EIGEN OPERATORS OF ERGODIC TRANSFORMATIONS(1)

BY

ANATOLE BECK(2)

Introduction. In a recent paper by A. Beck and J. T. Schwartz [2], mention is made of the following problem: Let $\mathfrak{X}$ be a B-space and T a bounded linear operator on $\mathfrak{X}$ with $\|T\| \leqq 1$. Let (S, Σ, m) be a measure space, with an ergodic measure-preserving transformation h defined on it. Let X be a strongly measurable function, not identically 0, from (S, Σ, m) into $\mathfrak{X}$ and suppose that

$$X(h(s)) = T(X(s)) \text{ a.e. in } S. \tag{1}$$

Then we shall call the operator T an *eigen operator* of h. The problem is to determine when these conditions can be satisfied. One is struck by the resemblance of equation (1) to the eigenvalue equation i.e. λ is defined to be an eigenvalue of the m.p.t. h if there is a nonzero complex-valued measurable function f on S such that

$$f(h(s)) = \lambda f(s) \text{ a.e. in } S. \tag{2}$$

The importance of eigenvalues as a tool in the study of m.p.t.'s suggests that eigen operators, which include eigenvalues as a special case, may prove a yet finer tool in this work.

In this paper, we will investigate the nature of eigen operators and their relation to their corresponding ergodic transformations. It will be shown that the solutions of this equation (1) depend on the theory of ergodic translations in a monothetic measure group.

DEFINITIONS. We take (S, Σ, m) to be a σ-finite measure space in the sense of Halmos [3]. A *measure-preserving transformation* (m.p.t.) h is a one-to-one mapping of S onto itself so that if $A_1 = h(A_2) = \{h(s) \mid s \in A_2\}$, then A_1 is Σ-measurable if and only if A_2 is and $m(A_1) = m(A_2)$. If $h(A) = A$, $A \in \Sigma$, necessarily implies that either $m(A) = 0$ or $m(S - A) = 0$, then h is said to be *ergodic*.

Let $\mathfrak{X}$ be a complex B-space. Then the set of all open sets in $\mathfrak{X}$ generates a σ-field of subsets of $\mathfrak{X}$ known as the *Borel field* of $\mathfrak{X}$, and each set in the Borel field is called a *Borel set*. A mapping X from S into $\mathfrak{X}$ is called *strongly measurable* if

Presented to the Society, December 29, 1956 under the title *On a functional equation in Banach space*; received by the editors March 28, 1957 and, in revised form, December 31, 1958.

(1) The material of this paper represents a major part of the dissertation presented by the author to Yale University in partial fulfillment of the requirements for the Doctorate of Philosophy. The author wishes to express his indebtedness to Professor Shizuo Kakutani for the invaluable assistance and kind encouragement with which he furthered this work.

(2) ONR Research Associate, Tulane University, 1956–1957.

1°. there is a separable subset $\mathfrak{X}_0 \subset \mathfrak{X}$ and a set $S_0 \in \Sigma$ such that $m(S_0)=0$ and $X(s) \in \mathfrak{X}_0$ for all $s \in S - S_0$,

2°. for every Borel set U,

$$X^{-1}(U) = \{s \mid X(s) \in U\} \in \Sigma.$$

It is easily shown that this definition of strong measurability is equivalent to the others (e.g. cf. E. Hille [4]). We shall find it convenient to use it in this form.

$K(x, r)$, the sphere about x with radius r, is defined for each $x \in \mathfrak{X}$ and $r>0$ by

$$K(x, r) = \{y \mid \|y - x\| < r\}.$$

If X is a strongly measurable function, we shall say that $y \in \mathfrak{R}(X)$ if for every $\epsilon>0$, $m(X^{-1}(K(y, \epsilon)))>0$. $\mathfrak{R}(X)$, defined in this way, is called the *essential range* of X.

For each $x \in \mathfrak{X}$, the closure of the set of all $T^i(x)$, $i=1, 2, \cdots$, is called the *positive orbit* of x (under T), and denoted P_x i.e. $P_x = \mathrm{Cl}(\{T^i(x)\}_{i=1}^{\infty})$(3).

We shall use the ordinary definition of *topological group* and *Haar measure* (cf. Halmos [3]). A topological group is called *monothetic* if there is an element a in it whose positive and negative powers are dense in the group. The element a is called a *topological generator* of the group, or simply a *generator*.

