ON INTEGRATION OF QUASI-LINEAR PARABOLIC EQUATIONS BY EXPLICIT DIFFERENCE METHODS(1)

BY

J. WOLFGANG SMITH(2)

Introduction. Our subject begins, in a sense, with the classic paper of Courant, Friedrichs and Lewy [1] on the difference equations of mathematical physics. These authors considered certain linear difference equations which formally represent the basic partial differential equations of second order. We would say, following present terminology, that the difference equations satisfied compatibility or consistency conditions determined by the corresponding differential equation. The question was to determine whether the solutions of these difference problems tend, in some sense, to a solution of the corresponding differential problem, as the mesh size tends to zero. It was found that this does happen, provided that the difference equation satisfies an additional requirement which, in present terminology, would be referred to as a condition of stability. This condition was required to establish certain a priori estimates for the solutions of the difference problem, on the basis of which one can prove the existence of a subsequence of difference solutions which converges to a solution of the corresponding differential problem. It is important to note that the existence of the latter was not assumed, but rather was established by the argument.

The property of difference equations which Courant, Friedrichs and Lewy had required, and which had led (in the parabolic and hyperbolic cases) to certain algebraic conditions on the ratios of the mesh constants, was later made the defining property for the notion of stability appropriate to difference equations. This was done by Fritz John [5] in his important paper on the integration of parabolic equations by difference methods. Roughly speaking, Fritz John defined stability for initial value difference problems as follows: Let there be given a sequence of lattices on which the difference problem is defined, with mesh size tending to zero. The difference equation (which depends on certain ratios of the mesh constants) is said to be stable if the amplification of arbitrary initial data (in some norm) is bounded uniformly

Received by the editors November 22, 1957.

(1) This work was done while the author was University Fellow in mathematics at Columbia University. The material in this paper is contained in a thesis prepared under the direction of Professor Francis J. Murray.

(2) At present in the Department of Mathematics, Massachusetts Institute of Technology.

for all lattices of the given sequence. The importance of this stability notion is evidenced by the theorems which follow. First it is shown that if a difference equation is stable and compatible with a parabolic differential equation of the given kind, and if the latter has a solution which is sufficiently regular, then the sequence of difference solutions determined by the given initial data will converge to the given solution of the differential problem. From this result Fritz John derives a uniqueness theorem for the differential equation. Finally, using again the fact that the latter may be represented by a stable difference equation, he establishes an existence theorem for the differential equation by an argument which is essentially that of Courant, Friedrichs and Lewy.

The relationship between stability (in this sense) and convergence of the difference solutions has been further investigated by P. D. Lax and R. D. Richtmyer [7] for linear initial value problems of considerable generality. Utilizing certain notions and theorems from the theory of Banach spaces, these authors have proved that stability is in a certain sense a necessary and sufficient condition for convergence of the difference solutions corresponding to arbitrary initial data.

These papers contain the elements of a general theory of linear initial value problems, in which stable difference equations play the central rôle. Stability of the difference equation not only implies convergence, but what is more, one may expect that it leads as well to existence and uniqueness theorems for the differential equation. Furthermore we note that given a stable difference representation and sufficiently regular coefficients, it is possible, in principle, to calculate the solution of the differential equation to any desired degree of accuracy by solving the difference problem for a sufficiently small value of the mesh size. The question of rounding error, which has been much emphasized by some authors and has inspired alternative definitions of stability for difference equations(3), need not be troublesome. If the difference equation is stable in the present sense, and if the rounding error at each step is accounted for by adding an inhomogeneous term to the difference equation, the effect of the latter on the difference solution may be estimated by Duhamel's principle. Thus one may control the effect of rounding error by increasing the number of decimal places in some inverse proportion to the mesh size of the calculation.

The chief theoretical difficulty lies in the verification of stability, i.e. in obtaining suitable a priori estimates for the solutions of a difference problem. This has so far been accomplished only for some special cases, e.g. for equations with constant coefficients. For such equations one is led, by elementary arguments, to a simple necessary condition for stability, referred to as the von Neumann condition(4), which in many cases is also sufficient. When the

(3) See, for instance, O'Brien, Hyman and Kaplan [8].

(4) See P. D. Lax and R. D. Richtmyer [7]. See also J. Douglas, Jr. [2].

coefficients of the difference equation are arbitrary functions of the independent variables, these elementary methods become inapplicable, and there exists at present no general theory which can cope with the problem. As a matter of practical expediency, it has been found that the von Neumann condition (or a slight modification thereof) may be applied locally to yield adequate stability criteria, and this has led to the conjecture that a local condition of this type is generally applicable to difference equations with nonconstant coefficients. In spite of much effort this conjecture has so far received verification only in two special cases(5). Apart from this, certain techniques have been applied successfully to establish stability for certain types of difference equations which are used in applications to physics and engineering(6).

Some attempts have been made to extend this kind of analysis to initial value problems of nonlinear type. Perhaps the most notable result in this connection is Fritz John's treatment [5] of certain semi-linear problems, in which the inhomogeneous term is allowed to depend on the solution. Beyond this, a general theoretical development along the indicated lines seems to be missing in the nonlinear case. It is the aim of this paper to supply such a development for a class of quasi-linear initial value problems which (in the limit) involve partial differential equations of parabolic type. The heart of the theory lies again in certain a priori estimates for the solutions of the difference equation, which now depend critically on an application of a fixed point theorem.

Specifically, we shall have to deal with a pure initial value problem for the quasi-linear difference equation(7)

$$u(x, t+\Delta t) = \sum_{r=-\rho}^{\rho} c^r(x, t, u; \Sigma) u(x + r\Delta x, t) + \Delta t d(x, t, u; \Sigma) \tag{1}$$

defined on a rectangular lattice Σ, which consists of points (x, t) with

$$\begin{aligned} x &= 0, \pm\Delta x, \pm 2\Delta x, \cdots, \\ t &= 0, \Delta t, \cdots, \nu\Delta t \leqq \tau, \\ \Delta t &= \lambda \Delta x^2. \end{aligned}$$

Here ρ is a positive integer; λ, τ & Δx are positive numbers; and the unknown function u, as well as the coefficients c^r & d, are real-valued. §1 contains a stability analysis for the given difference problem. In the succeeding sections,

(5) Fritz John [5] and Peter Lax [6].

(6) Many linear hyperbolic problems of mathematical physics may be handled by the stability theorem of K. O. Friedrichs [4] for positive operators. J. Douglas, Jr. [3] has established the stability of an implicit difference scheme for the numerical solution of mixed boundary-initial value problems involving quasi-linear parabolic equations. See also M. E. Rose [9].

(7) The case where the coefficients c^r do not depend on u has been fully investigated by Fritz John [5].

equation (1) is regarded as representing a quasi-linear partial differential equation of parabolic type. Convergence of the difference solutions is established in §2. Existence and uniqueness theorems for the differential equation are obtained in §3. According to the author's knowledge of the literature, the conditions involved in these theorems are not covered by previously known results.

