ANALYTIC GROUP KERNELS AND LIE ALGEBRA KERNELS(1)

BY

R. A. MACAULEY

1. **Introduction.** In this paper we develop a kernel theory for analytic groups (that is, connected Lie groups) analogous to the kernel theory developed for discrete groups by Eilenberg and MacLane in [3]. Although the methods employed in this paper are necessarily somewhat more delicate than those in [3], our development in §4 and the early part of §3 is modeled after that given by Eilenberg and MacLane.

The initial difficulty encountered is in suitably defining an "analytic" kernel. This situation is complicated by the fact that the group of inner automorphisms of an analytic group need not be a closed subgroup of the group of all continuous and open automorphisms of the given analytic group. However, we are able to circumvent this latter obstacle and we show in the early portion of §3 that we may indeed form the group $\mathrm{Ker}(C, H, d)$ of equivalence classes of analytic H-kernels with centers C and inducing H operators on C identical with those given by the continuous homomorphism $d: H \rightarrow A(C)$ = group of continuous automorphisms of C. We recall that in the discrete case $\mathrm{Ker}(C, H, d)$ is the standard interpretation for the three dimensional cohomology group of H in C.

In the following our notation is that X' denotes the Lie algebra of a Lie group X and if f is an analytic homomorphism, then f' is the induced Lie algebra homomorphism.

We associate with each analytic H-kernel a corresponding Lie algebra H'-kernel and show that this association induces a homomorphism of $\mathrm{Ker}(C, H, d)$ into the group $\mathrm{Ker}(C', H', d')$ of equivalence classes of finite dimensional Lie algebra H'-kernels with centers C' and inducing H' operators on C' identical with those given by $d': H' \rightarrow D(C')$ = algebra of derivations of C'. The latter part of §3 is concerned with showing that if H is simply connected, then the above mentioned homomorphism is in fact an isomorphism of $\mathrm{Ker}(C, H, d)$ onto $\mathrm{Ker}(C', H', d')$. Actually we obtain a somewhat better result than this in Theorem 3.18.

In §4 we establish, under the condition that H is simply connected, an isomorphism of $\mathrm{Ker}(C, H, d)$ into the three dimensional cohomology group

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of H in C which is computed using only cochains that are also analytic maps and using the customary coboundary operator.

We conclude with §5 which contains a result guaranteeing under a certain condition the existence of analytic H-kernels compatible with the situation $d: H \rightarrow A(C)$. The condition imposed is satisfied, for example, if H is semisimple and with mild conditions if H is compact.

2. **Preliminaries.** In order to avoid digressions within the development of the general theory, we collect here a number of elementary results and notions that shall be required later. We point out that our terminology and arguments concerning Lie groups are based on the theory of Lie groups as developed in [1] and that we shall adopt the following conventions.

If X is a topological group, X_0 denotes the component of the identity element in X; if X is a Lie group, X' denotes the Lie algebra of X and if f is a continuous homomorphism from one Lie group to another, f' denotes the induced Lie algebra homomorphism; if X is an analytic group, X^* denotes a simply connected covering group of X. Finally, we emphasize that a subgroup of a topological group need not be a closed subgroup but only a subgroup of the underlying algebraic structure.

If G is a connected topological group, then the diagonal D of $G \times G$ is a closed connected subgroup of $G \times G$. This is immediate for D is closed since it is the inverse image of $1 \in G$ under the continuous mapping of $G \times G$ into G that sends (x, y) to xy^{-1} and D is connected since D is the image of G under the continuous mapping of G into $G \times G$ that sends x to (x, x).

LEMMA 2.1. *Let K and L be Lie groups that satisfy the second axiom of countability and let G be an analytic group. Let f and g be continuous homomorphisms of K and L, respectively, onto G. Denote by $f \times g$ the continuous homomorphism of $K \times L$ onto $G \times G$ that sends (x, y) to $(f(x), g(y))$. Then there is a unique closed connected subgroup E of $K \times L$ that satisfies:*

(1) *$f \times g$ maps E onto the diagonal D of $G \times G$.*

(2) *E contains every subgroup of $K \times L$ which is connected in the induced topology and which is mapped into D by $f \times g$.*

(3) *The natural projections of E into K and L are continuous homomorphisms onto K_0 and L_0, respectively.*

(4) *If L and the kernel of f are connected or if K and the kernel of g are connected, then E is precisely the inverse image of D under $f \times g$.*