2. Preliminaries.

LEMMA 1. *If $m(S)>0$, then for every strongly measurable function X defined on S into $\mathfrak{X}$, $\mathfrak{R}(X)$ is not empty.*

Proof. Let S_0 be a set in Σ such that $X(S-S_0)$ is separable and $m(S_0)=0$. Suppose $\mathfrak{R}(X)$ is empty. Then for each $x \in \mathfrak{X}$, there is an $\epsilon_x>0$ such that $m(X^{-1}(K(x, \epsilon_x)))=0$. Then the spheres $K(x, \epsilon_x)$ clearly cover $X(S-S_0)$. Since $X(S-S_0)$ is separable, it can be covered by countably many of the $K(x, \epsilon_x)$, by the Lindelöf covering theorem. Call these K_i, $i=1, 2, \cdots$. Then, by construction, $m(X^{-1}(K_i))=0$, all $i>0$. Thus,

$$\begin{aligned} 0 &= \sum_{i=1}^{\infty} m(X^{-1}(K_i)) \geqq m\left(\bigcup_{i=1}^{\infty} X^{-1}(K_i)\right) \\ &= m\left(X^{-1}\left(\bigcup_{i=1}^{\infty} K_i\right)\right) \geqq m(X^{-1}(X(S - S_0))) \\ &= m(S - S_0). \end{aligned}$$

Therefore, $m(S)=m(S-S_0)+m(S_0)=0$. Thus, if $m(S)>0$, $\mathfrak{R}(X)$ is not empty. Q.E.D.

LEMMA 2. *For every strongly measurable X, $\mathfrak{R}(X)$ is closed.*

(3) "Cl" means closure.

Proof. Let X be defined from (S, Σ, m) into $\mathfrak{X}$. Let $x_i \in \mathfrak{R}(X)$, $i=1, 2, \cdots$ and let $x_n \to x \in \mathfrak{X}$(4). Choose any $\epsilon > 0$. Then for some $j > 0$, $\|x_j - x\| < \epsilon/2$. Thus, $K(x, \epsilon) \supset K(x_j, \epsilon/2)$ and $X^{-1}(K(x, \epsilon)) \supset X^{-1}(K(x_j, \epsilon/2))$. Therefore, $m(X^{-1}(K(x, \epsilon))) \geqq m(X^{-1}(K(x_j, \epsilon/2))) > 0$, since $x_j \in \mathfrak{R}(X)$. Therefore, $x \in \mathfrak{R}(X)$. Q.E.D.

We shall also need the following well-known facts:

LEMMA 3. *If h is a m.p.t. in the measure space (S, Σ, m) and if $m(S)=1$ or h is ergodic, then every eigen value has absolute value* 1.

Proof. Omitted.

LEMMA 4. *If G is a compact monothetic group and $(G, \mathcal{B}, \mu)$ is the Haar measure space in G, then translation by a topological generator in G is an ergodic m.p.t. in $(G, \mathcal{B}, \mu)$.*

Proof. Omitted.

LEMMA 5. *A necessary and sufficient condition that h be ergodic is that for every pair of measurable sets A and B which are not of measure* 0, *there is an integer i such that $m(h^i(A) \cap B) > 0$.*

Proof. Omitted.

If, in the statement above, i is restricted to being positive, then h is said to be *positive ergodic*. The only ergodic transformation which is not positive ergodic is the shift transformation in the Haar measure space of the integers. In the remainder of this paper, we shall take *ergodic* as meaning *positive ergodic*.

3. A necessary condition. Let $\mathfrak{X}$ be a B-space and let T be a bounded linear operator in $\mathfrak{X}$ with $\|T\| \leqq 1$. Let (S, Σ, m) be a measure space and h an ergodic m.p.t. in S. Let X be a strongly measurable function from S into $\mathfrak{X}$ and assume that $T(X(s)) = X(h(s))$ a.e. in S.

LEMMA 6. *If $x, y \in \mathfrak{R}(X)$, $\epsilon > 0$, then there is an $i > 0$ such that $\|T^i(x) - y\| < \epsilon$.*

Proof. Let $A = X^{-1}(K(x, \epsilon/2))$, $B = X^{-1}(K(y, \epsilon/2))$. Then, since $x, y \in \mathfrak{R}(X)$, $m(A) > 0$, $m(B) > 0$. It follows from the (positive) ergodicity of h that there is an $i > 0$ such that $m(h^i(A) \cap B) > 0$. Let

$$S_1 = \{s \mid T^j(X(h^{-j}(s))) = X(s) \text{ for all } j > 0\}.$$

Then $m(S - S_1) = 0$ and $m(h^i(A) \cap B \cap S_1) > 0$. Let $s_1 \in h^i(A) \cap B \cap S_1$. Then $T^i(X(h^{-i}(s_1))) = X(s_1)$. Therefore,

$$\begin{aligned} \|X(s_1) - T^i(x)\| &= \|T^i(X(h^{-i}(s_1))) - T^i(x)\| \\ &\leqq \|X(h^{-i}(s_1)) - x\| < \epsilon/2, \end{aligned}$$

(4) Convergence here, and throughout this paper, is as $n \to \infty$, unless otherwise noted.

since $s_1 \in h^i(A)$, so that $h^{-i}(s_1) \in A$. Also, $\|X(s_1) - y\| < \epsilon/2$, since $s_1 \in B$. Thus, $\|T^i(x) - y\| < \epsilon$.

COROLLARY 7. *For each* $x \in \mathfrak{R}(X)$, $\exists \{i_n\} \to \infty$ *such that* $T^{i_n}(x) \to x$.

