A THEOREM ON TRANSLATION KERNELS IN n DIMENSIONS(1)

BY
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1. In 1952, G. Szegö ([4], see also [5, §5.5]) proved the following theorem concerning Toeplitz matrices: *If* $f(\theta)>0$ *in* $(0, 2\pi)$ *and* $f'(\theta)$ *satisfies a Lipschitz condition* (*with exponent* α, $0<\alpha\leqq 1$) *then*

$$\lim_{n\to\infty} \frac{D_n(f)}{[G(f)]^{n+1}} = \exp\left\{\frac{1}{4}\sum_1^\infty n\,|\,k_n\,|^2\right\}$$

where

$$\sum_{n=0}^{\infty} k_n z^n = \frac{1}{2\pi}\int_0^{2\pi} \log f(\theta)\,\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,d\theta,$$

$$G(f) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log f(\theta)d\theta\right\}$$

and $D_n(f)$ *denotes the determinant of the Toeplitz matrix*

$$(c_{j-k}) \qquad\qquad 0\leqq j,\,k\leqq n$$

with

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}d\theta.$$

In 1954 M. Kac [3] obtained a continuous analogue of Szegö's result: *Let* $\rho(x)$ *be real and even,*

$$\int_{-\infty}^{\infty}(1+|\,x\,|)\,|\,\rho(x)\,|\,dx<\infty,$$

and assume

$$F(y) = \int_{-\infty}^{\infty} e^{ixy}\rho(x)dx$$

belongs to $L(-\infty,\,\infty)$. *Then for sufficiently small* λ

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$$\lim_{a\to\infty} \frac{D_a(\lambda)}{\exp\left\{\frac{a}{\pi}\int_{-\infty}^{\infty}\log\left[1-\lambda F(y)\right]dy\right\}}$$

$$= \exp\left(\int_0^{\infty} x\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left[1-\lambda F(y)\right]e^{ixy}dy\right\}^2 dx\right)$$

where $D_a(\lambda)$ is the Fredholm determinant of the integral equation

$$\int_{-a}^{a}\rho(x-y)\phi(y)dy = \lambda\phi(x).$$

Kac noticed that Szegö's result was equivalent to a certain theorem concerning sums of discrete random variables, the analogue of this for continuous random variables being the tool Kac used to obtain his result. Both the discrete and continuous versions of the random variable theorem were consequences of the following combinatorial lemma: *Let $a_1, \cdots, a_n$ be real numbers. Then*

$$\sum_{\sigma}\max\left(0, a_{\sigma 1}, a_{\sigma 1}+a_{\sigma 2}, \cdots, a_{\sigma 1}+\cdots+a_{\sigma n}\right)$$

$$= \sum_{\sigma}\sum_{k=1}^{n}\frac{1}{k}\max\left(0, a_{\sigma 1}+\cdots+a_{\sigma k}\right)$$

where in both sums σ runs over all permutations of $1, \cdots, n$.

In this paper we extend Kac's result to n dimensions.

THEOREM. *Let $\rho(x)=\rho(x^1, \cdots, x^n)$ be real and even,*

$$\int_{E_n}(1+|x|)|\rho(x)|dx < \infty$$

and assume

$$F(y) = \int_{E_n} e^{ix\cdot y}\rho(x)dx$$

belongs to $L(E_n)$. Let K be a compact subset of E_n which is the closure of its interior, and denote by $\lambda_1(a), \lambda_2(a), \cdots$ the eigenvalues of the integral equation

$$\int_{aK}\rho(x-y)\phi(y)dy = \lambda\phi(x).$$

Then for sufficiently small λ we have, as $a\to\infty$,

$$\sum_{j=1}^{\infty} \log [1 - \lambda\lambda_j(a)] = a^n V(K) \frac{1}{(2\pi)^n} \int_{E_n} \log [1 - \lambda F(y)] dy$$

$$+ \frac{1}{2} a^{n-1} \int_{\partial K} d\sigma \int_{x\cdot\xi \geq 0} x\cdot\xi \left\{ \frac{1}{(2\pi)^n} \int_{E_n} \log [1 - \lambda F(y)] e^{ix\cdot y} dy \right\}^2 dx + o(a^{n-1}),$$

where $V(K)$ denotes the volume of K, ∂K the boundary of K, $d\sigma$ the surface element on ∂K, and ξ the unit outer normal at a point of ∂K.