Proof. Let P be the inverse image of D under $f \times g$. Since D is closed in $G \times G$, P is closed in $K \times L$. Let E denote the component of 1 in P. Since P is a Lie group, E is open in P. Also, since P and D are locally compact and satisfy the second axiom of countability, the continuous homomorphism of P onto D induced by $f \times g$ is an open homomorphism [11]. Hence $f \times g$ maps E onto an open neighborhood of 1 in D and since D is connected, E is mapped onto D by $f \times g$. This proves (1). If F is any subgroup of $K \times L$ which is connected in the induced topology and which is mapped into D by $f\times g$, then F must be contained in $P_0=E$; hence (2) is proved. The natural projection of $K\times L$ onto L induces a continuous homomorphism of P onto L. Since P and L are Lie groups satisfying the second axiom of countability, this homomorphism is also open. Hence E must map onto a neighborhood of 1 in L and, consequently, E maps onto L_0. Similarly we show the projection of E into K is onto K_0. It remains to prove (4). The kernel of the continuous open homomorphism of P onto L, given by projecting P onto L, is isomorphic with the kernel of f. Since both L and this kernel are connected, P is connected [1]. Hence $E=P$. The remainder of (4) is proved similarly.

Since we have occasion to deal with nonconnected Lie groups, we will accept the following definition of $A(K)$ as a topological group. This definition is valid for every locally compact topological group K and it is shown in [6] that this definition coincides with the one given in [1] in the case where K is an analytic group.

DEFINITION 2.2. Let K be a locally compact topological group and let $A(K)$ be the group of all continuous open automorphisms of K. For each compact subset C of K and each neighborhood V of 1 in K, define $N(C, V)$ as the set of all elements a in $A(K)$ for which both $a(x)x^{-1}$ and $a^{-1}(x)x^{-1}$ belong to V for all x in C. $A(K)$ is then given the structure of a topological group by taking the set of all such $N(C, V)$ as a fundamental system of neighborhoods of the identity element in $A(K)$.

It follows easily from this definition that the map of $A(K)\times K$ into K that sends (a, x) to $a(x)$ is continuous. If K is an analytic group, then $A(K)$ is a Lie group [1] and the above map when restricted to $A_0(K)\times K$ is analytic, where $A_0(K)$ is the component of 1 in $A(K)$.

In the situation that G is a covering group of an analytic group K, we shall always identify G' with K'.

Whenever we have occasion to deal with an analytic group K and its simply connected covering group K^*, we shall regard $A(K)$ as the closed subgroup of $A(K^*)$ that consists of all elements of $A(K^*)$ that map the fundamental group of K onto itself [1]. Since the Lie algebra of $A(K^*)=A(K')$ can be identified with the derivation algebra $D(K')$ of K', the Lie algebra of $A(K)$ can be identified with a subalgebra of $D(K')$. In view of this identification it follows from the adjoint representation of K that the Lie algebra of the group $I(K)$ of all inner automorphisms of K is the ideal $J(K')$ of all inner derivations of K'.

The following results, although elementary, are basically important to us. These results are surely well known and in any case they are easily established.

Let G be an analytic group, let K be a closed subgroup of G, and let A be an analytic subgroup of $A(G)$ such that each element of A maps K into itself. Then the restriction homomorphism of A into $A(K)$ is continuous.

Hence the corresponding homomorphism of A into $A(K_0)$ is analytic and induces the corresponding restriction homomorphism of A' into $D(K')$.

If in the above K is also normal in G, then each element of A induces in a natural way a continuous automorphism of $G/K=H$ and this correspondence is an analytic homomorphism f of A into $A(H)$. Further, f' is the corresponding natural Lie algebra homomorphism of A' into $D(H')$.

It follows from the first of these two results that if K is a closed normal analytic subgroup of an analytic group G, the natural homomorphism of G into $A(K)$, which is the "adjoint representation" of G followed by the restriction homomorphism of $I(G)$ into $A(K)$, is analytic and induces the corresponding natural Lie algebra homomorphism of G' into $D(K')$. This result will be of importance when comparing group and algebra kernels derived from a corresponding set of extension sequences.

3. **Analytic group kernels.** Since the group $I(K)$ of all inner automorphisms of an analytic group K is not necessarily a closed subgroup of the group $A(K)$ of all continuous automorphisms of K [6], we shall regard the factor group $A(K)/I(K)$ as a group without topological structure throughout this paper.

We refer to a pair (K, f) as an H-kernel if H and K are analytic groups and if f is (algebraically) a homomorphism of H into $A(K)/I(K)$. We single out—by means of the following definition—those H-kernels (K, f) for which f satisfies a continuity-like requirement.