1. **Local stability.** Let the quantities λ, τ & Δx be given. This determines a lattice Σ. Let X denote the set $\{0, \pm\Delta x, \pm 2\Delta x, \cdots\}$ and let $\mathfrak{A}$ & $\mathfrak{B}$ denote, respectively, the class of real-valued functions defined on X & Σ. Given an element v in $\mathfrak{A}$, one can solve(8) Equation (1) on Σ, subject to initial values v. This determines a unique element u of $\mathfrak{B}$, and defines thus a transformation T of $\mathfrak{A}$ into $\mathfrak{B}$. Given an element u of $\mathfrak{B}$, we define its norm:

$$\|u\| = \operatorname*{l.u.b.}_{(x,t)\in\Sigma} \{ \, | u(x, t) | \, \},$$

the norm being infinite when the least upper bound does not exist. Let β denote a set of $(p+1)$ positive numbers $\beta_0, \beta_1, \cdots, \beta_p$. We define a subset $\mathfrak{B}(\beta)$ of $\mathfrak{B}$ by the conditions

$$\|u_x^{(n)}\| \leqq \beta_n, \qquad n = 0, 1, \cdots, p; \tag{1.1}$$

where(9)

$$u_x^{(n)}(x, t) = \frac{1}{\Delta x^n} \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} u(x + \nu\Delta x, t).$$

Let v be an element of $\mathfrak{A}$ and u an element of $\mathfrak{B}$. Substituting u into the coefficients of Equation (1), one obtains a linear difference equation which can be solved on Σ subject to initial values v. This determines a transformation S_v of $\mathfrak{B}$ into itself. We can now state a result which is basic for the subsequent discussion.

Lemma 1. *Let v be an element of $\mathfrak{A}$. If*

$$S_v[\mathfrak{B}(\beta)] \subset \mathfrak{B}(\beta), \tag{1.2}$$

then Tv belongs to $\mathfrak{B}(\beta)$.

One easily sees that Tv belongs to $\mathfrak{B}(\beta)$ if and only if S_v has a fixed point in $\mathfrak{B}(\beta)$, and the lemma is therefore essentially a fixed point theorem. In the special case of explicit difference equations, with which we are presently concerned, it turns out to be a trivial fixed point theorem, as will be apparent from the argument(10) given below. For general implicit difference equations,

(8) It will be assumed, for simplicity, that the coefficients c^r & d are defined and real-valued for all real values of u.

(9) The quantity $u_x^{(n)}$ is the nth divided difference of u, multiplied by $n!$.

(10) This argument was suggested to the author by the referee.

on the other hand, the corresponding result is essentially equivalent to the Brouwer theorem, and may be established with its aid[11].

To prove the lemma, let us set $w=Tv$ and suppose that

$$\left| w_x^{(n)}(x, (\nu - 1)\Delta t) \right| \leqq \beta_n, \qquad n = 0, 1, \cdots, p; \tag{1.3}$$

for all $x \in X$ and for some integer ν such that $0<\nu\Delta t \leqq \tau$. It is clear from Equation (1.2) that Equation (1.3) holds for at least $\nu=1$. Let u be an arbitrary function in $\mathfrak{B}(\beta)$ which coincides with w for $t=(\nu-1)\Delta t$. The function $S_\nu u$ will then coincide with w for $t=\nu\Delta t$. By Equation (1.2) it follows that

$$\left| w_x^{(n)}(x, \nu\Delta t) \right| \leqq \beta_n, \qquad n = 0, 1, \cdots, p;$$

for $x \in X$. By induction we conclude that w belongs to $\mathfrak{B}(\beta)$, as was to be proved.

Let α denote a set of $(p+1)$ positive numbers $\alpha_0, \alpha_1, \cdots, \alpha_p$; and let $\mathfrak{A}(\alpha)$ denote the subset of $\mathfrak{A}$ which is determined by the conditions

$$\left\| v_x^{(n)} \right\| \leqq \alpha_n, \qquad n = 0, 1, \cdots, p;$$

where

$$\|w\| = \underset{x \in X}{\text{l.u.b.}} \left\{ \,| w(x) |\, \right\} \qquad \text{for } w \in \mathfrak{A}.$$

A slight generalization of the ordinary stability concept for difference equations may now be stated in the following terms[12]: Let there be given a sequence of lattices Σ with λ & τ fixed and Δx tending to zero. It will be remembered that the sets $\mathfrak{A}$, $\mathfrak{B}$, etc. and the transformation T are functions of Σ. Given α, one requires that there exist a β independent of Δx such that

$$T[\mathfrak{A}(\alpha)] \subset \mathfrak{B}(\beta) \tag{1.4}$$

for all Σ of the given sequence. This is a property which a difference problem may have for certain values of λ and τ. Unfortunately no such values will exist for general quasi-linear difference problems of the given type, and one is therefore led to reformulate the stability condition along the following lines: Given α, there shall exist a β and positive numbers λ^* & τ^*, all independent of Δx, such that Equation (1.4) is satisfied for every lattice Σ with $\lambda \leqq \lambda^*$ and $\tau \leqq \tau^*$. Since τ, in particular, is restricted by a choice of α, the theory will assume a local character which is entirely analogous to the classical theory of nonlinear ordinary differential equations. Stability, in this sense, may therefore be referred to as *local stability*. Although the essential

[11] See §1 of the author's thesis: *Stability of quasi-linear difference equations*, Columbia University, 1957. Here one must assume that the c^r & d depend continuously on u.

[12] Ordinary stability corresponds roughly to the case $p=0$. The reason for considering general p-values will presently be pointed out.

idea of local stability is contained in the statement as given, we shall prefer a slightly stronger and more elaborate definition of this term, for reasons which will become apparent in the next section.

Before stating the definition in its final form, we would like to point out the general significance of Lemma 1 for the stability theory of quasi-linear difference equations. Let L_u denote the transformation of $\mathfrak{A}$ into $\mathfrak{B}$ which maps an element v in $\mathfrak{A}$ on $S_v u$, where u is a fixed element of $\mathfrak{B}$. We ask now whether, for an arbitrary value of α, there exist quantities β, λ^* & τ^* independent of Δx such that

$$L_u[\mathfrak{A}(\alpha)] \subset \mathfrak{B}(\beta) \tag{1.5}$$

for all Σ with $\lambda \leqq \lambda^*$ & $\tau \leqq \tau^*$ and all u in $\mathfrak{B}(\beta)$. This is a problem in the stability theory of linear difference equations. If the quantities β, λ^* & τ^* exist, then, according to Lemma 1, the quasi-linear difference problem will be locally stable in the present sense. The theorems which must be applied to establish the existence of β, λ^* & τ^* will frequently assume that the coefficients of the linear difference equation representing L_u are defined at all points (x, t) in a continuous region of the plane and have bounded derivatives in x up to some positive order p. We cite Fritz John's Theorem 3.1([13]) as an example corresponding to the case $p=2$. It is for this reason that a comprehensive study of quasi-linear difference problems along the present lines will require Lemma 1 in its full generality. For the remainder of this paper we shall be concerned with the special case([14]) $p=0$, which is not only of some interest in itself, but suggests also methods and results which can be expected to hold in the general case.

To state the notion of local stability in its final form, one requires linear operators $L_{u,m}$ mapping $\mathfrak{A}$ into $\mathfrak{B}$ which are analogous to certain operators defined by Fritz John. Let c_u^r & d_u denote, respectively, the functions of x, t & Σ which result when a given element u of $\mathfrak{B}$ is substituted into the coefficients c^r & d of Equation (1). Let v be an arbitrary element of $\mathfrak{A}$ and let m be a non-negative integer. The operator $L_{u,m}$ maps v on w, where w shall satisfy the homogeneous equation

$$w(x, t + \Delta t) = \sum_r c_u^r(x, t; \Sigma) w(x + r\Delta x, t) \tag{1.6}$$

at points of Σ with $t \geqq m\Delta t$, and shall assume values

$$w(x, t) = \begin{cases} v(x) & \text{for } t = m\Delta t, \\ 0 & \text{for } t < m\Delta t \end{cases}$$

at the remaining points of Σ. Let $\mathfrak{R}(\lambda^*, \tau^*, \Delta x^*)$ denote the class of rectangular

([13]) Fritz John [5, p. 166].