We shall prove the theorem under either of two additional assumptions, namely that K is a polyhedron or convex. In either of these cases integrals of the form $\int_{\partial K} f(\xi) d\sigma$, with ξ the unit outer normal at a point of ∂K, are easily interpreted.

As will be seen the difficulties in extending Kac's result to n dimensions are essentially geometric; the analysis is identical and in the end it is the very same combinatorial lemma mentioned above which gives the result.

The author was aided greatly in this work by conversations with H. Pollard and F. Spitzer; in fact the theorem and its proof in the first nontrivial case, when K is a disc in two dimensions, arose out of these conversations.

2. Beginning the proof of the theorem, we note that $\sum \lambda_j(a)^k$ converges for $k=1, 2, \cdots$ (that $\sum |\lambda_j(a)| < \infty$ follows from Mercer's theorem and a simple device; see [3, pp. 506–507]) and, denoting by χ_{aK} the characteristic function of aK,

$$\sum_{j=1}^{\infty} \lambda_j(a)^k = \int_{E_n} \cdots \int_{E_n} \rho(x_1 - x_2)\rho(x_2 - x_3) \cdots \rho(x_k - x_1)\chi_{aK}(x_1) \cdots \chi_{aK}(x_k) dx_1 \cdots dx_k. \tag{1}$$

Setting $R(y_1, \cdots, y_{k-1}) = \rho(y_1) \cdots \rho(y_{k-1})\rho(y_1 + \cdots + y_{k-1})$ we obtain from (1) by changing variables

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j(a)^k &= \int_{E_n} \cdots \int_{E_n} R(y_1, \cdots, y_{k-1})\chi_{aK}(y_0)\chi_{aK}(y_0 + y_1) \cdots \chi_{aK}(y_0 + \cdots + y_{k-1}) dy_0 \cdots dy_{k-1} \\ &= \int_{E_n} \cdots \int_{E_n} R(y_1, \cdots, y_{k-1}) V(aK \cap aK - y_1 \cap \cdots \cap aK - y_1 - \cdots - y_{k-1}) dy_1 \cdots dy_{k-1}. \end{aligned} \tag{2}$$

3. We consider first the case when K is a polyhedron. We denote the (closed) n dimensional simplices whose union is K by K_n and those $n-1$ dimensional simplices *whose union is* ∂K by K_{n-1}^ν; the unit outer normal to K_{n-1}^ν we denote by ξ^ν; and v denotes $n-1$ dimensional volume.

LEMMA 1. *For any points $x_1, \cdots, x_k \in E_n$ we have*

$$V(aK) - V(aK \cap aK - x_1 \cap \cdots \cap aK - x_k) \leqq a^{n-1} v(\partial K) \max_i |x_i|.$$

LEMMA 2. *For any points* $x_1, \cdots, x_k \in E_n$ *we have, as* $a \to \infty$,

$$V(aK \cap aK - x_1 \cap \cdots \cap aK - x_k)$$
$$= a^n V(K) - a^{n-1} \sum_\nu v(K_{n-1}^\nu) \max (0, x_1 \cdot \xi^\nu, \cdots, x_k \cdot \xi^\nu) + O(a^{n-2}).$$

Before proving the lemmas it is convenient to divide by a^n and write $\epsilon = a^{-1}$. Thus Lemma 1 becomes

$$V(K \ominus K - \epsilon x_1 \cap \cdots \cap K - \epsilon x_k) \leqq \epsilon v(\partial K) \max_i |x_i|, \tag{3}$$

where $\ominus$ denotes relative complement, and Lemma 2 becomes

$$\begin{aligned} V(K \ominus K - \epsilon x_1 \cap \cdots \cap K - \epsilon x_k) \\ = \epsilon \sum_\nu v(K_{n-1}^\nu) \max (0, x_1 \cdot \xi^\nu, \cdots, x_k \cdot \xi^\nu) + O(\epsilon^2) \end{aligned} \tag{4}$$

as $\epsilon \to 0$.