Proof. Clear.

This corollary can be written

COROLLARY 7′. $x \in P_x$.

We shall now see just how much information is implicit in the assertion that $x \in P_x$. Suppose we have a B-space $\mathfrak{X}$ and a bounded linear operator T defined in $\mathfrak{X}$ with $\|T\| \leqq 1$. Suppose there is an element $x \in \mathfrak{X}$ such that $x \in P_x = \mathrm{Cl}(\{T^i(x)\}_{i=1}^{\infty})$. Denote $T^i(x)$ by t_i, $i = 1, 2, \cdots$.

LEMMA 8. $\|x\| = \|t_i\|$, *all* $i > 0$.

Proof. Clearly, $\|x\| \geqq \|t_1\| \geqq \|t_2\| \geqq \cdots$. Since there is a sequence $\{t_{i_n}\}$ converging to x, the lemma is obvious. Q.E.D.

LEMMA 9. $\|x\| = \|y\|$, *all* $y \in P_x$.

Proof. The elements t_i are dense in P_x. Q.E.D.

LEMMA 10. $\|x - t_i\| = \|t_k - t_{i+k}\|$, *all* i, $k > 0$.

Proof. $\|t_k - t_{i+k}\| = \|T^k(x - t_i)\| \leqq \|x - t_i\|$. Also for all $j > 0$, $\|t_k - t_{i+k}\| \geqq \|t_{k+j} - t_{i+k+j}\|$. When $\|t_{k+j} - x\| < \epsilon$, $\|t_{i+k+j} - t_i\| < \epsilon$. Thus $\|t_{k+j} - t_{k+j+i}\|$ becomes arbitrarily close to $\|x - t_i\|$. Q.E.D.

LEMMA 11. $\|x - t_i\| = \|y - T^i(y)\|$, *all* $y \in P_x$.

Proof. Immediate from Lemma 10.

LEMMA 12. *If* y, $z \in P_x$, $\|y - z\| = \|T(y) - T(z)\|$.

Proof. By Lemma 10, the theorem is true if y, z are of the form t_i. Since the t_i are dense in P_x, the result is clear. Q.E.D.

LEMMA 13. *For each* $y \in P_x$, *there is precisely one* $y_0 \in P_x$ *such that*

$$T(y_0) = y.$$

Proof. Let y be the limit of the sequence $\{t_{i_n}\}$, where $i_n > 0$, $n = 1, 2, \cdots$. By Lemma 12, $\{t_{i_n - 1}\}$ is a Cauchy sequence. Since P_x is closed and thus complete, $\{t_{i_n - 1}\}$ has a limit. Call it y_0. Then $T(y_0) = y$. Suppose that $T(y_1) = y$, $y_1 \in P_x$. Then

$$\|y_1 - y_0\| = \|T(y_1) - T(y_0)\| = \|y - y\| = 0. \qquad \text{Q.E.D.}$$

LEMMA 14. *Let* y, $z \in P_x$. *Let* $t_{j_n} \to y$, $t_{k_n} \to z$. *Then* $\{t_{j_n + k_n}\}$ *is a Cauchy sequence.*

Proof. This follows from the fact that

$$\begin{aligned}\| t_{j_m+k_m} - t_{j_n+k_n}\| &\leqq \|t_{j_m+k_m} - t_{j_m+k_n}\| + \|t_{j_m+k_n} - t_{j_n+k_n}\| \\ &= \|t_{k_m} - t_{k_n}\| + \|t_{j_m} - t_{j_n}\|. \qquad \text{Q.E.D.}\end{aligned}$$

LEMMA 15. *If $y, z \in P_x$, $t_{j_n} \to y$, $t_{k_n} \to z$, $t_{j_n'} \to y$, $t_{k_n'} \to z$, $t_{j_n+k_n} \to u$, then $t_{j_n'+k_n'} \to u$.*

Proof. Follows directly from the observation that

$$\begin{aligned}\limsup \|t_{k_n'+j_n'} - u\| \leqq &\limsup \|t_{k_n'+j_n'} - t_{k_n+j_n'}\| \\ &+ \limsup \|t_{k_n+j_n'} - t_{k_n+j_n}\| \\ &+ \limsup \|t_{k_n+j_n} - u\| = 0. \qquad \text{Q.E.D.}\end{aligned}$$

DEFINITION 16. If y, z, and u are as in Lemma 15, we write $yz = u$.

LEMMA 17. *For all $y, z, w \in P_x$,*
1°. $w(yz) = (wy)z$,
2°. $yz = zy$,
3°. $yt_i = T^i(y)$,
4°. $t_i = (t_1)^i$,
5°. $yx = y$.

Proof. Clear.