([14]) The sets α & β will consist now of a single element, and it will simplify the notation to identify α & β with α_0 & β_0, respectively.

lattices Σ with $\lambda \leqq \lambda^*$, $\tau \leqq \tau^*$ & $\Delta x \leqq \Delta x^*$. The quasi-linear difference equation (1) is henceforth called *locally stable,* when the following two conditions are satisfied: Given $\beta > 0$ and $Q > 1$, there exist positive numbers λ^*, τ^{**} & Δx^* such that

$$\|L_{u,m}v\| \leqq Q\|v\| \tag{1.7}$$

for $\Sigma \in \mathfrak{L}(\lambda^*, \tau^{**}, \Delta x^*)$, $u \in \mathfrak{B}(\beta)$, $v \in \mathfrak{A}$ & $m \geqq 0$. There exists also a positive number γ, such that

$$\max_{|r| \leqq p} \left\{ \|\overset{r}{c}_u\|, \|d_u\| \right\} \leqq \gamma \tag{1.8}$$

for $\Sigma \in \mathfrak{L}(\lambda^*, \tau^{**}, \Delta x^*)$ & $u \in \mathfrak{B}(\beta)$.

The connection between local stability and the condition stated earlier is given by the following result.

LEMMA 2. *Let Equation* (1) *be locally stable. Given α, there exist positive numbers β, λ^*, τ^* & Δx^* such that Equation* (1.4) *holds for every lattice Σ in* $\mathfrak{L}(\lambda^*, \tau^*, \Delta x^*)$.

To show this, we note first that the solution $L_u v$ of the inhomogeneous equation can be written in the form

$$L_u v = L_{u,0} v + \Delta t \sum_{m>0} L_{u,m} d_{u,m-1}, \tag{1.9}$$

where $d_{u,m}$ denotes the function in $\mathfrak{A}$ which assumes values $d_u(x, m\Delta t; \Sigma)$ on X. Given a positive number α, we choose $\beta > \alpha$ and $Q > 1$ such that $Q\alpha < \beta$. If Equation (1) is locally stable, one may apply Equations (1.7), (1.8) and (1.9) to obtain

$$\|L_u v\| \leqq Q(\alpha + \tau\gamma)$$

for $\Sigma \in \mathfrak{L}(\lambda^*, \tau^{**}, \Delta x^*)$, $u \in \mathfrak{B}(\beta)$ & $v \in \mathfrak{A}(\alpha)$. Equation (1.5) holds now for $\Sigma \in \mathfrak{L}(\lambda^*, \tau^*, \Delta x^*)$ & $u \in \mathfrak{B}(\beta)$, where

$$\tau^* = \min \left\{ \tau^{**}, \frac{\beta - Q\alpha}{Q\gamma} \right\}.$$

This implies by Lemma 1 that Equation (1.4) is satisfied for

$$\Sigma \in \mathfrak{L}(\lambda^*, \tau^*, \Delta x^*).$$

We consider now an important difference equation for which local stability can be established by elementary methods.

LEMMA 3. *Let the difference equation*[15]

$$u_t(x, t) = a_0(x, t, u; \Sigma) u_{xx}(x - \Delta x, t) + a_1(x, t, u; \Sigma) u_x(x, t) + a_2(x, t, u; \Sigma) \tag{1.10}$$

[15] w_t denotes the forward divided difference in t; w_{xx} & w_x denote $w_x^{(2)}$ & $w_x^{(1)}$, respectively.

be defined for $\Sigma \in \mathfrak{L}(\lambda', \tau', \Delta x')$. *Given* $\beta > 0$, *we assume that there exist positive numbers* γ & $\delta < 1$ *such that*

$$\max_i \{ | a_i(x, t, u; \Sigma) | \} \leqq \gamma \ [\&] \ a_0(x, t, u; \Sigma) \geqq \delta \tag{1.11}$$

identically for $|u| \leqq \beta$. *Equation* (1.10) *is then locally stable.*

Equation (1.10) has the form of Equation (1), with coefficients given by

$$\begin{aligned} \overset{r}{c} &= \overset{r}{c}_0 + \Delta x \overset{r}{c}_1, \\ \overset{r}{c}_0 &= \overset{r}{\delta}_0 + \lambda(\overset{r}{\delta}_1 - 2\overset{r}{\delta}_0 + \overset{r}{\delta}_{-1})a_0, \\ \overset{r}{c}_1 &= \lambda(\overset{r}{\delta}_1 - \overset{r}{\delta}_0)a_1, \\ d &= a_2. \end{aligned} \tag{1.12}$$

Given $\beta > 0$ & $Q > 1$, we set $\lambda^* = \min \{\lambda', \gamma^{-1}(1-\delta)/2\}$ & $\tau^{**} = \tau'$. By Equations (1.11) and (1.12),

$$\overset{0}{c}_0(x, t, u; \Sigma) \geqq \delta \ [\&] \ \overset{\pm 1}{c_0}(x, t, u; \Sigma) \geqq \lambda\delta$$

for $\Sigma \in \mathfrak{L}(\lambda^*, \tau^{**}, \Delta x')$ & $u \in \mathfrak{B}(\beta)$. Hence there must exist a positive number $\Delta x^* \leqq \Delta x'$ such that $c^r(x, t, u; \Sigma) \geqq 0$ for $\Sigma \in \mathfrak{L}(\lambda^*, \tau^{**}, \Delta x^*)$ & $u \in \mathfrak{B}(\beta)$. For Σ & u thus restricted, the operators $L_{u,m}$ may be estimated as follows: From Equation (1.6) one obtains

$$| w(x, t + \Delta t) | \leqq \left\{ \sum_r | \overset{r}{c}_u | \right\} \ \underset{x \in X}{\text{l.u.b.}} \{ | w(x, t) | \} \cdot$$

Since the coefficients c_u^r are now non-negative, we have by Equation (1.12) $\sum_r |c_u^r| = 1$, which implies

$$\|L_{u,m} v\| \leqq \|v\| < Q\|v\|.$$

The remaining condition of local stability is obviously satisfied.

2. **Convergence of the difference solutions.** Let there be given a quasi-linear parabolic differential equation

$$\frac{\partial u}{\partial t} = a_0(x, t, u) \frac{\partial^2 u}{\partial x^2} + a_1(x, t, u) \frac{\partial u}{\partial x} + a_2(x, t, u) \tag{2.1}$$

for which the coefficients a_i are defined on the region $\mathfrak{R}$: $-\infty < x < \infty$, $0 \leqq t \leqq \tau_0$, $-\infty < u < \infty$. In this section it will be assumed that Equation (2.1) has a solution $u(x, t)$ which is sufficiently regular. We shall establish conditions which imply that the solutions $u(x, t; \Sigma)$ of Equation (1), determined by initial values $u(x, 0)$, will converge to $u(x, t)$ as Δx tends to zero.