For each K_{n-1}^ν construct the sum of K_{n-1}^ν and the line segment joining the origin to the point $-\epsilon \max(0, x_1 \cdot \xi^\nu, \cdots, x_k \cdot \xi^\nu)\xi^\nu$; this is just the prism with base K_{n-1}^ν, height $\epsilon \max(0, x_1 \cdot \xi^\nu, \cdots, x_k \cdot \xi^\nu)$, and entering the (unique) K_n of which K_{n-1}^ν is a face. The union of these prisms we call P. Let $x \in K \ominus K - \epsilon x_1 \cap \cdots \cap K - \epsilon x_k$. Then x belongs to some K_n, and for some i $(1 \leqq i \leqq k)$ we have $x + \epsilon x_i \notin K$, so the line segment joining x with $x + \epsilon x_i$ must pierce some K_{n-1}^ν, an $n-1$ dimensional face of K_n which is part of ∂K. It follows that the distance from x to K_{n-1}^ν is less than $\epsilon x_i \cdot \xi^\nu$, so that $x \in P$. Thus $K \ominus K - \epsilon x_1 \cap \cdots \cap K - \epsilon x_k$ is contained in P. Moreover since the volume of P is at most the sum of the volumes of its constituent prisms, we have (3), and so Lemma 1.

As for (4), it is not hard to see that the points of $K \ominus K - \epsilon x_1 \cap \cdots \cap K - \epsilon x_k$ not contained in P have volume at most $O(\epsilon^2)$. (The details, which are completely elementary although somewhat tedious, are left to the reader.) Moreover, with an error at most $O(\epsilon^2)$ the volume of P is the sum of the volumes of the constituent prisms. This gives (4), and so Lemma 2.

Before continuing with the proof of the theorem, we need some simple bounds. Let

$$A = \int_{E_n} |\rho(x)| dx, \qquad B = \int_{E_n} |x| |\rho(x)| dx, \qquad M = \max |\rho(x)|.$$

LEMMA 3. *Denote by* $\rho_*^{(k)}(x)$ *the k-fold convolution of* $|\rho(x)|$ *with itself. Then*

$$\max \rho_*^{(k)}(x) \leqq A^{k-1} M, \qquad \int_{E_n} |x| \rho_*^{(k)}(x) dx \leqq (1 + \cdots + A^{k-1}) B.$$

This last lemma is trivial and its proof is omitted.

4. To continue now, we obtain from (2)

$$\sum_{j=1}^{\infty} \lambda_j(a)^k$$
$$= a^n V(K) \frac{1}{(2\pi)^n} \int_{E_n} F(y)^k dy - a^{n-1} \int_{E_n} \cdots \int_{E_n} R(y_1, \cdots, y_{k-1})$$
$$\cdot \left\{ \frac{V(aK) - V(aK \cap aK - y_1 \cap \cdots \cap aK - y_1 - \cdots - y_{k-1})}{a^{n-1}} \right\} dy_1 \cdots dy_{k-1}. \tag{5}$$

By Lemma 2 the quotient appearing in the last integral approaches, as $a \to \infty$,

$$\sum_\nu v(K_{n-1}^\nu) \max(0, y_1 \cdot \xi^\nu, \cdots, y_1 \cdot \xi^\nu + \cdots + y_{k-1} \cdot \xi^\nu)$$

for all $y_1, \cdots, y_{k-1}$. Using Lemmas 1 and 3 we see that we can in fact take the limit under the integrals, so

$$\sum_{j=1}^{\infty} \lambda_j(a)^k = a^n V(K) \frac{1}{(2\pi)^n} \int_{E_n} F(y)^k dy$$
$$- a^{n-1} \sum_\nu v(K_{n-1}^\nu) \int_{E_n} \cdots \int_{E_n} R(y_1, \cdots, y_{k-1}) \tag{6}$$
$$\cdot \max(0, y_1 \cdot \xi^\nu, \cdots, y_1 \cdot \xi^\nu + \cdots + y_{k-1} \cdot \xi^\nu) dy_1 \cdots dy_{k-1} + o(a^{n-1}).$$