LEMMA 18. *If $y, z, y', z' \in P_x$, then $\|yz - y'z'\| \leqq \|y - y'\| + \|z - z'\|$.*

Proof. Let $t_{j_n} \to y$, $t_{k_n} \to z$, $t_{j_n'} \to y'$, $t_{k_n'} \to z'$. Then

$$\begin{aligned}\|yz - y'z'\| \leqq &\limsup \|yz - t_{j_n+k_n}\| + \limsup \|t_{j_n+k_n} - t_{j_n'+k_n}\| \\ &+ \limsup \|t_{j_n'+k_n} - t_{j_n'+k_n'}\| + \limsup \|t_{j_n+k_n'} - y'z'\| \\ = &\|y - y'\| + \|z - z'\|. \qquad \text{Q.E.D.}\end{aligned}$$

DEFINITION 19. By Lemma 13, there is for each $y \in P_x$ a unique $y_0 \in P_x$ such that $T(y_0) = y$. Denote y_0 as $T^{-1}(y)$ and define $T^{-n}(y)$ by iteration. Let $t_i = T^i(x)$, $-\infty < i < \infty$.

LEMMA 20. *If $t_{j_n} \to y \in P_x$, then the sequence $\{t_{-j_n}\}$ is Cauchy.*

Proof. Follows directly from the fact that

$$\|t_{-j_m} - t_{-j_n}\| = \|t_{j_n} - t_{j_m}\|. \qquad \text{Q.E.D.}$$

LEMMA 21. *If $t_{j_n} \to y \in P_x$, $t_{j_n'} \to y$, then $\{t_{-j_n}\}$ and $\{t_{-j_n'}\}$ have the same limit.*

Proof. Let $k_n = j_n$ when n is even, $k_n = j_n'$ when n is odd. Then $t_{k_n} \to y$. Therefore, $\{t_{-k_n}\}$ converges. Since $\{t_{-j_n}\}$ and $\{t_{-j_n'}\}$ have subsequences which are subsequences of $\{t_{-k_n}\}$ and all these sequences converge, $\lim t_{-j_n} = \lim t_{-j_n'}$. Q.E.D.

DEFINITION 22. If $t_{j_n} \to y \in P_x$, define $y^{-1} = \lim t_{-j_n}$.

LEMMA 23. *$yy^{-1} = t_0 = x$, all $y \in P_x$.*

Proof. Clear.

LEMMA 24. $\|y-z\| = \|y^{-1}-z^{-1}\|$, *all* $y, z \in P_x$.

Proof. Clear.

THEOREM 25. *Let* $\mathfrak{X}$ *be a Banach space and let* T *be a bounded linear operator in* $\mathfrak{X}$ *with* $\|T\| \leqq 1$. *Let* $x \in \mathfrak{X}$ *and set* $P_x = \mathrm{Cl}(\{T^i(x)\}_{i=1}^{\infty})$, *the positive orbit of* x. *If* $x \in P_x$, *then Definitions* 16 *and* 22 *define in* P_x *a multiplication and an inverse under which* P_x *becomes a monothetic abelian topological group with identity* x *and generator* $T(x)$, *under the topology induced in* P_x *by the metric in* $\mathfrak{X}$.

Proof. By Lemma 15, the product defined in Definition 16 is unique. By Lemma 17, this multiplication is associative, commutative, and has a unit element, x. By Lemma 23, there is an inverse, defined in Definition 22. Therefore, P_x is a group under the defined operations. By Lemma 18, the product is a continuous function with respect to the metric topology inherited from $\mathfrak{X}$. By Lemma 24, the inverse is likewise continuous. By Lemma 21, $(t_1)^i = t_i = T^i(x)$. Since these latter are dense in P_x by definition of P_x, it is clear that the group is monothetic with generator $t_1 = T(x)$. Q.E.D.

4. **A necessary and sufficient condition.** Applying the very strong results of Theorem 25 and particularly Lemma 12 to the conclusion of Corollary 7, we see that

LEMMA 26. *If* $\mathfrak{X}$, T, (S, Σ, m), h, *and* X *give a solution of equation* (1), *and* $x \in \mathfrak{R}(X)$, *then*

1°. $\mathfrak{R}(X) = P_x$,

2°. T *is an isometry in* $\mathfrak{R}(X)$.

Proof. Since $T^i(x) \in \mathfrak{R}(X)$ for all $i = 1, 2, \cdots$, and $\mathfrak{R}(X)$ is closed, $\mathfrak{R}(X) \supset \mathrm{Cl}(\{T^i(x)\}_{i=1}^{\infty}) = P_x$. By Lemma 6, $\mathfrak{R}(X) \subset P_x$. Since, by Corollary 7′, $x \in P_x$, T is an isometry in $P_x = \mathfrak{R}(X)$ by Lemma 12. Q.E.D.

