SPACES OF FUNCTIONS WITH VALUES IN A BANACH ALGEBRA[1]

BY

G. PHILIP JOHNSON[2]

Introduction. In the following pages certain spaces of abstract-valued functions are examined. Throughout the paper A will denote a commutative Banach algebra. In analogy to the group algebra $L^1(G)$ of a locally compact Abelian group G, that is, the space of absolutely integrable complex-valued functions on G, we form the set $B^1 = B^1(G, A)$ of Bochner integrable functions defined to A from G. B^1 is first of all a Banach space and it becomes a commutative Banach algebra if multiplication of two elements $f, g \in B^1$ is defined by the convolution formula, $(f*g)(x) = \int_G f(xy)g(y^{-1})dy$.

For the theory of the Bochner integral we shall rely mainly on the presentation in Hille's book [2, pp. 35–49]. Although the development there uses the Lebesgue measure for finite dimensional Euclidean spaces, the theorems which we shall need hold as well for more general measure spaces, in particular for a locally compact group with Haar measure. The calculus for the generalized convolution carries over directly from that for numerical functions and will be assumed. We note that the convolution of a function in B^1 and a function in $L^1(G)$ is well defined and in B^1.

To the greatest possible extent the notation and definitions are those of Loomis [3]. "Maximal ideal" means regular maximal ideal throughout. Special conventions are as follows: $f = f(x)$, $g = g(x)$, $\cdots$ denote elements of B^1. For the most part complex-valued functions are assigned Greek letters regardless of the set on which they are defined. χ is always used for the characteristic function of a set and μ is the Haar measure on G. $\mathfrak{M}_B$ denotes the maximal ideal space of B^1, $\mathfrak{M}_A$ that of A, and $\hat{G}$, the character group of G, is the maximal ideal space of L^1. Typical elements are M_B, M, and α respectively. Subscripts distinguish the various norms. For example, $\|f\|_B = \int \|f(x)\|_A dx$, $f \in B^1$.

For any complex or A-valued function on G and $x \in G$, the subscript x applied to the function denotes its translate by x. The symbol $\hat{}$ used with

Presented to the Society, April 13, 1956 under the title of *Spaces of functions with values in a normed ring*; received by the editors May 7, 1957 and, in revised form, February 2, 1958.

[1] Many of the results in the first three sections were obtained independently by Alvin Hausner and were presented at a different meeting of the Society at the same time as the author presented his results (see Bull. Amer. Math. Soc. vol. 62 (1956) pp. 383–384).

[2] This paper is based on a doctoral thesis submitted to the Graduate Faculty of the University of Minnesota in the summer of 1956. The author is much indebted to Professor B. R. Gelbaum who suggested the topic and gave unstintingly of his time in guiding preparation of the thesis.

an element of a Banach algebra denotes its abstract Fourier transform and E' is the complement of the set E.

The principal tool in this development is the Gelfand representation and there is a consequent preoccupation with the maximal ideals of B^1. In the first section some of the L^1 results connecting translation invariance and ideals are shown to hold in B^1. An analogue of the theorem stating that a convolution is in the span of translates of either of its arguments is proved. (This theorem has been proved for almost periodic functions by von Neumann [4, p. 457] and its statement for elements of L^1 is given by Segal [6, p. 94].) These results are applied in the second section in an analysis of the structure of $\mathfrak{M}_B$. It is found that a maximal ideal of B^1 is an ordered pair (α, M), $\alpha \in \hat{G}$, $M \in \mathfrak{M}_A$; and, in fact, $\mathfrak{M}_B$ is homeomorphic to $\hat{G} \times \mathfrak{M}_A$ with the weak topologies. The connection between the pairs (α, M) and the M_B is a generalized integral formula for the Fourier transform. In the third section it is shown that if A is a group algebra, B^1 is (isometric and isomorphic to) a group algebra. The fourth section is concerned with an isometric isomorphism T of $B^1(G, A)$ onto a like algebra $B^1(\tilde{G}, \tilde{A})$. We obtain conditions under which T gives rise to isomorphisms of G onto $\tilde{G}$ and of A onto $\tilde{A}$ in terms of which T can be expressed in a particularly simple fashion. An example shows that in general neither G and $\tilde{G}$ nor A and $\tilde{A}$ need be isomorphic.

1. **Approximation of convolution and translation invariance in ideals.** Let f be a continuous A-valued function with compact carrier C and let V be a measurable open neighborhood of the identity e. There exists a finite set $\{a_i V\}$, $i = 1, 2, \cdots, n$, of translates of V which cover C. Let $E_1 = C \cap a_1 V$, $E_2 = C \cap a_2 V \cap E_1'$, $\cdots$, $E_n = C \cap a_n V \cap (\bigcup_{i=1}^{n-1} E_i)'$. These are disjoint measurable sets whose union is C and for each i, $E_i \subset a_i V$. Form a simple function $f_V = \sum_i f(x_i)\chi_{E_i}$, where x_i is arbitrary in E_i. (If E_i is empty, the choice of x_i is immaterial.)

LEMMA 1.1. *If f is continuous with compact carrier C and $\epsilon > 0$, there exists a measurable open neighborhood V_ϵ of e such that for any measurable open neighborhood V of e contained in V_ϵ and corresponding f_V,*

$$\left\| \int f(x)dx - \int f_V(x)dx \right\|_A = \left\| \int f(x)dx - \sum_i f(x_i)\mu(E_i) \right\|_A < \epsilon.$$

Proof. If $f \neq 0 \in B^1$, there exists, by the uniform continuity of f, a neighborhood W of e such that $xy^{-1} \in W$ implies $\|f(x) - f(y)\|_A < \epsilon/\mu(C)$. Choose V_ϵ such that $V_\epsilon V_\epsilon^{-1} \subset W$. If $x \in C$, then $x \in E_i$ for some i, whence $xx_i^{-1} \in VV^{-1} \subset W$. Therefore

$$\left\| \int f(x)dx - \int f_V(x)dx \right\|_A \leqq \int \|f(x) - f_V(x)\|_A dx < \epsilon.$$

The following lemma contains the essentials of the convolution approximation referred to in the introduction. The lemma will also be used in the second section.

LEMMA 1.2. *Let V_0 be a measurable open neighborhood of e and let $\epsilon > 0$ be given. If $f \in B^1$ and g is a continuous A-valued function with compact carrier C_g, then there exists a measurable open neighborhood V of e contained in V_0 such that for a measurable subdivision $\{E_i\}$ of C_g obtained as above from V and $y_i \in E_i$, $i = 1, \cdots, L$,*

$$\left\| f * g - \sum_i f_{y_i^{-1}} g(y_i)\mu(E_i) \right\|_B < \epsilon.$$

Proof. We assume that neither f nor g is the zero element of B^1, for otherwise the assertion is trivial.

CASE 1. f is continuous with compact carrier C_f. Then $f * g$ is continuous and

$$(f * g)(x) = \int_{C_g} f(xy^{-1})g(y)dy.$$

If $y \in C_g$ and $x \notin C_g C_f$, then $xy^{-1} \notin C_f$, so that $f * g$ vanishes outside the compact set $C = C_g C_f$.

Let $\{E_i\}$ be any finite collection of disjoint measurable sets whose union is C_g and take $y_i \in E_i$. By virtue of the uniform continuity of f and $f * g$, one can find a neighborhood U of e such that $x_1 x_2^{-1} \in U$ implies

$$\left\| \sum_i [f(x_1 y_i^{-1}) - f(x_2 y_i^{-1})] g(y_i)\mu(E_i) \right\|_A$$
$$\leqq \sum_i \|f(x_1 y_i^{-1}) - f(x_2 y_i^{-1})\|_A \|g(y_i)\|_A \mu(E_i) < \epsilon/3\mu(C)$$

and

$$\|(f * g)(x_1) - (f * g)(x_2)\|_A < \epsilon/3\mu(C).$$

Let U_0 be a measurable open neighborhood of e such that $U_0 U_0^{-1} \subset U$. As before, construct from the translates of U_0 a finite collection $\{W_n\}$, $n = 1, \cdots, N$, of disjoint measurable sets whose union is C. Pick $x_n \in W_n$. Since for each n, $f(x_n y^{-1})g(y)$ is continuous and vanishes outside C_g, there exists by the preceding lemma a neighborhood V_n of e such that

$$\left\| \int f(x_n y^{-1})g(y)dy - \sum_i f[x_n (y_i^{(n)})^{-1}] g(y_i^{(n)})\mu(E_i^{(n)}) \right\|_A < \epsilon/3\mu(C)$$

where $\{E_i^{(n)}\}$ is a subdivision of C_g provided by V_n and $y_i^{(n)} \in E_i^{(n)}$, $i = 1, \cdots, L_n$. Let $V = (\bigcap_n V_n) \cap V_0$. Then $V \subset V_0$ and for $\{E_i\}$, a subdivision of C_g given by V, $y_i \in E_i$, $i = 1, \cdots, L$, and for every n,

$$\|(f*g)(x_n) - h(x_n)\|_A < \epsilon/3\mu(C), \quad \text{where} \quad h = \sum_i f_{y_i^{-1}} g(y_i)\mu(E_i).$$

Now if $x \in C$, there exists an n such that $x \in W_n$. Since W_n is contained in a translate of U_0, $xx_n^{-1} \in U_0 U_0^{-1} \subset U$. Hence

$$\|(f*g)(x) - h(x)\|_A \leqq \|(f*g)(x) - (f*g)(x_n)\|_A + \|(f*g)(x_n) - h(x_n)\|_A + \left\| \sum_i [f(x_n y_i^{-1}) - f(x y_i^{-1})] g(y_i)\mu(E_i) \right\|_A < \epsilon/\mu(C)$$

for all $x \in C$. Since for every i, $f_{y_i^{-1}}$ vanishes outside C,

$$\|f*g - h\|_B < \mu(C)\cdot\epsilon/\mu(C) = \epsilon.$$

CASE 2. f is arbitrary in B^1. It is possible to select a continuous function $f^{(1)}$ with compact carrier and from some $V \subset V_0$ to construct, as in Case 1, a function $h^{(1)} = \sum_i f^{(1)}_{y_i^{-1}} g(y_i)\mu(E_i)$ such that $\|f - f^{(1)}\|_B$ and $\|f^{(1)} * g - h^{(1)}\|_B$ are arbitrarily small. Let h be the function obtained from $h^{(1)}$ by replacing $f^{(1)}$ by f. Then since g is bounded, $\|h - h^{(1)}\|_B$ and, consequently, $\|f*g - h\|_B$ are small.