DEFINITION 3.1. Let H^* be a simply connected covering group of an analytic group H and let h be a covering homomorphism. An H-kernel (K, f) is called an analytic H-kernel if there exists an analytic map f_0 (not necessarily a homomorphism) of H^* into $A(K)$ such that the following diagram is commutative:

$$\begin{array}{ccc} H^* & \xrightarrow{f_0} & A(K) \\ \downarrow h & & \downarrow q \\ H & \xrightarrow{f} & A(K)/I(K) \end{array}$$

where q is the natural homomorphism.

The reasons for demanding that f be lifted analytically with domain H^* rather than H are two-fold. One, we prefer to place as lax a condition as possible on the kernels and two, since extendible kernels are essential to our theory, we wish to have that every extendible kernel is analytic. Our proof of this last statement will not yield the desired analytic map from H into $A(K)$ unless H is simply connected.

If (K, f) is an analytic H-kernel and if C is the center of K, then this kernel structure induces a continuous homomorphism d of H into $A(C)$. For f_0, followed by the continuous map obtained by sending an element of $A(K)$ to its restriction to C, is a continuous map of H^* into $A(C)$ and is the same as the homomorphism obtained by following fh by the "restriction" of $A(K)/I(K)$ to C. Hence the analytic kernel structure (K, f) gives rise to a continuous homomorphism of H^* into $A(C)$ and since the fundamental group of H is in the kernel of this homomorphism, we see that (K, f) induces a continuous homomorphism d of H into $A(C)$.

The following theorem, which is due to Malcev [10], is of the utmost technical importance to us. For a proof of this theorem we refer the reader to [4].

THEOREM 3.2. *Let E be a simply connected analytic group and let K be a connected normal analytic subgroup of E. Then K is a closed subgroup of E and, further, if $H=E/K$ and p is the natural homomorphism of E onto H, then there exists an analytic map g of H into E such that pg is the identity map on H; i.e., g is inverse to p.*

Using the analytic maps p and g, we can decompose E as a manifold into a product manifold isomorphic with the manifold $K\times H$. Hence we have the following corollary.

COROLLARY 3.3. *Let E, K, and H be as in the above theorem. Then both K and H are simply connected.*

COROLLARY 3.4. *Let p be an analytic homomorphism of an analytic group E onto a simply connected analytic group H. Then there exists an analytic map g of H into E inverse to p.*

Proof. Let E^* be a simply connected covering group of E and let e be a covering homomorphism. Then pe is an analytic homomorphism of E^* onto H. The kernel of pe is connected since H is simply connected. Then the analytic map of H into E^* inverse to pe (obtained by applying Theorem 3.2 to E^* and the kernel of pe) followed by e yields the desired analytic map g of the corollary.

Since one of our objectives is to exhibit a relation between analytic group kernels and Lie algebra kernels we shall now reformulate the definition of an analytic kernel in such a way as to make this relationship more apparent.

Let (K, f) be an H-kernel—not necessarily an analytic H-kernel. Denote by $A_0(K)$ the component of the identity element in $A(K)$. Let A be a simply connected covering group of $A_0(K)$ and let p be a covering homomorphism. Since A is simply connected and since $(p')^{-1}(J(K'))$ is an ideal of A', the analytic subgroup I of A with Lie algebra $(p')^{-1}(J(K'))$ is a closed normal analytic subgroup of A and, further, p maps I onto $I(K)$. Hence A/I is an analytic group (and in fact, is simply connected by Corollary 3.3) and p induces a natural (algebraic) homomorphism p_1 of A/I into $A(K)/I(K)$.

DEFINITION 3.5. An H-kernel (K, f) is said to be coverable if there exists an analytic homomorphism f^* of H^* into A/I such that the following diagram is commutative.

$$\begin{array}{ccc} H^* & \xrightarrow{f^*} & A/I \\ \downarrow h & & \downarrow p_1 \\ H & \xrightarrow{f} & A(K)/I(K) \end{array}$$

We remark for later use that the kernel of p_1 is totally disconnected in the topology induced by A/I. This follows because the topology of every analytic group satisfies the second axiom of countability and therefore, as a discrete subgroup of A, the fundamental group of $A_0(K)$ is countable; then, since the kernel of p_1 is the image of the fundamental group of $A_0(K)$ under the natural homomorphism of A onto A/I, we see that the kernel of p_1 is a countable subgroup of A/I and every countable subgroup of a locally Euclidean group is certainly totally disconnected.

THEOREM 3.6. *An H-kernel (K, f) is an analytic H-kernel if and only if it is coverable.*

Proof. Suppose (K, f) is coverable. Theorem 3.2 yields an analytic map g of A/I into A inverse to the natural homomorphism of A onto A/I. Then $f_0 = pgf^*$ is an analytic map of H^* into $A(K)$ satisfying the requirement that (K, f) be an analytic H-kernel.

