VANISHING ALGEBRAS(1)

BY

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1. **Introduction.** Recent papers by Arens [1], Arens and Singer [2], and Hoffman and Singer [6] deal, in part, with a certain subalgebra of a group algebra consisting of those functions which vanish outside of a given semigroup.

Throughout the literature, we find theorems concerning particular subalgebras of this type. For example, it is known that if G is the integers [reals] and L^+ is the algebra of all (complex-valued) summable functions which vanish on the negative elements of G, then the uniform closure of the set of (Fourier) transforms of members of L^+ form the set of all continuous functions on the circle [reals] with absolutely convergent Taylor series expansions on the closed disc [upper-half plane]. (See Arens and Singer [2] and Naimark [9]).

Again, Wermer [14] has proved the following: Let G be a discrete Abelian ordered group, let $L^1(G)$ be the group algebra of G, and let L^+ be the subalgebra of $L^1(G)$ consisting of those functions which are identically zero on the negative elements of G. Then L^+ is a maximal subalgebra of $L^1(G)$ if and only if the ordering in G is Archimedean.

These papers and results suggest that such subalgebras of group algebras may be treated in a unified manner. This paper is an initial step in such a treatment.

Throughout, we let G be a locally compact group and we denote by $L^1(G)$ the group algebra of G.

In the section entitled *Vanishing algebras*, we define a vanishing set L_S as the set of all summable functions in $L^1(G)$ which vanish (a.e.) outside of the given measurable subset S of G. In case L_S is an algebra, we call it a vanishing algebra. We then determine some of the properties of S and L_S which induce corresponding properties in the other. It is shown that if S is a semigroup a.e., then L_S is a vanishing algebra (Theorem 3.10); and if L_S is a vanishing algebra, then there is a measurable set $T \subset G$ such that the closure of T is a semigroup and $L_S = L_T$ (Theorem 3.12). The set L_S is a self adjoint algebra if and only if S is a subgroup a.e. of G (Theorem 3.15). Now

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let L_S be maximal with respect to being a proper vanishing algebra in $L^1(G)$. It is proved (Theorem 3.19) that either S is dense in G or S can be replaced by a proper closed semigroup T, i.e., $L_S = L_T$.

Part 4 concerns itself with the set of (Fourier) transforms of elements belonging to a vanishing algebra, where G now is a locally compact *Abelian* group. An interesting result (Theorem 4.3) states that L_S is a separating set if and only if S "topologically generates" G. Separating sets have recently been studied by Rudin [10]. It follows from the above mentioned theorem that the paper on *Generalized analytic functions* by Arens and Singer [2] deals exclusively with separating vanishing algebras. The remainder of §4 is devoted to showing that while no proper vanishing set is contained in a proper closed ideal of $L^1(G)$, the closure (in $L^1(\hat{G})$) of its set of transforms is an ideal of $L^1(\hat{G})$, where $\hat{G}$ is the dual (or character) group of G. (See Theorem 4.5 and Theorem 4.6.2.)

§5 is another application of the methods developed here. We prove (Theorem 5.1) that if S is a subsemigroup (of positive measure) of a compact Abelian group, then S is an open and closed subgroup of G. We also exhibit (Corollary 5.6) a class of groups which admit only two vanishing algebras; $\{0\}$ and $L^1(G)$.

Several questions remain unanswered, among which are: (1) Under what conditions is a maximal vanishing algebra a maximal subalgebra of $L^1(G)$? (See Theorem 3.23 and the remark at the end of this paper.) (2) If L_S is a vanishing algebra, is there a subsemigroup T of G for which $L_S = L_T$?

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2. **Preliminaries.** Throughout this paper, the symbol G will stand for a locally compact topological group. We will write our groups multiplicatively with 1 denoting the identity. As usual, if S and T are subsets of G, then $ST = \{st: s \in S \text{ and } t \in T\}$ and $S^{-1} = \{s^{-1}: s \in S\}$.

In any locally compact group, there exists an (essentially) unique left-invariant Haar measure μ and we will designate the modular function by Δ; i.e., $\mu(Sx) = \Delta(x)\mu(S)$ for all measurable $S \subset G$ and all $x \in G$. If G is Abelian, then $\Delta \equiv 1$ and μ is both left and right invariant.

Let $L^1 = L^1(G)$ be the set of all complex valued, measurable (with respect to μ) functions f on G with the property that $\int |f| d\mu < \infty$. Then L^1 forms a Banach algebra under the norm $\|f\| = \int |f| d\mu$; the multiplication being convolution defined by $f^*g(x) = \int f(xy)g(y^{-1})d\mu(y)$. This algebra is called the group algebra of G (see Segal [12]). The group algebra also has a continuous involution $f \to f^*$ defined by $f^*(x) = [f(x^{-1})]^{-}\Delta(x^{-1})$.

2.1. Definition. Let $x \in G$. We say that x is a *density point* of a measurable subset S of G, if every measurable open neighborhood of x meets S in a set of positive measure. We denote the set of all density points of S by the symbol $D(S)$. If $S \subset D(S)$, then S is said to have the *density property.*

These density points are not to be confused with points of metric density 1. It is true, however, that the points of metric density 1 are also density points.

