ON THE EMBEDDING OF VECTOR LATTICES IN F-RINGS[1]

BY
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1. **Introduction.** In this paper the term *F-space* is used to denote a σ-complete vector lattice with a weak order unit. (Other authors [8] do not require that a vector lattice with a weak order unit be σ-complete in order to be an F-space.) The term F-ring [2] denotes a σ-complete lattice-ordered rings with an identity which is positive and is a weak order unit. From [8] and [3] it follows that an F-space L with weak order unit u determines a unique (up to an isomorphism) regular F-ring $R(L, u)$ such that L is isomorphically embedded in $R(L, u)$ and u becomes the identity of $R(L, u)$ under the embedding.

In [7] it is shown that an F-space L with weak order unit u can be isomorphically embedded in a ring $N(L)$ of operators on L. This ring is referred to by Nakano as the ring of dilatators on L. (See [7, §43] for definition of dilatator.) §2 is devoted to showing that $N(L)$ and $R(L, u)$ are isomorphic, as well as showing that if u and v are different weak order units of L, then $R(L, u) \cong R(L, v)$.

In §3 certain propositions are proved which follow from the embeddability of an F-space in an F-ring.

It is known [3] that the class of idempotents of an F-ring R forms a σ-complete Boolean algebra. If this Boolean algebra supports a countably additive measure μ, then the set $U = \{f \in R \mid \mu(\bigvee_{n=1}^{\infty} (1 \wedge n|f|)) = 0\}$ is a closed ideal of R. U is the generalization for F-rings of the family of measurable functions which vanish except on a set of measure zero. In §4 properties of the quotient F-ring $R-U$ are discussed.

Finally in §5 we characterize the class of all bounded linear functionals defined on a regular F-ring, as well as the class of linear functionals on an F-space L with weak order unit u which can be extended to bounded linear functionals on the regular F-ring $R(L, u)$ mentioned in the first paragraph of this Introduction.

The notation of [2] and [3] is used here; in particular: $x^+ = x \vee 0$, $x^- = (-x)^+$, $|x| = x^+ + x^-$, $\bar{e}_x = \bigvee_{n=1}^{\infty} (1 \wedge n|x|)$, and $e_x = 1 - \bar{e}_x$. If L is an F-space with weak order unit u, then $B(L, u) = \{e \in L \mid e \wedge (u-e) = 0\}$ is well known [8] to be a σ-complete Boolean algebra. If L is an F-ring and u is the identity of L, then $e \wedge (u-e) = 0$ if and only if $e^2 = e$, and hence in this case

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$B(L, u)$ is the idempotent algebra [3] of L. An [1] l-ideal J of an F-space L is said to be *closed* if $a_n \in J$ for $n \geqq 1$ and $\bigvee_{n=1}^{\infty} a_n \in L$ imply $\bigvee_{n=1}^{\infty} a_n \in J$. A ring ideal J of an F-ring R is said to be *closed* if it is an l-ideal which is closed. An F-ring R is regular if for each $a \in R$ there is an x such that $axa = a$. It is easy to verify that for regular F-rings and bounded F-rings (those for which the identity 1 is a strong unit [1]) all ring ideals are also l-ideals. The converse here is not valid. Indeed if L is an F-space with weak order unit u and if $R(L, u)$ is a proper extension of L it can be shown (Theorem 3.1) that L is an l-ideal of $R(L, u)$ which need not be a ring ideal. In §3 it is shown (Theorem 3.2) that a closed l-ideal of an F-ring is a ring ideal.

A maximal l-ideal (ring ideal) N of an F-space L (F-ring R) is *real* if the quotient space $L-N$ (quotient ring $R-N$) is isomorphic to the ordered group (ring) of real numbers. An ideal of a Boolean algebra is said to be *closed* if it is closed with respect to countable sup's. A set $\{e_\gamma \mid \gamma \in \Gamma\}$ of non-negative elements of an F-space is said to be *orthogonal* if $e_{\gamma_1} \wedge e_{\gamma_2} = 0$ whenever $\gamma_1 \neq \gamma_2$. An F-space L is *orthogonally complete* [7, p. 156] if for each orthogonal sequence $\{f_n\}$ of non-negative elements of L the supremum $\bigvee_{n=1}^{\infty} f_n$ belongs to L. If e is an element of a Boolean algebra then $\bar{e}$ denotes the complement of e.

A linear functional ξ on an F-space L is said to be *continuous* if for each nonincreasing sequence $\{a_n\}$ of elements of L such that $\bigwedge_{n=1}^{\infty} a_n = 0$ we have $\lim_n \xi(a_n) = 0$. If B is a σ-complete Boolean algebra, by a *measure on B* we mean a functional μ on B which satisfies the following conditions: (i) if $e_1, e_2 \in B$ and $e_1 \wedge e_2 = 0$, then $\mu(e_1) + \mu(e_2) = \mu(e_1 \vee e_2)$, (ii) $\mu(e) \geqq 0$ for all $e \in B$. A measure μ on B is *countably additive* if when $\{e_n\}$ is an orthogonal sequence of elements of B, then $\mu(\bigvee_{n=1}^{\infty} e_n) = \sum_{n=1}^{\infty} \mu(e_n)$. A measure μ on B is *normal* if $\mu(1) = 1$. It is clear that any nontrivial measure μ on B can be normalized by dividing by $\mu(1)$.