A first set of conditions is obtained in the following way. Let the functions

$$u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t}$$

be uniformly continuous and bounded on the infinite strip $R(\tau_0)$: $-\infty < x < \infty$, $0 \leqq t \leqq \tau_0$. One then obtains by Taylor's theorem

$$(2.2)\quad u(x + r\Delta x, t) = u(x, t) + r\Delta x \frac{\partial u(x, t)}{\partial x} + \frac{1}{2} r^2 \Delta x^2 \frac{\partial^2 u(x, t)}{\partial x^2} + o(\Delta t),$$

$$(2.3)\quad u(x, t + \Delta t) = u(x, t) + \Delta t \frac{\partial u(x, y)}{\partial t} + o(\Delta t).$$

Substituting these expressions into Equation (1), one obtains

$$(2.4)\quad \begin{aligned} \frac{\partial u}{\partial t} = &\left\{\frac{1}{\lambda} \sum_r \frac{1}{2} r^2 c^r\right\} \frac{\partial^2 u}{\partial x^2} + \left\{\frac{1}{\lambda \Delta x} \sum_r r c^r\right\} \frac{\partial u}{\partial x} \\ &+ \left\{\frac{1}{\lambda \Delta x^2} \sum_r (c^r - \delta_0^r)\right\} u + d + \epsilon(\Delta t), \end{aligned}$$

where $\epsilon(\Delta t)$ denotes a function which tends to zero with Δt, uniformly on $R(\tau_0)$. We may regard Equation (2.4) as holding at every point (x, t) of a lattice Σ, the coefficients c^r & d being evaluated at the points $(x, t, u(x, t); \Sigma)$. In order that Equation (1) should represent the given differential equation as Δx tends to zero, one requires that

$$(2.5)\quad \lim_{\Delta x \to 0} \begin{cases} \dfrac{1}{\lambda} \displaystyle\sum_r \frac{1}{2} r^2 c^r = a_0, \\ \dfrac{1}{\lambda \Delta x} \displaystyle\sum_r r c^r = a_1 \ [\&]\ d(x, t, u; \Sigma) = a_2(x, t, u), \\ \dfrac{1}{\lambda \Delta x^2} \displaystyle\sum_r (c^r - \delta_0^r) = 0. \end{cases}$$

Following Fritz John, we shall suppose that the coefficients c^r are given in the form

$$(2.6)\quad c^r(x, t, u; \Sigma) = c_0^r(x, t, u, \lambda) + \Delta x c_1^r(x, t, u, \lambda) + \Delta x^2 c_2^r(x, t, u, \lambda, \Delta x)$$

where the functions c_i^r are defined for $(x, t, u) \in \Re$, and

$$(2.7)\quad \lim_{\Delta x \to 0} c_2^r(x, t, u, \lambda, \Delta x) = c_2^r(x, t, u, \lambda, 0),$$

uniformly when u is bounded. We assume further that the coefficient d does not depend on Σ. Equations (2.5), (2.6) and (2.7) lead to the relations

$$
(2.8)\qquad
\begin{aligned}
\sum_r \frac{1}{2} r^2 \overset{r}{c}_0(x, t, u, \lambda) &= \lambda a_0(x, t, u),\\
\sum_r r\overset{r}{c}_0(x, t, u, \lambda) &= 0,\\
\sum_r \overset{r}{c}_0(x, t, u, \lambda) &= 1,\\
\sum_r r\overset{r}{c}_1(x, t, u, \lambda) &= \lambda a_1(x, t, u),\\
\sum_r \overset{r}{c}_1(x, t, u, \lambda) &= 0,\\
\sum_r \overset{r}{c}_2(x, t, u, \lambda, 0) &= 0,\\
d(x, t, u; \Sigma) &= a_2(x, t, u)
\end{aligned}
$$

which are referred to by Fritz John as the *compatibility conditions* for Equation (1).

Given a lattice Σ, one may solve Equation (1) on Σ, subject to initial values $u(x, 0)$, and determine thus an element $u(\Sigma)$ in $\mathfrak{B}$. The given function $u(x, t)$, defined on $R(\tau_0)$, may likewise be regarded as an element u in $\mathfrak{B}$ when $\tau \leqq \tau_0$. The solutions $u(\Sigma)$ of Equation (1) are said to *converge locally* and *uniformly* to the function $u(x, t)$, if there exist positive numbers λ^* & $\tau_* \leqq \tau_0$ and a function $\epsilon(\Delta x)$ tending to zero with Δx, such that $\|u(\Sigma) - u\| \leqq \epsilon(\Delta x)$ for $\Sigma \in \mathfrak{L}(\lambda^*, \tau_*, \Delta x)$. Let $\mathfrak{R}(\beta)$ denote the following subset of $\mathfrak{R}$: $-\infty < x < \infty$, $0 \leqq t \leqq \tau_0$, $-\beta \leqq u \leqq \beta$.

THEOREM I. *Let the coefficients a_i of Equation* (2.1) *satisfy a Lipschitz condition in u, uniformly on every subset $\mathfrak{R}(\beta)$*([16]). *Let $u(x, t)$ be a solution of Equation* (2.1) *for which*

$$
(2.9)\qquad u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t}
$$

are uniformly continuous and bounded on $R(\tau_0)$. Let the difference equation (1) *be locally stable and satisfy the compatibility conditions* (2.8) *identically. The solutions $u(\Sigma)$ of Equation* (1), *determined by initial values $u(x, 0)$, will then converge locally and uniformly to $u(x, t)$.*

To prove the theorem, we note first that there exists a positive number α such that $|u(x, t)| \leqq \alpha$ on $R(\tau_0)$. One can select numbers $\beta > \alpha$ and $Q > 1$ such that $Q\alpha < \beta$. Since Equation (1) is locally stable, there exist positive numbers λ^*, $\tau^{**} \leqq \tau_0$ & Δx^* such that

$$
(2.10)\qquad \|L_{u,m} v\| \leqq Q\|v\|
$$

([16]) This means that there exists a positive number L such that $|a_i(x, t, u+w) - a_i(x, t, u)| \leqq L|w|$ for $(x, t, u) \in \mathfrak{R}(\beta)$.

for $\Sigma \in \mathfrak{L}(\lambda^*, \tau^{**}, \Delta x^*)$ & $u \in \mathfrak{B}(\beta)$. It follows again [17] by Lemma 1 that there exists a positive number $\tau^* \leqq \tau^{**}$ such that $u(\Sigma)$ belongs to $\mathfrak{B}(\beta)$ for every Σ in $\mathfrak{L}(\lambda^*, \tau^*, \Delta x^*)$. We agree henceforth that all equations, whose terms depend on Σ, are asserted to hold for every lattice in $\mathfrak{L}(\lambda^*, \tau^*, \Delta x^*)$. Thus we may say, by the second condition for local stability, that there exists a positive number γ such that

$$(2.11)\qquad \max_{|r| \leqq \rho} \left\{ \left\| \overset{r}{c}_u \right\|, \left\| \overset{r}{c}_{u(\Sigma)} \right\|, \left\| d_u \right\|, \left\| d_{u(\Sigma)} \right\| \right\} \leqq \gamma.$$