THEOREM 1.1. *If f, $g \in B^1$ and $\epsilon > 0$, there exists a finite sum h of A multiples of translates of f such that $\|f*g - h\|_B < \epsilon$.*

Proof. For some continuous function $g^{(1)}$ with compact carrier and, by Lemma 1.2, for a finite sum h of A multiples of translates of f, $\|g - g^{(1)}\|_B$ and $\|f*g^{(1)} - h\|_B$ are small, whence $\|f*g - h\|_B$ is small.

THEOREM 1.2. *Any closed translation invariant subspace of B^1 which admits multiplication by elements of A is an ideal.*

Proof. Let I be the subspace and take $f \in I$, $g \in B^1$. By Theorem 1.1, $f*g$ is the limit of sums of A multiples of translates of f. Each of these is in I and therefore, since I is closed, $f*g \in I$.

The converse of this theorem holds if the ideal is closed and regular. We shall use an approximate identity in the proof.

DEFINITION. A directed system $\{u\}$ of functions in L^1 is called an approximate identity for B^1 in case $\lim_u \|u*f - f\|_B = 0$ for every $f \in B^1$.

The following theorem, a paraphrase of Theorem 3.7.1 of [2], asserts that approximate identities for B^1 exist.

THEOREM 1.3. *Let $\epsilon > 0$ be given. Corresponding to $f \in B^1$ there exists a neighborhood V of the identity such that if u is any non-negative function of L^1 vanishing outside V and $\int u\,dx = 1$, then $\|u*f - f\|_B < \epsilon$.*

THEOREM 1.4. *A regular ideal in B^1 is translation invariant. A closed ideal admits multiplication by elements of A.*

Proof. Let I be a regular ideal in B^1 and u an identity modulo I. Under the homomorphism of B^1 induced by I, u is carried into the identity $\bar{u}$ of B^1/I. Let $f \in I$. Since $f * u_x = f_x * u$, we have in B^1/I, $0 = \bar{f}\bar{u}_x = \bar{f}_x\bar{u} = \bar{f}_x$. Hence $f_x \in I$.

Suppose I is a closed ideal. For $a \in A$ and each $u \in \{u\}$, an approximate identity, $u * af = au * f \in I$, and $\lim_u u * af = af$. Since I is closed, $af \in I$.

2. **The maximal ideals of B^1.** The fact that each pair (α, M), $\alpha \in \hat{G}$, $M \in \mathfrak{M}_A$, gives rise to a maximal ideal of B^1 is easily arrived at, and indeed follows immediately from the next result.

THEOREM 2.1. *For any character $\alpha \in \hat{G}$ the mapping $f \to \int f(x)\bar{\alpha}(x)dx$ is a continuous homomorphism of B^1 onto A.*

Proof. The mapping is obviously linear and we have by the Fubini theorem

$$f * g \to \int\int f(xy)g(y^{-1})dy\bar{\alpha}(x)dx = \int\int f(xy)\bar{\alpha}(xy)dxg(y^{-1})\bar{\alpha}(y^{-1})dy$$

$$= \int f(x)\bar{\alpha}(x)dx \int g(y)\bar{\alpha}(y)dy.$$

Since $\|\int f\bar{\alpha}dx\|_A \leqq \|f\|_B$, the homomorphism is continuous. To show that it maps B^1 onto A we take $\lambda \in L^1$ such that $\hat{\lambda}(\alpha) = 1$. If $a \in A$, we have $a\lambda \in B^1$ and $a\lambda \to a\int\lambda(x)\bar{\alpha}(x)dx = a\hat{\lambda}(\alpha) = a$.

Thus if M is any maximal ideal of A, the product homomorphism $f \to \int f(x)\bar{\alpha}(x)dx \to (\int f(x)\bar{\alpha}(x)dx)^\wedge(M)$ is a homomorphism of B^1 onto the complex numbers. It follows that to each pair (α, M) there corresponds a maximal ideal $M_B^{(\alpha,M)}$ of B^1 where

$$\hat{f}(M_B^{(\alpha,M)}) = \left(\int f(x)\bar{\alpha}(x)dx\right)^\wedge (M)$$

for every $f \in B^1$. Because the homomorphism given by M is a bounded linear transformation on A,

$$(*) \qquad \hat{f}(M_B^{(\alpha,M)}) = \int f(x)^\wedge(M)\bar{\alpha}(x)dx, \qquad f \in B'.$$

THEOREM 2.2. *If (α_1, M_1) is distinct from (α_2, M_2), then the corresponding maximal ideals of B^1 are distinct.*

Proof. If $M_1 \neq M_2$, take $a \in M_1 \cap M_2'$ and $\lambda \in L^1$ such that $\hat{\lambda}(\alpha_2) \neq 0$. Then

$$(a\lambda)^\wedge(M_B^{(\alpha_1,M_1)}) = \hat{\lambda}(\alpha_1)\hat{a}(M_1) = 0 \neq \hat{\lambda}(\alpha_2)\hat{a}(M_2) = (a\lambda)^\wedge(M_B^{(\alpha_2,M_2)}).$$

If $\alpha_1 \neq \alpha_2$, for $a \notin M_2$ and $\lambda \in L^1$ such that $\hat{\lambda}(\alpha_1) = 0$, $\hat{\lambda}(\alpha_2) \neq 0$,

$$(a\lambda)^\wedge(M_B^{(\alpha_1,M_1)}) = 0 \neq (a\lambda)^\wedge(M_B^{(\alpha_2,M_2)}).$$

The integral formula (*) thus defines a one-to-one correspondence between $\hat{G}\times\mathfrak{M}_A$ and a subset of $\mathfrak{M}_B$. We have yet to prove that all $\mathfrak{M}_B$ is accounted for in this way, or equivalently, that for each $M_B\in\mathfrak{M}_B$, a character and a maximal ideal of A can be found which are related to M_B through the integral formula. The generalization of a well-known result due to Raikov [5] provides a candidate for the character.

THEOREM 2.3. *For any $M_B\in\mathfrak{M}_B$ and $f\in B^1\cap M_B'$, the function α_B defined by $\alpha_B(x)=\hat{f}_x(M_B)/\hat{f}(M_B)$ is independent of f and is a character of G.*

The proof in [3, p. 135] applies virtually unchanged.

If A has an identity e, then $\lambda e\in B^1$ whenever $\lambda\in L^1$. In fact, if we assume, as we may, that $\|e\|_A=1$, then L^1 is isometric and isomorphic to the subalgebra L^1e of B^1. Since this feature considerably facilitates working in B^1, several of the next results assume the presence of e in A.

THEOREM 2.4. *If A has an identity and $M_B\in\mathfrak{M}_B$, then $M_B\cap L^1e$ is a maximal ideal of L^1e and $\{\lambda:\lambda e\in M_B\cap L^1e\}$ is the maximal ideal of L^1 corresponding to α_B.*

Proof. Let $\{u\}$ be an approximate identity. Since not every element ue can be in M_B, $M_B\cap L^1e\neq L^1e$ and hence is a maximal ideal.

Let α be the character corresponding to $\{\lambda:\lambda e\in M_B\cap L^1e\}$. If $\nu\in L^1$ is such that $\hat{\nu}(\alpha)=1$, clearly νe is an identity modulo $M_B\cap L^1e$. Since $\nu_x=\hat{\nu}_x(\alpha)\nu+\lambda_0$, where $\hat{\lambda}_0(\alpha)=0$, $(\nu_xe)\hat{}\,(M_B)=(\nu_xe)\hat{}\,(M_B\cap L^1e)=\hat{\nu}_x(\alpha)$. Therefore

$$\alpha_B(x) = (\nu_xe)\hat{}\,(M_B)/(\nu e)\hat{}\,(M_B) = \hat{\nu}_x(\alpha)/\hat{\nu}(\alpha) = \alpha(x)$$

for every $x\in G$.