2. **Extensions of F-spaces.** Let L be an F-space with weak order unit u. In [8] a lattice-ordered ring $R(L, u)$ is constructed such that L is isomorphic to an F-subspace of $R(L, u)$ and the image of u under this isomorphism is the identity of $R(L, u)$. In [3] we show that $R(L, u)$ is a regular F-ring, and then it is clear from the results of [3; 8] that $B(L, u)$ is mapped onto the idempotent algebra of $R(L, u)$ by this embedding isomorphism. We also show in [3] that the mapping which carries a regular F-ring R with identity 1 into its idempotent algebra $B(R, 1)$ is a mapping which is one to one (up to an isomorphism) from the set of regular F-rings onto the set of σ-complete Boolean algebras. In addition we show that every regular F-ring can be faithfully represented as the σ-homomorphic image of an M-ring [2]. The "σ" preceding the word "homomorphism" indicates that this homomorphism preserves countable sup's and inf's.

Nakano considers another method of extending an F-space. He defines [7, §33] what he calls a completion of a σ-complete vector lattice. This completion is an extension of the vector lattice which is unique up to an isomorphism. If the vector lattice is an F-space L, then the completion $N(L)$ of L is isomorphic to the dilatator ring of L. This result is embodied in [7, Theorem 45.7]. The dilatators of L are discussed in [7, §§44, 45]. In order to prove that $N(L)$ and $R(L, u)$ are isomorphic, we first consider the following lemma.

LEMMA 2.1. *If L is an F-space with weak order units u and v, then $B(L, u)$ and $B(L, v)$ are isomorphic Boolean algebras and therefore the F-rings $R(L, u)$ and $R(L, v)$ are isomorphic.*

Proof. Consider $R(L, u)$ and identify L with its isomorphic copy in $R(L, u)$. For the remainder of this proof all elements considered are considered as elements of $R(L, u)$. In particular any products discussed are products as defined in $R(L, u)$. From [8] it follows that $B(R(L, u), u)=B(L, u)$, and $B(L, v)$ is a subset of $R(L, u)$.

The mapping $e \rightarrow ve$ is an isomorphism of $B(L, u)$ onto $B(L, v)$. Indeed, if $e \in B(L, u)$, then $ve \wedge (v-ve)=v[e \wedge (u-e)]=0$ and hence $ve \in B(L, v)$. Since v is a weak order unit of L it is a weak order unit of $R(L, u)$ and hence $v^{-1} \in R(L, u)$. The first clause of the preceding sentence follows because every element of $R(L, u)$ is a sup of disjoint elements of L and the second clause follows from regularity together with the statement (valid in $R(L, u)$): If $a \geqq 0$, $b \geqq 0$, then $a \wedge b=0 \Leftrightarrow ab=0$.

The existence of $v^{-1} \in R(L, u)$ ensures that every element of $B(L, v)$ is the image of a unique element of $B(L, u)$ and hence the mapping $e \rightarrow ve$ is one-to-one and onto. The lattice operations are preserved because of the distributivity of multiplication with respect to the lattice operations and because the complement of $e \in B(L, v)$ is of the form $v-e$.

That $R(L, u) \cong R(L, v)$ follows from the results of [3].

THEOREM 2.2. *If L is an F-space with weak order unit u, the dilatator ring $N(L)$ of L is an F-ring which is isomorphic to $R(L, u)$.*

Proof. Since $R(L, u)$ satisfies conditions (1) through (4) in [7, §33] it follows that $R(L, u)$ is a completion of L. Moreover the F-spaces $R(L, u)$ and $N(L)$ are isomorphic because $N(L)$ is a completion and [7, Theorem 33.4] all completions of L are isomorphic. Let ρ be the isomorphism of $R(L, u)$ onto $N(L)$. Now $\rho(u)$ is a weak order unit of $N(L)$, and $B(N(L), \rho(u)) \cong B(R(L, u), u) \cong B(L, u)$. It follows from [3, Theorem 7] and from the remark in the first paragraph of this section that the F-rings $R(N(L), \rho(u))$ and $R(L, u)$ are isomorphic.

From [7, Theorems 29.9 and 44.1] we deduce that the dilatator ring $N(L)$ of L possesses a multiplicative identity $1 \geqq 0$ which is a weak order unit. Therefore the dilatator ring $N(L)$ is an F-ring. Now (Lemma 2.1) F-rings $R(N(L), \rho(u))$ and $R(N(L), 1)$ are isomorphic. From [7, Theorem 44.1] and the properties of Nakano's proper spaces, it follows that every principal ideal of $N(L)$ is generated by an idempotent. Hence $N(L)$ is regular and the F-rings $N(L)$ and $R(N(L), 1)$ are isomorphic. Thus

$$R(L, u) \cong R(N(L), \rho(u)) \cong R(N(L), 1) \cong N(L).$$

COROLLARY 2.3. *A necessary and sufficient condition for an F-space L to be orthogonally complete is that it be possible to define a product in L which turns L into a regular F-ring.*

Proof. If L is orthogonally complete, then by the definition of completion [7, §33] it is clear that the identity mapping on L is a completion of L. Since [7, Theorem 33.4] the completion of L is unique, it follows that L is its own dilatator ring. Therefore a product may be defined in L with respect to which L forms a regular F-ring.

Conversely, if R is a regular F-ring, $R \cong R(R, 1)$ and Theorem 2.2 implies $R \cong N(R)$ and hence R is orthogonally complete.