Now suppose that (K, f) is an analytic H-kernel. We may assume the analytic map f_0 of H^* into $A(K)$ is such that $f_0(1) = 1$; otherwise, compose the given f_0 with multiplication by $f_0(1)^{-1}$. Since H^* is simply connected, we may lift f_0 uniquely to an analytic map f_1 of H^* into A such that $f_1(1) = 1$. Then f^*, defined to be f_1 followed by the natural homomorphism of A into A/I, is an analytic map of H^* into A/I such that $p_1 f^* = fh$. Hence, in order to show that (K, f) is coverable, we need only show that f^* is actually a homomorphism.

Consider the analytic map of $H^* \times H^*$ into A/I that sends a pair (x, y) to $f^*(x)f^*(y)f^*(xy)^{-1}$. Since p_1 and $p_1 f^*$ are homomorphisms, we see that the image of $H^* \times H^*$ under the above map is in the kernel of p_1, which is totally disconnected. Therefore, since $H^* \times H^*$ is connected and since the pair $(1, 1)$ is mapped to 1, the image of $H^* \times H^*$ in A/I is the identity element. Hence f^* is a homomorphism and therefore (K, f) is coverable.

REMARK. If (K, f) is an H-kernel for which $I(K)$ is closed in $A(K)$, then (K, f) is an analytic H-kernel if and only if f is an analytic homomorphism. This result follows immediately by considering the property of coverability for (K, f) and noticing that in this case A/I is a (simply connected) covering group of $A_0(K)/I(K)$.

Let (K, f) be an analytic H-kernel. Since (K, f) is coverable, there exists an analytic homomorphism f^* of H^* into A/I such that $p_1 f^* = fh$. Let g^* denote any analytic homomorphism of H^* into A/I satisfying $p_1 g^* = fh$. Then the map of H^* into A/I which sends an element x to $f^*(x)g^*(x)^{-1}$ is analytic and the image of H^* is contained in the kernel of p_1. As in the proof of the above theorem, we conclude that $f^*(x)g^*(x)^{-1}=1$ for all x in H^*; that is, $f^*=g^*$. Hence we see that there exists one and only one analytic homomorphism f^* of H^* into A/I such that $p_1f^*=fh$ for any given analytic H-kernel (K, f).

It now follows that we may uniquely associate with each analytic H-kernel (K, f) a Lie algebra H'-kernel (K', f'), where f' is the Lie algebra homomorphism of H' into $D(K')/J(K')$ which is induced by f^*. We shall refer to (K', f') as the Lie algebra kernel associated with (K, f) or, perhaps, as the Lie algebra kernel induced by (K, f).

DEFINITION 3.7. A pair (P, p) is called an extension of an H-kernel (K, f) if P is an analytic group containing K and if p is an analytic homomorphism of P onto H with kernel K such that, for every x in P, the inner automorphism effected by x on P when restricted to K belongs to the coset $^f(p(x))$. An H-kernel which admits an extension is called extendible.

LEMMA 3.8. *Let r be an analytic homomorphism of an analytic group L onto an analytic group H with kernel F. Then each H-kernel (K, f) can be lifted to an L-kernel (K, fr) and we obtain the following results*:

(1) *If (K, f) is an analytic H-kernel, then (K, fr) is an analytic L-kernel and the associated Lie algebra kernel is $(K', f'r')$.*

(2) *If (P, p) is an extension of an H-kernel (K, f), we can lift (P, p) to an extension (Q, q) of the L-kernel (K, fr) such that Q contains a closed normal subgroup which is mapped isomorphically onto F by q. In particular, it follows that we obtain an extendible L-kernel on lifting an extendible H-kernel* [4].

Proof. (1) is an immediate consequence of the following commutative diagram, where r^* is obtained by the natural lifting of r.

$$\begin{array}{ccccc} L^* & \xrightarrow{r^*} & H^* & \xrightarrow{f^*} & A/I \\ \downarrow & & \downarrow & & \downarrow \\ L & \xrightarrow{r} & H & \xrightarrow{f} & A(K)/I(K) \end{array}$$

Let (P, p) be an extension of the H-kernel (K, f). Consider the analytic homomorphism $p\times r$ of $P\times L$ onto $H\times H$. Let Q be the inverse image under $p\times r$ of the diagonal of $H\times H$. Since L and the kernel of p are connected, we have by Lemma 2.1 that Q is connected. Let q be the projection of Q onto L. Then the kernel of q can be identified with K and it is easily seen that (Q, q) is an extension of the L-kernel (K, fr). Further, Q contains the closed normal subgroup $(1)\times F$ which is mapped isomorphically onto F by q.