Now by the combinatorial lemma stated in §1 we have, using the fact the R is a symmetric function of $y_1, \cdots, y_{k-1}$,

$$\int_{E_n} \cdots \int_{E_n} R(y_1, \cdots, y_{k-1}) \cdot \max(0, y_1 \cdot \xi^\nu, \cdots, y_1 \cdot \xi^\nu + \cdots + y_{k-1} \cdot \xi^\nu) dy_1 \cdots dy_{k-1}$$
$$= \sum_{r=1}^{k-1} r^{-1} \int_{E_n} \cdots \int_{E_n} R(y_1, \cdots, y_{k-1}) \max(0, y_1 \cdot \xi^\nu + \cdots + y_r \cdot \xi^\nu) dy_1 \cdots dy_{k-1}$$
$$= \sum_{r=1}^{k-1} r^{-1} \int_{x \cdot \xi^\nu \geq 0} x \cdot \xi^\nu dx \int_{E_n} \cdots \int_{E_n} \rho(y_1) \cdots \rho(y_{r-1}) \rho(x - y_1 - \cdots - y_{r-1})$$
$$\cdot \rho(y_{r+1}) \cdots \rho(y_{k-1}) \rho(x + y_{r+1} + \cdots + y_{k-1}) dy_1 \cdots dy_{r-1} dy_{r+1} \cdots dy_{k-1}$$
$$= \sum_{r=1}^{k-1} r^{-1} \int_{x \cdot \xi^\nu \geq 0} x \cdot \xi^\nu \rho^{(r)}(x) \rho^{(k-r)}(x) dx,$$

where $\rho^{(r)}(x)$ denotes the r-fold convolution of $\rho(x)$ with itself. Thus from (6)

$$\lim_{a\to\infty} a^{-n+1}\left\{\sum_{j=1}^{\infty} k^{-1}\lambda_j(a)^k - a^n V(K)\frac{1}{(2\pi)^n}\int_{E_n} k^{-1}F(y)^k dy\right\} \tag{7}$$
$$= -\frac{1}{2}\sum_{\nu} v(K_{n-1}^{\nu})\int_{x\cdot\xi^{\nu}\geq 0} x\cdot\xi^{\nu}\sum_{r=1}^{k-1}\frac{\rho^{(r)}(x)\rho^{(k-r)}(x)}{r(k-r)}dx.$$

Now by (5), using Lemmas 1 and 3, we obtain the bound

$$a^{-n+1}\left|\sum_{j=1}^{\infty} k^{-1}\lambda_j(a)^k - a^n V(K)\frac{1}{(2\pi)^n}\int_{E_n}\frac{1}{k}F(y)^k dy\right| \leq v(\partial K)A^{k-1}BM,$$

so if $|\lambda| < A^{-1}$ we may conclude from (7) that

$$\lim_{a\to\infty} a^{-n+1}\left\{\sum_{j=1}^{\infty}\log\left[1-\lambda\lambda_j(a)\right] - a^n V(K)\frac{1}{(2\pi)^n}\int_{E_n}\log\left[1-\lambda F(y)\right]dy\right\}$$
$$= \frac{1}{2}\sum_{k=1}^{\infty}\lambda^k\sum_{\nu} v(K_{n-1}^{\nu})\int_{x\cdot\xi^{\nu}\geq 0} x\cdot\xi^{\nu}\sum_{r=1}^{k-1}\frac{\rho^{(r)}(x)\rho^{(k-r)}(x)}{r(k-r)}dx$$
$$= \frac{1}{2}\sum_{\nu} v(K_{n-1}^{\nu})\int_{x\cdot\xi^{\nu}\geq 0} x\cdot\xi^{\nu}\left\{\frac{1}{(2\pi)^n}\int_{E_n}\log\left[1-\lambda F(y)\right]e^{ix\cdot y}dy\right\}^2 dx,$$

where the interchange of $\sum_k$ and $\int\cdots dx$ is justified by Lemma 3.

Thus the theorem is proved in case K is a polyhedron.

5. It is clear from the above discussion that the theorem holds for any set $K\subset E_n$ for which suitable analogues of Lemmas 1 and 2 hold. We shall state, and prove in detail, these analogues in case K, in addition to being compact and the closure of its interior, is convex.

Lemma 1'. $V(aK)-V(aK\cap aK-x_1\cap\cdots\cap aK-x_k)\leq a^{n-1}S(K)\max|x_i|$, *where* $S(K)$ *denotes the surface of* K.

Proof. We may clearly assume $a=1$. Set $N=\max|x_i|$ and assume $x\in K\ominus K-x_1\cap\cdots\cap K-x_k$. If the sphere with center x and radius N were contained in K we would have $x+x_i\in K$ $(1\leq i\leq k)$ which is not true. Thus x is within N of ∂K. Since

$$\{x\in K\mid \operatorname{dist}(x,\partial K)\leq N\}$$

has volume at most $NS(K)$, Lemma 1' is proved.