LEMMA 27. *Under the hypotheses of Lemma* 26, *if* $m(S) < \infty$, *then* $\mathfrak{R}(X)$ *is compact.*

Proof. Suppose that $\mathfrak{R}(X)$ is not compact. Then for some $\epsilon > 0$, $\exists \{y_i\}$, $i = 1, 2, \cdots$ in $\mathfrak{R}(X)$ such that $\|y_i - y_j\| > \epsilon$ if $i \neq j$. Let n_i be so chosen that $\|T^{n_i}(x) - y_i\| < \epsilon/4$, $i = 1, 2, \cdots$. Then $K(T^{n_i}(x), \epsilon/4) \subset K(y_i, \epsilon/2)$. These latter are disjoint in pairs. Since T is an isometry in $\mathfrak{R}(X)$, $X^{-1}(K(T^{n_i}(x), \epsilon/4))$ differs from $h^{n_i}(X^{-1}(K(x, \epsilon/4)))$ by a set of measure 0. Therefore,

$$m(S) \geqq m\left(\bigcup_{i=1}^{\infty} X^{-1}(K(T^{n_i}(x), \epsilon/4))\right)$$

$$= \sum_{i=1}^{\infty} m(X^{-1}(K(T^{n_i}(x), \epsilon/4))) = \sum_{i=1}^{\infty} m(X^{-1}(K(x, \epsilon/4))).$$

Since $x \in \mathfrak{R}(X)$, $m(X^{-1}(K(x, \epsilon/4))) > 0$. Thus $m(S) = \infty$, contrary to hypothesis, which completes the proof. Q.E.D.

THEOREM 28. *If $\mathfrak{X}$ is a B-space and T is a bounded linear operator in $\mathfrak{X}$ with $\|T\| \leqq 1$, then T can be represented as an eigen operator of an ergodic transformation in a finite non-null measure space if and only if there exists in $\mathfrak{X}$ an element $x \neq 0$ for which the positive orbit $P_x = \mathrm{Cl}(\{T^i(x)\}_{i=1}^{\infty})$ has the following two properties:*

1°. $x \in P_x$,

2°. *P_x is compact.*

Proof. The necessity of the condition has already been shown (cf. Corollary 7′ and Lemma 27). Suppose x is an element for which 1° and 2° hold. Then we let P denote the group generated in P_x as in the previous section. Thus, adding 2° to Theorem 25, we have P a compact monothetic metric abelian group. Let $(P, \mathcal{B}, \mu)$ be the Haar measure space of P. Let us consider the ergodic mapping (cf. Lemma 4) in P which is translation by the generator $t_1 = T(x)$. Let us consider also the function I which maps every point of P_x, considered as an element of $(P, \mathcal{B}, \mu)$ into itself considered as an element of $\mathfrak{X}$. This function I is strongly measurable since the range, P_x, is separable and the pre-image of any Borel set is a Borel set and thus μ-measurable. By part 3° of Lemma 17, $T(I(y)) = T(y) = t_1 y = I(t_1 y)$ for all $y \in P_x$. Thus, T can be represented as an eigen operator of the translation by t_1, an ergodic transformation in the Haar measure space $(P, \mathcal{B}, \mu)$. Q.E.D.

5. **Uniqueness.** Suppose we start with some (S, Σ, m), h, $\mathfrak{X}$, T, and X giving us a solution of equation (1). Suppose further that $0 < m(S) < \infty$. Let us fix a point $x \in \mathfrak{R}(X)$. Then, as we have proved, $x \in P_x = \mathfrak{R}(X)$, and P_x is compact. If we construct the measure space $(P, \mathcal{B}, \mu)$ and the solution of equation (1) exhibited in Theorem 28, what is the relationship between these two solutions? Our answer is that they are, in some sense, essentially the same.

THEOREM 29. *Under the conditions above, define a measure on the Borel subsets of $\mathfrak{R}(X)$ by*

$$\lambda(U) = \frac{m(X^{-1}(U))}{m(S)}.$$

Then $\lambda(U) = \mu(U)$, for every Borel subset U of $\mathfrak{R}(X)$.

Proof. For every $U \in \mathcal{B}$, we have (in the notation of the group P)

$$\begin{aligned} \lambda(t_1 U) \cdot m(S) &= \lambda(T(U)) \cdot m(S) = m(X^{-1}(T(U))) \\ &= m(h(X^{-1}(U))) \\ &= m(X^{-1}(U)) \\ &= \lambda(U) \cdot m(S). \end{aligned}$$

Thus, λ-measure is invariant under translation by t_1 and therefore also by $(t_1)^i = t_i$. If C is a closed set and V a sufficiently small neighborhood of x, then $\lambda(CV)$ is arbitrarily close to $\lambda(C)$, by the total additivity of m. Since the t_i are dense in $\mathfrak{R}(X)$, we obtain $\lambda(yC) = \lambda(C)$ for all $y \in \mathfrak{R}(X)$, C closed in $\mathfrak{R}(X)$. It follows immediately that λ is Haar measure, and thus that $\lambda = \mu$. Q.E.D.

Thus, the function X maps (S, Σ, m) and h respectively into $(\mathfrak{R}(X), \mathfrak{B}, \lambda)$ and T. By this theorem, the image of this map is unique, depending only on T and the point $x \in \mathfrak{X}$.