The regularity conditions on $u(x, t)$ imply that Equations (2.2) and (2.3) are satisfied. Using Equations (2.2), (2.7), (2.8) and (2.11), one obtains

$$\sum_r \overset{r}{c}_u(x, t; \Sigma) u(x + r\Delta x, t) = u(x, t) + \Delta t \left\{ a_0(x, t, u(x, t)) \frac{\partial^2 u}{\partial x^2} + a_1(x, t, u(x, t)) \frac{\partial u}{\partial x} \right\} + o(\Delta t).$$

Since $u(x, t)$ satisfies Equations (2.1) and (2.3), this becomes

$$(2.12)\qquad u(x, t + \Delta t) = \sum_r \overset{r}{c}_u(x, t; \Sigma) u(x + r\Delta x, t) + \Delta t d_u(x, t; \Sigma) + o(\Delta t).$$

Subtracting Equation (2.12) from Equation (1), one obtains

$$(2.13)\qquad \begin{aligned} \delta u(x, t + \Delta t) = &\sum_r \overset{r}{c}_{u(\Sigma)}(x, t; \Sigma) \delta u(x + r\Delta x, t) \\ &+ \sum_r \delta c^r(x, t; \Sigma) u(x + r\Delta x, t) + \Delta t \delta d(x, t; \Sigma) + o(\Delta t) \end{aligned}$$

where

$$\begin{aligned} \delta u &= u(\Sigma) - u, \\ \delta c^r &= \overset{r}{c}_{u(\Sigma)} - \overset{r}{c}_u, \\ \delta d &= d_{u(\Sigma)} - d_u. \end{aligned}$$

It will be necessary to obtain suitable estimates for the second group of terms on the right side of Equation (2.13). For this purpose we use Equation (2.2) once more, to obtain

$$\sum_r \delta c^r(x, t; \Sigma) u(x + r\Delta x, t) = \sum_r \left\{ \delta \overset{r}{c}_0 + \Delta x \delta \overset{r}{c}_1 + \Delta x^2 \delta \overset{r}{c}_2 \right\} \left\{ u + r\Delta x \frac{\partial u}{\partial x} + \frac{1}{2} r^2 \Delta x^2 \frac{\partial^2 u}{\partial x^2} + o(\Delta t) \right\},$$

where the δc_i^r are defined in the obvious way. Applying again the compatibility conditions (2.8), one finds that the terms of zero'th and first order in Δx

[17] See proof of Lemma 2.

vanish, and what remains is

$$\Delta t\left\{\delta a_0 \frac{\partial^2 u}{\partial x^2} + \delta a_1 \frac{\partial u}{\partial x}\right\} + o(\Delta t),$$

where the δa_i are functions of x, t & Σ, determined by the relation

$$\delta a_i = \left\{ a_i(x, t, u(x, t; \Sigma)) - a_i(x, t, u(x, t))\right\}.$$

The functions δu therefore satisfy an equation

$$(2.14) \qquad \delta u(x, t + \Delta t) = \sum_r \overset{r}{c}_{u(\Sigma)}(x, t; \Sigma)\delta u(x + r\Delta x, t) + \Delta t e + o(\Delta t)$$

with initial values zero, where

$$e = \delta a_0 \frac{\partial^2 u}{\partial x^2} + \delta a_1 \frac{\partial u}{\partial x} + \delta d.$$

So far u, $u(\Sigma)$ & δu have been regarded as given elements of $\mathfrak{B}$, and it has been shown that δu satisfies Equation (2.14) with certain initial values. We shall now set

$$\delta a_i(x, t, w) = \left\{ a_i(x, t, u(x, t) + w) - a_i(x, t, u(x, t))\right\}$$

and regard δa_i as a function of x, t & w, where w is an arbitrary element of $\mathfrak{B}$; and we do likewise for δd. The term e becomes thus a function of x, t & w. Let v denote the zero element of $\mathfrak{A}$, and let S_v denote the transformation of $\mathfrak{B}$ into itself which maps w on w', where w' is determined by the linear equation

$$(2.15) \quad w'(x, t + \Delta t) = \sum_r \overset{r}{c}_{u(\Sigma)}(x, t; \Sigma)w'(x + r\Delta x, t) + \Delta t e(x, t, w) + o(\Delta t)$$

and initial values v. If w is replaced by w' in Equation (2.15), there results a quasi-linear difference equation which determines a transformation T of $\mathfrak{A}$ into $\mathfrak{B}$. The given function δu satisfies Equation (2.14) with initial values v, which implies that $\delta u = Tv$. We are now ready to apply Lemma 1 once more to obtain the desired estimate for δu. For this purpose we note that the term $o(\Delta t)$ has the form $\Delta t\epsilon(x, t; \Sigma)$, where $\epsilon(x, t; \Sigma)$ is majorized by a monotone function $\epsilon(\Delta x)$ which tends to zero with Δx. The bound for S_v may now be estimated in terms of the given bounds for the original operators $L_{u,m}$. Applying Equations (2.10) and (1.9), one obtains

$$\|S_v w\| \leqq Q_T \|e(x, t, w) + \epsilon(x, t; \Sigma)\|.$$

Since the coefficients a_i satisfy a Lipschitz condition in u, uniformly on $\mathfrak{R}(\beta)$, and the quantities (2.9) are bounded on $R(\tau_0)$, there exists a positive number k such that $\|e(x, t, w)\| \leqq k\|w\|$. Hence

$$\|S_v w\| \leqq Q_T\left\{ k\|w\| + \epsilon(\Delta x)\right\}$$

which implies that

$$S_\nu[\mathfrak{B}(\epsilon(\Delta x))] \subset \mathfrak{B}(\epsilon(\Delta x))$$

for $\Sigma \in \mathfrak{L}(\lambda^*, \tau_*, \Delta x)$, where $\tau_* = \min \{\tau^*, (Q(k+1))^{-1}\}$. It follows by Lemma 1 that $\|Tv\| \leqq \epsilon(\Delta x)$ for $\Sigma \in \mathfrak{L}(\lambda^*, \tau_*, \Delta x)$, as was to be proved.

3. **Uniqueness and existence theorems for parabolic equations.** With the differential equation (2.1) we now associate the difference equation

$$(3.1) \quad u_t(x, t) = a_0(x, t, u)u_{xx}(x - \Delta x, t) + a_1(x, t, u)u_x(x, t) + a_2(x, t, u)$$

which is a special case of Equation (1.10). Equation (3.1) satisfies the compatibility conditions (2.8) and is locally stable, by Lemma 3, provided Equations (1.11) are satisfied. Theorem I leads then to a local uniqueness theorem for the parabolic equation (2.1), by a simple argument which is due to Fritz John[18]. Let $u(x, t)$ be a solution of Equation (2.1) for which the quantities (2.9) are uniformly continuous and bounded on $R(\tau_0)$, and let the coefficients a_i satisfy the required Lipschitz condition. There exist then, by Theorem I, positive numbers λ^* & τ_* such that $\|u(\Sigma) - u\|$ tends to zero with Δx for Σ in $\mathfrak{L}(\lambda^*, \tau_*, \Delta x)$. Let (x, t) be a point of $R(\tau_*)$. One can choose a sequence $\{\Sigma_\nu\}$ of lattices with Σ_ν in $\mathfrak{L}(\lambda^*, \tau_*, \Delta x_\nu)$ and Δx_ν tending to zero, such that (x, t) belongs to Σ_ν for every value of ν. Since $u(x, t; \Sigma_\nu)$ is uniquely determined by the initial values $u(x, 0)$ and

$$u(x, t) = \lim_{\nu \to \infty} u(x, t; \Sigma_\nu),$$

one obtains the following result:

THEOREM II. *Let the coefficients a_i of Equation* (2.1) *be bounded and uniformly Lipschitz continuous in u, and let a_0 be positive and bounded away from zero, on every subset $\mathfrak{R}(\beta)$ of $\mathfrak{R}$. Let $u(x, 0)$ be defined for $-\infty < x < \infty$. There exists a positive number $\tau_* \leqq \tau_0$ such that the initial values $u(x, 0)$ uniquely determine a smooth solution of Equation* (2.1) *on $R(\tau_*)$. This means, more precisely, that there can exist at most one solution $u(x, t)$, defined on $R(\tau_*)$ and assuming the given initial values, for which the quantities* (2.9) *are uniformly continuous and bounded on $R(\tau_*)$.*

We proceed now to establish a local existence theorem for quasi-linear parabolic equations, using the fact that the differential equation can be approximated by a difference equation which is locally stable. This has been carried out by Fritz John for the linear case[19], and our argument will deviate from his only in the method whereby the difference solutions $u(\Sigma)$ and their divided differences are estimated. This is done in the quasi-linear case by application of Lemma 1.

---

(18) Fritz John [5, p. 171].

(19) Fritz John [5, Theorem 4.1, p. 175].

Let $C^i(\tau)$ denote the class of functions $g(x, t)$, for which

$$\frac{\partial^l g}{\partial x^l}, \qquad l = 0, 1, \cdots, i;$$

are uniformly continuous and bounded on $R(\tau)$. Let $\mathfrak{C}^i$ denote the class of functions $g(x, t, u)$, for which

$$\frac{\partial^{l+m} g}{\partial x^l \partial u^m}, \qquad l + m = 0, 1, \cdots, i;$$

are uniformly continuous and bounded on every subset $\mathfrak{R}(\beta)$ of $\mathfrak{R}$.

THEOREM III. *Let the coefficients a_i of Equation* (2.1) *belong to $\mathfrak{C}^6$, let a_0 be positive and bounded away from zero on every subset $\mathfrak{R}(\beta)$ of $\mathfrak{R}$, and let the quantities*

$$\frac{\partial a_0}{\partial t}, \quad \frac{\partial a_1}{\partial t}, \quad \frac{\partial a_2}{\partial t}$$

belong to $\mathfrak{C}^0$. Let $u(x, 0)$ belong to $C^6(0)$. There exists a positive number $\tau' \leqq \tau_0$ and a solution $u(x, t)$ of Equation (2.1), *which belongs to $C^4(\tau')$ and assumes the initial values $u(x, 0)$.*

Before proving this result, it will be convenient to introduce a new lemma.

LEMMA 4. *Let $g(x, t, u)$ belong to $\mathfrak{C}^p$, and let λ'', $\tau'' \leqq \tau_0$, $\Delta x''$, $\beta_0, \beta_1, \cdots, \beta_p$ be positive numbers. Let Σ be an arbitrary lattice in $\mathfrak{L}(\lambda'', \tau'', \Delta x'')$, let (x, t) be a point of Σ and $w(y)$ a function defined for $y = x, x+\Delta x, \cdots, x+p\Delta x$; such that*

$$\left| w_y^{(j)}(x) \right| \leqq \beta_j, \qquad j = 0, 1, \cdots, p.$$

Let $h(y)$ denote the function $g(y, t, w(y))$. There exists a positive number M, independent of Σ, (x, t) & $w(y)$, such that $\left| h_y^{(p)}(x) \right| \leqq M$.

This follows by an elementary argument which we relegate to the appendix.

With the differential equation (2.1) we associate again the difference equation (3.1). The latter may be solved on every lattice Σ with $\tau \leqq \tau_0$, subject to initial values $u(x, 0)$, and determines thus a function $u(x, t; \Sigma)$. We shall establish first the existence of positive numbers λ', τ', $\Delta x'$, $\beta_0, \beta_1, \cdots, \beta_6$ such that

$$(3.2) \qquad \left\| u_x^{(j)}(\Sigma) \right\| \leqq \beta_j, \qquad j = 0, 1, \cdots, 6$$

for $\Sigma \in \mathfrak{L}(\lambda', \tau', \Delta x')$. We then obtain bounds for certain additional differences of $u(\Sigma)$, and apply this information finally to establish the existence of $u(x, t)$ on $R(\tau')$.

Since $u(x, 0)$ belongs to $C^0(0)$, there exists a positive number α_0 such that $|u(x, 0)| \leqq \alpha_0$ identically. Equation (3.1) clearly satisfies the conditions of Lemma 3 and is therefore locally stable. It follows by Lemma 2 that there exist positive numbers β_0, λ_0^*, $\tau_0^* \leqq \tau_0$ & Δx_0^*, such that $\|u(\Sigma)\| \leqq \beta_0$ for $\Sigma \in \mathfrak{L}(\lambda_0^*, \tau_0^*, \Delta x_0^*)$.

We shall show now that $u_x^{(j)}(\Sigma)$ satisfies a difference equation

$$
\begin{aligned}
w_t(x, t) = {} & a_0^{(j)}(x, t, w; \Sigma) w_{xx}(x - \Delta x, t) \\
& + a_1^{(j)}(x, t, w; \Sigma) w_x(x, t) + a_2^{(j)}(x, t, w; \Sigma).
\end{aligned} \tag{3.3}
$$

This is clearly the case for $u_x^{(0)}(\Sigma)$, which is determined by Equation (3.1). In general, let us suppose that $u_x^{(j)}(\Sigma)$ satisfies an Equation (3.3) for $j = q$. Then $u_x^{(q+1)}(\Sigma)$ will satisfy the difference equation

$$
\begin{aligned}
w_t(x, t) = {} & a_0^{(q)}(x, t, u_x^{(q)}; \Sigma) w_{xx}(x - \Delta x, t) \\
& + \left\{ a_1^{(q)}(x, t, u_x^{(q)}; \Sigma) + \frac{\Delta}{\Delta x} a_0^{(q)}(x, t, u_x^{(q)}; \Sigma) \right\} w_x(x, t) \\
& + \left\{ \frac{\Delta}{\Delta x} a_2^{(q)}(x, t, u_x^{(q)}; \Sigma) + u_x^{(q)}(x + \Delta x, t; \Sigma) \frac{\Delta}{\Delta x} a_1^{(q)}(x, t, u_x^{(q)}; \Sigma) \right\},
\end{aligned} \tag{3.4}
$$

where $\Delta/\Delta x$ denotes the "total" difference operator, defined by the functional equation

$$
\frac{\Delta}{\Delta x} f(x, t, w(x, t)) = \frac{1}{\Delta x} \{ f(x + \Delta x, t, w(x + \Delta x, t)) - f(x, t, w(x, t)) \}.
$$

We now replace $u_x^{(q)}(x + \Delta x, t; \Sigma)$ by $\{u_x^{(q)}(x, t; \Sigma) + \Delta x w\}$ in the expression for $(\Delta/\Delta x) a_i^{(q)}$ and regard $u_x^{(q)}$ as a given function of Σ & (x, t). The coefficients of Equation (3.4) thus become functions of Σ, (x, t) & w, which we denote by $a_i^{(q+1)}$. The function $u_x^{(q+1)}(\Sigma)$ then satisfies Equation (3.3) for $j = q+1$.