We are now in a position to prove easily the following theorem.

THEOREM 3.9. *Every extendible H-kernel is an analytic H-kernel.*

Proof. Let (K, f) be an extendible H-kernel and let h be the covering homomorphism of H^* onto H. Then (K, fh) is an extendible H^*-kernel. Let (P, p) be an extension of (K, fh). Since H^* is simply connected, Corollary 3.4 yields an analytic map g of H^* into P inverse to p. Then g followed by the natural analytic homomorphisms of P into $I(P)$ and of $I(P)$ into $A(K)$ yields the desired analytic map f_0 in Definition 3.1 of an analytic H-kernel.

We now begin the rather lengthy verification that the method of composition and the equivalence relation used by Eilenberg and MacLane will yield the desired similar results for analytic kernels. The main difficulty is to replace by analytic maps, at each stage, the arbitrary selection of elements made by Eilenberg and MacLane. At the same time, we must take care to establish the connections with the composition and equivalence for the corresponding Lie algebra kernels.

We define the composite of two kernels over a common closed subgroup of the centers rather than the whole center. Reasons for this will appear near the end of this section.

Let K and L be two analytic groups with centers C_1 and C_2, respectively. Suppose that U is a closed subgroup of both C_1 and C_2. Form $K\times L$ and let S be the subgroup consisting of all elements (u, u^{-1}), with u in U. Since S is the inverse image of 1 in U under the continuous mapping of $U\times U$ into U that sends a pair (u, v) to uv, S is closed in $U\times U$ and therefore in $K\times L$. Consequently, $(K\times L)/S$, with the induced topology, is an analytic group and the center is easily seen to be $(C_1\times C_2)/S$. Since the natural homomorphism of $C_1\times C_2$ onto $(C_1\times C_2)/S$ is certainly a continuous and open homomorphism and since the inverse image of the subgroup of $(C_1\times C_2)/S$ consisting of all elements of the form $(u, 1)S$, with u in U, is exactly the closed subgroup $U\times U$ of $C_1\times C_2$, this subgroup of $(C_1\times C_2)/S$ is closed. The map of U onto this subgroup, which sends an element u to the coset $(u, 1)S$, is easily checked to be both a group isomorphism and a homeomorphism. We denote by $K\times_U L$ the analytic group obtained upon embedding U as a closed subgroup of the center of $(K\times L)/S$ by the above isomorphism.

The Lie algebra S' of S is a subalgebra of the direct sum $U'+U'$. Suppose (u', v') is an element of S'. Then, for all real numbers t, $\exp t(u', v') = (\exp tu', \exp tv')$ is in S. Hence $\exp tv' = (\exp tu')^{-1} = \exp t(-u')$ for all t. Consequently, $v' = -u'$. Similarly we can show that all $(u', -u')$ with u' in U' are in S'. Therefore S' is exactly the ideal of $U'+U'$ consisting of all elements $(u', -u')$. As in the case of groups, we denote by $K'\times_{U'}L'$ the Lie algebra obtained upon embedding U' into the center of $(K'+L')/S'$ by the algebra isomorphism which sends an element u' to the coset $(u', 0)+S'$. Since the exact sequence

$$(1) \to S \to K \times L \to K\times_U L \to (1)$$

induces the algebra sequence

$$(0) \to S' \to K' + L' \to (K\times_U L)' \to (0),$$

we see that $(K \times_U L)'$ can be identified with $K' \times_{U'} L'$.

Now let (K, f) and (L, g) be two H-kernels such that U is a closed H-subgroup of the centers of both K and L. Also suppose that the H operators defined on U by f are identical with those defined by g. For each x in H, choose elements $a(x)$ and $b(x)$ in the cosets $f(x)$ and $g(x)$, respectively. Then the pair $(a(x), b(x))$ defines a continuous open automorphism on $K \times L$ and since, by assumption, $a(x)$ and $b(x)$ are identical on U, $(a(x), b(x))$ induces a continuous open automorphism $c(x)$ on $K \times_U L$. The map of H into $A(K \times_U L)/I(K \times_U L)$ which sends x to the coset containing $c(x)$ is easily verified to be a homomorphism independent of the choices for $a(x)$ and $b(x)$. This kernel structure on $K \times_U L$ is denoted by $(K, f) \times_U (L, g)$.