6. Infinite measure spaces. In the preceding two sections, we have presented a fairly full picture of which operators can be eigen operators of ergodic transformations in finite non-null measure spaces. In null measure spaces, the entire question is without meaning. This leaves open the question of infinite measure spaces. Here, the situation is not yet so well crystallized. It is clear that any operator T which can be represented as an eigen operator in a finite measure space can be so represented in an infinite measure space as well. This is easily done by assigning a new measure which is 0 on sets of measure 0, and ∞ on sets of measure greater than 0. Thus, the conditions of Theorem 28 are sufficient in this case. Actually, we can do better than that.

THEOREM 30. *If $\mathfrak{X}$ is a B-space and T is a bounded linear operator in $\mathfrak{X}$ with $\|T\| \leqq 1$, then T can be represented as an eigen operator of an ergodic transformation in an infinite measure space if and only if there exists in $\mathfrak{X}$ an element $x \neq 0$ for which $x \in P_x = \mathrm{Cl}(\{T^i(x)\}_{i=1}^{\infty})$.*

Proof. It should be noted that all the work up to and including Lemma 27 makes no mention of compactness or finite measure and is thus applicable here *in toto*. In particular, we have the necessity of the condition above by Lemma 7. Now let us construct a measure space $(P, \mathfrak{B}, \mu)$ in P_x by taking $\mathfrak{B}$ the Borel field of P_x and defining $\mu(U) = 0$ if U is of first category in P_x, $\mu(U) = \infty$ otherwise. Using the group structure built up in §3 and especially Theorem 25, we take for our ergodic m.p.t. multiplication by t_1, i.e. T. If we take I for our strongly measurable function, as in Theorem 25, then $T(I(y)) = T(y) = t_1 y = I(t_1 y)$ for all $y \in P_x$. Furthermore, $\mu(P) = \infty$, since P_x is closed, and thus a complete metric space, and thus not of first category in itself. It remains only to show ergodicity. Let U and V be Borel subsets of P_x which are of second category. Then there is an open set $G \subset P_x$ such that $G - U$ is of first category in P_x. Then $\bigcup_{i=1}^{\infty} T^i(G) = P_x$. Therefore,

$$\begin{aligned}\bigcup_{i=1}^{\infty} T^i(U) &\supset \bigcup_{i=1}^{\infty} T^i(G \cap U)\\ &\supset \bigcup_{i=1}^{\infty} T^i(G) - \bigcup_{i=1}^{\infty} T^i(G - U)\\ &= P_x - W,\end{aligned}$$

where W is a set of first category. Therefore, $\bigcup_{i=1}^{\infty}(T^i(U)\cap V)=(\bigcup_{i=1}^{\infty}T^i(U))\cap V\supset V-W$ is a set of second category. Thus, $T^i(U)\cap V$ is of second category for some $i>0$. That is, $\mu((t_1)^iU\cap V)=\infty>0$ for some $i>0$. Q.E.D.

The above two examples of measure spaces offend the intuition in the same way: every set has measure 0 or ∞. We would like to be able to find a solution to our problem which has some "local finiteness" property:

DEFINITION 31. A strongly measurable function X is said to be *locally finite* if it satisfies condition (F) below.

(F) For every $x\in\mathfrak{R}(X)$, there is an $\epsilon>0$ such that $m(X^{-1}(K(x,\epsilon)))<\infty$.

It will be shown that there are no solutions in an infinite measure space under condition (F). Since T in any solution, finite or infinite, must be an isometry on $\mathfrak{R}(X)$, this will be equivalent to showing that there are no solutions under the weaker condition (F′):

(F′) There is an open set $U\subset\mathfrak{X}$ such that

$$0<m(X^{-1}(U))<\infty.$$

LEMMA 32. *If* $\mathfrak{X}$, T, (S,Σ,m), h, *and* X *give us a solution of equation* (1) *and* $m(S)=\infty$ *and* X *satisfies condition* (F), *then* $\mathfrak{R}(X)$ *is not compact.*

Proof. Suppose $\mathfrak{R}(X)$ is compact. Let $x\in\mathfrak{R}(X)$. Then, by condition (F) there is an $\epsilon>0$ such that $m(X^{-1}(K(x,\epsilon)))<\infty$. The spheres $K(T^i(x),\epsilon)$, $i=1,2,\cdots$, cover $\mathfrak{R}(X)$ by Lemma 6. Since $\mathfrak{R}(X)$ is assumed compact, $\mathfrak{R}(X)$ can be covered by finitely many such spheres. But these spheres have pre-images of equal finite measure, as shown in Lemma 27. Thus, the pre-image of $\mathfrak{R}(X)$ has finite measure. But $m(X^{-1}(\mathfrak{R}(X)))=m(S)=\infty$. This is a contradiction and therefore the assumption that $\mathfrak{R}(X)$ is compact must be false. Q.E.D.

