WEIGHTED QUADRATIC NORMS AND ULTRASPHERICAL POLYNOMIALS, I(1)

BY

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1. **Introduction.** Let $\nu \geqq 0$ be fixed and let(2)

$$2^n\left(\nu+\frac{1}{2}\right)_n W_\nu(n, x) = (-1)^n(1-x^2)^{-\nu+1/2}\left(\frac{d}{dx}\right)^n[(1-x^2)^{n+\nu-1/2}]$$

be the ultraspherical polynomial of degree n and index ν normalized by the condition

$$W_\nu(n, 1) = 1 \qquad (n = 0, 1, \cdots).$$

If we define

$$d\Omega_\nu(x) = (1-x^2)^{\nu-1/2}dx,$$

$$\omega_\nu(n) = \frac{(2\nu)_n(n+\nu)\Gamma(\nu)}{n!\Gamma(\nu+1/2)},$$

then

$$\int_{-1}^{1} W_\nu(n, x)W_\nu(m, x)d\Omega_\nu(x) = \delta(n, m)/\omega_\nu(n),$$

where $\delta(n, m)$ is 1 if $n=m$ and is 0 if $n\neq m$. For all such formulas see [2, vol. 2, Chapter X]. The harmonic analysis of ultraspherical polynomials rests upon the dual convolution structure due to Lewitan [7] and Bochner [1] and described below. For a detailed discussion of the implications of the following formulas, as well as a general survey of the present subject, see [6]. Let $f(x)$ be a measurable function on $[-1 \leqq x \leqq 1]$ and let us write $f(x) \in B_\nu$ if

$$\|f\|_1 = \int_{-1}^{1} |f(x)| d\Omega_\nu(x)$$

is finite. For $f \in B_\nu$ we define the transform $f^\frown(n)$ of $f(x)$ by

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(2) $(\alpha)_r = \Gamma(\alpha+r)/r!$

$$f^\frown(n) = \int_{-1}^{1} f(x) W_\nu(n, x) d\Omega_\nu(x) \qquad (n = 0, 1, \cdots).$$

We then have the (formal) inversion formula

$$f(x) = \sum_{n=0}^{\infty} f^\frown(n) W_\nu(n, x) \omega_\nu(n).$$

Let us set

$$C_\nu(x, y, z)$$
$$= 2^{1-2\nu}\Gamma(2\nu)\Gamma(\nu)^{-2}(1 - x^2 - y^2 - z^2 + 2xyz)^{\nu-1}[(1 - x^2)(1 - y^2)(1 - z^2)]^{(1/2)-\nu}$$

if $(1-x^2-y^2-z^2+2xyz)>0$, otherwise let $C_\nu(x, y, z)=0$. By a formula of Gegenbauer [1, vol. 2, p. 177]

$$\int_{-1}^{1} C_\nu(x, y, z) W_\nu(n, z) d\Omega_\nu(z) = W_\nu(n, x) W_\nu(n, y). \tag{1}$$

Starting from this result it is possible to show that if $f_1(x), f_2(x) \in B_\nu$ and if

$$f_1 * f_2 \cdot (x) = \int_{-1}^{1}\int_{-1}^{1} f_1(y) f_2(z) C_\nu(x, y, z) d\Omega_\nu(y) d\Omega_\nu(z), \tag{2}$$

then $f_1 * f_2 \cdot (x) \in B_\nu$ and $(f_1 * f_2)^\frown(n) = f_1^\frown(n) f_2^\frown(n)$. Let $F(n)$ be defined for $[n=0, 1, \cdots]$ and let us write $F(n) \in b_\nu$ if

$$\|F\|_1 = \sum_{n=0}^{\infty} |F(n)| \omega_\nu(n)$$

is finite. For $F \in b_\nu$ we define the transform $F^\frown(x)$ by

$$F^\frown(x) = \sum_{m=0}^{\infty} F(n) W_\nu(n, x) \omega_\nu(n) \qquad [-1 \leqq x \leqq 1].$$

The inversion formula is then

$$F(n) = \int_{-1}^{1} F^\frown(x) W_\nu(n, x) d\Omega_\nu(x).$$

Let

$$c_\nu(k, j, n) = \frac{\pi 2^{1-2\nu}(\nu)_{\sigma-k}(\nu)_{\sigma-j}(\nu)_{\sigma-n}}{\Gamma(\nu)^2(\sigma - k)!(\sigma - j)!(\sigma - n)!} \, \frac{k! \quad j! \quad n!}{(2\nu)_k(2\nu)_j(2\nu)_n} \, \frac{(2\nu)_\sigma}{(\nu)_\sigma} \, \frac{1}{\sigma + \nu}$$

if $k+j+n$ is even and if $\max(k, j, n) \leqq \sigma$ where $2\sigma = k+j+n$; otherwise let $c_\nu(k, j, n)$ be 0. A formula of Dougall [8] asserts that

$$\sum_{n=0}^{\infty} c_\nu(k, j, n) W_\nu(n, x) \omega_\nu(n) = W_\nu(k, x) W_\nu(j, x), \tag{3}$$

and from this it can be shown that if $F_1(n), F_2(n) \in b_\nu$ and if

$$F_1 * F_2 \cdot (n) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} F_1(k) F_2(j) c_\nu(k, j, n) \omega_\nu(k) \omega_\nu(j), \tag{4}$$

then $F_1 * F_2 \cdot (n) \in b_\nu$ and $(F_1 * F_2)^\frown(x) = F_1^\frown(x) F_2^\frown(x)$.

We shall be concerned in the present paper with those linear transformations T of functions on $[-1, 1]$ into functions on $[-1, 1]$ which commute with the convolution operation (2); that is

$$(Tf_1) * f_2 = f_1 * (Tf_2).$$

It is easily seen that to every such transformation there corresponds a function $t(n)$ defined on $n = 0, 1, \cdots$ such that

$$(Tf)^\frown(n) = f^\frown(n) t(n).$$

An equivalent formulation is that if

$$f(x) = \sum_{n=0}^{\infty} f^\frown(n) W_\nu(n, x) \omega_\nu(n)$$

then (formally)

$$Tf(x) = \sum_{n=0}^{\infty} f^\frown(n) t(n) W_\nu(n, x) \omega_\nu(n).$$

Such transformations are called "multiplier" transformations.

For $f(x)$ a real measurable function on $[-1, 1]$ we set

$$\mathfrak{N}^\nu_{\beta,\alpha}[f] = \left[\int_{-1}^{1} f(x)^2 (1 + x)^\beta (1 - x)^\alpha d\Omega_\nu(x) \right]^{1/2}, \quad \left(-\frac{1}{2} < \alpha, \beta < \frac{1}{2} \right).$$

We shall also use $\mathfrak{N}^\nu_{\beta,\alpha}$ to denote the space of functions $f(x)$ for which $\mathfrak{N}^\nu_{\beta,\alpha}[f]$ is finite.

