem 4 follows now from Theorem 3.

**THEOREM 5.** *If  $f \in \mathfrak{F} \cup \mathfrak{G}$ , then  $\mathfrak{L}_f = \mathfrak{R}_f$ .*

**Proof.** Theorem 4 implies  $\mathfrak{R}_f \subseteq \mathfrak{L}_f$ . The reverse inclusion follows from the fact that (3.10) holds uniformly in compact subsets of  $S_k$ .

**5. Some subclasses of  $\mathfrak{F}$  and  $\mathfrak{G}$ .** 5.1. *A subclass of  $\mathfrak{F}$ .* Let

$$(5.1) \quad f(z) = z^m P(z) e^{Q(z)}$$

where  $Q(z) = \sum_{k=0}^q b_k z^{q-k}$  and  $P$  is a canonical product of genus  $p$ ,  $0 \leq p \leq q-2$ . In order that  $f \in \mathfrak{F}$ , with  $c = b_0$  and  $d = b_1$ , it suffices that  $f$  have at most a finite set of zeros in some set  $E(b_0, \rho)$ . The lemma of [9, p. 53] serves well as the basis of a proof.

5.2. *A subclass of  $\mathcal{G}$ .* Let  $f$  have the form (5.1) with  $Q$  of degree 2 and  $P$  of genus 1. Suppose that  $b_0 > 0$ , that  $f$  has but a finite number of zeros in some  $E(1, \rho)$ , and that  $\sum_{k=1}^{\infty} |\operatorname{Re} (1/a_k)| < \infty$ , where  $\{a_k\}$  is the sequence of zeros of  $f$ . Then  $f \in \mathcal{G}$  with  $c = b_0$  and  $d = b_1 + \sum_{k=1}^{\infty} \operatorname{Re} (1/a_k)$ . The lemma mentioned above again serves well in the proof.

6. **A class outside  $\mathcal{F} \cup \mathcal{G}$ .** In the description of the subclass of  $\mathcal{F}$  given above, let the requirement that in some  $E(b_0, \rho)$   $f$  has a finite set of zeros be replaced by the requirement that  $f$  has in each disk in  $E(b_0, \rho)$  at most one zero, counted according to multiplicity. (A finite set of zeros may be excepted.) It is possible to find  $\mathcal{L}_f$  in this instance by a modification of the methods used for  $\mathcal{F}$  and  $\mathcal{G}$ . In fact,  $\mathcal{R}_f \subseteq \mathcal{L}_f$ , but the inclusion may be proper.  $\mathcal{L}_f$  contains the half-line  $S_k \cap \{z: \operatorname{Im} [z/\alpha_k(b_0)] = \eta\}$  if and only if for every  $\delta > 0$  the half-strip

$$S_k \cap \{z: \eta - \delta \leq \operatorname{Im} [(qb_0z - (q-1)b_1)(q^2b_0\alpha_k(b_0))^{-1}] \leq \eta + \delta\}$$

contains an infinite set of zeros of  $f$ . These additional points in  $\mathcal{L}_f$  arise from the presence of zeros of  $f$  near the path on which

$$\max_{u \in D(z, r)} |\exp [b_0u^q + b_1u^{q-1}]|, \quad z \in S_k,$$

occurs.

It seems reasonable to conjecture that for each function  $f$  of the form (5.1), without restriction on the location of the zeros but with  $0 \leq p \leq q-2$ , the set  $\mathcal{L}_f$  will have the description just given.

#### REFERENCES

1. Tord Ganelius, *Sequences of analytic functions and their zeros*, Ark. Mat. vol. 3 (1954) pp. 1-50.
2. W. K. Hayman, *A generalisation of Stirling's formula*, J. Reine Angew. Math. vol. 196 (1956) pp. 67-95.
3. A. F. Kay, *Distribution of zeros of sequences of polynomials of unbounded degree*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 571-582.
4. Leo Moser and Max Wyman, *Asymptotic expansions*, Canad. J. Math. vol. 8 (1956) pp. 225-233.
5. ———, *Asymptotic expansions II*, Canad. J. Math. vol. 9 (1957) pp. 194-209.
6. G. Pólya, *Über die Nullstellen sukzessiver Derivierten*, Math. Z. vol. 12 (1922) pp. 36-60.
7. ———, *On the zeros of the derivatives of a function and its analytic character*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 178-191.
8. E. C. Titchmarsh, *Theory of functions*, 2d ed., Oxford, 1939.
9. G. Valiron, *Theory of integral functions*, New York, 1949.
10. E. B. Van Vleck, *On the location of roots of polynomials and entire functions*, Bull. Amer. Math. Soc. vol. 35 (1929) pp. 643-683.

DUKE UNIVERSITY,  
DURHAM, N. C.