LEMMA 33. *Under the same hypotheses,* $\mathfrak{R}(X)$ *is locally compact.*

Proof. Let $x\in\mathfrak{R}(X)$. Choose $\epsilon>0$ so that $m(X^{-1}(K(x,\epsilon)))<\infty$. We will show that $\mathrm{Cl}(K(x,\epsilon/2))$ is compact. Suppose not. Then there is a $\delta>0$ and a sequence $\{y_i\}$ such that $\|y_i-y_j\|>\delta$ if $i\neq j$. Since $\|y_i-x\|<\epsilon/2$, all i, we see that $\delta<\epsilon$. Thus, the spheres $K(y_i,\delta/2)$ are pairwise disjoint and contained in $K(x,\epsilon)$. The result now follows from the method of Lemma 27. Q.E.D.

LEMMA 34. *If* $\mathfrak{X}$, T, (S,Σ,m), h, *and* X *give a solution of equation* (1), *and* $m(S)=\infty$, *then* X *does not satisfy condition* (F).

Proof. Assume the contrary. Then, by Theorem 25 and Lemmas 33 and 34, the group P set up in $\mathfrak{R}(X)$ is a monothetic abelian metric group which is locally compact but not compact. It follows by a theorem of H. Anazai and S. Kakutani [1] that P is isometrically isomorphic with the discrete group of integers. Since we have taken "ergodic" to mean positive ergodic, this group cannot be the group P, as is seen from Lemma 6. Q.E.D.

Thus, we see that there are no locally finite solutions.

7. **Classification by eigen operators.** Now that we have some idea of the dependence of eigen operator theory on the theory of topological groups, we turn to the question of the generality of eigen operators as compared with eigen values. It is easily seen that 1 is an eigen value of every m.p.t. Similarly, we note that in any B-space, the identity I is an eigen operator of every m.p.t.

DEFINITION 35. A m.p.t. with only the eigen value 1 is said to have *trivial point spectrum.*

DEFINITION 36. Let (S, Σ, m) be a measure space and h an ergodic m.p.t. in S. Suppose that equation (1) is only solved for h with X (almost) identically constant. Then we say that h has *trivial point spectrum* (*eigen operators*).

THEOREM 37. *If (S, Σ, m) is a finite measure space and h an ergodic m.p.t. in S, then h has trivial point spectrum (eigen operators) if and only if h has trivial point spectrum.*

Proof. If h does not have trivial point spectrum, then there is a complex number λ and a (nonconstant) complex measurable function f for which

$$\lambda f(s) = f(h(s)) \text{ a.e. in } S.$$

Choose any $x_0 \neq 0$ from $\mathfrak{X}$. Define $T = \lambda I$, where I is the identity operator in $\mathfrak{X}$. Let $X(s) = f(s)x_0$ for all $s \in S$. Then

$$T(X(s)) = \lambda I(f(s)x_0) = \lambda f(s)x_0 = f(h(s))x_0 = X(h(s))$$

a.e. in S. Since $|\lambda| = 1$, $\|T\| = 1$. Since f is not constant, neither is X.

On the other hand, let us assume there is a solution to equation (1) in which X is not constant. Choose any element $x \in \mathfrak{R}(X)$. We thus have the fact $\mathfrak{R}(X)$ does not consist of x alone. Let us represent $\mathfrak{R}(X)$ as a compact abelian group with identity x and generator $T(x)$, after the manner of §3. Then, by the Peter-Weyl theorem, we know that there is a continuous homomorphism $\hat{x}$ of $\mathfrak{R}(X)$ (considered as a topological group) into the multiplicative group of complex numbers of norm 1, $\hat{x} \not\equiv 1$. Since $T(x)$ is a topological generator of $\mathfrak{R}(X)$, we have $\hat{x}(T(x)) \neq 1$. Then we define $f(s) = \hat{x}(X(s))$ if $X(s) \in \mathfrak{R}(X)$, $f(s) = 0$ otherwise. Set $\lambda = \hat{x}(T(x))$. Then

$$\begin{aligned} \lambda f(s) &= \hat{x}(T(x))\hat{x}(X(s)) \\ &= \hat{x}(T(x)X(s)) \\ &= \hat{x}(T(X(s))) \\ &= \hat{x}(X(h(s))) \\ &= f(h(s)) \text{ a.e. in } S. \end{aligned}$$

Thus, h does not have trivial point spectrum. Q.E.D.

This theorem limits sharply the possibilities of eigen operators as a tool

in the classification of ergodic m.p.t.'s. It should be noted that while $|\lambda| = 1$ is a consequence of λ satisfying the eigen value equation, the assumption $\|T\| \leqq 1$ is necessary. In fact, if we allow $\|T\|$ to exceed 1, we can construct examples in which equation (1) is satisfied in a finite measure space and yet ess sup $\|X(s)\| = \infty$. The construction is lengthy and will not appear here.