We suppose now that there exist positive numbers $\beta_0, \beta_1, \cdots, \beta_q, \lambda_q^*, \tau_q^* \leqq \tau_0$ & Δx_q^* such that

$$
\|u_x^{(j)}(\Sigma)\| \leqq \beta_j, \qquad j = 0, 1, \cdots, q;
$$

for $\Sigma \in \mathfrak{L}(\lambda_q^*, \tau_q^*, \Delta x_q^*)$. Let the coefficients a_i of Equation (3.1) belong to $\mathfrak{C}^{q+1}$, and let $u(x, 0)$ belong to $C^{q+1}(0)$. We would like to show that Equation (3.3) for $j = q+1$ is locally stable. This difference equation is defined for Σ in $\mathfrak{L}(\lambda_q^*, \tau_q^*, \Delta x_q^*)$ and the coefficients depend continuously on w. Since

$$
a_0^{(q+1)}(x, t, w; \Sigma) = a_0(x, t, u(x, t; \Sigma))
$$

and $|u(x, t; \Sigma)| \leqq \beta_0$ identically for $\Sigma \in \mathfrak{L}(\lambda_q^*, \tau_q^*, \Delta x_q^*)$, $a_0^{(q+1)}$ is certainly positive and bounded away from zero. Let a positive number β be given. We show that there exists a positive number γ such that

$$\max_i \{ | \overset{(q+1)}{a_i}(x, t, w; \Sigma) | \} \leqq \gamma$$

identically for $|w| \leqq \beta$. Let Σ be an arbitrary lattice in $\mathfrak{L}(\lambda_q^*, \tau_q^*, \Delta x_q^*)$, let (x, t) be a point of Σ and w a real number such that $|w| \leqq \beta$. We define a function $w(y)$ for $y=x, x+\Delta x, \cdots, x+(q+1)\Delta x$; such that

$$\overset{(j)}{w_y}(x) = \overset{(j)}{u_x}(x, t; \Sigma), \qquad j = 0, 1, \cdots, q; \qquad \overset{(q+1)}{w_y}(x) = w.$$

Then $a_i^{(q+1)}(x, t, w; \Sigma)$ is a linear function of the quantities

$$\left(\frac{\Delta}{\Delta x}\right)^p a_j(x, t, w(x)), \qquad j = 0, 1, 2;\ p = 0, 1, \cdots, q+1; \tag{3.5}$$

with coefficients whose bounds do not depend on Σ, (x, t) & w. It remains then to establish similar bounds for the quantities (3.5). But this is an immediate consequence of Lemma 4, since the coefficients a_i belong to $\mathfrak{C}^p$ for $p \leqq q+1$. The local stability of Equation (3.3) for $j=q+1$ may now be inferred by Lemma 3.

Since $u(x, 0)$ belongs to $C^{q+1}(0)$, there exists a positive number α_{q+1} such that $|u_x^{(q+1)}(x, 0)| \leqq \alpha_{q+1}$ identically. The function $u_x^{(q+1)}(\Sigma)$ satisfies a locally stable difference equation and assumes initial values $u_x^{(q+1)}(x, 0)$. It follows by Lemma 2 that there exist positive numbers β_{q+1}, $\lambda_{q+1}^* \leqq \lambda_q^*$, $\tau_{q+1}^* \leqq \tau_q^*$ & $\Delta x_{q+1}^* \leqq \Delta x_q^*$, such that $\|u_x^{(q+1)}(\Sigma)\| \leqq \beta_{q+1}$ for $\Sigma \in \mathfrak{L}(\lambda_{q+1}^*, \tau_{q+1}^*, \Delta x_{q+1}^*)$.

This consideration shows that, under the assumptions of Theorem III, there exist positive numbers λ', $\tau' \leqq \tau_0$, $\Delta x'$, $\beta_0, \beta_1, \cdots, \beta_6$ such that Equation (3.2) holds for $\Sigma \in \mathfrak{L}(\lambda', \tau', \Delta x')$. We agree henceforth to consider only lattices Σ belonging to this class. Our next task is to obtain bounds for the quantities

$$u_t(\Sigma),\ u_{tx}(\Sigma),\ \cdots,\ u_{txxxx}(\Sigma),\ u_{tt}(\Sigma).$$

The estimate for $u_t(\Sigma)$ is obtained directly from Equation (3.1), since $u(\Sigma)$, $u_x(\Sigma)$ & $u_{xx}(\Sigma)$ are bounded by β_0, β_1 & β_2, respectively, and the coefficients a_i belong to $\mathfrak{C}^0$. Operating on both sides of Equation (3.1) by $\Delta/\Delta x$, one obtains an equation for $u_{tx}(\Sigma)$. Since the quantities $u(\Sigma)$, $u_x(\Sigma)$, $u_{xx}(\Sigma)$ & $u_{xxx}(\Sigma)$ are bounded by β_0, β_1, β_2 & β_3, respectively, and the coefficients a_i belong to $\mathfrak{C}^1$, one obtains an estimate for $u_{tx}(\Sigma)$. In this way one can estimate $u_{txx}(\Sigma)$, $u_{txxx}(\Sigma)$ & $u_{txxxx}(\Sigma)$. Finally one operates on both sides of Equation (3.1) by the "total" difference operator $\Delta/\Delta t$, which yields an equation for $u_{tt}(\Sigma)$. Since estimates for $u(\Sigma)$, $u_x(\Sigma)$, $u_{xx}(\Sigma)$, $u_t(\Sigma)$, $u_{tx}(\Sigma)$ & $u_{txx}(\Sigma)$ are known, and the functions $\partial a_i/\partial t$ belong to $\mathfrak{C}^0$, one obtains an estimate for $u_{tt}(\Sigma)$.

From this point the proof proceeds exactly as in the linear case, and the argument becomes identical with that of Fritz John [5, pp. 174–175]. One can select a sequence $\{\Sigma_n\}$ of lattices in $\mathfrak{L}(\lambda', \tau', \Delta x')$ such that Σ_n is a subset of Σ_m for $n<m$, and the union Σ_∞ of all these sets is everywhere dense in $R(\tau')$. Since the functions $u(x, t; \Sigma_n)$ are uniformly bounded and Σ_∞ is denumerable, there exists a set S of positive integers n such that

$$\lim_{n\in S; n\to\infty} u(x, t; \Sigma_n) = u(x, t) \tag{3.6}$$

exists on Σ_∞. It follows from the uniform boundedness of $u_x(\Sigma_n)$ that

$$u(x', t; \Sigma_n) - u(x, t; \Sigma_n) = \Delta x_n \sum_{\nu=1}^{(x'-x)/\Delta x_n} u_x(x + (\nu - 1)\Delta x_n, t; \Sigma_n) = O(x' - x),$$

where (x, t) and (x', t) are points of Σ_∞ and n is large enough to insure that $u(x, t; \Sigma_n)$ and $u(x', t; \Sigma_n)$ are both defined. This implies by Equation (3.6) that

$$u(x', t) - u(x, t) = O(x' - x) \text{ on } \Sigma_\infty.$$

By a similar argument, utilizing the uniform boundedness of $u_t(\Sigma_n)$, one obtains

$$u(x, t') - u(x, t) = O(t' - t) \text{ on } \Sigma_\infty.$$

Since $u(x, t)$ is thus uniformly continuous and bounded on Σ_∞, it can be continued into the whole set $R(\tau')$ such that the resulting function (which we again denote by $u(x, t)$) belongs to $C^0(\tau')$.