Our objective in the present paper is to find rather general sufficient conditions which will insure that $T = \{t(n)\}_0^\infty$ be a bounded linear transformation of $\mathfrak{N}^\nu_{\alpha,\beta}$ into itself. The dual theory, in which the roles of F and f are interchanged, will be dealt with in the subsequent paper. We shall there be concerned with those linear transformations t of functions on $[n = 0, 1, \cdots]$ into functions on $[n = 0, 1, \cdots]$ which commute with the convolution operation (4); that is

$$tF_1 * F_2 = F_1 * tF_2.$$

To every such transformation there corresponds a measurable function $t(x)$ defined on $-1 \leqq x \leqq 1$ such that

$$[tF]^\frown(x) = F^\frown(x) t(x).$$

A (formally) equivalent definition is that if

$$F(n) = \int_{-1}^{1} F^{\frown}(x) W_\nu(n, x) d\Omega_\nu(x)$$

then

$$(tF)(n) = \int_{-1}^{1} F^{\frown}(x) t(x) W_\nu(n, x) d\Omega_\nu(x).$$

We set

$$\mathfrak{N}_\alpha^\nu[F] = \left\{ \sum_{n=0}^{\infty} F(n)^2 \omega_\nu(n)(n+1)^{2\alpha} \right\}^{1/2} \left(-\frac{1}{2} < \alpha < \frac{1}{2} \right).$$

We also use $\mathfrak{N}_\alpha^\nu$ to denote the space of functions $F(n)$ for which $\mathfrak{N}_\alpha^\nu[F]$ is finite. In the succeeding paper we shall find rather general sufficient conditions on $t = t(x)$ which will insure that t is a bounded transformation of $\mathfrak{N}_\alpha^\nu$ into itself. These papers thus complete investigations initiated in [4].

2. **Weighted quadratic norms.** Let $\nu > 0$ be fixed and let $r(k)$ be a non-negative function defined for $k = 1, 2, \cdots$ such that $\sum_1^\infty r(k) < \infty$. We define

$$s(x) = \sum_{k=1}^{\infty} [1 - W_\nu(k, x)] r(k),$$

$$S(m, n) = \sum_{k=1}^{\infty} c_\nu(m, n, k) r(k).$$

We write $f(x) \in L^1$ if $\int_{-1}^{1} |f(x)| d\Omega_\nu(x) < \infty$.

THEOREM 2a. *If $s(x)$ and $S(m, n)$ are defined as above and if $f(x) \in L^1$ then*

$$\int_{-1}^{1} f(x)^2 s(x) d\Omega_\nu(x) = \frac{1}{2} \sum_{m,n=0}^{\infty} [f^{\frown}(n) - f^{\frown}(m)]^2 S(n, m) \omega_\nu(n) \omega_\nu(m) \tag{1}$$

provided that the left hand side is finite.

The assumption $f \in L^1$ is, of course, necessary to insure that $f^{\frown}(n)$ is defined. Let us suppose first that

$$\int_{-1}^{1} f(x)^2 d\Omega_\nu(x) < \infty, \tag{2}$$

a restriction that will be removed at the end of the proof. We expand $[f^{\frown}(n) - f^{\frown}(m)]^2$ so that the right hand side of (1) splits into three terms,

$$I_1 = \frac{1}{2} \sum_{m,n=0}^{\infty} f^{\frown}(n)^2 S(m, n) \omega_\nu(n) \omega_\nu(m),$$

$$I_2 = \frac{1}{2} \sum_{m,n=0}^{\infty} f^{\frown}(m)^2 S(m, n) \omega_\nu(n) \omega_\nu(m),$$

$$I_3 = -\sum_{m,n=0}^{\infty} f^{\frown}(n)f^{\frown}(m)S(m, n)\omega_\nu(n)\omega_\nu(m).$$

Because the summand in I_1 is non-negative we have

$$I_1 = \frac{1}{2}\sum_{n=0}^{\infty} f^{\frown}(n)^2\omega_\nu(n) \sum_{m=0}^{\infty} S(n, m)\omega_\nu(m).$$

Now

$$\begin{aligned}\sum_{m=0}^{\infty} S(n, m)\omega_\nu(m) &= \sum_{m=0}^{\infty} \omega_\nu(m) \sum_{k=1}^{\infty} c_\nu(n, m, k)r(k)\\ &= \sum_{k=1}^{\infty} r(k) \sum_{m=0}^{\infty} c_\nu(n, m, k)\omega_\nu(m).\end{aligned}$$

By (3) of §1 with $x=1$

$$\sum_{m=0}^{\infty} c_\nu(n, m, k)\omega_\nu(m) = 1$$

and thus

$$I_1 = \frac{1}{2}\left[\sum_{n=1}^{\infty} f^{\frown}(n)^2\omega_\nu(n)\right]\left[\sum_{k=1}^{\infty} r(k)\right].$$

Using Parseval's equality we obtain

$$I_1 = \frac{1}{2}\left[\int_{-1}^{1} f(x)^2 d\Omega_\nu(x)\right]\left[\sum_{k=1}^{\infty} r(k)\right].$$

Similarly

$$I_2 = \frac{1}{2}\left[\int_{-1}^{1} f(x)^2 d\Omega_\nu(x)\right]\left[\sum_{k=1}^{\infty} r(k)\right].$$

As we have seen the infinite sums defining I_1 and I_2 converge absolutely. Now we have

$$I_3 = -\sum_{n,m=0}^{\infty} f^{\frown}(n)f^{\frown}(m)\omega_\nu(n)\omega_\nu(m) \sum_{k=1}^{\infty} c_\nu(n, m, k)r(k), \tag{3}$$

and the inequality

$$|f^{\frown}(n)f^{\frown}(m)| \leqq \frac{1}{2}f^{\frown}(n)^2 + \frac{1}{2}f^{\frown}(m)^2$$

shows that the sum in (3) also converges absolutely; thus

$$I_3 = -\sum_{n=0}^{\infty}\{f^\frown(n)\}\left\{\sum_{k=1}^{\infty} r(k)\sum_{m=0}^{\infty} f^\frown(m)c_\nu(n, m, k)\omega_\nu(m)\right\}\omega_\nu(n). \tag{4}$$

We have

$$\sum_{m=0}^{\infty} f^\frown(m)c_\nu(n, m, k)\omega_\nu(m) = \sum_{m=0}^{\infty}\int_{-1}^{1} f(x)c_\nu(n, m, k)\omega_\nu(m)W_\nu(m, x)d\Omega_\nu(x).$$

The formal infinite sum is actually finite here and thus the order of summation and integration can be interchanged. Using (3) of §1 we obtain

$$\sum_{m=0}^{\infty} f^\frown(m)c_\nu(n, m, k)\omega_\nu(m) = \int_{-1}^{1} f(x)W_\nu(n, x)W_\nu(k, x)d\Omega_\nu(x).$$

Since $|W_\nu(k, x)| \leqq 1$, $k=0, 1, \cdots, -1\leqq x\leqq 1$ it is easy to see that

$$\begin{aligned}\sum_{k=1}^{\infty} r(k)\sum_{m=0}^{\infty} f^\frown(m)c_\nu(n, m, k)\omega_\nu(m) &= \int_{-1}^{1} f(x)\left\{\sum_{k=1}^{\infty} W_\nu(k, x)r(k)\right\}W_\nu(n, x)d\Omega_\nu(x),\\ &= \int_{-1}^{1} f(x)\left\{\sum_{1}^{\infty} r(k) - s(x)\right\} W_\nu(n, x)d\Omega_\nu(x).\end{aligned}$$

Making use of this and of the definition of $f^\frown(n)$ and employing the general form of Parseval's equality in (4) we find that

$$I_3 = -\int_{-1}^{1} f(x)^2\left\{\sum_{1}^{\infty} r(k) - s(x)\right\}d\Omega_\nu(x).$$

Combining our evaluations of I_1, I_2, and I_3 our theorem is proved, under the additional assumption (2).