For the analogue of Lemma 2 we shall have to use surface integrals on ∂K. These may be obtained as follows. (See [1, Chapter X, §§1, 2].) Choose point p in the interior of K, and denote by Σ the surface of the unit sphere with center p. For any $x\in\partial K$ let πx denote the intersection with Σ of the ray beginning at p and passing through x; thus π projects ∂K radially onto Σ. For almost every $x\in\Sigma$ there is a unique tangent plane to ∂K at $\pi^{-1}x$; ξ, the unit outer normal to ∂K at this point is then well defined. Now let S be a Borel set on ∂K. Then we have

$$\sigma(S) = \int_{\pi S} \frac{|x - p|^{n-1}}{(x - p)\cdot\xi} d\bar{\sigma}(x)$$

where $\bar{\sigma}$ represents the surface element on Σ. The measure σ on ∂K is independent of the particular choice of the point p of the interior of K.

LEMMA 2′. *There is a set $Z \subset E_n$ of measure zero such that, as $a \to \infty$,*

$$V(aK - x_1 \cap \cdots \cap aK - x_k)$$
$$= a^n V(K) - a^{n-1} \int_{\partial K} \max(x_1\cdot\xi, \cdots, x_k\cdot\xi)d\sigma + o(a^{n-1})$$

as long as no $x_i - x_j$ with $i \neq j$ belongs to Z.

The proof of Lemma 2′ is rather lengthy and will be given in stages. For a point $x \in E_n$ we denote by K_x the set of $p \in K$ such that the line through p and parallel to the vector x (i.e., the set of all $p + \mu x$, $-\infty < \mu < \infty$) does not meet the interior of K. Clearly K_x is a subset of ∂K.

SUBLEMMA 1. *Let $Z = \{x \in E_n | \sigma(K_x) > 0\}$. Then Z has measure zero.*

Proof. Assume Z has positive measure. Since $\sigma(\partial K) = S(K) < \infty$, the set of supporting hyperplanes H which intersect ∂K in a set of positive measure is at most countable. For any such H let H_0 be the hyperplane parallel to H and passing through the origin. Then $Z_1 = Z - \bigcup H_0$ has positive measure. I claim that if $y_1, \cdots, y_{n-1} \in Z_1$ are linearly independent, then

$$\sigma(K_{y_1} \cap \cdots \cap K_{y_{n-1}}) = 0.$$

For let S be the set of singular points of ∂K (i.e., the set of points at which there is not a unique supporting hyperplane), and assume $x \in K_{y_1} \cap \cdots \cap K_{y_{n-1}} \ominus S$. Let $\tilde{H}$ be the unique supporting hyperplane containing x. Since the line through x and parallel to y_i does not meet the interior of K, this line contains a supporting hyperplane, i.e., $\tilde{H}$. Thus $\tilde{H}$ is a supporting hyperplane, and parallel to the hyperplane $\overline{H}$ passing through $0, y_1, \cdots, y_{n-1}$. Since $y_i \in Z_1$, $\overline{H}$ is not an H_0. But there are exactly two supporting hyperplanes parallel to $\overline{H}$, each intersecting ∂K in a set of measure zero. Therefore $\sigma(K_{y_1} \cap \cdots \cap K_{y_{n-1}} \ominus S) = 0$. Since the projection of S on any sphere has measure zero, $\sigma(S) = 0$, so also $\sigma(K_{y_1} \cap \cdots \cap K_{y_{n-1}}) = 0$.

Now let Z_2 be a subset of Z_1 maximal with respect to the property: any $n-1$ points of Z_2 are linearly independent. If Z_2 were countable, the set of hyperplanes determined by 0 and $n-1$ points of Z_2 would be countable, and so the union would have measure zero, which implies there is a point of Z_1 not in this union; but then this point could be added to Z_2 preserving its defining property, which would contradict maximality. Hence Z_2 is uncountable, and $y_1, \cdots, y_{n-1} \in Z_2$ implies $\sigma(K_{y_1} \cap \cdots \cap K_{y_{n-1}}) = 0$. Since each $\sigma(K_y) > 0$ for $y \in Z_2$, and

$$\sum_{y \in Z_2} \sigma(K_y) \leqq (n-2)\left(\bigcup_{y \in Z_2} K_y\right) < \infty,$$

we have arrived at a contradiction. Thus Z has measure zero.