8. **About the group P_x.** When $m(S) < \infty$, the group P_x constructed in §3 is compact, metric, monothetic, and abelian. There is nothing more that we can say about P_x, in general, for we can realize any compact metric monothetic abelian group in this way. Thus, let G be such a group, and let a be a generator. Let $(G, \mathcal{B}, \mu)$ be the Haar measure space in G. Define a strongly measurable mapping X from $(G, \mathcal{B}, \mu)$ into $L_2(G)$, the Hilbert space of square-summable complex functions on $(G, \mathcal{B}, \mu)$, by $g_0 \to X(g_0) = f_{g_0}$, where f_{g_0} is defined by $f_{g_0}(g) = d(g_0, g)$. Let the unitary operator U be defined in $L_2(G)$ by

$$Uf(g) = f(a^{-1}g).$$

THEOREM 38. *Using the notation defined above, $U(X(g)) = X(ag)$ a.e. in G, and $\mathfrak{R}(X)$ as a topological group is topologically isomorphic with G.*

Proof. We first mention that, under the above hypotheses, multiplication by a is an ergodic transformation in $(G, \mathcal{B}, \mu)$ and that $\|U\| = 1$, since U is unitary. Since G is compact, the functions f_{g_0} are uniformly bounded by $\max_{g_1, g_2 \in G} d(g_1, g_2)$. The norm

$$\begin{aligned} |||f_{g_1} - f_{g_2}|||_2 &= \left(\int_G |f_{g_1}(g) - f_{g_2}(g)|^2 \mu(dg) \right)^{1/2} \\ &= \left(\int_G |d(g_1, g) - d(g_2, g)|^2 \mu(dg) \right)^{1/2} \\ &\leqq \left(\int_G |d(g_1, g_2)|^2 \mu(dg) \right)^{1/2} \\ &= d(g_1, g_2). \end{aligned}$$

Consider now the set of all $f_g \in L_2(G)$. Call it G_1. Then G_1 is the 1-1 continuous image of a compact space. Thus the mapping is a homeomorphism. Also, G_1, which is the range of X, is actually equal to $\mathfrak{R}(X)$ and $G \cong G_1$ as a group. Finally,

$$\begin{aligned} U(X(g_0)) &= U(f_{g_0}(\cdot)) = f_{g_0}(a^{-1}(\cdot)) = d(g_0, a^{-1}(\cdot)) \\ &= d(ag_0, \cdot) = f_{ag_0}(\cdot) = X(ag_0) \qquad \text{for all } g_0 \in G. \qquad \text{Q.E.D.} \end{aligned}$$

COROLLARY 39. *If we take $L_\infty(G)$ instead of $L_2(G)$ in Theorem 38, the mapping $G \to G_1$ becomes an isometric isomorphism.*

Proof. Clear.

9. **Unsolved problems.** We come now to a review of the principal problems left unsolved in the foregoing.

(1) We have seen that there are certain very severe limits to the degree by which the theory of eigen operators can extend the results of the theory of eigenvalues. It would be interesting to know whether the added generality of the definition can add to the solution of the classification problem for ergodic transformations.

(2) Is it possible to derive some sort of spectral theory for B-spaces of vector-valued measurable functions using eigen operators?

(3) Very little has been done under the hypothesis $\|T\|>1$. It can be shown that there are solutions under such a hypothesis which are different from those under $\|T\|\leqq 1$ in very basic ways, but no general theory of such solutions exists.

(4) It is not known whether, in the case $\|T\|\leqq 1$, $m(S)=\infty$, there exist any solutions other than the two given in this paper. More specifically, it is not known whether a solution can be found in a measure space containing any sets of measure other than 0 or ∞. Under the condition of ergodicity, this is the same as asking whether there exist solutions in σ-finite measure spaces.

(5) Finally, we should like to be able to pair the compact monothetic metric groups with the B-spaces in which they can be embedded, in the sense of P_x. The general problem of when a given compact monothetic metric group can be embedded in a given B-space so that the translation in the group by a generator can be extended to an isometric linear operator in the B-space is completely open.

Bibliography

1. H. Anzai and S. Kakutani, *Bohr compactifications of a locally compact abelian group*, I and II, Proceedings of the Imperial Academy of Tokyo vol. 19 (1943) pp. 476–480; 533–539.

2. A. Beck and J. T. Schwartz, *A vector-valued random ergodic theorem*, Proc. Amer. Math. Soc. vol. 8 (1957) pp. 1049–1059.

3. P. R. Halmos, *Measure theory*, New York, 1950.

4. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, New York, 1948.

5. E. Hopf, *Ergodentheorie*, Berlin, 1937.

6. S. Kakutani, *Mean ergodic theorem in abstract (L)-spaces*, Proceedings of the Imperial Academy of Tokyo, vol. 15 (1939) pp. 121–123.

7. L. Pontrjagin, *Topological groups*, Princeton, 1939.

8. M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc. Colloquium Publications, vol. 15, New York, 1932.

Yale University,
New Haven, Conn.
Tulane University,
New Orleans, La.