Suppose that (2) is not satisfied. We set

$$f_j(x) = \begin{cases} j & f(x) > j,\\ f(x) & -j \leqq f(x) \leqq j,\\ -j & f(x) < -j.\end{cases}$$

Then

$$\int_{-1}^{1} f_j(x)^2 d\Omega_\nu(x) < \infty \qquad j = 1, 2, \cdots,$$

and thus by what we have already proved

$$\int_{-1}^{1} f_j(x)^2 s(x)d\Omega_\nu(x) = \frac{1}{2}\sum_{m,n=0}^{\infty}[f_j^\frown(n) - f_j^\frown(m)]^2 S(m, n)\omega_\nu(m)\omega_\nu(n).$$

Now

$$\lim_{j\to\infty}\int_{-1}^{1} f_j(x)^2 s(x)\,d\Omega_\nu(x) = \int_{-1}^{1} f(x)^2 s(x)\,d\Omega_\nu(x)$$

while, since $f_j^\frown(n)\to f^\frown(n)$ as $j\to\infty$ $(n=0, 1, \cdots)$, we have

$$\liminf_{j\to\infty} \sum_{m,n=0}^{\infty} [f_j^\frown(n) - f_j^\frown(m)]^2 S(m, n)\omega_\nu(m)\omega_\nu(n)$$

$$\geqq \sum_{m,n=0}^{\infty} [f^\frown(n) - f^\frown(m)]^2 S(m, n)\omega_\nu(m)\omega_\nu(n).$$

Thus

$$(5)\qquad \int_{-1}^{1} f(x)^2 s(x)\,d\Omega_\nu(x) \geqq \frac{1}{2}\sum_{m,n=0}^{\infty} [f^\frown(n) - f^\frown(m)]^2 S(m, n)\omega_\nu(m)\omega_\nu(n).$$

If

$$\int_{-1}^{1} f(x)^2 s(x)\,d\Omega_\nu(x) < \infty$$

then given $\epsilon>0$ we can choose functions $g(x)$ and $h(x)$ such that

$$f(x) = g(x) + h(x),$$

$$\int_{-1}^{1} g(x)^2 d\Omega_\nu(x) < \infty,$$

$$\int_{-1}^{1} g(x)^2 s(x)\,d\Omega_\nu(x) \geqq \int_{-1}^{1} f(x)^2 s(x)\,d\Omega_\nu(x) - \epsilon,$$

$$\int_{-1}^{1} h(x)^2 s(x)\,d\Omega_\nu(x) < \epsilon.$$

From the inequality

$$a^2 \leqq (1 + e^{-\rho})(a + b)^2 + (1 + e^\rho)b^2,$$

valid for any value of ρ, and the relation $f^\frown(n) = g^\frown(n) + h^\frown(n)$, we obtain

$$(1 + e^{-\rho}) \sum_{m,n=0}^{\infty} [f^\frown(n) - f^\frown(m)]^2 S(m, n)\omega_\nu(m)\omega_\nu(n)$$

$$\geqq \sum_{m,n=0}^{\infty} [g^\frown(n) - g^\frown(m)]^2 S(m, n)\omega_\nu(m)\omega_\nu(n)$$

$$- (1 + e^\rho) \sum_{m,n=0}^{\infty} [h^\frown(n) - h^\frown(m)]^2 S(m, n)\omega_\nu(m)\omega_\nu(n).$$

Using (1) for $g(x)$ and (5) for $h(x)$ we have

$$\frac{1}{2}(1+e^{-\rho})\sum_{m,n=0}^{\infty}[f^\frown(m)-f^\frown(n)]^2 S(m,n)\omega_\nu(m)\omega_\nu(n)$$

$$\geqq \int_{-1}^{1} f(x)^2 s(x)d\Omega_\nu(x)-(2+e^\rho)\epsilon.$$

Taking first ϵ arbitrarily small and then ρ arbitrarily large we see that

$$\frac{1}{2}\sum_{m,n=0;m\neq n}^{\infty}[f^\frown(m)-f^\frown(n)]^2 S(m,n)\omega_\nu(m)\omega_\nu(n)\geqq\int_{-1}^{1}f(x)^2 s(x)d\Omega_\nu(x). \tag{6}$$

The inequalities (5) and (6) together imply our desired result.

THEOREM 2b. *If* $f(x)\in L^1$, $g(x)\in L^1$ *and if*

$$\int_{-1}^{1}f(x)^2 s(x)d\Omega_\nu(x)<\infty,\qquad \int_{-1}^{1}g(x)^2 s(x)d\Omega_\nu(x)<\infty,$$

then

$$\int_{-1}^{1}f(x)g(x)s(x)d\Omega_\nu(x)$$

$$=\frac{1}{2}\sum_{m,n=0;m\neq n}^{\infty}[f^\frown(n)-f^\frown(m)][g^\frown(n)-g^\frown(m)]S(m,n)\omega_\nu(m)\omega_\nu(n).$$

This follows from Theorem 2a using the standard device of writing out (1) for $f(x)+g(x)$ and for $f(x)-g(x)$ and then subtracting the results.

3. **Approximations.** Let $\nu>0$ be fixed[3]. We set

$$s_\alpha(x)=\sum_{1}^{\infty}[1-W_\nu(n,x)]n^{-2\alpha-1}, \tag{1}$$

$$S_\alpha(m,n)=\sum_{1}^{\infty}c_\nu(k,m,n)k^{-2\alpha-1}. \tag{2}$$

Let us write $A(y)\approx B(y)$ for $y\in Y$ if there exist finite positive constants C_1 and C_2 such that $A(y)\leqq C_1B(y)$ and $B(y)\leqq C_2A(y)$ for $y\in Y$.

LEMMA 3a. *If* $0<\alpha<1/2$ *and if* $s_\alpha(x)$ *is defined by* (1) *then*

$$s_\alpha(x)\approx(1-x)^\alpha\qquad(-1\leqq x\leqq 1).$$

We have, see [2, vol. 2, p. 175],

$$W_\nu(n,\cos\theta)=\frac{n!}{(2\nu)_n}\sum_{m=0}^{n}\frac{(\nu)_m(\nu)_{n-m}}{m!(n-m)!}\cos(n-2m)\theta.$$

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(3) The case $\nu=0$ requires certain small changes and is left to the reader.