SUBLEMMA 2. *Let $x_1, \cdots, x_k \in E_n$ be such that no $x_i - x_j$ with $i \neq j$ belongs to Z, and denote by P the convex hull of $\{x_1, \cdots, x_k\}$. Then*

$$V[(K + \epsilon P) \ominus (K + \epsilon x_1 \cup \cdots \cup K + \epsilon x_k)]$$

is $o(\epsilon)$ as $\epsilon \to 0$.

Proof. Let $p \in (K+\epsilon P) \ominus (K+\epsilon x_1 \cup \cdots \cup K+\epsilon x_k)$. Then $p = q + \epsilon \sum \lambda_i x_i$ ($\sum \lambda_i = 1$, $\lambda_i \geqq 0$) with $q \in K$; and $q = p - \epsilon \sum \lambda_i x_i \in p - \epsilon P = Q$, say. No vertex of Q lies in K, for $p - \epsilon x_i \in K$ would imply $p \in K + \epsilon x_i$. Thus p is within $d(P)\epsilon$ of a point $q \in Q \cap K$, where Q is a translate of $-\epsilon P$ none of whose vertices lie in K. ($d(P)$ denotes the diameter of P.)

Given points $p, x \in E_n$ denote by $h_{p,x}$ the line through p and parallel to the vector x. I claim that for some $q_0 \in Q \cap K$ and some x_i, x_j $(i \neq j)$ we have

$$\left| h_{q_0, x_i - x_j} \cap K \right| \leqq d(P)\epsilon \tag{8}$$

where $| \cdots |$ denotes length. Let m be the dimension of Q, $\tilde{K}$ the intersection of K with the m dimensional plane through Q. If $\tilde{K} \subset Q$ then $d(\tilde{K}) \leqq d(Q) = \epsilon d(P)$, so (8) is satisfied if we take any $q_0 \in \tilde{K}$ and the pair x_i, x_j corresponding to the vertices of Q (i.e., the vertices $p - \epsilon x_i$, $p - \epsilon x_j$ of Q) whose distance is $d(Q)$. If $\tilde{K} \not\subset Q$, $\tilde{K}$ meets the complement of Q. Since also $\tilde{K}$ meets Q, $\tilde{K}$ has a point in common with Q_{m-1}, a closed $m-1$ face of Q. Denote by H_{m-1} the $m-1$ dimensional plane through Q_{m-1}. If $H_{m-1} \cap \tilde{K} \subset Q_{m-1}$ we may take, to satisfy (8), any $q_0 \in H_{m-1} \cap \tilde{K}$ and the pair x_i, x_j corresponding to vertices of Q_{m-1} whose distance is $d(Q_{m-1})$. If $H_{m-1} \cap \tilde{K} \not\subset Q_{m-1}$ then $\tilde{K}$ has a point in common with Q_{m-2}, a closed $m-2$ face of Q_{m-1}. Clearly we may repeat the process. If we reach the point where $\tilde{K}$ has a point in common with Q_1 but $H_1 \cap \tilde{K} \not\subset Q_1$, then $\tilde{K}$ would contain a vertex of Q_1. Thus we must have $H_1 \cap \tilde{K} \subset Q_1$ and (8) is satisfied for any $q_0 \in H_1 \cap \tilde{K}$, and x_i, x_j corresponding to the end-point of Q_1.

Changing notation, we have shown that p is within $2d(P)\epsilon$ of a point $q \in K$ such that

$$\left| h_{q, x_i - x_j} \cap K \right| \leqq d(P)\epsilon$$

for some pair x_i, x_j. Next we show that this q is within $A\epsilon$ of a point of $K_{x_i - x_j}$. (Here we use A to denote any constant which depends only on K and $x_1, \cdots, x_k$.) Let h_0 be a fixed line segment in K parallel to the vector $x_i - x_j$, and Π the two dimensional plane through h_0 and $h_{q, x_i - x_j}$. Since

$$\left| h_{q, x_i - x_j} \cap K \right| < \left| h_0 \right|$$

if ϵ is sufficiently small, the two line segments determine a triangle lying in

Π with h_0 as one of its sides, and the end-points of $h_{q,x_i-x_j}\cap K$ lying on the other sides; denote by v the vertex of this triangle.