Theorem 3.10. *Let (K, f) and (L, g) be two analytic H-kernels. Suppose that U is a closed H-subgroup of the centers of both K and L and that f and g define identical H operators on U. Then $(K, f) \times_U (L, g)$ is an analytic H-kernel and $(K', f') \times_{U'} (L', g')$ is the associated Lie algebra H'-kernel.*

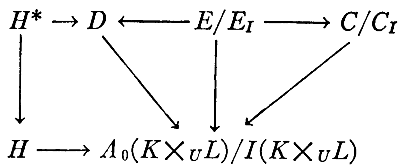
Proof. Using Definition 3.1 of an analytic H-kernel, we could quite easily show that $(K, f) \times_U (L, g)$ is an analytic H-kernel. However, since we are also interested in the associated algebra kernel, we show simultaneously that the composite $(K, f) \times_U (L, g)$ is coverable and that the induced algebra kernel is $(K', f') \times_{U'} (L', g')$.

If, in the diagrams to follow, the reader will replace each analytic group by its Lie algebra and replace each analytic homomorphism by the induced algebra homomorphism, he will see that exactly the analogous constructions for Lie algebras are induced by our constructions on the groups.

To begin the proof, let A and B be simply connected covering groups of $A_0(K)$ and $A_0(L)$, respectively, and let A_I and B_I denote the closed normal analytic subgroups of A and B with Lie algebras $J(K')$ and $J(L')$, respectively. The analytic homomorphism (f^*, g^*) of H^* into $A/A_I \times B/B_I$ which is given by $(f^*, g^*)(x) = (f^*(x), g^*(x))$ maps H^* onto an analytic subgroup D of $A/A_I \times B/B_I$. Let E_1 be the analytic subgroup of $A \times B$ that contains $A_I \times B_I$ such that $E_1/(A_I \times B_I) = D = (f^*, g^*)(H^*)$ and choose E to be a simply connected covering group of E_1 and let E_I be the closed normal analytic subgroup of E with Lie algebra $J(K' + L')$. Then E/E_I is a simply connected covering group of D. If we regard E as operating on $K \times L$ by means of the composite of the covering homomorphisms of E onto E_1 and of $A \times B$ onto $A_0(K) \times A_0(L)$, then the subgroup S, consisting of all pairs (u, u^{-1}) with u in U, is invariant under E—for the automorphism of $K \times L$ effected by an element in E belongs to a coset $(f(x), g(x))$ for some x in H and the invariance of S follows because f and g define identical H operators on U. Consequently we can map the operators E analytically and naturally into $A_0(K \times_U L)$. If C is a simply connected covering group of $A_0(K \times_U L)$ and C_I is the closed normal analytic subgroup of C with Lie algebra $J((K \times_U L)')$, then the homomorphism of E into C obtained by lifting the above homomorphism of E into $A_0(K\times_U L)$ is such that E_I maps onto C_I and thereby induces a homomorphism of E/E_I into C/C_I. Also, since the inverse image in E of $A_I\times B_I$ under the covering homomorphism of E onto E_1 is clearly contained in the kernel of the composite homomorphism $E\rightarrow A_0(K\times_U L)\rightarrow A_0(K\times_U L)/I(K\times_U L)$ this composite homomorphism induces a homomorphism of

$$D\rightarrow A_0(K\times_U L)/I(K\times_U L).$$

Upon factoring by the groups E_I and C_I we are in the situation described by the following commutative diagram.



The lower homomorphism is that which defines the H-kernel structure $(K, f)\times_U(L, g)$.

Since E/E_I is a covering group of D, we can lift $H^*\rightarrow D$ to $H^*\rightarrow E/E_I$. This homomorphism composed with $E/E_I\rightarrow C/C_I$ yields the commutative diagram:

$$\begin{array}{ccc} H^* & \longrightarrow & C/C_I \\ \downarrow & & \downarrow \\ H & \rightarrow & A_0(K\times_U L)/I(K\times_U L). \end{array}$$

Hence we see that $(K, f)\times_U(L, g)$ is an analytic H-kernel. Further, as mentioned at the beginning of the proof, by replacing in the diagrams each group by its Lie algebra and each homomorphism by its induced Lie algebra homomorphism we see that the Lie algebra kernel associated with $(K, f)\times_U(L, g)$ is indeed $(K', f')\times_{U'}(L', g')$. For verification of this fact one requires the result mentioned in §2 which shows that the homomorphism of E into $A_0(K\times_U L)$ induces the natural algebra homomorphism of E' into $D(K'\times_{U'}L')$.

This completes the proof of Theorem 3.10.