Setting $\theta=0$ we see that

$$1 = \frac{n!}{(2\nu)_n} \sum_{m=0}^{n} \frac{(\nu)_m(\nu)_{n-m}}{m!(n-m)!}$$

and thus

$$1 - W_\nu(n, \cos\theta) = \frac{n!}{(2\nu)_n} \sum_{m=0}^{n} \frac{(\nu)_m(\nu)_{n-m}}{m!(n-m)!} [1 - \cos(n-2m)\theta].$$

Summing separately over n even and n odd we have

$$\sum_{n=1}^{\infty} [1 - W_\nu(2n, \cos\theta)](2n)^{-1-2\alpha}$$
$$= 2\sum_{j=1}^{\infty} [1 - \cos 2j\theta] \sum_{n\geq j} \frac{(\nu)_{n-j}(\nu)_{n+j}}{(n-j)!(n+j)!} \frac{(2n)!}{(2\nu)_{2n}} (2n)^{-1-2\alpha},$$

$$\sum_{n=0}^{\infty} [1 - W_\nu(2n+1, \cos\theta)](2n+1)^{-1-2\alpha}$$
$$= 2\sum_{j=0}^{\infty} [1 - \cos(2j+1)\theta] \sum_{n\geq j} \frac{(\nu)_{n-j}(\nu)_{n+j+1}(2n+1)!}{(n-j)!(n+j+1)!(2\nu)_{2n+1}} (2n+1)^{-1-2\alpha}.$$

Since $(\nu)_r/r! \approx r^{\nu-1}$ we have that

$$\sum_{n\geq j} \frac{(\nu)_{n-j}(\nu)_{n+j}(2n)!}{(n-j)!(n+j)!(2\nu)_{2n}} (2n)^{-1-2\alpha} \approx \sum_{n\geq j} n^{-2\nu-2\alpha}(n-j)^{\nu-1}(n+j)^{\nu-1}$$
$$\approx \Sigma_1 + \Sigma_2$$

where Σ_1 corresponds to the range $j \leqq n \leqq 2j$ and Σ_2 to the range $2j<n$. Now

$$\Sigma_1 \approx j^{-\nu-2\alpha-1} \sum_{n=j}^{2j} (n-j)^{\nu-1} \approx j^{-2\alpha-1},$$
$$\Sigma_2 \approx \sum_{n>2j} n^{-2\alpha-2} \approx j^{-2\alpha-1},$$

and thus

$$\sum_{n=1}^{\infty} [1 - W_\nu(2n, \cos\theta)](2n)^{-1-2\alpha} \approx \sum_{j=1}^{\infty} [1 - \cos 2j\theta](2j)^{-2\alpha-1}.$$

Similarly

$$\sum_{n=0}^{\infty} [1 - W_\nu(2n+1, \cos\theta)](2n+1)^{-1-2\alpha}$$
$$\approx \sum_{j=0}^{\infty} [1 - \cos(2j+1)\theta](2j+1)^{-2\alpha-1}.$$

Combining these results we see that

$$s_\alpha(\cos\theta) \approx \sum_{j=1}^{\infty} [1 - \cos j\theta] j^{-2\alpha-1}.$$

Now

$$\frac{d}{d\theta}\sum_{j=1}^{\infty} [1 - \cos j\theta] j^{-2\alpha-1} = \sum_{j=1}^{\infty} (\sin j\theta) j^{-2\alpha},$$

$$\frac{d}{d\theta}\sum_{j=1}^{\infty} [1 - \cos j\theta] j^{-2\alpha-1} \sim \theta^{-1+2\alpha} \qquad (\theta \to 0+).$$

For this last step see Zygmund [11, p. 114]. Integrating, we find that

$$\sum_{j=1}^{\infty} [1 - \cos j\theta] j^{-2\alpha-1} \approx (1 - \cos\theta)^\alpha \qquad (0 \leqq \theta \leqq \pi);$$

that is

$$s_\alpha(x) \approx (1 - x)^\alpha \qquad (-1 \leqq x \leqq 1).$$

LEMMA 3b. *If* $0<\alpha<1/2$ *and if* $S_\alpha(m, n)$ *is defined by* (2) *then*

$$S_\alpha(m, n) \approx (n + 1)^{-2\nu} \cdot (n - m)^{-1-2\alpha} \qquad (n > m).$$

If $n>m$ then

$$S_\alpha(m, n) = \sum_{j=0}^{m} c_\nu(n - m + 2j, m, n)(n - m + 2j)^{-1-2\alpha}.$$

It is easily verified from this, using the relation $(\alpha)_r/r! \approx (r+1)^{\alpha-1}$ that

$$S_\alpha(m, n) \approx (m + 1)^{1-2\nu}(n + 1)^{-\nu}$$
$$\cdot \sum_{j=0}^{m} (m - j + 1)^{\nu-1}(n - m + j)^{\nu-1}(j + 1)^{\nu-1}(n - m + 2j)^{-2\nu-2\alpha}.$$

We must distinguish between two cases, $n \geqq 3m/2$ and $n<3m/2$. Suppose $n \geqq 3m/2$. Then $(n-m+j+1) \approx n+1$, $(n-m+2j) \approx n+1$, and

$$S_\alpha(m, n) \approx (m + 1)^{1-2\nu}(n + 1)^{-1-2\nu-2\alpha} \sum_{j=0}^{m} (m - j + 1)^{\nu-1}(j + 1)^{\nu-1}.$$

Now

$$\sum_{j=0}^{m} (m - j + 1)^{\nu-1}(j + 1)^{\nu-1} \approx (m + 1)^{2\nu-1},$$

and thus, since $n+1 \approx n-m$ if $n \geqq 3m/2$,

$$S_\alpha(m, n) \approx (n + 1)^{-2\nu}(n - m)^{-1-2\alpha} \qquad (n \geqq 3m/2).$$

If $n<3m/2$ then we have

$$S_\alpha(m,n) \approx \Sigma_1 + \Sigma_2 + \Sigma_3$$

where Σ_1 corresponds to the range $0\leqq j<n-m$, Σ_2 to the range $n-m\leqq j<m/2$, and Σ_3 to the range $m/2\leqq j\leqq m$. If $0\leqq j<n-m$ (and if $m<n<3m/2$) then $(m-j+1)\approx(n+1)$, $(n-m+j)\approx(n-m)$, $(n-m+2j)\approx(n-m)$, $m+1\approx n+1$, and thus

$$\Sigma_1 \approx (n+1)^{-2\nu}(n-m)^{-1-\nu+2\alpha}\sum_{0\leqq j<n-m}(j+1)^{\nu-1},$$
$$\approx (n+1)^{-2\nu}(n-m)^{-1-2\alpha}.$$