In Π, there are two supporting lines for $\Pi\cap K$ parallel to h_0; let h be the one on the same side of h_0 as h_{q,x_i-x_j}, and choose a $q_0\in h\cap K$. Clearly q_0 must belong to the small triangle determined by v and $h_{q,x_i-x_j}\cap K$ and, furthermore, $q_0\in K_{x_i-x_j}$. Let q_1 be the intersection of the line segment vq_0 with h_{q,x_i-x_j} and r the intersection of vq_0 with h_0. Then

$$\frac{|v-q_1|}{|v-r|}=\frac{|h_{q,x_i-x_j}\cap K|}{|h_0|}\leqq\frac{d(P)\epsilon}{|h_0|},$$

so

$$|v-q_1|\leqq|v-r|\frac{d(P)\epsilon}{|h_0|}\leqq\frac{d(K)d(P)}{|h_0|}\epsilon.$$

Therefore certainly

$$|q_0-q_1|\leqq\frac{d(K)d(P)}{|h_0|}\epsilon,$$

and so

$$|q_0-q|\leqq\frac{d(K)+1}{|h_0|}d(P)\epsilon.$$

Thus q is within $A\epsilon$ of q_0 and so, since $|p-q|\leqq 2d(P)\epsilon$, also p is within $A\epsilon$ of q_0; and $q_0\in K_{x_i-x_j}$. Therefore if

$$C=\bigcup_{i\neq j}K_{x_i-x_j}$$

we have $\operatorname{dist}(p, C)\leqq A\epsilon$. Now C is homeomorphic, via some homeomorphism ϕ such that ϕ and ϕ^{-1} satisfy a Lip 1 condition, to a subset ϕC of E_{n-1}. (This is seen as follows: The projection π of ∂K onto a sphere Σ from an interior point of K is such that π, $\pi^{-1}\in\text{Lip }1$. This is easily verified. Since C is a proper closed subset of ∂K [note that $\sigma(C)=0$] we can then map πC into E_{n-1} by means of a ψ such that ψ and ψ^{-1} are, in fact, infinitely differentiable. We then take $\phi=\psi\pi$.) Since $\sigma(C)=0$, $\phi(C)$ will have $n-1$ dimensional Lebesgue measure zero. Set

$$M=\sup_{x,y\in C}\frac{|x-y|}{|\phi x-\phi y|}\cdot$$

Then, since ϕC has measure zero, ϕC can be covered by $o(\epsilon^{-n+1})$ spheres of radius $A\epsilon/M$, so C can be covered by $o(\epsilon^{-n+1})$ spheres of radius $A\epsilon$. Since, as we have shown, any point $p\in(K+\epsilon P)\ominus(K+\epsilon x_1\cup\cdots\cup K+\epsilon x_k)$ is within $A\epsilon$ of C, this set is covered by $o(\epsilon^{-n+1})$ spheres of radius $2A\epsilon$, and so its volume is $o(\epsilon)$.

SUBLEMMA 3. *Under the hypothesis of sublemma 2 we have, as* $a \to \infty$,

$$V(aK + x_1 \cup \cdots \cup aK + x_k)$$
$$= a^n V(K) + a^{n-1} \int_{\partial K} \max(x_1 \cdot \xi, \cdots, x_k \cdot \xi) d\sigma + o(a^{n-1}).$$

Proof. Using the notion of mixed volumes [2, §29] we can write

$$V(aK + P) = a^n V(K) + n a^{n-1} V(P, K, \cdots, K) + o(a^{n-1}) \tag{9}$$

where P is the convex hull of $\{x_1, \cdots, x_k\}$. If K is a polyhedron with $n-1$ dimensional faces K_{n-1}^{ν} and corresponding outer normals ξ^{ν},

$$V(P, K, \cdots, K) = \frac{1}{n} \sum_{\nu} v(K_{n-1}^{\nu}) \max(x_1 \cdot \xi^{\nu}, \cdots, x_k \cdot \xi^{\nu})$$

([2, formula (3), §29] applied to the case of a polyhedron P). A simple approximation argument yields, in the case of general convex K,

$$V(P, K, \cdots, K) = \frac{1}{n} \int_{\partial K} \max(x_1 \cdot \xi, \cdots, x_k \cdot \xi) d\sigma$$

so from (9),

$$V(aK + P) = a^n V(K) + a^{n-1} \int_{\partial K} \max(x_1 \cdot \xi, \cdots, x_k \cdot \xi) d\sigma + o(a^{n-1}).$$

But by Sublemma 2 with $\epsilon = a^{-1}$,

$$V(aK + x_1 \cup \cdots \cup aK + x_k) = V(aK + P) + o(a^{n-1}),$$

so the sublemma is proved.