Theorem 3.11. *Let (K, f) and (L, g) be two H-kernels. Suppose U is a closed H-subgroup of the centers of both K and L and that f and g induce identical H operators on U. If both (K, f) and (L, g) are extendible, then $(K, f)\times_U(L, g)$ is an extendible H-kernel.*

Proof. Let (P, p) and (Q, q) be extensions of (K, f) and (L, g) respectively. Let E be the inverse image of the diagonal of $H\times H$ under the corresponding analytic homomorphism of $P\times Q$ onto $H\times H$. Since Q and the kernel of p are both connected, we have by Lemma 2.1 that E is connected. Clearly

$K \times L$ is contained in E and the natural homomorphism of E into H is onto H. Therefore we have the exact sequence

$$(1) \to K \times L \to E \to H \to (1).$$

Since this extension defines the given H operators coordinate-wise on $K \times L$ and since the subgroup S, consisting of all pairs (u, u^{-1}), with u in U, is an H-subgroup of $K \times L$, S is normal in E. Further, S is closed in E since S is closed in $K \times L$, which is closed in E. Hence, upon factoring by S, we obtain the exact sequence

$$(1) \to K \times_U L \to E/S \to H \to (1).$$

Therefore E/S is an extension of $(K, f) \times_U (L, g)$ and the theorem is proved.

We shall next construct from any analytic H-kernel (K, f) another analytic H-kernel (K^-, f^-), called the canonical inverse kernel of (K, f). The underlying analytic manifold of K^- is taken to be the underlying manifold of K. We define the product $x * y$ of any two elements x and y of K^- by $x * y = yx$, where yx is the product of y and x in K. Clearly K^- is an analytic group and the center C of K can be identified with the center of K^-; that is, C is also the center of K^-. To each automorphism a in $A(K)$ we associate an automorphism a^- in $A(K^-)$ by defining $a^-(x) = a(x)$ for all x in K^-. This correspondence is both a group and topological isomorphism of $A(K)$ with $A(K^-)$. We define f^- to be f followed by the induced isomorphism of $A(K)/I(K)$ with $A(K^-)/I(K^-)$. Hence we have an H-kernel (K^-, f^-). By taking for $(f^-)_0$ the analytic map of H^* into $A(K^-)$ obtained by following f_0 with the above isomorphism of $A(K)$ onto $A(K^-)$, we see that (K^-, f^-) is an analytic H-kernel.

The following theorem justifies calling (K^-, f^-) an inverse kernel of (K, f).

THEOREM 3.12. *Let (K, f) be any analytic H-kernel and let C be the center of K. Then the composite kernel $(K, f) \times_C (K^-, f^-)$ is an extendible H-kernel.*

Proof. Let A be a simply connected covering group of $A_0(K)$ and let I be the closed normal analytic subgroup of A with Lie algebra $J(K')$. Since (K, f) is an analytic H-kernel, there is an analytic homomorphism f^* of H^* into A/I covering f. The image of H^* in A/I is an analytic subgroup of A/I. By considering the natural homomorphism of A onto A/I and the corresponding homomorphism on the Lie algebras, we see that there is an analytic subgroup A_1 of A containing I such that H^* is mapped onto A_1/I by f^*. Using the natural homomorphism of A_1 onto A_1/I and f^* coordinate-wise, we obtain an analytic homomorphism of $A_1 \times H^*$ onto $A_1/I \times A_1/I$. Since H^* and the kernel I of the homomorphism operating on the first coordinate are both connected, we have by Lemma 2.1 that the inverse image of the diagonal of $A_1/I \times A_1/I$ under the above homomorphism is a closed analytic subgroup B_1 of $A_1 \times H^*$. It is clear that B_1 is precisely the set of all pairs (a, x) in

$A_1 \times H^*$ for which a belongs to the coset $f^*(x)$. Let B be the image in $A_0(K) \times H$ of B_1 under the homomorphism induced by the covering homomorphisms. Then B is an analytic subgroup of $A_0(K) \times H$ and consists of exactly all pairs (a, x) in $A_0(K) \times H$ for which a belongs to the coset $f(x)$.

We define a group structure on the product manifold $E = K \times B$ by

$$(u, a, x) \cdot (v, b, y) = (ua(v), ab, xy)$$

for all such triples in E. It is routine to check that E is a group with this product and since all operations are analytic, we conclude that E is an analytic group. The composite of the projections of E onto B and of B onto H is an analytic homomorphism p of E onto H. The kernel L of p is exactly the set of all elements $(k, a, 1)$ of E for which a belongs to $I(K)$.