THEOREM 2.5. *If A has an identity and if $\lambda e\in L^1e$ is an identity modulo $M_B\in\mathfrak{M}_B$, then the mapping $a\to(a\lambda)\hat{}\,(M_B)$ is a homomorphism of A onto the complex numbers. It is independent of λ, and the corresponding maximal ideal $M^{(B)}\in\mathfrak{M}_A$ is such that for every $f\in B^1$*

$$\hat{f}(M_B) = \int f(x)\hat{}\,(M^{(B)})\bar{\alpha}_B(x)dx.$$

Proof. If a different identity νe modulo M_B is used, then $\lambda e-\nu e\in M_B$ and by Theorem 1.4, $a\lambda-a\nu\in M_B$ for every $a\in A$. Hence the mapping is independent of λ. In particular, since $(\lambda*\lambda e)\hat{}\,(M_B)=[(\lambda e)\hat{}\,(M_B)]^2=1$, $(a\lambda)\hat{}\,(M_B)=(a\lambda*\lambda)\hat{}\,(M_B)$. Let $\bar{a}=(a\lambda)\hat{}\,(M_B)$. $a\to\bar{a}$ is linear and for $a_1, a_2\in A$

$$\begin{aligned}\bar{a}_1\bar{a}_2 &= (a_1\lambda)\hat{}\,(M_B)(a_2\lambda)\hat{}\,(M_B) = (a_1a_2\lambda*\lambda)\hat{}\,(M_B)\\ &= (a_1a_2\lambda)\hat{}\,(M_B) = (a_1a_2)^-.\end{aligned}$$

$a\to\bar{a}$ is therefore a homomorphism of A onto the complexes.

Now let E be a set of finite measure in G and let $f=a\chi_E$, $a\in A$. Since

$\chi_E \in L^1$, we may write $\chi_E = \hat{\chi}_E(\alpha_B)\lambda + \lambda_0$, where $\hat{\lambda}_0(\alpha_B) = 0$. Then $f(x) = \hat{\chi}_E(\alpha_B)a\lambda + a\lambda_0$, and $\int f(x)^\wedge (M^{(B)})\bar{\alpha}_B(x)dx = \hat{\chi}_E(\alpha_B)\hat{a}(M^{(B)})$. On the other hand, since $a\lambda_0 \in M_B$, $\hat{f}(M_B) = \hat{\chi}_E(\alpha_B)(a\lambda)^\wedge(M_B) = \hat{\chi}_E(\alpha_B)\hat{a}(M^{(B)})$. Hence $\hat{f}(M_B) = \int f(x)^\wedge (M^{(B)})\bar{\alpha}_B(x)dx$. The integral formula has thus been proved for functions of the form $a\chi_E$. Since such functions constitute a fundamental set of B^1 and since the left and right hand sides of the formula are each linear functionals on B^1 which are equal on this set, they are equal throughout B^1. This completes the proof.

If A does not have an identity, it can be embedded in a Banach algebra $A_e = \{a + ce : a \in A, c \text{ complex}\}$ which does. Let $B^1(e) = B^1(G, A_e)$. An element of $B^1(e)$ is the sum of an A-valued function and a complex-valued function multiplied by e. If $f + \lambda e$ is one such, then

$$\|f + \lambda e\|_{B(e)} = \int \|f + \lambda e\|_{A_e} dx = \|f\|_B + \|\lambda\|_1.$$

Thus $B^1(e)$ consists of the functions $f + \lambda e$, where $f \in B^1$, $\lambda \in L^1$. We note that B^1 is a closed ideal of $B^1(e)$. This follows from Theorem 1.2; for B^1 is a translation invariant subspace which admits multiplication by elements of A_e, and it is closed because it is complete with respect to the $B^1(e)$ norm.

LEMMA 2.1. *If $f \in B^1$, $\lambda \in L^1$, and $M_B \in \mathfrak{M}_B$, then* $(\lambda * f)^\wedge(M_B) = \hat{\lambda}(\alpha_B)\hat{f}(M_B)$.

Proof. Since M_B is a closed subspace of B^1, it is a closed subspace of $B^1(e)$. By Theorem 1.4, it is translation invariant and admits multiplication by elements of A_e. Hence by Theorem 1.2, M_B is an ideal of $B^1(e)$. Thus if $f \in M_B$, $\lambda * f = \lambda e * f \in M_B$. Therefore $(\lambda * f)^\wedge(M_B) = 0 = \hat{\lambda}(\alpha_B)\hat{f}(M_B)$, which proves the formula for this case.

If $f \notin M_B$, then $\alpha_B(x) = \hat{f}_x(M_B)/\hat{f}(M_B)$. Suppose λ is a continuous function with compact carrier C and let $\epsilon > 0$ be given. Since Lemma 1.1 applies to $\lambda\bar{\alpha}_B$, there exists an open measurable neighborhood V_ϵ such that

$$\left| \int \lambda(y)\bar{\alpha}_B(y)dy - \sum_i \lambda(y_i)\bar{\alpha}_B(y_i)\mu(E_i) \right| < \epsilon/2,$$

where $\{E_i\}$ is a subdivision of C obtained from V_ϵ and $y_i \in E_i$, $i = 1, \cdots, n$. Because λe is a continuous A_e-valued function with compact carrier, there exists, by Lemma 1.2, a neighborhood $V \subset V_\epsilon$ such that for a subdivision $\{E_j\}$ obtained from V and $y_j \in E_j$, $j = 1, \cdots, L$,

$$\left\| \lambda * f - \sum_j f_{y_j^{-1}}\lambda(y_j)\mu(E_j) \right\|_B < \frac{\epsilon}{2} \left| \hat{f}(M_B) \right|.$$

Therefore

$$\left| \frac{(\lambda * f)^\wedge(M_B)}{\hat{f}(M_B)} - \sum_j \bar{\alpha}_B(y_j)\lambda(y_j)\mu(E_j) \right| < \frac{\epsilon}{2}.$$

Since $V\subset V_\epsilon$,

$$\left|\frac{(\lambda * f)^\wedge(M_B)}{\hat f(M_B)} - \int \lambda(y)\bar\alpha_B(y)dy\right| < \epsilon.$$

Thus $(\lambda * f)^\wedge(M_B)=\hat\lambda(\alpha_B)\hat f(M_B)$.

The two sides of the formula are continuous functions of λ and are equal on a dense subset of L^1. Therefore $(\lambda * f)^\wedge(M_B)=\hat\lambda(\alpha_B)\hat f(M_B)$ for every $\lambda\in L^1$.

For $M_B\in\mathfrak{M}_B$ the linear mapping $f+\lambda e\rightarrow\hat f(M_B)+\hat\lambda(\alpha_B)$ is also multiplicative, for by the lemma

$$\begin{aligned}(f+\lambda e) * (g+\nu e) &\rightarrow (f * g + \lambda * g + \nu * f)^\wedge(M_B) + (\lambda * \nu)^\wedge(\alpha_B)\\ &= [\hat f(M_B)+\hat\lambda(\alpha_B)][\hat g(M_B)+\hat\nu(\alpha_B)].\end{aligned}$$

The mapping is therefore a homomorphism of $B^1(e)$ onto the complex numbers. Let $M_{B(e)}$ be the associated maximal ideal. Since $M_{B(e)}\cap L^1e=\{\lambda e:\hat\lambda(\alpha_B)=0\}$, $\alpha_{B(e)}=\alpha_B$ (see Theorem 2.4). Therefore, by Theorem 2.5, there exists a maximal ideal $M^{(B(e))}$ of A_e such that

$$\begin{aligned}(f+\lambda e)^\wedge(M_{B(e)}) &= \int [f(x)+\lambda(x)e]^\wedge(M^{(B(e))})\bar\alpha_B(x)dx\\ &= \int f(x)^\wedge(M^{(B)})\bar\alpha_B(x)dx + \hat\lambda(\alpha_B),\end{aligned}$$

where $M^{(B)}=M^{(B(e))}\cap A$. Note that $M^{(B)}$ cannot be A; for if it were, all B^1 would be mapped by M_B into zero. Hence $M^{(B)}$ is a maximal ideal of A and $\hat f(M_B)=\int f(x)^\wedge(M^{(B)})\bar\alpha_B(x)dx$.

The following characterization of $\mathfrak{M}_B$ has been completely demonstrated.

THEOREM 2.6. *The mapping $(\alpha, M)\rightarrow M_B^{(\alpha,M)}$ is a one-to-one mapping of $\hat G\times\mathfrak{M}_A$ onto $\mathfrak{M}_B$, and*

$$\hat f(M_B^{(\alpha,M)}) = \int f(x)^\wedge(M)\bar\alpha(x)dx$$

for all $f\in B^1$.

The remainder of this section is given to showing that in the sense of the weak topology the correspondence between $\hat G\times\mathfrak{M}_A$ and $\mathfrak{M}_B$ is a homeomorphism.