A careful comparison between the "density operator" D, Kuratowski's closure operator and the category operator should prove very interesting, but does not suit our purpose here. We content ourselves with exhibiting several properties of D without calling attention to the corresponding properties of the above mentioned operators.

2.2. LEMMA. *For any measurable subsets S and T of G we have*: (1) *$D(S)$ is closed*, (2) *$S^0 \subset D(S) \subset \overline{S}$*, (3) *if $\mu(S) > 0$, then $S \cap D(S) \neq \emptyset$*, (4) *if $\mu(S) = 0$, then $D(S) = \emptyset$*, (5) *if $S \subset T$, then $D(S) \subset D(T)$, and* (6) *if $S \subset D(S)$, then every open set which meets S meets it in a set of positive measure.*

Proof. (1) If $x \notin D(S)$, then there is an open measurable set U containing x with $\mu(U \cap S) = 0$. Then it must be that $U \cap D(S) = \emptyset$.

(2) Let $x \in S^0$. Then every open set U about x contains an open set V such that $x \in V \subset S$ and $\mu(V) > 0$, hence $x \in D(S)$. Clearly every point of $D(S)$ is in the closure of S.

(3) Since Haar measure is regular, we may approximate the measure of S by a compact subset $C \subset S$. If no point of S were a density point, we could cover C with a finite number of sets of measure zero and it follows that $\mu(C) = 0$; $\mu(S) = 0$.

Statements (4), (5), and (6) are proved in a similar manner.

The following are immediate consequences of 2.2.

2.3. COROLLARY. *Let S be a measurable subset of G. If S is closed, then $D(S) \subset S$. If S is open, then $S \subset D(S) = \overline{S}$.*

2.4. COROLLARY. *Let S and T be measurable subsets of G. Then $D(S) - D(T) \subset D(S-T)$. Therefore, if the symmetric difference, $(S-T) \cup (T-S)$, of S and T has measure zero, then $D(S) = D(T)$.*

The fact that $D(S) \subset D(T)$ does not imply $\mu(S-T) = 0$ is seen by taking S to be the unit interval and T to be an open dense subset of S with measure less than 1. Then $\mu(S-T) > 0$ and by 2.3, $D(S) = D(T)$.

2.5. THEOREM. *If S is measurable, then $S \cap D(S)$ is measurable and $\mu(S - D(S)) = 0$. Hence, if S is measurable and closed, then $D(S)$ is measurable.*

Proof. Let C_n, $n = 1, 2, \cdots$, be a sequence of compact sets which cover S (see [4, p. 26]). Since $D(S)$ is closed, $D(S) \cap C_n$ is measurable for each n. Therefore $S \cap D(S) = \bigcup_{n=1}^{\infty} [D(S) \cap C_n \cap S]$ is measurable. Now $S - D(S)$

$=S-(D(S)\cap S)$ is measurable and if it had positive measure, then $D(S-D(S))\cap(S-D(S))\neq\emptyset$, by 2.2 (3). But $D(S-D(S))\subset D(S)$, hence $\mu(S-D(S))=0$. The last statement now follows from Lemma 2.2 (2).

By the above theorem, all of the measure of S is contained in $D(S)$ and so the following corollaries are clear.

2.6. COROLLARY. *If S and T are measurable, then* (1) $\mu(S)=\mu(S\cap D(S))$, (2) $\mu(S\cap T)=\mu(S\cap T\cap D(S))=\mu(S\cap T\cap D(T))$, *and* (3) $S\cap D(S)$ *has the density property.*

2.7. COROLLARY. *For any measurable open set U and any measurable set S, the following are equivalent*: (1) $\mu(U\cap S)=0$, (2) $U\cap D(S)=\emptyset$, (3) $U\cap S\cap D(S)=\emptyset$, (4) $\mu(U\cap S\cap D(S))=0$. *Thus* $D(D(S))=D(S)$.

2.8. COROLLARY. *If S is measurable, then* $(S\cap D(S))^-=(D(S))^-=D(S)=D(S\cap D(S))$.

We now exhibit some algebraic properties of D.

2.9. LEMMA. *Let S and T be measurable subsets of G. Then* (1) $(D(S))^{-1}=D(S^{-1})$, *and* (2) $[(\overline{S})(D(T))]\cup[(\overline{T})(D(S))]\subset D(ST)$.

Proof. The first statement follows from the fact that $\mu(U^{-1}\cap S)>0$ if and only if $\mu(U\cap S^{-1})>0$ for any two subsets U and S. Now let $x=ab$, where $a\in\overline{S}$ and $b\in D(T)$, and let U be any open neighborhood of x. Find V and W open sets about a, respectively, b such that $VW\subset U$. Then, for $s\in V\cap S$, we have $\mu(s(W\cap T))>0$, $s(W\cap T)\subset(U\cap ST)$, and thus $x\in D(ST)$. It follows from this argument that assertion (2) is true.