3. **Some properties of F-spaces.** This section is devoted to the study of some properties of F-spaces which follow from their embeddability in F-rings.

Let L stand for an F-space with weak order unit 1. Since L is an F-subspace of $R(L, 1)$, certain pairs a, b of elements of L have the property that their product ab in $R(L, 1)$ also belongs to L. Again we identify L with its isomorphic copy in $R(L, 1)$.

THEOREM 3.1. *The F-space L is an l-ideal of $R(L, 1)$, and if $a, b \in L$ and $|a| \leqq \lambda \cdot 1$ for some $\lambda > 0$, then $ab \in L$.*

Proof. L is clearly a subspace of $R(L, 1)$; in order to show that it is an l-ideal we must prove that $a \in R(L, 1)$, $b \in L$, and $0 \leqq a \leqq |b|$ imply $a \in L$. The element $a \in R(L, 1)$ is the sup of a sequence of elements of L [1, p. 251], and since L is σ-complete, $a \in L$. Therefore L is an l-ideal of $R(L, 1)$.

To prove the second part of the theorem, note that $\lambda|b| \in L$ and $|ab| \leqq \lambda|b|$. Hence $ab \in L$ because L is an l-ideal of $R(L, 1)$.

THEOREM 3.2. *Let M be a closed l-ideal of L and let $b \in M$. If $a \in L$ and $ab \in L$, then $ab \in M$; hence closed l-ideals of F-rings are ring ideals.*

Proof. Assume $a \geqq 0$ and $b \geqq 0$. Then for each integer $n \geqq 0$,

$$a_n = a[e_{(a-n)^+} - e_{(a-n+1)^+}]\, b \leqq nb.$$

Since a_n is dominated by an element $nb \in M$, $a_n \in M$ for all $n \geqq 0$. Therefore $ab = \bigvee_{n=1}^{\infty} a_n b$ belongs to M.

If a and b are not non-negative, then $ab \in L$ implies $|ab| \in L$. Since a^+b^+, a^-b^-, a^+b^-, and a^-b^+ are all dominated by $|ab| \in L$, it follows that each belongs to L and therefore each belongs to M.

THEOREM 3.3. *The correspondence ϕ: $M \rightarrow M \cap L$ is a one-to-one mapping of the closed maximal ideals of $R(L, 1)$ onto the closed maximal l-ideals of L. The* *inverse of ϕ can be written ϕ^{-1}: $M \to (M \cap B(L, 1))R(L, 1)$ where $AR(L, 1)$ stands for the (ring) ideal of $R(L, 1)$ generated by the set $A \subseteq R(L, 1)$.*

Proof. For the remainder of this proof we use the notation R for $R(L, 1)$ and B for $B(L, 1)$, and again L is identified with its isomorphic copy in R. Let Ω stand for the class of closed maximal ideals of R and let Ω' stand for the class of closed maximal l-ideals of L.

In order to show $\phi(\Omega) \subseteq \Omega'$ consider $L - M \cap L$ where $M \in \Omega$; this ordered group is isomorphic to the ordered group $[L+M]-M$ because M is an l-ideal. In [2, p. 677] it is shown that $\overline{R}+M=R$ where $\overline{R}$ signifies the F-ring composed of those elements $b \in R$ for which a $\lambda>0$ exists such that $|b| \leqq \lambda \cdot 1$. Since $L \supseteq \overline{R}$, it follows that $L+M=R$ and $[L+M]-M$ is an ordered group [2, Theorem 5] which is isomorphic to G, the ordered group of real numbers. Therefore $L - M \cap L$ and G are ordered group-isomorphic, and since G has no proper l-ideals, $M \cap L \in \Omega'$.

To show $\phi(\Omega) \supseteq \Omega'$ and $\phi \circ \phi^{-1}$ is the identity mapping on Ω', let $M' \in \Omega'$. Then $M' \cap B$ is a closed maximal ideal of B. By [4, Lemma 2] $M=(M' \cap B)R$ is a closed maximal ideal of R. Both propositions to be proved are valid if the sets $M \cap L$ and M' are equal. Since R is regular, every element $a \in R$ has the property $a \in M$ if and only if $\bar{e}_a \in M' \cap B$, and because of Theorem 3.2 every element $a \in L$ has the property $a \in M'$ if and only if $\bar{e}_a \in M' \cap B$. From these two statements it is easy to deduce that $M \cap L = M'$.

Since it is a trivial consequence of [4, Lemma 2] that $\phi^{-1}(\Omega') \subseteq \Omega$, the theorm follows if it can be shown that $\phi^{-1}(\Omega') \supseteq \Omega$ and $\phi^{-1} \circ \phi$ is the identity mapping on Ω. Let $M \in \Omega$. Then $M \cap L \in \phi(\Omega)$ and $\phi^{-1}(M \cap L) = (M \cap L \cap B)R = (M \cap B)R = M$ by [4, Lemma 2]. Therefore $\phi^{-1}(\Omega') \supseteq \Omega$, $\phi^{-1} \circ \phi(M) = M$ for all $M \in \Omega$, and the theorem follows.

COROLLARY 3.4. *If M' is a closed maximal l-ideal of L, then $L - M'$ is isomorphic to the ordered group G of real numbers.*

Proof. By Theorem 3.3, $M' = M \cap L$ where M is a closed maximal ideal of $R(L, 1)$. In the proof of Theorem 3.3 it is shown that $L - M \cap L$ is isomorphic to G.