LEMMA 2.2. *Let $U\subset\hat G$ be open with respect to the weak topology of $\hat G$. Then $\{M_B:\alpha_B\in U\}$ is open in the weak topology of $\mathfrak{M}_B$.*

Proof. Let $M_B^{(0)}=(\alpha_0, M_0)$ be an arbitrary element in $\{M_B:\alpha_B\in U\}$. It suffices to show that there exists a neighborhood N of $M_B^{(0)}$ such that $N\subset\{M_B:\alpha_B\in U_1\}$, where $U_1=\{\alpha:|\hat\lambda(\alpha)-\hat\lambda(\alpha_0)|<\epsilon\}$, $\lambda\in L^1$. Therefore pick

$f \in B^1$ such that $|\hat{f}(M_B^{(0)})| > 1$ and let

$$N_1 = \{M_B: |\hat{f}(M_B)| > 1\}.$$

N_1 is then an open neighborhood of $M_B^{(0)}$ as are

$$N_2 = \{M_B: |(\lambda * f)^\wedge(M_B) - (\lambda * f)^\wedge(M_B^{(0)})| < \epsilon/2\},$$

and

$$N_3 = \{M_B: |\hat{f}(M_B) - \hat{f}(M_B^{(0)})| < \epsilon/2 \max(|\hat{\lambda}(\alpha_0)|, 1)\}.$$

By Lemma 2.1, for all $M_B \in N = N_1 \cap N_2 \cap N_3$

$$\begin{aligned} |\hat{\lambda}(\alpha_B) - \hat{\lambda}(\alpha_0)| &< |\hat{\lambda}(\alpha_B)\hat{f}(M_B) - \hat{\lambda}(\alpha_0)\hat{f}(M_B)| \\ &\leqq |(\lambda * f)^\wedge(M_B) - (\lambda * f)^\wedge(M_B^{(0)})| + |\hat{\lambda}(\alpha_0)| \, |\hat{f}(M_B^{(0)}) - \hat{f}(M_B)| \\ &< \epsilon. \end{aligned}$$

LEMMA 2.3. *If $f \in B^1$, then $\int f(x)\bar{\alpha}(x)dx$ is in the weak topology a uniformly continuous A-valued function over $\hat{G}$.*

The proof is omitted since the proof in [3, 34C] can be applied readily.

THEOREM 2.7. *With the weak topologies on $\hat{G}$, $\mathfrak{M}_A$, and $\mathfrak{M}_B$, $\hat{G} \times \mathfrak{M}_A$ and $\mathfrak{M}_B$ are homeomorphic.*

Proof. We first choose a $\hat{G} \times \mathfrak{M}_A$ sub-basis element $N(\alpha_0, M_0)$ about a point (α_0, M_0) and find an $\mathfrak{M}_B$-neighborhood of $M_B^{(0)} = (\alpha_0, M_0)$ contained in it. For $\lambda \in L^1$, $a \in A$, let

$$N(\alpha_0, M_0) = \{(\alpha, M): |\hat{\lambda}(\alpha) - \hat{\lambda}(\alpha_0)| < \epsilon, \ |\hat{a}(M) - \hat{a}(M_0)| < \epsilon\}.$$

If $\nu \in L^1$ is such that $|\hat{\nu}(\alpha_0)| > 1$, then, by Lemma 2.2,

$$N_1 = \{M_B: |\hat{\nu}(\alpha_B)| > 1\}$$

is a neighborhood of $M_B^{(0)}$. Similarly

$$\begin{aligned} N_2 &= \{M_B: |(a\nu)^\wedge(M_B) - (a\nu)^\wedge(M_B^{(0)})| < \epsilon/2\}, \\ N_3 &= \{M_B: |\hat{\nu}(\alpha_B) - \hat{\nu}(\alpha_0)| < \epsilon/2 \max(|\hat{a}(M_0)|, 1)\}, \\ N_4 &= \{M_B: |\hat{\lambda}(\alpha_B) - \hat{\lambda}(\alpha_0)| < \epsilon\}, \end{aligned}$$

are neighborhoods of $M_B^{(0)}$. Let $N(M_B^{(0)})$ be the intersection of these four sets. Since $(a\nu)^\wedge(M_B) = \hat{a}(M^{(B)})\hat{\nu}(\alpha_B)$, it follows, as in the proof of Lemma 2.2, that $N(M_B^{(0)}) \subset N(\alpha_0, M_0)$.

Conversely, let

$$N(M_B^{(0)}) = \{M_B: |\hat{f}(M_B) - \hat{f}(M_B^{(0)})| < \epsilon\},$$

for some $f \in B^1$, represent a sub-basis element of the weak topology of $\mathfrak{M}_B$. By Lemma 2.3, there exists an open $\hat{G}$-neighborhood $N(\alpha_0)$ of α_0 such that

$\alpha \in N(\alpha_0)$ implies $\|\int f(x)[\bar{\alpha}(x)-\bar{\alpha}_0(x)]dx\|_A < \epsilon/2$. If

$$N(M_0) = \left\{M: \left| \int [f(x)\hat{}(M) - f(x)\hat{}(M_0)]\bar{\alpha}_0(x)dx \right| < \epsilon/2 \right\},$$

then $N(M_0)$ is open in $\mathfrak{M}_A$. Therefore $N(\alpha_0, M_0) = N(\alpha_0) \times N(M_0)$ is open in $\hat{G} \times \mathfrak{M}_A$ and it contains (α_0, M_0). For each $(\alpha, M) \in N(\alpha_0, M_0)$ we have

$$\left| \hat{f}(M_B^{(\alpha,M)}) - \hat{f}(M_B^{(0)}) \right| \leqq \left| \int f(x)\hat{}(M)[\bar{\alpha}(x) - \bar{\alpha}_0(x)]dx \right| + \left| \int [f(x)\hat{}(M) - f(x)\hat{}(M_0)]\bar{\alpha}_0(x)dx \right| < \epsilon.$$

Thus $N(\alpha_0, M_0) \subset N(M_B^{(0)})$.

3. **Semi-simplicity and the case where A is a group algebra.** Some of the properties of L^1 can be given B^1 by assuming these properties for A. Statements like the next theorem have been proved in the author's thesis for regularity and the Tauberian condition (the set of elements whose Fourier transforms have compact carriers is dense) by extensive use of Theorems 2.6 and 2.7. We shall deal here with semi-simplicity.

THEOREM 3.1. *B^1 is semi-simple if and only if A is semi-simple.*

Proof. Let B^1 be semi-simple and let $a \in \bigcap_{\mathfrak{M}_A} M$. Let $\lambda \in L^1$ be such that $\|\lambda\|_1 \neq 0$. By Theorem 2.6, $(a\lambda)\hat{}(M_B) = \hat{a}(M^{(B)})\hat{\lambda}(\alpha_B) = 0$ for every $M_B \in \mathfrak{M}_B$. Thus $a\lambda \in \bigcap_{\mathfrak{M}_B} M_B = 0$ so that $0 = \|a\lambda\|_B = \|a\|_A \|\lambda\|_1$, whence $a = 0$.

Now suppose A is semi-simple and let $f \in \bigcap_{\mathfrak{M}_B} M_B$, where f is continuous. Then $\hat{f}(M_B) = 0$ for all M_B and hence $\int f(x)\hat{}(M)\bar{\alpha}(x)dx = 0$ for all $\alpha \in \hat{G}$, $M \in \mathfrak{M}_A$. $f(x)\hat{}(M)$ is for each M a continuous function in L^1. Its Fourier transform is identically zero and thus, because L^1 is semi-simple, $f(x)\hat{}(M) = 0$ a.e. (almost everywhere). In fact, since it is continuous, $f(x)\hat{}(M) \equiv 0$. Hence for each x, $f(x) \in \bigcap_{\mathfrak{M}_A} M = 0$, so $f = 0$.

We now consider any $f \in \bigcap_{\mathfrak{M}_B} M_B$ and assume that $f(x)$ is not a.e. zero. By Theorem 1.3, there exists a uniformly continuous function $u \in L^1$ such that $\|u * f\|_B > 0$. Since each M_B is an ideal of $B^1(e)$ (see proof of Lemma 2.1), $u * f = ue * f \in M_B$ for all $M_B \in \mathfrak{M}_B$. Furthermore $u * f$ is continuous. This contradicts the preceding paragraph and completes the proof.

We shall now treat the special case where A is a group algebra. Let $A = L^1(H)$, where H is a locally compact Abelian group. A function representing an element of $B^1(G, L^1(H))$ takes at each $x \in G$ a value which is an element of $L^1(H)$ and this element can be represented by a function over H. Thus elements of B^1 may be thought of as functions over $G \times H$. This double representation of elements of B^1 does not, however, lead directly to an identification of B^1 with $L^1(G \times H)$. Suppose, for example, that G and H are the reals and let S be Sierpinski's nonmeasurable plane set which intersects each line in at most two points [7]. The characteristic function of S represents no element of $L^1(G\times H)$, but in the above sense quite properly represents the zero element of $B^1(G, L^1(H))$. It is nevertheless true that the two algebras are isomorphic. The difficulty we have mentioned is easily avoided for elements of a certain dense set and an isomorphism can be established by means of an extension. In the following proof the symbol $\tilde{}$ will be used to distinguish a function from the equivalence class to which it belongs except when the representative is a characteristic function. For any function $\tilde{\lambda}$, $N(\tilde{\lambda})$ means $\{x: \tilde{\lambda}(x)\neq 0\}$.