Now $D(S)$ is always contained in $\overline{S}$ and hence, by 2.9 (2), $(D(S))(D(T))\subset D(ST)$. If we now substitute S for T, we have

2.10. THEOREM. *If S is a semigroup, then $D(S)$ is also a semigroup.*

Throughout the remainder of this note, we use the letters P, Q, S, and T to denote measurable subsets of G. We also use the convenient notation "$S\subset T$ a.e." to mean that $\mu(S-T)=0$ and "$S=T$ a.e." when the symmetric difference of S and T has measure zero; i.e., when $\mu((S-T)\cup(T-S))=0$. By "S is a semigroup a.e." we mean there exists a semigroup $T\subset G$ such that $S=T$ a.e. The symbol χ_S will denote the characteristic function of the set S.

3. **Vanishing algebras.** We define here for each measurable subset S of G a subset L_S of L^1 and determine which properties of S and L_S induce corresponding properties in the other.

3.1. DEFINITION. Let $f\in L^1$. Then S is said to be a *support* for f if $f=0$ a.e. outside of S. An equivalent statement would be that $\int_{G-S}|f|\,d\mu=0$.

3.2. DEFINITION. Let us designate by L_S the set of all members of L^1 which have S for a support. If L_S forms an algebra (a subalgebra of L^1), we call it a *vanishing algebra* and write L_S is a v.a.

From the definition of the set L_S it is evident that the following result is true.

3.3. LEMMA. (1) *L_S is always a closed linear subspace of L^1, and* (2) *$S \subset T$ a.e. if and only if $L_S \subset L_T$; $S = T$ a.e. if and only if $L_S = L_T$.*

The measurable set $S \subset G$ is itself a measure space and as such it is clear that L_S can be considered as identical with $L^1(S)$.

3.4. THEOREM. *The set L_S is a vanishing algebra if and only if $\chi_P * \chi_Q \in L_S$ for every pair of sets P, Q contained in S having finite measure.*

Proof. The condition is surely necessary. From the above remark we see that the subspace spanned by such characteristic functions is dense in L_S. Then the sufficiency follows from the fact convolution is jointly continuous and L_S is closed.

We now proceed to examine $\chi_P * \chi_Q$ and exhibit some conditions on S which are necessary in order that $\chi_P * \chi_Q \in L_S$.

3.5. DEFINITION. For any two sets P and Q with finite measure and for any $y \in G$, let $\omega_{P,Q}(y) = \mu(y^{-1}P \cap Q^{-1})$ and let $W_{P,Q} = \{y \in G: \omega_{P,Q}(y) > 0\}$.

We remark that $\omega_{P,Q}$ is a measurable function and that $W_{P,Q}$ is a measurable set (see [4, p. 261 and p. 79]).

3.6. LEMMA. *If P and Q have finite measure, then $W_{P,Q} \subset PQ$. If Q has the density property* (*see* 2.1) *and P is open, then $W_{P,Q} = PQ$.*

Proof. From the definitions above it follows that if $y \in W_{P,Q}$, then $y^{-1}P \cap Q^{-1} \neq \emptyset$ and hence $y \in PQ$. Now if Q has the density property, then surely so does Q^{-1}, and therefore every open set which meets Q^{-1} meets it in a set of positive measure. Consequently, $W_{P,Q} = \{y: y^{-1}P \cap Q^{-1} \neq \emptyset\} = PQ$.

3.7. LEMMA. *If P and Q are subsets of S with finite measure, then* (1) *$\chi_P * \chi_Q(y) = \omega_{P,Q}(y)$ for almost all y, and* (2) *$\mu(P)\mu(Q) = \int_{W_{P,Q}} \chi_P * \chi_Q d\mu$.*

Proof. By definition, for almost all y, we have

$$\chi_P * \chi_Q(y) = \int \chi_P(yx)\chi_Q(x^{-1})d\mu(x) = \int_{Q^{-1}} \chi_P(yx)d\mu(x)$$

$$= \int_{Q^{-1}} \chi_{y^{-1}P}(x)d\mu(x) = \mu(y^{-1}P \cap Q^{-1})$$

and (1) is proved. By the Average Theorem [4, p. 261] and part (1), we have $\mu(P)\mu(Q) = \int \chi_P * \chi_Q d\mu = \int \omega_{P,Q} d\mu = \int_{W_{P,Q}} \omega_{P,Q} d\mu$ and the lemma is proved.

The next two lemmas are essential in obtaining the main results of this section.

3.8. LEMMA. *If P and Q are subsets of S having finite measure, then the* *following are equivalent*: (1) $\chi_P * \chi_Q \in L_S$, (2) $(W_{P,Q}-S)$ *has measure zero, and* (3) $\mu(Q)\mu(P)=\int_{W_{P,Q}\cap S}\chi_P * \chi_Q d\mu$.