COROLLARY 3.5. *An F-space L is isomorphic to an F-space of measurable functions if and only if the intersection of the class Ω' of all its closed maximal l-ideals is the zero ideal.*

Proof. Consider the class $\phi^{-1}(\Omega') = \Omega$. The set $I = \bigcap \phi^{-1}(\Omega')$ is an l-ideal of $R(L, 1)$. Therefore $a \in I$ implies $|a| \wedge 1 \in I$, but $|a| \wedge 1$ is an element of L and hence $|a| \wedge 1 \in \bigcap \Omega'$. By hypothesis $|a| \wedge 1 = 0$, and since 1 is a weak order unit of $R(L, 1)$ we have $a=0$. Therefore $\bigcap \Omega = \{0\}$ and [2, Theorem 7] $R(L, 1)$ is isomorphic to an F-ring of measurable functions. The corollary then follows immediately.

An F-space L (Boolean algebra B) is *m-complete*, where m is a cardinal number larger than or equal to $\aleph_0$, provided every subset A of $L(B)$ which is bounded above by an element of $L(B)$ and has power less than or equal to m has a supremum in $L(B)$.

THEOREM 3.6. *If L is an F-space with weak order unit* 1, *then* $B(L, 1)$ *is an m-complete Boolean algebra if and only if L is an m-complete F-space.*

Proof. If $B(L, 1)$ is m-complete, then it can be verified directly from Olmsted's definition of $R(L, 1)$ that $R(L, 1)$ is an m-complete F-ring. Since L is an l-ideal (Theorem 3.1) of $R(L, 1)$ it follows that the supremum of any set A ($|A| \leqq m$) of elements of L bounded above by an element of L belongs to L and hence L is m-complete.

It is clear that the m-completeness of L implies the m-completeness of $B(L, 1)$.

An F-space L (Boolean algebra B) is *complete* provided $L(B)$ is m-complete for each cardinal number m.

COROLLARY 3.7. *If L is an F-space with weak order unit* 1, *then* $B(L, 1)$ *is complete if and only if K is complete.*

4. Functionals on F-rings and measures on Boolean algebras. Let L be an F-space with weak order unit 1. If μ is a non-negative (continuous) linear functional on L, then the restriction of μ to $B(L, 1)$ is a (countably additive) measure on $B(L, 1)$. Of course not every (countably additive) measure on $B(L, 1)$ is a restriction of a non-negative (continuous) linear functional on L. It is sometimes of interest to consider those F-spaces L for which a countably additive measure μ can be defined on $B(L, 1)$ and to form the quotient space of L by the l-ideal U composed of those elements $f \in L$ which are the abstract counterpart of functions nonvanishing on a set of measure zero. Let $U = \{g \in L \mid \mu(\bar{e}_g) = 0\}$. It is clear that if L is an F-space of measurable functions, then U is the set of all elements of L which vanish on the complement of a set A such that $\mu(x_A) = 0$.

Let R be an F-ring and let $B = B(R, 1)$. Assume μ is a countably additive normal measure on B. It is clear that there is no loss in generality in assuming μ normal.

THEOREM 4.1. *If* $U = \{a \in R \mid \mu(\bar{e}_a) = 0\}$, *then the following statements are valid*:

(1) *U is a closed ideal of R.*

(2) $R - U$ *is a complete regular F-ring.*

(3) *If* $\hat{a}$ *stands for the image of* $a \in R$ *under the natural homomorphism* $R \rightarrow R - U$ *and if by definition* $\hat{\mu}(\hat{a}) = \mu(a)$ *where* $a \in \hat{a} \cap B$, *then* $\hat{\mu}$ *is a normal positive countably additive measure on* $B(R - U, \hat{1})$.

REMARK. A measure μ on a Boolean algebra B is *positive* if $\mu(a) = 0$ if and only if $a = 0$.

Proof. (1) First we show that U is an l-ideal. By definition $a \in U$ if and only if $\bar{e}_a \in U$. Therefore since $a \in U$, $b \in R$ such that $|b| \leqq |a|$ implies $b \in U$. Since for $\alpha \neq 0$ the equation $\bar{e}_a = \bar{e}_{\alpha a}$ is valid, $a \in U$ implies $\alpha a \in U$ for all real numbers α. Let f, $g \in U$. Then $|f+g| \leqq 2(|f| \vee |g|)$ and hence $\bar{e}_{(f+g)} \leqq \bar{e}_f \vee \bar{e}_g$. Therefore $\mu(\bar{e}_{(f+g)}) = 0$ and U is an l-ideal.

To show U is closed let $f_n \in U$ for $n \geqq 1$ and let $f = \bigvee_{n=1}^{\infty} f_n$ belong to R. It is a matter of direct verification to show that $\bigvee_{n=1}^{\infty} \bar{e}_{f_n} = \bar{e}_f$. Since $\mu(\bar{e}_f) \leqq \sum_{n=1}^{\infty} \mu(\bar{e}_{f_n})$, it follows that $\mu(\bar{e}_f) = 0$ and $f \in U$.

That U is a closed ring ideal of R then follows from Theorem 3.2.

(2) The results of [3, Theorem 2] imply that $R-U$ is a regular F-ring. Since the quotient Boolean algebra $B/B \cap U$ and $B(R-U, \hat{1})$ are isomorphic, it follows from [9, Theorem 4.7] that $B(R-U, \hat{1})$ is a complete Boolean algebra. Corollary 3.7 yields the result that $R-U$ is complete.