THEOREM 3.2. *If H is a locally compact Abelian group, then $B^1(G, L^1(H))$ is isometric and isomorphic to $L^1(G\times H)$.*

Proof. Let B_0 be the subspace of B^1 generated by elements of the form $\eta\lambda$, $\eta\in L^1(H)$, $\lambda\in L^1(G)$. Since B_0 contains the simple functions, it is dense in B^1. For $f\in B_0$, $f=\sum_{i=1}^n \eta_i\lambda_i$, we pick representatives $\tilde{\eta}_i$ of η_i, $\tilde{\lambda}_i$ of λ_i and define the function $(Tf)^{\sim}$ by the equation

$$(Tf)^{\sim}(x, y) = \sum_i \tilde{\eta}_i(y)\tilde{\lambda}_i(x).$$

We shall show that the equivalence class Tf to which it belongs is in $L^1(G\times H)$.

Let $\tilde{\eta}_i'(y)=\tilde{\eta}_i(y)$ a.e. and $\tilde{\lambda}_i'(x)=\tilde{\lambda}_i(x)$ a.e. Now

$$\{(x, y): \tilde{\eta}_i(y)\tilde{\lambda}_i(x) \neq \tilde{\eta}_i'(y)\tilde{\lambda}_i'(x)\}$$
$$\subset [N(\tilde{\lambda}_i)\cup N(\tilde{\lambda}_i')]\times N(\tilde{\eta}_i - \tilde{\eta}_i')\cup N(\tilde{\lambda}_i - \tilde{\lambda}_i')\times[N(\tilde{\eta}_i)\cup N(\tilde{\eta}_i')]$$

and the larger set has zero measure in $G\times H$. Hence Tf is independent of the representatives chosen for the η_i and λ_i. For $\tilde{\eta}_i$ real and c real

$$\{(x, y): \chi_{N(\tilde{\lambda}_i)}(x)\tilde{\eta}_i(y)\neq 0\}\cap\{(x, y): \chi_{N(\tilde{\lambda}_i)}(x)\tilde{\eta}_i(y) < c\}$$
$$= N(\tilde{\lambda}_i)\times[N(\tilde{\eta}_i)\cap\{y: \tilde{\eta}_i(y) < c\}]$$

is measurable. If $\tilde{\eta}_i$ is not real, this argument applies to its real and imaginary parts separately. Thus $\chi_{N(\tilde{\lambda}_i)}\tilde{\eta}_i$ and, similarly, $\chi_{N(\tilde{\eta}_i)}\tilde{\lambda}_i$ are measurable functions, whence $\tilde{\eta}_i\tilde{\lambda}_i$ is measurable. Consequently for any $f\in B_0$, $(Tf)^{\sim}$ is measurable and Tf is in $L^1(G\times H)$; for

$$\|Tf\|_{L^1(G\times H)} = \int_G\int_H |(Tf)^{\sim}(x, y)|\,dydx$$
$$= \int_G \|\tilde{f}(x)\|_{L^1(H)}dx = \|f\|_B.$$

T is therefore an isometric linear mapping of the dense subspace B_0 into $L^1(G\times H)$. The image of B_0 is dense since it contains all L^1 classes represented by functions of the form $\sum_{i=1}^n\sum_{j=1}^m c_{ij}\chi_{E_j}(y)\chi_{E_i}(x)$. Hence T extends uniquely to an isometric linear mapping of B^1 onto $L^1(G\times H)$.

It remains to be shown that T preserves convolution. Let f, $g\in B_0$, $f=\sum_{i=1}^{n}\eta_i\lambda_i$, $g=\sum_{j=1}^{m}\eta_j'\lambda_j'$. Then $f*g$ is represented by

$$(f*g)^{\sim}=\left(\sum_i \tilde{\eta}_i\tilde{\lambda}_i\right)*\left(\sum_j \tilde{\eta}_j'\tilde{\lambda}_j'\right)=\sum_i\sum_j(\tilde{\eta}_i*\tilde{\eta}_j')(\tilde{\lambda}_i*\tilde{\lambda}_j')$$

and so $f*g\in B_0$. Thus

$$\begin{aligned}[T(f*g)]^{\sim}(x,y)&=\sum_i\sum_j\int_H\tilde{\eta}_i(yu)\tilde{\eta}_j'(u^{-1})du\int_G\tilde{\lambda}_i(xv)\tilde{\lambda}_j'(v^{-1})dv\\&=\int_G\int_H\sum_i\tilde{\eta}_i(yu)\tilde{\lambda}_i(xv)\sum_j\tilde{\eta}_j'(u^{-1})\tilde{\lambda}_j'(v^{-1})dudv\\&=(Tf*Tg)^{\sim}(x,y)\text{ a.e.,}\end{aligned}$$

and hence $T(f*g)=Tf*Tg$. Since convolution is continuous in both its arguments, $T(f*g)=Tf*Tg$ throughout B^1.

There is no further need for distinguishing functions from the elements they represent so the customary practice will be resumed.

Note that a maximal ideal of $B^1(G,L^1(H))$ is of the form (α_1,α_2), $\alpha_1\in\hat{G}$, $\alpha_2\in\hat{H}$. For $f\in B_0$

$$\hat{f}(\alpha_1,\alpha_2)=\int_G f(x)^{\wedge}(\alpha_2)\bar{\alpha}_1(x)dx=\int_G\int_H(Tf)(x,y)\bar{\alpha}_2(y)\bar{\alpha}_1(x)dydx,$$

which is the integral form of the Fourier transform of $Tf\in L^1(G\times H)$ at $(\alpha_1,\alpha_2)\in\hat{G}\times\hat{H}$. The fact that $\hat{f}(\alpha_1,\alpha_2)=(Tf)^{\wedge}(\alpha_1,\alpha_2)$ for all $f\in B^1$ follows immediately.

4. **Isomorphisms of Banach algebras of the form B^1.** An isomorphism between two topological groups or Banach algebras will mean an algebraic isomorphism which is also a homeomorphism. G and $\tilde{G}$ will be locally compact Abelian groups and A and $\tilde{A}$ commutative Banach algebras. If G is isomorphic to $\tilde{G}$ and A is isomorphic to $\tilde{A}$, isomorphisms of $B^1(G,A)$ and $B^1(\tilde{G},\tilde{A})$ can be easily constructed (Theorem 4.1); but the determination of conditions under which the converse is true is, as we shall see, more difficult.

THEOREM 4.1. *Let τ be an isomorphism of G onto $\tilde{G}$ and $\mathfrak{J}$ an isomorphism of A onto $\tilde{A}$. Let k denote the constant determined by $dx=kd\tau x$. Then for any $\alpha\in\hat{G}$, the mapping T defined by the equation*

$$(Tf)(\tau x)=k\alpha(x)\mathfrak{J}[f(x)]$$

is an isomorphism of $B^1(G,A)$ onto $B^1(\tilde{G},\tilde{A})$. T is an isometry if and only if $\mathfrak{J}$ is an isometry.

Proof. If $f\in B^1(G,A)$, Tf is measurable and

$$\int \|(Tf)(\tau x)\|_{\tilde{A}} d\tau x = \int \|\mathfrak{J}[f(x)]\|_{\tilde{A}} dx \leqq \|\mathfrak{J}\| \, \|f\|_B.$$

Thus $Tf \in B^1(\tilde{G}, \tilde{A})$ and, since T is clearly linear, it is continuous. If $\tilde{f} \in B^1(\tilde{G}, \tilde{A})$, then $f(\tau^{-1}\tilde{x}) = k^{-1}\bar{\alpha}(\tau^{-1}\tilde{x})\mathfrak{J}^{-1}[\tilde{f}(\tilde{x})]$ represents its unique pre-image and $\|f\|_B \leqq \|\mathfrak{J}^{-1}\| \, \|\tilde{f}\|_{\tilde{B}}$. T is therefore one-to-one, onto, and bicontinuous. Finally, for $f, g \in B^1(G, A)$

$$\begin{aligned}(T(f * g))(\tau x) &= k\alpha(x)\mathfrak{J} \int f(xy)g(y^{-1})dy \\ &= \int (Tf)(\tau x \tau y)(Tg)[(\tau y)^{-1}] d\tau y = (Tf * Tg)(\tau x).\end{aligned}$$

T is clearly an isometry if $\mathfrak{J}$ is. Conversely, if T is an isometry, for any $a \in A$ and $\lambda \in L^1(G)$ where $\|\lambda\|_1 = 1$, we have $(T(a\lambda))(\tau x) = k\alpha(x)\lambda(x)\mathfrak{J}a$ and

$$\|a\|_A = \|a\lambda\|_B = \|T(a\lambda)\|_{\tilde{B}} = \|\mathfrak{J}a\|_{\tilde{A}}.$$

In the special case of group algebras both A and $\tilde{A}$ are the complex numbers, $\mathfrak{J}$ is necessarily the identity transformation, and the converse of this theorem holds provided T is an isometry. The following result is due to Wendel [8].

If T is an isometric isomorphism of $L^1(G)$ onto $L^1(\tilde{G})$, there exist an isomorphism τ of G onto $\tilde{G}$ and a character $\alpha \in \tilde{G}$ such that for every $\lambda \in L^1(G)$, $T\lambda_x = \bar{\alpha} \cdot (T\lambda)_{\tau x}$ and $(T\lambda)(\tau x) = k\alpha(x)\lambda(x)$, where k is such that $dx = kd\tau x$.