Proof. If $\chi_P * \chi_Q \in L_S$, then it vanishes almost everywhere outside of S. But it is positive everywhere on $W_{P,Q}$ and therefore $\mu(W_{P,Q}-S)=0$. Clearly (2)$\Rightarrow$(3) in light of 3.7 (2). It remains to show (3) is sufficient for (1). Again, by the Average Theorem, we know $\int \chi_P * \chi_Q d\mu = \mu(P)\mu(Q)$. Then, by hypothesis, $\int \chi_P * \chi_Q d\mu = \int_{W_{P,Q}\cap S}\chi_P * \chi_Q d\mu$ and since $\chi_P * \chi_Q \geqq 0$ everywhere, it follows that its support is in S.

3.9. LEMMA. *If* $\mu(PQ-S)=0$ *for every pair of subsets* P, Q *contained in* S, *then* L_S *is a vanishing algebra. If* L_S *is a vanishing algebra, then* $\mu(PQ\cap S)>0$ *if* P *and* Q *are subsets of* S *with finite positive measure; and, if in addition,* Q *has the density property and* P *is open,* $\mu(PQ-S)=0$.

Proof. By Lemma 3.6, $W_{P,Q}\subset PQ$ and hence, by hypothesis, $\mu(W_{P,Q}-S)=0$. Therefore it follows from 3.8 and Theorem 3.4 that L_S is a v.a. and the first assertion is proved. If L_S is a v.a., then $\chi_P * \chi_Q \in L_S$ and $\mu(Q)\mu(P)=\int_{W_{P,Q}\cap S}\chi_P * \chi_Q d\mu$, by 3.8. Since $\chi_P * \chi_Q \geqq 0$ and $W_{P,Q}\cap S\subset(PQ\cap S)$ it follows that $0<\mu(P)\mu(Q)\leqq\int_{(PQ\cap S)}\chi_P * \chi_Q d\mu$; $\mu(PQ\cap S)>0$. If $Q\subset D(Q)$ and P is open, then $W_{P,Q}=PQ$ and from 3.8 (2) it is true that $\mu(PQ-S)=0$.

3.10. THEOREM. *If* S *is a semigroup a.e., then* L_S *is a vanishing algebra.*

Proof. By hypothesis, there is semigroup T such that $S=T$ a.e.; therefore $L_S=L_T$. The first part of Lemma 3.9 clearly implies that L_T is a v.a.

3.11. THEOREM. *If* L_S *is a vanishing algebra, then* $D(S)$ *is a semigroup.*

Proof. Let U be any measurable open set containing xy, where $x, y\in D(S)$. It is our intent to show $\mu(U\cap S)>0$; thus proving $xy\in D(S)$. Choose open sets V and W about x and y, respectively, which have finite measure and which also have the property $VW\subset U$. Let $P=V\cap S$ and $Q=W\cap S$. Then $(PQ\cap S)\subset(U\cap S)$; and by 3.9, $\mu(PQ\cap S)>0$ holds and the theorem is proved.

We now summarize several of the preceding results in the next two theorems.

3.12. THEOREM. *Let* $T=D(S)\cap S$. *Then* $T=S$ *a.e.,* $T^0=S^0$, T *has the density property and* T *is closed in* S. *Thus, if* L_S *is a vanishing algebra, then* S *can be replaced by a set* T *(i.e.,* $L_S=L_T$*) where* T *has all of the properties above and, in addition,* $\overline{T}$ *is a semigroup.*

3.13. THEOREM. *If* $D(S)\subset S$, *hence if* S *is closed, then* L_S *is a vanishing algebra if and only if* S *is a semigroup a.e. Explicitly,* $S=D(S)\cup(S-D(S))$, *where* $D(S)$ *is the semigroup and* $S-D(S)$ *has measure zero.*

An immediate consequence of Theorem 3.13 is

3.14. Corollary. *If G is discrete, then L_S is a vanishing algebra if and only if S is a semigroup.*

That these are the best possible results is unknown to the author. The obvious question "If L_S is a v.a., then is S a semigroup a.e.?" remains a mystery. The answer is affirmative in the special cases when G is discrete (3.14) or compact (shown in (5)). We conjecture that if G is not discrete, compact, or Abelian, then it is possible to exhibit a subset $S\subset G$ which is not a semigroup a.e. but has the property $(SS-S)$ is of measure zero. Then L_S is a vanishing algebra, by 3.9, thus supplying a negative answer to the question. As yet, we have not been able to produce such a set.

To establish a connection between the involution in L^1 and the group structure of G, we have

3.15. Theorem. *Let S be closed. Then L_S is a self adjoint vanishing algebra if and only if S is a group a.e.*

Proof. That the condition is sufficient is clear. For, L_S is a v.a. and if f has its support in S, then S^{-1} is a support for f^*, but $S=S^{-1}$ a.e. To prove necessity, we first establish that $(L_S)^*=L_{S^{-1}}$ by observing that $\Delta(x^{-1})$ is a positive number and hence $f^*(x)=0$ if and only if $f(x^{-1})=0$. Then, by hypothesis, $L_S=(L_S)^*=L_{S^{-1}}$; so $S=S^{-1}$ a.e. and by 2.4 and 2.9 (1) we have $D(S)=D(S^{-1})=(D(S))^{-1}$. Since L_S is a v.a., $D(S)$ is already a semigroup so $D(S)$ must be a group; S must be a group a.e.