(3) The functional $\hat{\mu}$ is well defined on $\hat{B} = B(R-U, \hat{1})$. Indeed for each $\hat{a} \in \hat{B}$ if a, $b \in \hat{a} \cap B$, then $|a-b| = (a - a \wedge b) + (b - a \wedge b) \in U$ and so $a - a \wedge b$ and $b - a \wedge b$ both belong to U. Since for each $a \in B$ we have $a = \bar{e}_a$, it follows that $\mu(a - a \wedge b) = \mu(b - a \wedge b) = 0$. Therefore $\mu(a) = \mu(a \wedge b) = \mu(b)$.

$\hat{\mu}$ is clearly normal, positive, and finitely additive. To prove that $\hat{\mu}$ is countably additive, let $\{\hat{e}_n\}$ be an orthogonal sequence of elements of $\hat{B}$. Since U is a closed ideal of R, an orthogonal sequence $\{e_n\}$ can be constructed in B such that e_n maps into $\hat{e}_n$ under the natural homomorphism of R onto $R-U$. Indeed, let $\{b_n\}$ be a sequence of idempotents of R such that $b_n \to \hat{e}_n$ under the natural homomorphism. Let ψ represent this natural homomorphism; then $\psi(b_n) = \hat{e}_n$. For each pair m, n $(m \neq n)$, $b_n \wedge b_n = k_{mn}$ and $k_{mn} = k_{nm}$ belongs to $U \cap B$. Therefore $\psi(b_n \bar{k}_{mn}) = \psi(b_n)$, $\psi(b_m \bar{k}_{mn}) = \psi(b_m)$, and $(b_n \bar{k}_{mn}) \wedge (b_m \bar{k}_{mn}) = 0$. Now $k = \bigvee_{n,m} k_{mn}$ belongs to U and $\bar{k} \leqq \bar{k}_{mn}$ for $m \geqq 1$, $n \geqq 1$. Therefore $\psi(b_n \bar{k}) = \psi(b_n)$ for all $n \geqq 1$ and $b_n \bar{k} \wedge b_m \bar{k} = 0$ if $m \neq n$. Thus if $e_n = b_n \bar{k}$, $\{e_n\}$ is the required orthogonal sequence.

It is now clear that

$$\hat{\mu}\left(\bigvee_{n=1}^{\infty} \hat{e}_n\right) = \hat{\mu}\left(\bigvee_{n=1}^{\infty} e_n\right)^{\wedge} = \mu\left(\bigvee_{n=1}^{\infty} e_n\right) = \sum_{n=1}^{\infty} \mu(e_n) = \sum_{n=1}^{\infty} \hat{\mu}(\hat{e}_n),$$

and hence $\hat{\mu}$ is countably additive.

We return to the consideration of non-negative linear functionals on an F-space L with weak order unit 1.

THEOREM 4.2. *If μ is a non-negative continuous linear functional on L, then*

$$\mu(|f|) = 0 \Leftrightarrow \mu(\bar{e}_f) = 0.$$

Proof. Suppose $\mu(|f|) = 0$. Then $\mu(n|f| \wedge 1) = 0$. The sequence $\bar{e}_f - n|f| \wedge 1$ is a nonincreasing sequence with infimum zero. Therefore $\lim_n \mu(\bar{e}_f - n|f| \wedge 1) = \lim_n [\mu(\bar{e}_f) - \mu(n|f| \wedge 1)] = \mu(\bar{e}_f) - \lim_n \mu(n|f| \wedge 1) = \mu(\bar{e}_f) = 0.$

Conversely, suppose $\mu(\bar{e}_f)=0$. Then $\mu(n|f|\wedge 1)=0$ for all n and hence $\mu(\alpha|f|\wedge 1)=0$ for all $\alpha\geqq 0$. This then implies that $\mu(|f|\wedge\alpha\cdot 1)=0$ for all $\alpha\geqq 0$. Again $|f|-|f|\wedge n\cdot 1$ is a nonincreasing sequence with infimum zero, so $\lim_n \mu(|f|-|f|\wedge n\cdot 1)=0$ and hence $\mu(|f|)=0$.

THEOREM 4.3. *If μ is a non-negative continuous linear functional on L, then*
(1) *$V=\{f\in L|\mu(|f|)=0\}$ is a closed l-ideal of L.*
(2) *$L-V$ is a complete F-space.*
(3) *If $\hat{a}\in L-V$ is the image of $a\in L$ under the natural homomorphism of L onto $L-V$, then if by definition $\hat{\mu}(\hat{a})=\mu(a)$, it follows that $\hat{\mu}$ is a continuous positive linear functional on $L-V$.*

Proof. (1) By Theorem 4.2, $V=\{f\in R(L,1)|\mu(\bar{e}_f)=0\}\cap L$ where L is identified with its isomorphic copy in $R(L,1)$. Therefore V is a closed l-ideal by Theorem 4.1.

(2) It is easy to verify that $L-V$ is an F-space with weak order unit $\hat{1}$. Since $V\cap B(L,1)$ is the set of elements of $B(L,1)$ with measure zero, $B(L,1)/V\cap B(L,1)$ is [9, Theorem 4.7] a complete Boolean algebra, and since $\hat{B}=B(L-V,\hat{1})$ is isomorphic to $B(L,1)/V\cap B(L,1)$, it follows that $\hat{B}$ is complete. Hence $L-V$ is complete (Corollary 3.7).