If $n-m\leqq j<m/2$ then $(m-j+1)\approx(n+1)$, $(n-m+j)\approx(j+1)$, $(n-m+2j)\approx(j+1)$, $(m+1)\approx(n+1)$, and thus

$$\Sigma_2 \approx (n+1)^{-2\nu}\sum_{n-m\leqq j<m/2}(j+1)^{-2\alpha-2},$$
$$\approx (n+1)^{-2\nu}(n-m)^{-1-2\alpha}.$$

If $m/2\leqq j\leqq m$ then $(n-m+j)\approx(n+1)$, $(j+1)\approx(n+1)$, $(n-m+2j)\approx(n+1)$, $(m+1)\approx(n+1)$ and

$$\Sigma_3 \approx (n+1)^{-1-3\nu-2\alpha}\sum_{m/2<j\leqq m}(m-j+1)^{\nu-1},$$
$$<\approx (n+1)^{-2\nu}(n-m)^{-1-2\alpha}.$$

Combining these estimates we see that

$$S_\alpha(m,n) \approx (n+1)^{-2\nu}(n-m)^{-1-2\alpha} \qquad (n<3m/2).$$

Our demonstration is now complete.

Lemmas 3a and 3b and Theorem 2a together imply the following result.

THEOREM 3c. *If $0<\alpha<1/2$, and if $f\in\mathfrak{N}^{\nu}_{0,\alpha}$ then*

$$\mathfrak{N}^{\nu}_{0,\alpha}[f]^2 \approx \sum_{m,n=0;m\neq n}^{\infty}[f^\frown(n)-f^\frown(m)]^2 S_\alpha(m,n)\omega_\nu(m)\omega_\nu(n).$$

4. **Some inequalities.** We begin by noting that the functions

$$W_\nu(n,\cos\theta)(\sin\theta)^{\nu}\omega_\nu^{1/2}(n) \qquad (n=0,1,\cdots),$$

are orthonormal and uniformly bounded on $0\leqq\theta\leqq\pi$, see [2, vol. 2, p. 174 and p. 206]. If $\phi_n(\theta)$, $n=0,1,\cdots$, are a uniformly bounded orthonormal set of functions on $[0,b]$ and if

$$a(n) = \int_0^b \psi(\theta)\phi_n(\theta)d\theta,$$

then if $0 \leqq \alpha < 1/2$

$$\sum_0^\infty a(n)^2(R(n)+1)^{-2\alpha} \leqq A(\alpha)\int_0^b \psi(\theta)^2\theta^{2\alpha}d\theta,$$

see [5]. Here $R(0)$, $R(1)$, $R(2)$, $\cdots$ is any rearrangement of 0, 1, 2, $\cdots$. We have

$$f^\frown(n) = \int_{-1}^1 f(s)W_\nu(n, x)d\Omega_\nu(x).$$

Setting $x = \cos\theta$ we find that

$$\omega_\nu(n)^{1/2}f^\frown(n) = \int_0^\pi \{f(\cos\theta)(\sin\theta)^\nu\}\{\omega_\nu(n)^{1/2}W_\nu(n, \cos\theta)(\sin\theta)^\nu\}d\theta$$

and thus

$$\sum_{n=0}^\infty f^\frown(n)^2\omega_\nu(n)[R(n)+1]^{-2\alpha} \leqq A(\alpha)\int_0^\pi \{f(\cos\theta)(\sin\theta)^\nu\}^2\theta^{2\alpha}d\theta,$$

$$\leqq A'(\alpha)\int_{-1}^1 f(x)^2(1-x)^\alpha d\Omega_\nu(x).$$

We have proved the following result.

THEOREM 4a. *If $0 \leqq \alpha < 1/2$ and if $R(0)$, $R(1)$, $R(2)$, $\cdots$ is any rearrangement of 0, 1, 2, $\cdots$ then*

$$\sum_{n=0}^\infty f^\frown(n)^2\omega_\nu(n)[R(n)+1]^{-2\alpha} \leqq A(\alpha)\mathfrak{N}^\nu_{0,\alpha}[f].$$

Let S_N be the multiplier transformation which carries

$$f(x) \sim \sum_{n=0}^\infty f^\frown(n)\omega_\nu(n)W_\nu(n, x)$$

into

$$S_Nf(x) \sim \sum_{n=0}^N f^\frown(n)\omega_\nu(n)W_\nu(n, x).$$

As a first application of our ideas we prove

THEOREM 4b. *If $0 \leqq \alpha < 1/2$ then*

$$\mathfrak{N}^\nu_{0,\alpha}[S_Nf] \leqq A(\alpha)\mathfrak{N}^\nu_{0,\alpha}[f].$$

We may suppose $\alpha > 0$ since the case $\alpha = 0$ follows from Parseval's equality. By Theorem 3c we have

$$\mathfrak{N}^{\nu}_{0,\alpha}[S_N f]^2 \leqq A \sum_{m,n\leqq N} [f^\frown(n) - f^\frown(m)]^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n)$$
$$+ A \sum_{m\leqq N; n>N} f^\frown(m)^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n).$$

A second application of this same result shows that

$$\sum_{m,n\leqq N} [f^\frown(n) - f^\frown(m)]^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n) \leqq A\mathfrak{N}^{\nu}_{0,\alpha}[f]^2. \tag{1}$$

Further, Lemma 3b implies that if $m \leqq N$ then

$$\sum_{n>N} S_\alpha(m, n)\omega_\nu(n) \approx \sum_{n>N} (n - m)^{-1-2\alpha} \leqq (N + 1 - m)^{-2\alpha}.$$

Thus

$$\begin{aligned} \sum_{m\leqq N; n>N} f^\frown(m)^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n) &\approx \sum_{m=0}^{N} f^\frown(m)^2\omega_\nu(m)(N + 1 - m)^{-2\alpha}, \\ &< A\mathfrak{N}^{\nu}_{0,\alpha}[f]^2, \end{aligned} \tag{2}$$

by Theorem 4a. The inequalities (1) and (2) together imply our desired result.

5. **Bounded multiplier transformations.** Let $b_\mu = 3\cdot 2^{\mu-2}$, $r_\mu = 2^{\mu-1}$, let σ_μ be the set of integers $b_\mu - r_\mu \leqq k < b_\mu + r_\mu$, and let

$$\rho_\mu(x) = [1 - r_\mu^{-2}(x - b_\mu)^2].$$

If

$$f(x) \sim \sum_{n=0}^{\infty} f^\frown(n)\omega_\nu(n) W_\nu(n, x)$$

then we set

$$E_\mu(x) = \sum_{n\in\sigma_\mu} f^\frown(n)\rho_\mu(n)\omega_\nu(n) W_\nu(n, x).$$

LEMMA 5a. *If* $0 \leqq \alpha < 1/2$ *then*

$$\sum_{\mu=2}^{\infty} \mathfrak{N}^{\nu}_{0,\alpha}[E_\mu]^2 \leqq A(\alpha)\mathfrak{N}^{\nu}_{0,\alpha}[f]^2.$$