3.16. Definition. A subset A of a topological space is said to be regularly closed if $A=[A^0]^-$. We define a subset $S\subset G$ to be *smooth* if (1) S is nonempty and proper, (2) S is measurable, and (3) S is regularly closed.

If S is smooth, it is clear that (1) S is closed, (2) $S^0\neq\emptyset$, (3) $\mu(S)>0$, (4) $G-S$ is a nonvoid open set, (5) xS is a smooth set for each $x\in G$, and (6) $S=D(S)$.

3.17. Lemma. *Let L_S be a vanishing algebra where $S^0\neq\emptyset$. If S is open [closed], then $\overline{S}$ $[S^0]$ is a semigroup. In either case $[S^0]^-$ is a smooth semigroup.*

Proof. If S is open, then $D(S)=\overline{S}$ (Corollary 2.3). By 3.11, $D(S)$, and hence $\overline{S}$, is a semigroup. If S is closed, then it follows from Lemma 2.2 that $(D(S))^0=S^0$ and we know that the former is a semigroup since it is the interior of a semigroup. The assertion $[S^0]^-$ is smooth is true for any set with a nonvoid interior.

The next lemma is needed for the proof of a later theorem (3.22) but it seems to fit in better here.

3.18. Lemma. *Let L_S be a vanishing algebra with S closed. Then $(D(S))(S^0)^-\subset[S^0]^-$. Thus, if $1\in[S^0]^-$, then $[S^0]^-=D(S)$.*

Proof. Now $S^0\subset D(S)\subset\overline{S}=S$ and since $D(S)$ is a semigroup, we have

$(D(S))(S^0)\subset S^0$; therefore $(D(S))(S^0)^-\subset[S^0]^-$. The second statement follows from the first and the fact that $[S^0]^-\subset D(S)$ always holds.

We now turn our attention to maximal vanishing algebras. A vanishing algebra L_S is called a *maximal vanishing algebra* if (1) it is proper, and (2) if L_T is a vanishing algebra and $L_S\subset L_T$, then $L_S=L_T$ or $L_T=L^1$.

Let L_S be a maximal v.a. Now $D(S)$ is a closed semigroup of G and $L_S\subset L_{D(S)}$. Therefore it is true that either S is dense in G or $L_S=L_{D(S)}$. We have just shown

3.19. THEOREM. *If L_S is a maximal vanishing algebra, then either S is dense in G or there is a proper closed semigroup $T\subset G$ such that $L_S=L_T$.*

If we let S be the right half line union the negative rationals we have that L_S is proper while S is dense. It should be noted that in this case there is a T such that $L_S=L_T$. It would be interesting to have an example in which S is dense and the set T of 3.19 does not exist.

Now if L_S is maximal where S is not dense, then we may, and do, assume that S is a proper closed semigroup. It may not be a maximal proper closed semigroup, but it will be maximal to within a set of measure zero as shown below.

3.20. THEOREM. *Let L_S be a maximal vanishing algebra and suppose S is not dense in G. If T is any proper closed semigroup and $S\subset T$ a.e., then $S=T$ a.e. Also, if S is a closed semigroup which enjoys this property, then L_S is properly contained in only those vanishing algebras L_T for which T is dense in G.*

Proof. If $S\subset T$ a.e., then $L_S\subset L_T$. Since T is proper and closed, it follows that $L_T\neq L^1$, $L_S=L_T$, and $S=T$ a.e. To prove the remaining statement, let L_S be contained in L_T, a v.a. It is true then that $L_S\subset L_{D(T)}$; $S\subset D(T)$ a.e.; and $D(T)$ is a closed semigroup. If $D(T)$ is proper, then, by hypothesis, $S=D(T)$ a.e. and $L_T\subset L_{D(T)}=L_S$. Thus, either T is dense or $L_S=L_T$.

It is interesting to note here that if G is compact and L_S is a maximal v.a., then L_S is self adjoint. For, by 3.19, we may replace S by a closed semigroup T. Consequently (see Iwasawa [7] or Wallace [13]), T is a closed subgroup of G and then, by 3.15, $L_S=L_T$ is self adjoint. (Note: In §5 we show the word "maximal" may be omitted if G is compact and Abelian.)

3.21. LEMMA. *If A is a proper open semigroup of G, then $\overline{A}\neq G$.*

Proof. If $1\in\overline{A}$, then by a result of Hille's [5] we have $(\overline{A})^0=A$. Thus $G\neq A=(\overline{A})^0$; $\overline{A}\neq G$.