(3) $\hat{\mu}$ is well defined on $L-V$ because $x,y\in\hat{x}$ implies $|x-y|\in\hat{0}$ and hence $\mu(|x-y|)=0$. Therefore $\mu(x)=\mu(y)$. The functional $\hat{\mu}$ is clearly linear and positive. To show it is continuous let $\{\hat{x}_n\}$ be a nonincreasing sequence of elements in $L-V$ such that $\bigwedge_{n=1}^{\infty}\hat{x}_n=0$. A sequence $\{x_n\}$ is contained in L such that $x_n\in\hat{x}_n$ and $x_n\geqq 0$ for all $n\geqq 1$.

Let $y_n=\bigwedge_{k=1}^{n}x_k$. Then it is clear that $\hat{y}_n=\hat{x}_n$ and that $y=\bigwedge_{n=1}^{\infty}y_n$ belongs to $\hat{0}$. Therefore $\{z_n\}$ where $z_n=y_n-y$ is a sequence of elements of L such that $z_n\geqq 0$, $z_n\geqq z_{n+1}$, $z_n\in\hat{x}_n$ for all $n\geqq 1$, and $\bigwedge_{n=1}^{\infty}z_n=0$. Now $\lim_n\mu(z_n)=0$ and hence $\lim_n\hat{\mu}(\hat{x}_n)=0$ and the continuity of $\hat{\mu}$ is established.

REMARK. Part (2) of the above proof can be established directly from part (3). Indeed, the positiveness of $\hat{\mu}$ can be shown to imply that B is complete. Furthermore, if L is an F-space with weak order unit 1 and $B(L,1)$ supports a positive measure, then L and $R(L,1)$ are complete [9, Theorem 4.7 and Corollary 3.7].

5. **Bounded linear functionals on regular F-rings.** A linear functional ξ defined on an F-space L is bounded if bounded sets of elements of L are carried by ξ into bounded sets of real numbers. It is well known [7] that every such bounded linear functional is the difference of two non-negative linear functionals.

Let μ be a nontrivial non-negative linear functional defined on a regular F-ring R with identity 1. The following lemmas are used for the characterization of the class of bounded linear functionals on R.

LEMMA 5.1. *If $\{e_n\}$ is an orthogonal sequence of idempotents of R, then $\mu(e_n)=0$ for all n larger than some fixed n_0.*

Proof. Suppose there is an orthogonal sequence $\{e_n\}$ of idempotents of R such that $\mu(e_{n(K)})\neq 0$ for an infinite sub-sequence $\{n(K)\}$ of the natural numbers. Since R is orthogonally complete (Corollary 2.3), the element

$$g = \bigvee_{K=1}^{\infty} e_{n(K)}/\mu(e_{n(K)})$$

belongs to R. Thus we find that $\mu(g)\geqq n$ for all $n\geqq 1$ which contradicts the hypothesis that μ is defined throughout R.

LEMMA 5.2. *The restriction of μ to $B=B(R, 1)$ is a countably additive measure.*

Proof. (This proof is essentially due to Mackey [6].) Let $\{e_n\}$ be an orthogonal sequence of idempotents of R. Then (Lemma 5.1) $\mu(e_n)=0$ for n larger than some natural number n_0. Since $\mu(\bigvee_{n=1}^{\infty} e_n-\sum_{n=1}^{n_0} e_n)=\mu(\bigvee_{n=1}^{\infty}e_n)-\sum_{n=1}^{n_0}\mu(e_n)$, it is only necessary to show that

$$\mu\left(\bigvee_{n=1}^{\infty} e_n - \sum_{n=1}^{n} e_n\right) = 0;$$

thus no generality is lost in considering the case where $\mu(e_n)=0$ for all $n\geqq 1$. The element $g=\bigvee_{n=1}^{\infty} ne_n$ exists in R because R is orthogonally complete (Corollary 2.3). Therefore

$$g - \sum_{k=1}^{m} ke_k \geqq m\left(\bigvee_{n=1}^{\infty} e_n - \sum_{k=1}^{m} e_k\right),$$

and hence $\mu(g)\geqq m\mu(\bigvee_{n=1}^{\infty} e_n)$ for all $m\geqq 1$. Thus $\mu(\bigvee_{n=1}^{\infty} e_n)=0$, and μ is countably additive.

Let $U_\mu=\{a\in R\mid \mu(\bar{e}_a)=0\}$. Since μ is a countably additive measure, U_μ is a closed ideal of R and hence $\hat{R}=R-U_\mu$ possesses the properties indicated in Theorem 4.1. Let $\hat{a}$ stand for the image of $a\in R$ under the natural homomorphism of R onto $\hat{R}=R-U_\mu$.

LEMMA 5.3. *The Boolean algebra $\hat{B}=B(\hat{R}, 1)$ is atomic and the set of all its atoms is finite.*

Proof. To show $\hat{B}$ is atomic, first note that ascending (descending) chains of elements of $\hat{B}$ have finite length. This follows from Lemma 5.1 because if $\{\hat{e}_n\}$ is an ascending chain of elements of $\hat{B}$, then $\{\hat{e}_{n+1}-\hat{e}_n\}$ is an orthogonal sequence and if $\hat{\mu}$ is the positive countably additive measure on $\hat{B}$ induced by μ (see Theorem 4.1), then $\hat{\mu}(\hat{e}_{n+1}-\hat{e}_n)=0$ for $n>n_0$. Therefore $\hat{e}_{n_0+1}=\hat{e}_{n_0+2}=\cdots$ and the chain $\{\hat{e}_n\}$ has finite length. A similar proof can be given for descending chains.