It suffices to assume that T is an isomorphism such that $\|T\| \leqq 1$ (see Helson [1]), in which case T must again be an isometry. However, isomorphism alone is not sufficient for there are nonisomorphic groups whose algebras are isomorphic [8]. Algebras of the form B^1 display even greater pathology since an appeal to isometry fails to produce isomorphisms of the component spaces. Indeed, we know that $B^1(G, L^1(H))$ is isometric and isomorphic to $L^1(G \times H) = B^1(G \times H, C)$, C being the complexes (Theorem 3.2), but in general the corresponding components are not isomorphic.

It is not even true that isomorphism between just one pair of the components can be forced by assuming isomorphism between the other pair along with an isometric isomorphism of $B^1(G, A)$ onto $B(\tilde{G}, \tilde{A})$. We proceed to counterexamples.

Consider an infinite-dimensional Hilbert space $\mathfrak{H}$. $\mathfrak{H}$ is isomorphic to the direct sum $\mathfrak{H} \times \mathfrak{H}$, and the additive group H of $\mathfrak{H}$ becomes locally compact when assigned the discrete topology. H and $H \times H$ are algebraically isomorphic by virtue of the Hilbert space isomorphism and homeomorphic because they are discrete spaces in one-to-one correspondence. $B^1(H, L^1(H))$ is then isometrically isomorphic to $L^1(H \times H)$ and therefore to $L^1(H)$. This is also true for the compact group $\hat{H}$ since it is isomorphic to $\hat{H} \times \hat{H}$. Hence even for compact groups G, $B^1(G, A)$ isomorphic to $L^1(G)$ does not imply that A is the complex numbers.

In regard to the case in which A is isomorphic to $\tilde{A}$, we note that for any G and H, $B^1(G\times H, L^1(H))$ is isometric and isomorphic to $L^1(G\times H\times H)$ and hence to $B^1(G, L^1(H\times H))$. The counterexample comes from taking H isomorphic to $H\times H$.

It is therefore by no means the case that every isomorphism T is of the triple form $(\mathfrak{J}, \tau, \alpha)$. However

THEOREM 4.2. *If the triples* $(\mathfrak{J}_1, \tau_1, \alpha_1)$ *and* $(\mathfrak{J}_2, \tau_2, \alpha_2)$ *are distinct, the corresponding isomorphisms* T_1 *and* T_2 *are distinct.*

Proof. We treat four cases. Subscripts are not written on corresponding components assumed to be equal.

CASE 1. $\tau_1\neq\tau_2$. Let $\tilde{x}\in\tilde{G}$ be such that $\tau_1^{-1}\tilde{x}\neq\tau_2^{-1}\tilde{x}$. Choose disjoint open neighborhoods U, V of $\tau_1^{-1}\tilde{x}$, $\tau_2^{-1}\tilde{x}$ such that $\mu(U)<\infty$, $\mu(V)<\infty$ and let $f\in B^1(G, A)$ be zero on U but not on V. Then $T_1f\neq T_2f$ on $\tau_1U\cap\tau_2V$, a set of positive measure.

CASE 2. $\mathfrak{J}_1\neq\mathfrak{J}_2$, $\tau_1=\tau_2$, $\alpha_1=\alpha_2$. If $a\in A$ is such that $\mathfrak{J}_1(a)\neq\mathfrak{J}_2(a)$ and $f=a$ on a set E of finite nonzero measure, then $T_1f\neq T_2f$ on τE.

CASE 3. $\alpha_1\neq\alpha_2$, $\mathfrak{J}_1=\mathfrak{J}_2$, $\tau_1=\tau_2$. Let E_1 be a set of finite nonzero measure contained in the open set $E=\{x: \alpha_1(x)\neq\alpha_2(x)\}$. If $f=a\neq 0$ on E_1, $T_1f\neq T_2f$ on τE_1 since $k\alpha_1(x)\mathfrak{J}(a)=k\alpha_2(x)\mathfrak{J}(a)$ implies $x\notin E$.

CASE 4. $\mathfrak{J}_1\neq\mathfrak{J}_2$, $\alpha_1\neq\alpha_2$, $\tau_1=\tau_2$. Let a be such that $\mathfrak{J}_1(a)\neq\mathfrak{J}_2(a)$ and let $E=\{x: \alpha_1(x)\neq\alpha_2(x)\}$. If $\mathfrak{J}_1(a)$ is not a scalar multiple of $\mathfrak{J}_2(a)$, the case is settled by $f=a$ on a set of finite nonzero measure in E. Suppose that $\mathfrak{J}_1(a)=c\mathfrak{J}_2(a)$, where $c\neq 1$ is complex. Either there exists $E_1\subset E$ of finite nonzero measure on which $\alpha_1\neq c\alpha_2$ or $E=\{x: \alpha_1\alpha_2^{-1}(x)=c\}$. In the first instance choose $f=a$ on E_1. In the second E' is open and nonempty since it contains e. Let $E_2\subset E'$ be of finite nonzero measure and take $f=a$ on E_2.

The purpose of the remainder of this paper is to obtain conditions under which an isomorphism T must be of the form $(\mathfrak{J}, \tau, \alpha)$. If M_B is a maximal ideal of $B^1(G, A)$, TM_B is a maximal ideal of $B^1(\tilde{G}, \tilde{A})$. Also $\hat{f}(M_B)=(Tf)\hat{}\,(TM_B)$, $f\in B^1(G, A)$, for the kernel of the homomorphism $f\rightarrow(Tf)\hat{}\,(TM_B)$ is clearly M_B. Since M_B is of the form (α, M), $\alpha\in\hat{G}$, $M\in\mathfrak{M}_A$, and TM_B is of the form $(\tilde{\alpha}, \tilde{M})$, $\tilde{\alpha}\in\tilde{G}\hat{}$, $\tilde{M}\in\mathfrak{M}_{\tilde{A}}$, T gives rise to correspondences between the characters of G and $\tilde{G}$ and the maximal ideals of A and $\tilde{A}$. However, these correspondences can be intractable. Consider, for example, the isomorphism T of $B^1(G, L^1(H))$ onto $L^1(G\times H)$. Since $\hat{f}(\alpha_1, \alpha_2)=(Tf)\hat{}\,(\alpha_1, \alpha_2)$, $\alpha_1\in\hat{G}$, $\alpha_2\in\hat{H}$, $f\in B^1$ (see remarks at end of §3), $T(\alpha_1, \alpha_2)=((\alpha_1, \alpha_2), (0))$, where (0) is the single ideal of the complexes. If α_1 is held fixed and α_2 is varied, the maximal ideal component of $T(\alpha_1, \alpha_2)$ cannot change but the character component does. To allow for this kind of behavior we shall use the notation $T(\alpha, M)=(\tilde{\alpha}_{\alpha,M}, \tilde{M}_{\alpha,M})$.

If T is a triple $(\mathfrak{I}, \tau, \alpha_0)$, the complications discussed in the last paragraph do not appear. For any $s \in G$, $f \in B^1(G, A)$,

$$(Tf_s)(\tau x) = k\alpha_0(x)\mathfrak{I}[f_s(x)] = \bar{\alpha}_0(s)(Tf)(\tau sx) = \bar{\alpha}_0(s)(Tf)_{\tau s}(\tau x).$$

Hence, if $f \notin M_B = (\alpha, M)$, by Theorem 2.3,

$$\begin{aligned}\tilde{\alpha}_{\alpha,M}(\tau x) &= \frac{(Tf)\hat{\ }_{\tau x}(TM_B)}{(Tf)\hat{\ }(TM_B)} = \frac{\alpha_0(x)(Tf_x)\hat{\ }(TM_B)}{(Tf)\hat{\ }(TM_B)} = \frac{\alpha_0(x)\hat{f}_x(M_B)}{\hat{f}(M_B)} \\ &= \alpha_0(x)\alpha(x).\end{aligned}$$

If λ is such that $\hat{\lambda}(\alpha) = 1$, then for any $a \in A$ the integral formula yields

$$\begin{aligned}\hat{a}(M) &= (a\lambda)\hat{\ }(M_B) = [T(a\lambda)]\hat{\ }(TM_B) \\ &= \int k\alpha_0(x)(\mathfrak{I}[a\lambda(x)])\hat{\ }(\tilde{M}_{\alpha,M})\bar{\tilde{\alpha}}_{\alpha,M}(\tau x)d\tau x \\ &= (\mathfrak{I}a)\hat{\ }(\tilde{M}_{\alpha,M})\int \alpha_0(x)\lambda(x)\bar{\alpha}_0(x)\bar{\alpha}(x)dx = (\mathfrak{I}a)\hat{\ }(\tilde{M}_{\alpha,M}).\end{aligned}$$

Hence for every $M \in \mathfrak{M}_A$, $\tilde{M}_{\alpha,M} = \mathfrak{I}M$ regardless of α and for every $\alpha \in \hat{G}$, $\tilde{\alpha}_{\alpha,M}(\tau x) = \alpha_0(x)\alpha(x)$ regardless of M. Conversely, if $B^1(\tilde{G}, \tilde{A})$ is semi-simple, these conditions guarantee that T is the triple $(\mathfrak{I}, \tau, \alpha_0)$. Before proving this we observe that the semi-simplicity of any one of the algebras A, $\tilde{A}$, $B^1(G, A)$, or $B^1(\tilde{G}, \tilde{A})$ implies that of the others (Theorem 3.1). In the statements of results requiring semi-simplicity we shall say merely that A is semi-simple.