3.22. THEOREM. *Let S be a smooth subset* (*see* 3.16) *of G with $S^0\neq S$. Then these are equivalent statements*: (1) *L_S is a vanishing algebra which is properly contained in only those vanishing algebras L_T for which T is dense*; (2) *S is a maximal proper smooth semigroup*; *and* (3) *S^0 is a maximal proper open semigroup.*

Proof. We first prove (1) implies (2). Let $S\subset T$ where T is a proper smooth semigroup. Then $L_S=L_T$ and $S=T$ a.e. It now follows that $S^0=T^0$ and hence $S=T$; the former being true because both S and T are closed; the latter is true since they are both smooth sets. That (2) is sufficient for (3) is seen as follows: Let $S^0\subset T$, an open proper semigroup. Then $\overline{T}$ is a smooth semigroup and is proper, by Lemma 3.21. Therefore $S=\overline{T}$, and we have $T\subset(\overline{T})^0=S^0$; thus $S^0=T$. It remains to show (3) implies (1). By hypothesis, S^0 is a maximal open semigroup which is not closed. Then, by a theorem due to F. B. Wright [15], we know that $1\in[S^0]^-=S$. Suppose, now, that $L_S\subset L_T$, a v.a., and T is not dense. Then $L_S\subset L_{D(T)}$ and we can assert $S^0\subset(D(T))^0$; $S^0=(D(T))^0$. It is then a consequence of Lemma 3.18 that $L_S=L_T$, for $1\in S=[(D(T))^0]^-$ and hence $S=[(D(T))^0]^-=D(D(T))=D(T)$.

We conclude this section with a result resembling that of Wermer's [14] spoken of in the introduction.

3.23. Theorem. *Let G be a discrete Abelian group and let $S\subset G$ be such that $S\cap S^{-1}=\{0\}$. Then the following are equivalent*: (1) *L_S is a maximal subalgebra of L^1*; (2) *L_S is a maximal vanishing algebra*; (3) *S is a maximal semigroup*; *and* (4) *G can be linearly ordered with S as the non-negative elements and the ordering will be Archimedean.*

Proof. That (1) implies (2) and, in the discrete case, (2) implies (3) is quite clear. The implication from (4) to (1) is part of Wermer's theorem. It remains only to show (4) is a necessary condition for (3). To this end, we observe that S is not a group (unless $G=\{0\}$, in which case everything is trivial) and is maximal with respect to being a semigroup which is not a group. It is proved in [3] that in this case $G=S\cup S^{-1}$. It is well known that under these conditions G can be given an Archimedean order with S as the non-negative members. (See, for example, F. B. Wright [16].)

4. **Transforms of vanishing algebras.** We endeavor here to examine some of the properties of the set of Fourier transforms of a vanishing algebra. In order to avail ourselves of the Pontrjagin duality theory we assume throughout this section that G is a locally compact *Abelian* group.

Recall that the Fourier transformation T is a continuous linear homomorphism from the group algebra to the algebra of continuous functions, vanishing at ∞, on the character group. It is defined by

$$Tf(q) = \int f(x)[q(x)]^- d\mu(x)$$

for each character q. In symbols, $T: L^1\to C(\hat{G})$, where $\hat{G}$ is the character group of G; $Tf=\hat{f}$; and $\hat{f}(q)=\int f\bar{q}$. The reader is referred to Loomis [8, Chapter 7] for further details.

We will denote the annihilator, in $L^\infty(G)$, of a subset A of L^1 by $A^\perp$; i.e.,

$$A^{\perp} = \left\{ p \in L^{\infty}(G): \int a(x)[p(x)]^{-} d\mu(x) = 0 \text{ for all } a \in A \right\} \cdot$$

The symbols S and L_S have the same connotation as before.

4.1. LEMMA. $(L_S)^{\perp} = \{p \in L^{\infty}(G): p = 0$ *a.e. on* $S\}$; *and hence is an ideal of* $L^{\infty}(G)$.

Proof. If the first assertion is true, the second is clear. If $p=0$ a.e. on S, then it is surely in $(L_S)^{\perp}$ and if S has measure zero, the reverse containment is obvious. Therefore, let $\mu(S)>0$ and suppose we have chosen T to be a subset of S with positive finite measure. Evidently, then, $\chi_T p \in L_S$ for any $p \in L^{\infty}(G)$. Now $\int (\chi_T p)\bar{p} = \int_T p\bar{p}$ and it follows if $p \in (L_S)^{\perp}$, then it must vanish a.e. on S and the proof is complete.

Let A be a subset of L^1 such that the transforms of members of A separate characters of G. Then we call A a *separating set* (defined by Rudin in [10]). For any subset S of G, we use the notation $[S]$ to denote the *smallest closed subgroup* of G containing S; i.e., S "topologically generates" $[S]$.

4.2. THEOREM. *Let q and r be two characters on G and let S have the density property* (*see* 2.1). *Then the following are equivalent*: (1) $q=r$ *on* $[S]$; (2) $\int f\bar{q} = \int f\bar{r}$ *for every* $f \in L_S$; *and* (3) $q=r$ *a.e. on* S.