Since descending chains have finite length, it follows that every element covers an atom of $\hat{B}$ and hence $\hat{B}$ is atomic. Similarly since ascending chains have finite length there can be at most a finite set of atoms in $\hat{R}$.

Now it is possible to characterize the bounded linear functionals on R.

THEOREM 5.4. *Every nontrivial bounded linear functional on R is a finite linear combination of ring-homomorphisms of R onto the real field.*

Proof. Since every bounded linear functional on R is the difference of two non-negative linear functionals, the theorem will follow in general if it can be proved for non-negative linear functionals. Let μ be a nontrivial non-negative linear functional. From Lemma 5.2 it follows that μ is a countably additive measure on $B(R, 1)$. If we adopt the conventions stated in the paragraph preceding Lemma 5.3, then it is clear from Lemma 5.3 that $\hat{R}=R-U_\mu$ is the F-ring of ordered n-tupples of real numbers for some fixed n and that the atoms $\hat{a}_1, \cdots, \hat{a}_n$ of $\hat{B}=B(\hat{R}, 1)$ form a basis of $\hat{R}$.

Let $\hat{\mu}(\hat{a})=\mu(a)$ by definition. Since we have not proved that μ is a continuous linear functional on R, we cannot use Theorem 4.3 to establish that $\hat{\mu}$ is a linear functional on $\hat{R}$. We can however show $\hat{\mu}$ has these properties by a slightly different method. First, $\hat{\mu}$ is well defined. Indeed, if $a, b\in\hat{a}$, then $|a-b|\in\hat{0}$ and hence $\mu(\bar{e}_{|a-b|})=0$. Since $|a-b|\bar{e}_{|a-b|}=|a-b|$, it follows by the Cauchy inequality that

$$0 \leqq \mu(|a-b|) \leqq (\mu(|a-b|^2))^{1/2}(\mu(\bar{e}_{|a-b|}))^{1/2} = 0$$

and hence $\mu(a)=\mu(b)$. The linearity of $\hat{\mu}$ is a direct consequence of the result that U_μ is an ideal.

Since the atoms $\hat{a}_1, \cdots, \hat{a}_n$ of $\hat{B}$ form a basis of $\hat{R}$ every element $\hat{a}$ of $\hat{R}$ can be written

$$\hat{a} = \sum_{k=1}^{n} \alpha_k\hat{a}_k,$$

and

$$\hat{\mu}(\hat{a}) = \sum_{k=1}^{n} \alpha_k\hat{\mu}(\hat{a}_k).$$

If we define $\phi_k(\hat{a})=\alpha_k$, then it is clear that each ϕ_k is a ring-homomorphism of $\hat{R}$ onto the real field. Thus $\hat{\mu}(\hat{a})$ is a finite linear combination of ring-homomorphisms:

$$\hat{\mu}(\hat{a}) = \sum_{k=1}^{n} \phi_k(\hat{a})\hat{\mu}(\hat{a}_k).$$

Let $\Phi_k(a)=\phi_k(\hat{a})$ where $a\in\hat{a}$. Then Φ_k is a ring-homomorphism of R onto the real field and by the definition of $\hat{\mu}$,

$$\mu(a) = \hat{\mu}(\hat{a}) = \sum_{k=1}^{n} \hat{\mu}(\hat{a}_k)\Phi_k(a).$$

Therefore μ is a finite linear combination of ring-homomorphisms and the proof is complete.

The following theorem is an important result in the theory of F-rings. A proof appears elsewhere [4] but the present proof is included because of its simplicity.

THEOREM 5.5. *If M is a real maximum ideal of R, then M is closed.*

Proof. Let $a(M)$ designate the real number associated with $a \in R$ under the homomorphism of R onto $R-M$. Then $\mu(a)=a(M)$ is a non-negative linear functional on R. By Lemma 5.2, μ is a countably additive two-valued measure on $B=B(R, 1)$. The ideal $M \cap B$ of B is therefore closed, and hence [4, Lemma 2] the ideal M is also closed.

From [2, Theorem 5] it follows that if R is a regular F-ring, then a maximal ideal of R is closed if and only if it is real. The following corollary is a direct consequence of this remark and Theorem 5.4.

COROLLARY 5.6. *If $B(R, 1)$ contains no closed maximal ideals, then there are no bounded linear functionals defined on R.*

It is possible to provide a large class of examples of regular F-rings which have only the trivial bounded linear functional defined on them. A measurable space $(\Omega, \mathfrak{F})$ is said to have property (U) if every nontrivial countably additive two-valued measure ν on the σ-algebra $\mathfrak{F}$ is *fixed*, that is, there exists a point $p \in \Omega$ such that $\nu(A)=1$ if and only if $p \in A$. Examples of measurable spaces with property (U) are:

(i) Ulam spaces, that is, those where $|\Omega|$ is nonmeasurable and $\mathfrak{F}$ is the set of all subsets of Ω.

(ii) The Lebesgue measurable space, that is the space where Ω designates the unit interval and $\mathfrak{F}$ the collection of all Lebesgue measurable subets of Ω. That the Lebesgue measurable space has property (U) follows from the nested intervals theorem.