Evidently we may suppose $\alpha > 0$. By Theorem 3c we have

$$\mathfrak{N}^{\nu}_{0,\alpha}[E_\mu]^2 \approx \Sigma_1 + \Sigma_2 + \Sigma_3$$

where

$$\Sigma_1 = \sum_{m,n\in\sigma_\mu;n>m} [\rho_\mu(n)f^\frown(n) - \rho_\mu(m)f^\frown(m)]^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n),$$
$$\Sigma_2 = \sum_{m\in\sigma_\mu;n>\sigma_\mu} \rho_\mu(m)^2 f^\frown(m)^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n),$$
$$\Sigma_3 = \sum_{n\in\sigma_\mu;m<\sigma_\mu} \rho_\mu(n)^2 f^\frown(n)^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n).$$

Let us begin with Σ_2. We have

$$\rho_\mu(m)^2 \leqq 4(b_\mu + r_\mu - m)^2 r_\mu^{-2} \qquad (m \in \sigma_\mu),$$

and

$$\sum_{n>\sigma_\mu} S_\alpha(m, n)\omega_\nu(n) \leqq A \sum_{n>\sigma_\mu} (n - m)^{-1-2\alpha}$$
$$\leqq A(b_\mu + r_\mu - m)^{-2\alpha}.$$

Making use of the inequalities

$$(b_\mu + r_\mu - m)^{2-2\alpha} \leqq A(m + 1)^{2-2\alpha} \qquad (m \in \sigma_\mu),$$
$$(m + 1) \leqq Ar_\mu \qquad (m \in \sigma_\mu),$$

we obtain

$$\Sigma_2 \leqq A \sum_{m\in\sigma_\mu} f^\frown(m)^2(m + 1)^{-2\alpha}\omega_\nu(m).$$

Exactly the same argument shows that

$$\Sigma_3 \leqq A \sum_{n\in\sigma_\mu} f^\frown(n)^2(n + 1)^{-2\alpha}\omega_\nu(n).$$

It remains to treat Σ_1. Since

$$\rho_\mu(n)f^\frown(n) - \rho_\mu(m)f^\frown(m) = [f^\frown(n) - f^\frown(m)]\rho_\mu(n) + f^\frown(m)[\rho_\mu(n) - \rho_\mu(m)]$$

and since $0 \leqq \rho_\mu(n) \leqq 1$, we have

$$\Sigma_1 < 2 \sum_{n,m\in\sigma_\mu;n>m} [f^\frown(n) - f^\frown(m)]^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n)$$
$$+ 2 \sum_{n,m\in\sigma_\mu;n>m} f^\frown(m)^2[\rho_\mu(n) - \rho_\mu(m)]^2 S_\alpha(m, n)\omega_\nu(m)\omega_\nu(n).$$

We assert that

$$\sum_{n=m+1}^{b_\mu+r_\mu} [\rho_\mu(n) - \rho_\mu(m)]^2 S_\alpha(m, n)\omega_\nu(n) \leqq A(m + 1)^{-2\alpha} \qquad (m \in \sigma_\mu).$$

To verify this note that

$$\rho_\mu(n) - \rho_\mu(m) = -(n - m)(n + m - 2b_\mu)r_\mu^{-2},$$
$$|\rho_\mu(n) - \rho_\mu(m)| \leqq A(n - m)r_\mu^{-1} \qquad (n, m \in \sigma_\mu).$$

It follows that

$$\sum_{n=m+1}^{b_\mu+r_\mu} [\rho_\mu(n) - \rho_\mu(m)]^2 S_\alpha(m, n)\omega_\nu(n) \leqq A r_\mu^{-2} \sum_{n=m+1}^{b_\mu+r_\mu} (n-m)^{1-2\alpha},$$
$$\leqq A r_\mu^{-2}(b_\mu + r_\mu - m)^{2-2\alpha},$$
$$\leqq A(m+1)^{-2\alpha},$$

as desired. Thus

$$\sum_{n,m\in\sigma_\mu;n>m} f^\frown(m)^2[\rho_\mu(n) - \rho_\mu(m)]^2 S_\alpha(m, n)\omega_\nu(n)\omega_\nu(m)$$
$$\leqq A \sum_{m\in\sigma_\mu} f^\frown(m)^2(m+1)^{-2\alpha}\omega_\nu(m).$$

Combining our results we have shown that

$$\mathfrak{N}_{0,\alpha}^{\nu}[E_\mu]^2 \leqq \sum_{n,m\in\sigma_\mu;n>m} [f^\frown(n) - f^\frown(m)]^2 S_\alpha(m, n)\omega_\nu(n)\omega_\nu(m)$$
$$+ A \sum_{m\in\sigma_\mu} f^\frown(m)^2(m+1)^{-2\alpha}\omega_\nu(m).$$

Since no integer belongs to more than three sets σ_μ we see that

$$\sum_{\mu=2}^{\infty} \mathfrak{N}_{0,\alpha}^{\nu}[E_\mu]^2 \leqq A \sum_{n>m} [f^\frown(n) - f^\frown(m)]^2 S_\alpha(m, n)\omega_\nu(n)\omega_\nu(m)$$
$$+ A \sum_{m} f^\frown(m)^2(m+1)^{-2\alpha}\omega_\nu(m).$$

Applying Theorems 3c and 4a we have proved our desired result.

Let S_μ be the set of integers $2^{\mu-1} \leqq k < 2^\mu$, $\mu = 2, 3, \cdots$.

LEMMA 5b. *If* $0 \leqq \alpha < 1/2$ *and if* $n_\mu \in S_\mu$ *then*

$$\sum_{\mu=2}^{\infty} \sum_{m\in S_\mu} f^\frown(m)^2[|m - n_\mu| + 1]^{-2\alpha} \omega_\nu(m) \leqq A(\alpha)\mathfrak{N}_{0,\alpha}^{\nu}[f]^2.$$

By Theorem 4a

$$\sum_{m\in\sigma_\mu} \rho_\mu(m)^2 f^\frown(m)^2[|m - n_\mu| + 1]^{-2\alpha}\omega_\nu(m) \leqq A\mathfrak{N}_{0,\alpha}^{\nu}[E_\mu]^2.$$

For $m \in S_\mu$ $\rho_\mu(m) \geqq A$ and thus

$$\sum_{m\in S_\mu} f^\frown(m)^2[|m - n_\mu| + 1]^{-2\alpha}\omega_\nu(m)$$
$$\leqq A \sum_{m\in\sigma_\mu} \rho_\mu(m)^2 f^\frown(m)^2[|m - n_\mu| + 1]^{-2\alpha}\omega_\nu(m).$$

These two inequalities together imply our desired result.