Proof. To show (1) implies (2), we have that if $f \in L_S$, then $\int f\bar{q} - \int f\bar{r} = \int f(\bar{q}-\bar{r}) = \int_S f(\bar{q}-\bar{r}) = 0$, since $q-r=0$ on S. That (2) implies (3) follows from the chain of equalities above and Lemma 4.1 (note characters are also members of $L^{\infty}(G)$). We now argue that (1) follows from (3) to complete the proof. Since S has the density property, any subset T of S such that $\mu(S-T)=0$ is necessarily dense in S. Thus, the set $T=\{s \in S: q(s)=r(s)\}$ is topologically dense in S; so $S \subset \overline{T}$. But q and r are continuous homomorphisms; therefore the set $H=\{x \in G: q(x)=r(x)\}$ is a closed subgroup of G. Hence, $S \subset H$; $[S] \subset H$, yielding (1).

4.3. THEOREM. *If S has the density property, then a necessary and sufficient condition that L_S be separating is that* $[S]=G$.

Proof. That the condition is necessary is a consequence of the previous theorem and the fact that if H is a proper closed subgroup of G, then there exists a character q on G such that $q\equiv 1$ on H but $q \not\equiv 1$ on G. To argue sufficiency we again cite the previous theorem, for if $[S]=G$ and $q \neq r$, then $q \neq r$ on $[S]$ and there must be some $f \in L_S$ with $\hat{f}q = \int f\bar{q} \neq \int f\bar{r} = \hat{f}r$ and L_S is separating.

The hypothesis "S has the density property" cannot be omitted: Let R be the additive group of reals and let G be the direct product of R with the two element group $\{0, 1\}$. Let S be the subgroup consisting of all those elements whose second co-ordinate is 0 and let T be S union the one element (0, 1). Then $L_S = L_T$; but T generates G while S does not.

If $f \in L^1$ and $y \in G$, we define a new function f_y by $f_y(x) = f(xy^{-1})$. The function f_y is clearly in L^1 and is called the y-translate of f. It is known (see Loomis [8]) that a closed linear subspace A of L^1 is an ideal if and only if A is invariant under translations; i.e., $f \in A$ and $y \in G$ imply $f_y \in A$ (recall that here G is Abelian). A subset K of L^1 is called an *approximate identity* (abbreviated a.i.) for L^1 if for every $f \in L^1$ and $\epsilon > 0$, there exists a $k \in K$ with $\|k\| \leqq 1$ such that $\|f - f * k\| < \epsilon$. The algebra L^1 always contains an a.i. (see Loomis [8, p. 125] or Segal [12]).

4.4. LEMMA. *If x is a density point of S (see 2.1), then L_S contains the x-translate of an approximate identity*

Proof. Let V be any open set about x and set $k_V = (1/\mu(V \cap S))\chi(V \cap S)$. It follows (see Loomis [8, p. 124]) that the collection $\{k_V\}$ obtained by consideration of all neighborhoods of x is an x-translate of an a.i.

4.5. THEOREM. *If $L_S \neq \{0\}$, then it is contained in no proper closed ideal of L^1.*

Proof. Since $L_S \neq \{0\}$, there must exist some $x \in D(S)$ and so L_S contains the x-translate of an a.i. Now if L_S were contained in a closed ideal A, then A must contain an a.i. since it is invariant. But then A would not be proper and the proof is complete.

We have just seen that no nontrivial L_S is contained in a proper closed ideal of L^1. The situation for the transforms is as follows:

4.6.1. THEOREM. *If 1 is a density point of S, then $\hat{L}_S$ is contained in no proper closed ideal of $C(\hat{G})$; but*

4.6.2. THEOREM. *The closure (in $L^1(\hat{G})$) of $\hat{L}_S \cap L^1(\hat{G})$ is an ideal of $L^1(\hat{G})$.*

The arguments are presented below.

We first make the claim: *if K is an approximate identity for L^1, then $\hat{K}$ is an approximate identity for $C(\hat{G})$.* For, let $g \in C(\hat{G})$ and find $f \in L^1$ such that $\|g - \hat{f}\| < \epsilon/4$ and pick $k \in K$ such that $\|\hat{f} - \hat{f}\hat{k}\|_\infty \leqq \|f - f * k\|_1 < \epsilon/2$. Then $\|g - g\hat{k}\| \leqq 2\|g - \hat{f}\| + \|\hat{f} - \hat{f}\hat{k}\| < \epsilon$. The first inequality follows from $\|k\| \leqq 1$ and our claim is esteblished.

Now if $1 \in D(S)$, then by 4.4, L_S has an a.i. and then so does its set of transforms. Clearly no proper closed ideal can contain an a.i. and Theorem 4.6.1 is proved.

Let r and q be any two characters on G and let $f \in L^1$. Then fq (pointwise product) is surely in L^1 and any support for f is also a support for fq. The equalities $(\hat{f})_q(r) = \hat{f}(r\bar{q}) = \int fq\bar{r} = (fq)^\frown(r)$ show that $(\hat{f})_q = (fq)^\frown$. It is now clear if $\hat{f} \in \hat{L}_S$ and if $q \in \hat{G}$, then $(\hat{f})_q \in \hat{L}_S$; $\hat{L}_S \cap L^1(\hat{G})$ is an invariant subspace; its closure is an ideal; and we have shown statement 4.6.2.