SUBLEMMA 4. *For any real numbers* $r_1, \cdots, r_k$ *we have*

$$\sum_{j=1}^{k} (-1)^j \sum_{i_1 < \cdots < i_j} \max(r_{i_1}, \cdots, r_{i_j}) = -\min(r_1, \cdots, r_k).$$

Proof. Let $\chi(r)$ be the characteristic function of $(0, \infty)$. Then

$$\chi_{[-\infty, \min(r_1, \cdots, r_k)]}(r) = \{1 - \chi(r - r_1)\} \cdots \{1 - \chi(r - r_k)\}$$
$$= 1 + \sum_{j=1}^{k} (-1)^j \sum_{i_1 < \cdots < i_j} \chi(r - r_{i_1}) \cdots \chi(r - r_{i_j})$$
$$= -\sum_{j=1}^{k} (-1)^j \sum_{i_1 < \cdots < i_j} \{1 - \chi(r - r_{i_1}) \cdots \chi(r - r_{i_j})\}$$
$$= -\sum_{j=1}^{k} (-1)^j \sum_{i_1 < \cdots < i_j} \chi_{[-\infty, \max(r_{i_1}, \cdots, r_{i_j})]}(r).$$

Integrating from $\min(r_1, \cdots, r_k)$ to ∞,

$$0 = -\sum_{j=1}^{k} (-1)^j \sum_{i_1<\cdots<i_j} \{\max(r_{i_1}, \cdots, r_{i_j}) - \min(r_1, \cdots, r_k)\}$$

$$= -\sum_{j=1}^{k} (-1)^j \sum_{i_1<\cdots<i_j} \max(r_{i_1}, \cdots, r_{i_j}) - \min(r_1, \cdots, r_k),$$

which gives the result.

We can now prove Lemma 2′. The set Z will be that defined in the statement of Sublemma 1. The standard inclusion-exclusion principle gives

$$V(aK - x_1 \cap \cdots \cap aK - x_k) = -\sum_{j=1}^{k} (-1)^j \sum_{i_1<\cdots<i_j} V(aK - x_{i_1} \cup \cdots \cup aK - x_{i_j})$$

so by Sublemma 3 we have

$$V(aK - x_1 \cap \cdots \cap aK - x_k) = a^n V(K) - a^{n-1} \int_{\partial K} \sum_{j=1}^{k} (-1)^j \sum_{i_1<\cdots<i_j} \max(-x_1\cdot\xi, \cdots, -x_k\cdot\xi) d\sigma + o(a^{n-1}).$$

But by Sublemma 4

$$\sum_{j=1}^{k} (-1)^j \sum_{i_1<\cdots<i_j} \max(-x_1\cdot\xi, \cdots, -x_k\cdot\xi) = -\min(-x_1\cdot\xi, \cdots, -x_k\cdot\xi)$$

$$= \max(x_1\cdot\xi, \cdots, x_k\cdot\xi),$$

so Lemma 2′ is proved.

Using Lemmas 1′ and 2′, as Lemmas 1 and 2 were used in §4, one now easily proves the main theorem for an arbitrary compact convex set K with interior.

References

1. A. D. Alexandrow, *Die innere Geometrie der konvexen Flächen*, Berlin, 1955.

2. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Berlin, 1934.

3. M. Kac, *Toeplitz matrices, translation kernels and a related problem in probability theory*, Duke Math. J. vol. 21 (1954) pp. 501–510.

4. G. Szegö, *On certain Hermitian forms associated with the Fourier series of a positive function*, Festskrift Marcel Riesz, Lund, 1952, pp. 228–238.

5. U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Berkeley-Los Angeles, University of California Press, 1958.

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