DEFINITION. $T=\{t(n)\}$ $(0\leqq n\leqq\infty)$ is said to belong to class $M(C)$ if

$$|t(n)| \leqq C \qquad (n = 0, 1, \cdots);$$

$$\sum_{2n}^{2n+1} |t(k) - t(k-1)| \leqq C \qquad (n = 0, 1, \cdots).$$

THEOREM 5c. *If* $0\leqq\alpha<1/2$ *and if*

1. $$f(x) \sim \sum_0^\infty f^\frown(n)\omega_\nu(n)W_\nu(n, x) \qquad f \in \mathfrak{N}^\nu_{0,\alpha},$$

2. $$T = \{t(n)\}_0^\infty \text{ belongs to } M(C),$$

3. $$Tf(x) \sim \sum_0^\infty f^\frown(n)t(n)\omega_\nu(n)W_\nu(n, x),$$

then

$$\mathfrak{N}^\nu_{0,\alpha}[Tf] \leqq A(\alpha)C\mathfrak{N}^\nu_{0,\alpha}[f].$$

We set

$$\delta_\mu(x) = \sum_{n\in S_\mu} f^\frown(n)t(n)\omega_\nu(n)W_\nu(n, x).$$

Let us begin by supposing that $t(0)=t(1)=0$; this restriction is unimportant and is made only for the sake of convenience. Let

$$F_M(x) = \sum_{\mu=2}^M \delta_\mu(x).$$

It will be sufficient to show that

$$\mathfrak{N}^\nu_{0,\alpha}[F_M] \leqq AC\mathfrak{N}^\nu_{0,\alpha}[f]$$

where A is independent of M. We have

$$\mathfrak{N}^\nu_{0,\alpha}[F_M]^2 \approx \int_{-1}^1 F_M(x)^2 s_\alpha(x)d\Omega_\nu(x),$$

and since

$$\int_{-1}^1 F_M(x)^2 s_\alpha(x)d\Omega_\nu(x) = \sum_{\mu=2}^M \int_{-1}^1 \delta_\mu(x)^2 s_\alpha(x)d\Omega_\nu(x) + \sum_{\lambda,\mu=2;\lambda\neq\mu}^M \int_{-1}^1 \delta_\mu(x)\delta_\nu(x)s_\alpha(x)d\Omega_\nu(x),$$

it is sufficient to show that

$$(1) \qquad \sum_{\mu,\lambda=2;\mu\neq\lambda}^{\infty} \left| \int_{-1}^{1} \delta_\mu(x)\delta_\lambda(x)s_\alpha(x)d\Omega_\nu(x) \right| \leqq AC^2 \mathfrak{N}_{0,\alpha}^{\nu}[f]^2,$$

and

$$(2) \qquad \sum_{\mu=2}^{\infty} \int_{-1}^{1} \delta_\mu(x)^2 s_\alpha(x)d\Omega_\nu(x) \leqq AC^2 \mathfrak{N}_{0,\alpha}^{\nu}[f]^2.$$

By Theorem 2b and the inequality $|ab| \leqq (a^2+b^2)/2$ we have

$$\begin{aligned} I_{\mu,\lambda} &= \int_{-1}^{1} \delta_\mu(x)\delta_\lambda(x)s_\alpha(x)d\Omega_\nu(x) \\ &= \sum_{n\in S_\mu;m\in S_\lambda} [t(n)f^\frown(n) - t(m)f^\frown(m)]^2 S_\alpha(m,n)\omega_\nu(n)\omega_\nu(m), \\ &\leqq 2C^2 \sum_{n\in S_\mu} f^\frown(n)^2\omega_\nu(n) \sum_{m\in S_\lambda} S_\alpha(m,n)\omega_\nu(m) + 2C^2 \sum_{m\in S_\lambda} f^\frown(m)^2\omega_\nu(m) \sum_{n\in S_\mu} S_\alpha(m,n)\omega_\nu(n). \end{aligned}$$

Thus

$$\sum_{\mu,\lambda=2;\mu\neq\lambda}^{\infty} |I_{\mu,\lambda}| \leqq AC^2 \sum_{\mu=2}^{\infty} \sum_{n\in S_\mu} f^\frown(n)^2\omega_\nu(n) \sum_{\lambda=2;\lambda\neq\mu}^{\infty} \sum_{m\in S_\lambda} S_\alpha(m,n)\omega_\nu(n).$$

Now, as is easily verified,

$$\sum_{\lambda=2;\lambda\neq\mu}^{\infty} \sum_{m\in S_\lambda} S_\alpha(m,n)\omega_\nu(m) \leqq A[|n-2^{\mu-1}|+1]^{-2\alpha} + A[|n-2^\mu|+1]^{-2\alpha}$$

so that

$$\sum_{\mu,\lambda=2;\mu\neq\lambda}^{\infty} |I_{\mu,\lambda}| \leqq AC^2 \sum_{\mu=2} \sum_{n\in S_\mu} f^\frown(n)^2\{[n-2^{\mu-1}+1]^{-2\alpha} + [2^\mu+1-n]^{-2\alpha}\}$$

so that using Lemma 5b (1) is seen to be valid.

Let us next consider

$$\int_{-1}^{1} \delta_\mu(x)^2 s_\alpha(x)d\Omega_\nu(x) \approx \mathfrak{N}_{0,\alpha}^{\nu}[\delta_\mu]^2.$$

For μ fixed we set

$$p(n,x) = \sum_{b_\mu-r_\mu}^{n} \rho_\mu(n)f^\frown(n)\omega_\nu(n)W_\nu(n,x) \qquad (n\in\sigma_\mu).$$

It follows from Theorem 4b that

$$\mathfrak{N}_{0,\alpha}^{\nu}[p(n,x)] \leqq A\mathfrak{N}_{0,\alpha}^{\nu}[E_\mu].$$

If $u(n)=t(n)/\rho_\mu(n)$, then

$$\delta_\mu(x) = \sum_{n\in S_\mu} u(n)[p(n,x) - p(n-1,x)].$$

Summing by parts this becomes

$$\delta_\mu(x) = \sum_{n\in S_\mu} p(n, x)[u(n) - u(n+1)] + u(2^\mu)p(2^\mu - 1, x) - u(2^{\mu-1})p(2^{\mu-1} - 1, x),$$

from which using Theorem 4b it follows that

$$\mathfrak{N}^\nu_{0,\alpha}[\delta_\mu] \leqq A\mathfrak{N}^\nu_{0,\alpha}[E_\mu]\left\{\sum_{n\in\sigma_\mu} |u(n) - u(n+1)| + |u(2^\mu)| + |u(2^{\mu-1})|\right\}.$$

Now it is easily verified that

$$\sum_{n\in\sigma_\mu} |u(n) - u(n+1)| + |u(2^\mu)| + |u(2^{\mu-1})| \leqq AC$$

and thus

$$\mathfrak{N}^\nu_{0,\alpha}[\delta_\mu] \leqq AC\mathfrak{N}^\nu_{0,\alpha}[E_\mu].$$

Squaring and summing over μ we see, using Lemma 5a, that (2) holds.