Lemma 4.1. *Let T be an isomorphism of $B^1(G, A)$ onto $B^1(\tilde{G}, \tilde{A})$, where A is semi-simple. Let $\mathfrak{I}$ be an isomorphism of A onto $\tilde{A}$, τ an isomorphism of G onto $\tilde{G}$, and $\alpha_0 \in \hat{G}$ such that for every $\alpha \in \hat{G}$, $M \in \mathfrak{M}_A$, $\tilde{\alpha}_{\alpha,M}(\tau x) = \alpha_0(x)\alpha(x)$ and $\tilde{M}_{\alpha,M} = \mathfrak{I}M$. Then T is the isomorphism $(\mathfrak{I}, \tau, \alpha_0)$.*

Proof. Let T_1 denote the isomorphism $(\mathfrak{I}, \tau, \alpha_0)$. If $f \in B^1(G, A)$, for any $M_B = (\alpha, M)$

$$\begin{aligned}(T_1 f)\hat{\ }(TM_B) &= \int \alpha_0(x)(\mathfrak{I}[f(x)])\hat{\ }(\mathfrak{I}M)\bar{\alpha}_0(x)\bar{\alpha}(x)dx \\ &= \int f(x)\hat{\ }(M)\bar{\alpha}(x)dx = \hat{f}(M_B) = (Tf)\hat{\ }(TM_B).\end{aligned}$$

Since every maximal ideal of the semi-simple algebra $B^1(\tilde{G}, \tilde{A})$ has the form TM_B, $T_1 f = Tf$ for every $f \in B^1(G, A)$.

Let us examine further the case where T is given as $(\mathfrak{I}, \tau, \alpha_0)$. Let (α_1, M_1) be a fixed but arbitrary maximal ideal of $B^1(G, A)$. Then $f \in \bigcap_{\mathfrak{M}_A} (\alpha_1, M)$ if and only if $\int f(x)\bar{\alpha}_1(x)dx \in \bigcap_{\mathfrak{M}_A} M$. Since

$$\int (Tf)(\tau x)\bar{\tilde{\alpha}}_{\alpha_1,M_1}(\tau x)d\tau x = \mathfrak{I}\int f(x)\bar{\alpha}_1(x)dx,$$

this is equivalent to

$$\int (Tf)(\tau x)\bar{\tilde{\alpha}}_{\alpha_1,M_1}(\tau x)d\tau x \in \bigcap_{\mathfrak{M}_A} \mathfrak{I}M = \bigcap_{\mathfrak{M}_{\tilde{A}}} \tilde{M},$$

and hence to $Tf\in\bigcap_{\mathfrak{M}_{\tilde{A}}} (\tilde{\alpha}_{\alpha_1,M_1}, \tilde{M})$. Thus

$$T\left[\bigcap_{\mathfrak{M}_A} (\alpha_1, M)\right] = \bigcap_{\mathfrak{M}_{\tilde{A}}} (\tilde{\alpha}_{\alpha_1,M_1}, \tilde{M}).$$

Also $f\in\bigcap_{\hat{G}} (\alpha, M_1)$ if and only if $f(x)\in M_1$ for almost every x, which is true if and only if $(Tf)(\tau x)\in\mathfrak{I}M_1=\tilde{M}_{\alpha_1,M_1}$ for almost every τx. Since the last is equivalent to $Tf\in\bigcap_{\tilde{G}^\wedge}(\tilde{\alpha}, M_{\alpha_1,M_1})$,

$$T\left[\bigcap_{\hat{G}} (\alpha, M_1)\right] = \bigcap_{\tilde{G}^\wedge} (\tilde{\alpha}, \tilde{M}_{\alpha_1,M_1}).$$

These are essentially the conditions which will be used to obtain a statement converse to Theorem 4.1. Their intrinsic appeal lies in the fact that they are basic requirements for the construction from T of a $\mathfrak{I}$ and an isomorphism of $L^1(G)$ onto $L^1(\tilde{G})$ by means of the Fourier transform. Note that these conditions do not hold for our isomorphism of $B^1(G, L^1(H))$ onto $L^1(G\times H)$.

DEFINITION. Let T be an isomorphism of $B^1(G, A)$ onto $B^1(\tilde{G}, \tilde{A})$. Condition I is said to hold for T at $\alpha_1\in\hat{G}$ if there exists $M_1\in\mathfrak{M}_A$ such that $T[\bigcap_{\mathfrak{M}_A}(\alpha_1, M)]=\bigcap_{\mathfrak{M}_{\tilde{A}}}(\tilde{\alpha}_{\alpha_1,M_1}, \tilde{M})$. Condition II is said to hold for T at $M_1\in\mathfrak{M}_A$ if there exists $\alpha_1\in\hat{G}$ such that $T[\bigcap_{\hat{G}}(\alpha, M_1)]=\bigcap_{\tilde{G}^\wedge}(\tilde{\alpha}, \tilde{M}_{\alpha_1,M_1})$.

LEMMA 4.2. *Let A be semi-simple. If T is an isomorphism for which Condition* I *holds at some $\alpha_1\in\hat{G}$, then there is an isomorphism $\mathfrak{I}_{\alpha_1}$ of A onto $\tilde{A}$, and for every $M\in\mathfrak{M}_A$*

$$\tilde{\alpha}_{\alpha_1,M} = \tilde{\alpha}_{\alpha_1,M_1}, \qquad \tilde{M}_{\alpha_1,M} = \mathfrak{I}_{\alpha_1}M.$$

Proof. Suppose that there exists $M_2\in\mathfrak{M}_A$ such that $\tilde{\alpha}_{\alpha_1,M_2}\neq\tilde{\alpha}_{\alpha_1,M_1}$. Choose $\tilde{\lambda}\in L^1(\tilde{G})$ such that $\tilde{\lambda}^\wedge(\tilde{\alpha}_{\alpha_1,M_1})=0$ and $\tilde{\lambda}^\wedge(\tilde{\alpha}_{\alpha_1,M_2})\neq 0$. If $\tilde{a}\notin\tilde{M}_{\alpha_1,M_2}$, then $\tilde{a}\tilde{\lambda}\notin(\tilde{\alpha}_{\alpha_1,M_2})\tilde{M}_{\alpha_1,M_2}=T(\alpha_1,M_2)$. But $\tilde{a}\tilde{\lambda}\in\bigcap_{\mathfrak{M}_{\tilde{A}}}(\tilde{\alpha}_{\alpha_1,M_1}, \tilde{M})$ and hence, by Condition I, $T^{-1}(\tilde{a}\tilde{\lambda})\in\bigcap_{\mathfrak{M}_A}(\alpha_1, M)$. In particular $T^{-1}(\tilde{a}\tilde{\lambda})\in(\alpha_1, M_2)$ so that $\tilde{a}\tilde{\lambda}\in T(\alpha_1, M_2)$, which is impossible. Therefore $\tilde{\alpha}_{\alpha_1,M}=\tilde{\alpha}_{\alpha_1,M_1}$ for every $M\in\mathfrak{M}_A$.

Theorem 2.1 asserts that for any $a\in A$ there exists $f\in B^1(G, A)$ such that $\int f(x)\bar{\alpha}_1(x)dx=a$, and we accordingly set

$$\mathfrak{I}_{\alpha_1}a = \int (Tf)(\tilde{x})\bar{\tilde{\alpha}}_{\alpha_1,M_1}(\tilde{x})d\tilde{x}.$$

If g is such that $\int g(x)\bar{\alpha}_1(x)dx = a$, then, by Condition I, $Tf - Tg \in \bigcap_{\mathfrak{M}_{\tilde{A}}}(\tilde{\alpha}_{\alpha_1,M_1}, \tilde{M})$, and since $\tilde{A}$ is semi-simple,

$$\int (Tf)(\tilde{x})\bar{\tilde{\alpha}}_{\alpha_1,M_1}(\tilde{x})d\tilde{x} = \int (Tg)(\tilde{x})\bar{\tilde{\alpha}}_{\alpha_1,M_1}(\tilde{x})d\tilde{x}.$$

Thus $\mathfrak{J}_{\alpha_1}$ is a well defined mapping of A into $\tilde{A}$. Because of the symmetry of Condition I and the semi-simplicity of A, the argument just used can be reversed, and therefore $\mathfrak{J}_{\alpha_1}$ is one-to-one. It follows directly from Theorem 2.1 and the fact that T is an isomorphism that $\mathfrak{J}_{\alpha_1}$ maps A onto $\tilde{A}$ and that it is linear and multiplicative. Since the topology of a commutative semi-simple Banach algebra is uniquely determined, $\mathfrak{J}_{\alpha_1}$ is, in addition, a homeomorphism.

If f is such that $\int f(x)\bar{\alpha}_1(x)dx = a$, the following four conditions are equivalent: $a \in M$; $f \in (\alpha_1, M)$; $Tf \in (\tilde{\alpha}_{\alpha_1,M}, \tilde{M}_{\alpha_1,M}) (= (\tilde{\alpha}_{\alpha_1,M_1}, \tilde{M}_{\alpha_1,M}))$; $\mathfrak{J}_{\alpha_1}a \in \tilde{M}_{\alpha_1,M}$. Hence $\tilde{M}_{\alpha_1,M} = \mathfrak{J}_{\alpha_1}M$ for every $M \in \mathfrak{M}_A$.