Of course, if $\hat{L}_S \cap L^1(\hat{G}) = \{0\}$, then Theorem 4.6.2 has no content. To insure this does not happen we need only to insist that L_S be a nontrivial v.a. For, $L_S \cap L^2(G)$ is dense in L_S and whenever $f, g \in L^1 \cap L^2(G)$, it follows that $(f * g)^\frown \in L^1(\hat{G})$ (see Loomis [8, p. 145]). We therefore may restate 4.6.2 as follows:

4.6.2′. Theorem. *If L_S is a nontrivial vanishing algebra, then the closure (in $L^1(\hat{G})$) of $\hat{L}_S \cap L^1(\hat{G})$ is a nontrivial ideal of $L^1(\hat{G})$.*

If G is discrete, then $\hat{G}$ is compact and hence every element of $(L^1)^\frown$ is also in $L^1(\hat{G})$. Also $\|\hat{f}\|_1 \leq \|\hat{f}\|_\infty$ for each $\hat{f} \in C(\hat{G})$. Therefore, the above theorem tells us that if G is discrete and if A is the collection of generalized analytic functions on $\hat{G}$ (for a given semigroup of G (see Arens and Singer [2])), then the closure of A in $L^1(\hat{G})$ is an ideal.

5. **Compact Abelian groups.** We have seen in §4 that if L_S contains more than one element, then it is contained in no proper closed ideal of L^1. Also, if $S \subset D(S)$, then a necessary and sufficient condition that L_S be separating is $[S] = G$ (recall that $[S]$ designates the smallest closed subgroup containing S).

In [11] Rudin states the following theorem: If G is compact and Abelian, then the only separating algebras are the maximal ideals of L^1.

Putting these three results together we are able to prove the following proposition:

5.1. Theorem. *If S is a subsemigroup of a compact Abelian group G and if S has positive measure, then S must be an open and closed subgroup of G.*

Note: In [3] we prove this theorem, by different methods, for the non-Abelian case also. We show it here because it is an interesting application of our theorems.

To prove 5.1 we first need two lemmas.

5.2. Lemma. *Let G be a compact Abelian group and let S be a subsemigroup which has the density property. Then $[S] = G$ if and only if $S = G$ a.e.*

Proof. Obviously, if $S = G$ a.e., then $G = \bar{S} \subset [S]$. Now suppose $[S] = G$. Then, by hypothesis and a previous result, L_S is a separating algebra. In light of Rudin's theorem and our remarks above, it must be $L_S = L^1 = L_G$; whence $S = G$ a.e. and the lemma is proved.

5.3. Lemma. *In any locally compact group G, if S is a semigroup of positive measure and if T is a subgroup of G containing S with the property $\mu(T - S) = 0$, then $S = T$.*

Proof. Suppose, by way of contradiction, that $x \in T - S$. Then $xS^{-1} \subset (T - S)$; but xS^{-1} has positive measure while $T - S$ does not. Thus $S = T$.

Note. It is reasonably clear that the same sort of proof goes through if S is assumed to be the group and T the semigroup.

Proof of 5.1. Since S is a semigroup of positive measure, we have $D(S)$ and $S\cap D(S)$ are also semigroups of positive measure and they both have the density property. Now $D(S)$ is closed, and hence compact. Thus $D(S)$ has an idempotent and it must be unique since we are inside a group. Now, by a well known theorem (see Iwasawa [7] or Wallace [13]) $D(S)$ is a group. It follows from the equality $(S\cap D(S))^- = D(S)$ that $[S\cap D(S)] = D(S)$. Hence, by 5.2, $S\cap D(S) = D(S)$ a.e. It is now a consequence of Lemma 5.3 and the note after it that $S = D(S)$. So S is a closed group of positive measure; but any subgroup of positive measure is open and the proof is complete.

5.4. COROLLARY. *If G is compact and Abelian, then a necessary and sufficient condition that L_S be a vanishing algebra is that S be an open and closed group a.e.*

Proof. The sufficiency is clear. If L_S is a v.a., then $D(S)$ is an open and closed (therefore compact) group. Now, as shown in the proof of 5.1, $S\cap D(S) = D(S)$ a.e. and hence $S = D(S)$ a.e.

The next three corollaries follow immediately from 5.4.

5.5. COROLLARY. *If G is compact and Abelian, then every vanishing algebra is self adjoint.*

5.6. COROLLARY. *If G is compact and Abelian, then G is connected if and only if there are no proper nonzero vanishing algebras.*

5.7. COROLLARY. *If G is compact and Abelian and S is a subset of positive measure with the property $\mu((SS)-S)=0$, then S must be an open and closed subgroup a.e.*

REMARK. Theorem 3.23 established that under certain conditions a maximal vanishing algebra is a maximal subalgebra of L^1. This is not true in general even when G is discrete. An example is as follows: Let I be the group of integers and let S be the subgroup of even integers. Then, obviously, L_S is a maximal v.a. Now let

$$M = \left\{ f \in L^1 \colon \sum_{x\,\mathrm{odd}} f(x) = 0 \right\} \cdot$$

The set M is a proper closed algebra containing L_S. (We might add that M is a maximal subalgebra of $L^1(I)$.)

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