6. **Multiplier transformations continued.** Let $p(\beta, \alpha)$ stand for the proposition that if $T\in M(C)$ then $\mathfrak{N}^\nu_{\beta,\alpha}[Tf]\leqq AC\mathfrak{N}^\nu_{\beta,\alpha}[f]$ where A depends only upon α, β and of course ν. Theorem 5c shows that $p(0, \alpha)$ is valid for $0\leqq\alpha<1/2$. In this section we shall show that $p(\beta, \alpha)$ is valid for $(-1/2<\beta, \alpha<1/2)$. The following general principles are easily established, see [4].

i. If $p(\beta, \alpha)$ is valid so is $p(\alpha, \beta)$.

ii. If $p(\beta, \alpha)$ is valid so is $p(-\beta, -\alpha)$.

iii. If $p(\beta_1, \alpha_1)$ and $p(\beta_2, \alpha_2)$ are valid so is $p(\beta, \alpha)$ where $\beta=\min(\beta_1, \beta_2)$, $\alpha=\min(\alpha_1, \alpha_2)$.

Using these it is easily shown that $p(\alpha, \beta)$ is valid if $-1/2<\alpha, \beta<1/2$ and if in addition $\alpha\beta\geqq 0$. To remove this restriction we require an additional argument.

LEMMA 6a. *If $-1/2<\beta\leqq 0\leqq\alpha<1/2$, if $T\in M(C)$, and if*

$$F(x) = (1-x)T[f(x)] - T[(1-x)f(x)],$$

then

$$\mathfrak{N}^\nu_{\beta,0}[F] \leqq AC\mathfrak{N}^\nu_{0,\alpha}[f].$$

The familiar recurrence formula for ultraspherical polynomials, see [2, vol. 2, p. 175], implies that

$$(1-x)W_\nu(n, x) = +[(2\nu+n)/2(n+\nu)][W_\nu(n, x) - W_\nu(n+1, x)] + [n/2(n+\nu)][W_\nu(n, x) - W_\nu(n-1, x)].$$

Supposing, as we may, that only finitely many $f^\frown(n)$ are not zero we find, after a short computation, that $F(x)=F_1(x)+F_2(x)$ where

$$F_1(x) = \sum_{n=0}^{\infty} \frac{n}{2(n+\nu)} f^\frown(n-1)[t(n)-t(n-1)]\omega_\nu(n)W_\nu(n,x),$$

$$F_2(x) = \sum_{n=0}^{\infty} \frac{2\nu+n}{2(n+\nu)} f^\frown(n+1)[t(n)-t(n+1)]\omega_\nu(n)W_\nu(n,x).$$

Let $g(x)\in\mathfrak{N}^{\nu'}_{-\beta,0}$ and let $g^\frown(n)$ be defined as usual. We have

$$\int_{-1}^{1} F_1(x)g(x)dx = \sum_{n=0}^{\infty} \frac{n}{2(n+\nu)} f^\frown(n-1)g^\frown(n)[t(n)-t(n-1)]\omega_\nu(n),$$

$$\left|\int_{-1}^{1} F_1(x)g(x)dx\right|$$

$$\leqq A\sum_{\mu=0}^{\infty}\sum_{n\in S_\mu} |f^\frown(n-1)|\,|g^\frown(n)|\,|t(n)-t(n-1)|\,(\omega_\nu(n)^{1/2}\omega_\nu(n-1)^{1/2}.$$

If $f^*(\mu)=\text{l.u.b. } |f^\frown(n-1)|\ \omega_\nu(n-1)^{1/2}$ for $n\in S_\mu$, and

$$g^*(\mu)=\text{l.u.b. } |g^\frown(n)|\ \omega(n)^{1/2}$$

for $n\in S_\mu$, then

$$\begin{aligned}\left|\int_{-1}^{1} F_1(x)g(x)dx\right| &\leqq A\sum_{\mu=0}^{\infty} f^*(\mu)g^*(\mu)\sum_{n\in S_\mu}|t(n)-t(n-1)|\\ &\leqq AC\sum_{\mu=0}^{\infty} f^*(\mu)g^*(\mu)\\ &\leqq AC\left[\sum_{\mu=0}^{\infty} f^*(\mu)^2\right]^{1/2}\left[\sum_{\mu=0}^{\infty} g^*(\mu)^2\right]^{1/2}\\ &\leqq AC\mathfrak{N}^{\nu}_{0,\alpha}[f]\mathfrak{N}^{\nu'}_{-\beta,0}[g]\end{aligned}$$

by Lemma 5b. Since this holds for every $g\in\mathfrak{N}^{\nu'}_{-\beta,0}$ it implies that $\mathfrak{N}^{\nu}_{\beta,0}[F_1] \leqq AC\mathfrak{N}^{\nu}_{0,\alpha}[f]$. Similarly we can show that $\mathfrak{N}^{\nu}_{\beta,0}[F_2]\leqq AC\mathfrak{N}^{\nu}_{0,\alpha}[f]$, and our lemma is established.

Using this we can now show that if $-1/2<\beta\leqq 0\leqq\alpha<1/2$ then $p(\beta,\alpha)$ is valid. We have

$$\mathfrak{N}^{\nu}_{\beta,\alpha}[Tf] \leqq A\mathfrak{N}^{\nu}_{0,\alpha}[Tf] + A\mathfrak{N}^{\nu}_{\beta,0}[(1-x)Tf].$$

Since $p(0,\alpha)$ is valid $\mathfrak{N}^{\nu}_{0,\alpha}[Tf] \leqq AC\mathfrak{N}^{\nu}_{0,\alpha}[f] \leqq AC\mathfrak{N}^{\nu}_{\beta,\alpha}[f]$ If $F(x)$ is defined as above then

$$\begin{aligned}\mathfrak{N}^{\nu}_{\beta,0}[(1-x)Tf] &= \mathfrak{N}^{\nu}_{\beta,0}[T\{(1-x)f(x)\} + F(x)] \\ &\leqq \mathfrak{N}_{\beta,0}[T\{(1-x)f(x)\}] + \mathfrak{N}^{\nu}_{\beta,0}[F(x)].\end{aligned}$$

By $p(\beta, 0)$,

$$\begin{aligned}\mathfrak{N}^{\nu}_{\beta,0}[T\{(1-x)f(x)\}] &\leqq AC\mathfrak{N}^{\nu}_{\beta,0}[(1-x)f(x)], \\ &\leqq AC\mathfrak{N}^{\nu}_{\beta,\alpha}[f(x)].\end{aligned}$$

Lemma 6a implies that

$$\mathfrak{N}^{\nu}_{\beta,0}[F(x)] \leqq AC\mathfrak{N}^{\nu}_{0,\alpha}[f] \leqq AC\mathfrak{N}^{\nu}_{\beta,\alpha}[f].$$

Combining these results we have our desired result.

THEOREM 6a. *$p(\alpha, \beta)$ is valid for* $-1/2<\alpha, \beta<1/2$.

This follows from the above.

The restriction $-1/2<\alpha, \beta<1/2$ is essential in Theorem 6a and the result is not otherwise true. See in this connection the discussion at the end of §6 of [4].

An application of Theorem 6a to the theory of fractional integration is described in [6]. Proofs for the special case $\nu=1/2$ are given in [4]. The modifications needed to adapt the proof to the case of general ν are slight.

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