If S is an isomorphism of $L^1(G)$ onto $L^1(\tilde{G})$ and $\alpha \in \hat{G}$, the maximal ideal to which α corresponds is carried by S into a maximal ideal of $L^1(\tilde{G})$. The corresponding character will be denoted by $S\alpha$. The next result parallels the last and provides a means for constructing the group isomorphism we seek. This time semi-simplicity is not required since our concern is with group algebras.

LEMMA 4.3. *If T is an isomorphism for which Condition* II *holds at some $M_1 \in \mathfrak{M}_A$, then there is an isomorphism S_{M_1} of $L^1(G)$ onto $L^1(\tilde{G})$, and for every $\alpha \in \hat{G}$*

$$\tilde{M}_{\alpha,M_1} = \tilde{M}_{\alpha_1,M_1}, \qquad \tilde{\alpha}_{\alpha,M_1} = S_{M_1}\alpha.$$

Proof. Because of the obvious similarity to Lemma 4.2, we shall carry through only the construction of S_{M_1}. If $f \in B^1(G, A)$, the function λ determined by $\lambda(x) = f(x)\hat{\ }(M_1)$ is in $L^1(G)$. A linear mapping of $B^1(G, A)$ onto $L^1(G)$ is thereby defined by M_1. The mapping is also a homomorphism; for if $f(x)\hat{\ }(M_1) = \lambda(x)$ a.e. and $g(x)\hat{\ }(M_1) = \nu(x)$ a.e., then

$$(f * g)(x)\hat{\ }(M_1) = \int f(xy)\hat{\ }(M_1)g(y^{-1})(M_1)dy = (\lambda * \nu)(x) \text{ a.e.}$$

$\tilde{M}_{\alpha_1,M_1}$ provides in like manner a homomorphism of $B^1(\tilde{G}, \tilde{A})$ onto $L^1(\tilde{G})$. Let $S_{M_1}\lambda$ be defined by the equation

$$(S_{M_1}\lambda)(\tilde{x}) = (Tf)(\tilde{x})\hat{\ }(\tilde{M}_{\alpha_1,M_1}),$$

where f is such that $f(x)\hat{\ }(M_1) = \lambda(x)$ a.e. Arguments much like those used above complete the proof.

If T is an isometry, or even such that $\|T\| \leq 1$, a technique which will be used in the proof of the next theorem shows that S_{M_1} is an isometry. Wendel's theorem then produces an isomorphism of the groups and a character. A question which has not been answered is whether these assumptions combined with those of Lemma 4.2 are sufficient to make T a triple. We shall instead assume that Conditions I and II hold everywhere.

THEOREM 4.3. *Let A be semi-simple and T an isometric isomorphism of $B^1(G, A)$ onto $B^1(\tilde{G}, \tilde{A})$ for which Condition* I *holds at every $\alpha \in \hat{G}$ and Condition* II *holds at every $M \in \mathfrak{M}_A$. Then there exist an isometric isomorphism $\mathfrak{J}$ of A onto $\tilde{A}$, an isomorphism τ of G onto $\tilde{G}$, and a character $\alpha_0 \in \hat{G}$ such that T is the isomorphism $(\mathfrak{J}, \tau, \alpha_0)$.*

Proof. Let α_1 and α_2 be arbitrary characters of G. Since the hypotheses of Lemma 4.2 are satisfied for both characters, there exist isomorphisms $\mathfrak{J}_{\alpha_1}$ and $\mathfrak{J}_{\alpha_2}$ of A onto $\tilde{A}$ such that for every $M \in \mathfrak{M}_A$, $\mathfrak{J}_{\alpha_1} M = \tilde{M}_{\alpha_1, M}$ and $\mathfrak{J}_{\alpha_2} M = \tilde{M}_{\alpha_2, M}$. Since Lemma 4.3 obtains for every M, $\tilde{M}_{\alpha_1, M} = \tilde{M}_{\alpha_2, M}$. Hence $\mathfrak{J}_{\alpha_1} M = \mathfrak{J}_{\alpha_2} M$ for every $M \in \mathfrak{M}_A$. Let $a \in A$. If $\tilde{M} \in \mathfrak{M}_{\tilde{A}}$, then $\mathfrak{J}_{\alpha_1}\mathfrak{J}_{\alpha_2}^{-1}\tilde{M} = \mathfrak{J}_{\alpha_2}\mathfrak{J}_{\alpha_2}^{-1}\tilde{M} = \tilde{M}$. Therefore

$$(\mathfrak{J}_{\alpha_1} a)^\wedge(\tilde{M}) = (\mathfrak{J}_{\alpha_1} a)^\wedge(\mathfrak{J}_{\alpha_1}\mathfrak{J}_{\alpha_2}^{-1}\tilde{M}) = \hat{a}(\mathfrak{J}_{\alpha_2}^{-1}\tilde{M}) = (\mathfrak{J}_{\alpha_2} a)^\wedge(\tilde{M}).$$

Thus for every $a \in A$, $(\mathfrak{J}_{\alpha_1} a)^\wedge \equiv (\mathfrak{J}_{\alpha_2} a)^\wedge$ and it follows from the semi-simplicity of $\tilde{A}$ that $\mathfrak{J}_{\alpha_1} a = \mathfrak{J}_{\alpha_2} a$. Hence $\mathfrak{J}_{\alpha_1}$ and $\mathfrak{J}_{\alpha_2}$ are identical isomorphisms. Since α_1 and α_2 are arbitrary, $\mathfrak{J} \equiv \mathfrak{J}_\alpha$, for any α, is independent of α, and $\mathfrak{J} M = \tilde{M}_{\alpha, M}$ for every $\alpha \in \hat{G}$, $M \in \mathfrak{M}_A$.

Let S_M, $M \in \mathfrak{M}_A$, be the isomorphism of $L^1(G)$ onto $L^1(\tilde{G})$ given by Lemma 4.3. In the fashion of the preceding paragraph we deduce that $S = S_M$ is independent of M, and $S\alpha = \tilde{\alpha}_{\alpha, M}$ for every $\alpha \in \hat{G}$, $M \in \mathfrak{M}_A$.

If $M \in \mathfrak{M}_A$ and u_0 is an identity modulo M, then

$$1 = \hat{u}_0(M) = |\hat{u}_0(M)| = \inf_{u \in u_0 + M} \|u\|_A.$$

Hence for $\epsilon > 0$, there exists an identity u_1 modulo M such that $\|u_1\|_A < 1 + \epsilon$. For $\lambda \in L^1(G)$ and $f = u_1\lambda$, we have $f(x)^\wedge(M) = \lambda(x)$. By the definition of S_M (see proof of Lemma 4.3) and because T is an isometry,

$$\begin{aligned} \|S_M\lambda\|_{L^1(\tilde{G})} &\leqq \|Tf\|_{\tilde{B}} = \|f\|_B = \|u_1\|_A \|\lambda\|_{L^1(G)} \\ &< (1+\epsilon)\|\lambda\|_{L^1(G)}. \end{aligned}$$

Since ϵ is arbitrary, $\|S_M\lambda\|_{L^1(\tilde{G})} \leqq \|\lambda\|_{L^1(G)}$. Thus $\|S\| \equiv \|S_M\| \leqq 1$. Wendel's theorem then applies. There exist an isomorphism τ of G onto $\tilde{G}$ and a character α_0 such that for every $\lambda \in L^1(G)$ and $x \in G$, $S\lambda_x = \bar{\alpha}_0 \cdot (S\lambda)_{\tau x}$. Let $\alpha \in \hat{G}$. If $\hat{\lambda}(\alpha) \neq 0$, then

$$\begin{aligned} \tilde{\alpha}_{\alpha, M}(\tau x) = (S\alpha)(\tau x) &= \frac{(S\lambda)^\wedge_{\tau x}(S\alpha)}{(S\lambda)^\wedge(S\alpha)} = \alpha_0(x)\frac{(S\lambda_x)^\wedge(S\alpha)}{(S\lambda)^\wedge(S\alpha)} \\ &= \alpha_0(x)\frac{\hat{\lambda}_x(\alpha)}{\hat{\lambda}(\alpha)} = \alpha_0(x)\alpha(x) \end{aligned}$$

for every $M \in \mathfrak{M}_A$. It follows from Lemma 4.1 that $T = (\mathfrak{J}, \tau, \alpha_0)$ and, therefore, from Theorem 4.1, that $\mathfrak{J}$ is an isometry.

Bibliography

1. H. Helson, *Isomorphisms of abelian group algebras*, Ark. Mat. vol. 2 (1954) pp. 475–487.

2. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, New York, 1948.

3. L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953.

4. J. von Neumann, *Almost periodic functions in a group*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 445–492.

5. D. A. Raikov, *Harmonic analysis on commutative groups with the Haar measure and the theory of characters*, Trav. Inst. Math. Stekloff vol. 14 (1945).

6. I. E. Segal, *The group algebra of a locally compact group*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 69–105.

7. W. Sierpinski, *Sur un problème concernant les ensembles measurables superficiellement*, Fund. Math. vol. 1 (1920) pp. 112–115.

8. J. Wendel, *On isometric isomorphism of group algebras*, Pacific J. Math. vol. 1 (1951) pp. 305–311.

University of Minnesota,
Minneapolis, Minnesota