From the uniform boundedness of $u_{xx}(\Sigma_n)$ it follows that

$$\frac{u(x', t; \Sigma_n) - u(x, t; \Sigma_n)}{x' - x} = u_x(x, t; \Sigma_n) + O(x' - x), \tag{3.7}$$

where (x, t) and (x', t) belong to Σ_∞ and n is sufficiently large. Since the left side of Equation (3.7) tends to a limit as n tends to infinity through values in S, the right side does likewise. Regarding the right side as a function $F(x', n; x, t)$, one can write

$$\lim_{n\in S; n\to\infty} F(x', n; x, t) = G(x'; x, t).$$

Now let $\{x_\nu\}$ be a sequence tending to x, such that (x_ν, t) belongs to Σ_∞ for all ν; and let a positive number ϵ be given. There exists an integer M such that $|O(x_\nu - x)| < \epsilon/3$ for $\nu > M$ and n sufficiently large. For $\nu, \mu > M$ one has

$$|G(x_\nu; x, t) - G(x_\mu; x, t)| = \lim_{n\in S; n\to\infty} |F(x_\nu, n; x, t) - F(x_\mu, n; x, t)| \leqq 2\epsilon/3 < \epsilon,$$

which shows that $\{G(x_\nu; x, t)\}$ is a Cauchy sequence. Let $v(x, t)$ denote its limit. One easily verifies that

$$\lim_{n\in S; n\to\infty} u_x(x, t; \Sigma_n) = v(x, t).$$

From the uniform boundedness of $u_{xx}(\Sigma_n)$ and $u_{tx}(\Sigma_n)$ one concludes again that $v(x, t)$ is uniformly continuous on Σ_∞, and that it may consequently be extended to a function of class $C^0(\tau')$. It follows by Equation (3.7) that

$$\frac{u(x', t) - u(x, t)}{x' - x} = v(x, t) + O(x' - x)$$

for (x, t) and (x', t) in Σ_∞. By continuity of $u(x, t)$ and $v(x, t)$ this equation holds for all (x, t) and (x', t) in $R(\tau')$, and one concludes that

$$\frac{\partial u(x, t)}{\partial x} = v(x, t) \text{ on } R(\tau').$$

Proceeding in this manner, one can prove that the quantities

$$u_{xx}(\Sigma_n),\ u_{xxx}(\Sigma_n),\ u_{xxxx}(\Sigma_n),\ u_t(\Sigma_n)$$

converge respectively to the corresponding derivatives of $u(x, t)$, which belong to $C^0(\tau')$. Hence $u(x, t)$ belongs to $C^4(\tau')$ and $\partial u/\partial t$ belongs to $C^0(\tau')$. From the difference equation (3.1) one concludes by passage to the limit that $u(x, t)$ satisfies the differential equation (2.1) on Σ_∞, and therefore on $R(\tau')$. This completes the proof of Theorem III.

The same argument leads to a more general result, viz.

THEOREM IV. *Let the coefficients a_i of Equation* (2.1) *belong to $\mathfrak{C}^{q+2}$, where $q \geqq 2$; let a_0 be positive and bounded away from zero on every subset $\mathfrak{R}(\beta)$ of $\mathfrak{R}$, and let the quantities $\partial a_i/\partial t$ belong to $\mathfrak{C}^0$. Let $u(x, 0)$ belong to $C^{q+2}(0)$. There exists a positive number $\tau' \leqq \tau_0$ and a solution $u(x, t)$ of Equation* (2.1), *which belongs to $C^q(\tau')$ and assumes the initial values $u(x, 0)$.*

Appendix. Proof of Lemma 4. Let $u(y)$ denote the pth order polynomial which coincides with $w(y)$ at the points $x, x+\Delta x, \cdots, x+p\Delta x$. Let $f(y) = g(y, t, u(y))$. Since g belongs to $\mathfrak{C}^p$, $f(y)$ has a continuous pth derivative. Hence there exists a number z in $(x, x+p\Delta x)$ such that

$$h_y^{(p)}(x) = \frac{d^p f(z)}{dy^p}\cdot$$

But the expression on the right is a polynomial in the quantities

$$\frac{\partial^\nu g}{\partial x^\alpha \partial u^\beta},\quad \frac{d^\nu u(z)}{dy^\nu}, \qquad \alpha + \beta = \nu;\ \nu = 0, 1, \cdots, p;$$

the partial derivatives of g being evaluated at the point $(z, t, u(z))$. Since g belongs to $\mathfrak{C}^p$, the desired estimate for $h_y^{(p)}(x)$ will follow when it has been established that the quantities

$$u(z), \frac{du(z)}{dy}, \cdots, \frac{d^p u(z)}{dy^p} \tag{1}$$

have bounds which depend on $\beta_0, \beta_1, \cdots, \beta_p$ & $\Delta x''$ alone.

Since $u(y)$ is a polynomial of degree p, one obtains

$$\frac{d^p u(y)}{dy^p} = w_y^{(p)}(x) \qquad \text{for all } y, \tag{2}$$

which implies

$$\left| \frac{d^p u(z)}{dy^p} \right| \leqq \beta_p.$$

Since $u(y)$ has a continuous derivative of order $(p-1)$, there exists a number z' in $(x, x+(p-1)\Delta x)$ such that

$$\frac{d^{p-1}u(z')}{dy^{p-1}} = w_y^{(p-1)}(x).$$

It follows now by (2) that

$$\left| \frac{d^{p-1}u(y)}{dy^{p-1}} \right| \leqq \beta_{p-1} + p\Delta x''\beta_p \qquad \text{for all } y \text{ in } (x, x + p\Delta x).$$

Continuing in this manner one obtains the desired bounds for the quantities (1).

Bibliography

1. R. Courant, K. O. Friedrichs, and H. Lewy, *Über die partiellen Differenzengleichungen der mathematischen Physik*, Math. Ann. vol. 100 (1928) pp. 243–255.

2. J. Douglas, Jr., *On the relation between stability and convergence in the numerical solution of linear hyperbolic and parabolic differential equations*, J. Soc. Indust. Appl. Math. vol. 4 (1956) pp. 20–37.

3. ———, *On the numerical integration of quasi-linear parabolic differential equations*, Pacific J. Math. vol. 6 (1956) pp. 35–42.

4. K. O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. vol. 7 (1954) pp. 345–392.

5. F. John, *On integration of parabolic equations by difference methods*, Comm. Pure Appl. Math. vol. 5 (1952) pp. 155–211.

6. P. Lax, *Difference approximation to solutions of linear differential equations—an operator theoretical approach*, Lecture Series Symposium on Partial Differential Equations, University of Kansas, 1957.

7. P. Lax, and R. D. Richtmyer, *Survey of the stability of linear finite difference equations*, Comm. Pure Appl. Math. vol. 9 (1956) pp. 267–293.

8. G. G. O'Brien, M. A. Hyman and S. Kaplan, *A study of the numerical solution of partial differential equations*, J. Math. Phys. vol. 29 (1951) pp. 223–251.

9. M. E. Rose, *On the integration of non-linear parabolic equations by implicit difference methods*, Quart. Appl. Math. vol. 14 (1956) pp. 237–248.

Columbia University,
New York, N. Y.