(iii) The measurable space where Ω is both a P-space and a Q-space [5] and $\mathfrak{F}$ is the σ-algebra of all open-closed subsets of Ω. This space has property (U) by Theorem 5.5 and [2, Theorem 5].

If $(\Omega, \mathfrak{F})$ is a measurable space with property (U) and $\mathfrak{A}$ is a closed ideal of $\mathfrak{F}$ with the property that the union of its elements is Ω, then the quotient Boolean algebra $\mathfrak{F}/\mathfrak{A}$ is σ-complete. The assumption of the existence of a closed maximal ideal of $\mathfrak{F}/\mathfrak{A}$ implies the existence of a closed maximal ideal $\mathfrak{M}$ of $\mathfrak{F}$ with the property that the union of the elements of $\mathfrak{M}$ is Ω, and this then implies that there exists a nonfixed countably additive two-valued measure on $\mathfrak{F}$. Let R be a regular F-ring for which $B(R, 1)$ is isomorphic to $\mathfrak{F}/\mathfrak{A}$. From [3] it follows that R is unique up to an isomorphism and (Corollary 5.6) there are no nontrivial bounded linear functionals on R.

The following theorem is a direct consequence of the above remarks.

THEOREM 5.7. *Let* $(\Omega, \mathfrak{F})$ *be a measurable space with property* (U) *and let* $\mathfrak{A}$ *be a closed ideal of* $\mathfrak{F}$ *such that* $\bigcup \mathfrak{A} = \Omega$. *If* R *is a regular F-ring with identity* 1 *such that* $B(R, 1)$ *is isomorphic to the quotient* σ*-complete Boolean algebra* $\mathfrak{F}/\mathfrak{A}$, *then* R *possesses only the trivial bounded linear functional.*

A characterization of those linear functionals on an F-space L with weak order unit 1 which can be extended to bounded linear functionals on $R(L, 1)$ can now be given. Let Ω' stand for the class of closed maximal l-ideals of L and let $F(\Omega')$ stand for the class of finite linear combinations of linear functionals γ_M on L with kernels M in Ω'. It is a simple matter to verify that $F(\Omega')$ forms a complete vector lattice under the order relation: $\sum_{i=1}^{n} \alpha_i \gamma_{M_i} \geqq 0$ if and only if $\alpha_i \geqq 0$ for $1 \leqq i \leqq n$.

THEOREM 5.8. *A linear functional* μ *on* L *can be extended to a bounded linear functional on* $R(L, 1)$ *if and only if* $\mu \in F(\Omega')$.

Proof. If $\mu \in F(\Omega')$, then $\mu = \sum_{i=1}^{n} \alpha_i \gamma_{M_i}$ where $M_i \in \Omega'$. By Theorem 3.3 there is a one-to-one mapping ϕ: $S \rightarrow S \cap L$ of the class of closed maximal ideals of $R(L, 1)$ onto Ω'. From Theorem 4.2 it follows that $\gamma_{M_i}(1) \neq 0$ for all $1 \leqq i \leqq n$ because $\gamma_{M_i}(1) = 0$ implies $\gamma_{M_i}(|f|) = 0$ for all $f \in L$. Let $\mu^\star$ be the bounded linear functional on $R(L, 1)$ which has the form:

$$\mu^\star = \sum_{i=1}^{n} \alpha_i \cdot \gamma_{M_i}(1) \cdot \gamma^\star_{\phi^{-1}(M_i)}$$

where $\gamma^{\star}_{\phi^{-1}(M_i)}$ designates the ring-homomorphism associated with the closed (real) maximal ideal $\phi^{-1}(M_i)$ of $R(L, 1)$.

The restriction of $\mu^\star$ to L is the functional μ. To show this, note that for $x, y \in L$, $x - y \in M_i$ if and only if $x - y \in \phi^{-1}(M_i)$ and that if $\gamma_i^\star$ is used to designate $\gamma^{\star}_{\phi^{-1}(M_i)}$, then $x - \gamma_i^\star(x) \cdot 1 \in \phi^{-1}(M_i)$. Here again L is identified with its isomorphic copy in $R(L, 1)$. Since $\gamma_i^\star(x) \cdot 1 \in L$, we have

$$x - \gamma_i^\star(x) \cdot 1 \in M_i$$

and hence $\gamma_{M_i}(x) = \gamma_i^\star(x) \gamma^\star_{M_i}(1)$. Thus for $x \in L$,

$$\mu^\star(x) = \sum_{i=1}^{n} \alpha_i \gamma_{M_i}(x) = \mu(x).$$

Conversely if μ is a nontrivial functional on L which has a bounded linear extention $\mu^\star$ to $R(L, 1)$, then there exists (Theorem 5.4) a set $\{M_1, M_2, \cdots, M_n\}$ of closed maximal ideals of $R(L, 1)$ such that

$$\mu^\star = \sum_{i=1}^{n} \alpha_i \gamma^\star_{M_i}.$$

Here $\gamma^{\bigstar}_{M_i}$ is the ring-homomorphism of $R(L, 1)$ associated with the closed ideal M_i. The restriction of $\gamma^{\bigstar}_{M_i}$ to L is of the form $\gamma_{\phi(M_i)}$ and hence $\mu = \sum_{i=1}^{n} \alpha_i \gamma_{\phi(M_i)}$. Therefore $\mu \in F(\Omega')$.

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