We now show that the H-kernel structure induced on L by the extension (E, p) of L by H can be identified with $(K, f) \times_C (K^-, f^-)$. Denote by $b(x)$ the inner automorphism effected on K by an element x of K. The map of K into L, which sends x to $(x, 1, 1)$, is analytic and the map of K^- into L, which sends y to $(y, b(y^{-1}), 1)$ is analytic. On combining these maps coordinate-wise, we obtain an analytic map of $K \times K^-$ into $L \times L$. We follow this map by the product map of $L \times L$ into L and thereby obtain the analytic map of $K \times K^-$ into L that sends a pair (x, y) to $(xy, b(y^{-1}), 1)$. This map is easily shown to be a homomorphism onto L with kernel S, where S consists of all pairs (x, x^{-1}) for x in C. Hence, this homomorphism induces an isomorphism of $K \times_C K^-$ with L. We wish to show this is an H-isomorphism. For each x in H, choose an element $a(x)$ in the coset $f(x)$. Denote by $a^-(x)$ the corresponding automorphism of K^- which is determined by $a^-(x)(u) = a(x)(u)$. Let (u, v) be an element of $K \times K^-$. The image of (u, v) in L is $(uv, b(v^{-1}), 1)$. The image of this element of L under the inner automorphism effected on E by $(1, a(x), x)$ is easily computed to be $(a(x)(uv), b(a(x)(v^{-1})), 1)$, which is the image in L of the element $(a(x)(u), a^-(x)(v))$ of $K \times K^-$. Hence we see that the above isomorphism of $K \times_C K^-$ with L is an H-isomorphism; that is, if we embed $K \times_C K^-$ in E by this isomorphism with L, then (E, p) is an extension of the H-kernel $(K, f) \times_C (K^-, f^-)$.

THEOREM 3.13. *Let (K, f) and (L, g) be two analytic H-kernels, both with the same center C, and inducing identical H operators on C. If both (K, f) and $(K, f) \times_C (L, g)$ are extendible, then (L, g) is extendible.*

Proof. Let (P, p) and (Q, q) be extensions of (K, f) and $(K, f) \times_C (L, g)$, respectively. We identify K with a closed normal subgroup of Q by the isomorphism which sends an element k to the coset $[(k, 1)]$, where in general $[(k, u)]$ denotes the element of $K \times_C L$ with (k, u) as a representative. It is clear that, by this identification, the homomorphism of Q into $A(K)$, obtained by restricting inner automorphisms of Q to K, induces the H-kernel structure (K, f). Combining, coordinate-wise, this homomorphism of Q into

$A(K)$ with q, we obtain an analytic homomorphism of Q into $A(K)\times H$. The image of Q under this homomorphism is clearly the analytic group B, which was obtained in the proof of the last theorem. In the same way, we obtain an analytic homomorphism of P onto B. Combining these homomorphisms we obtain an analytic homomorphism of $P\times Q$ onto $B\times B$. Let E be the closed analytic subgroup of $P\times Q$ which is mapped onto the diagonal of $B\times B$ and which contains every analytic subgroup of $P\times Q$ that is mapped into the diagonal of $B\times B$ (see Lemma 2.1). Since Q is connected, the natural projection of E into Q is onto Q (Lemma 2.1). Following this projection by q, we have an analytic homomorphism of E onto H. Let F denote the kernel of this homomorphism. We now have the exact sequence

$$(1)\to F\to E\to H\to (1).$$

If (x, y) is an element of F, then $p(x)=q(y)=1$; hence x is in K and y is in $K\times_C L$. Let $(k, [(k_1, u_1)])$ be an element of F. Since F is contained in E, the automorphisms effected on K by k and k_1 must be identical. Hence there is an element c in the center C of K such that $k_1=kc$. Let $u=cu_1$. Then $(k, [(k_1, u_1)])=(k, [(k, u)])$. Hence every element of F may be written uniquely in the form $(k, [(k, u)])$. On the other hand we claim that every element of this form is in F. For, consider the natural homomorphism of $K\times L$ onto $K\times_C L$; this yields an analytic homomorphism of $K\times L$ into Q. Also, the projection of $K\times L$ onto K yields an analytic homomorphism into P. Combining these homomorphisms, we obtain an analytic homomorphism of $K\times L$ into $P\times Q$. The image of $K\times L$ is an analytic subgroup of $P\times Q$ which is mapped into the diagonal of $B\times B$. Hence the image of $K\times L$ is contained in E. Since this image is exactly all elements of the form $(k, [(k, u)])$, we see that F is precisely this image. We also notice that this homomorphism is actually an isomorphism of $K\times L$ onto F. Let F_1 be the image in F of the closed subgroup $K\times(1)$ of $K\times L$. Then F_1 is closed in F and, therefore, is closed in E. Further, F_1 is normal in E since elements of F_1 have the form $(k, [(k, 1)])$ and if (x, y) is an element of E, the automorphisms effected on K by x and y are identical. Hence we have the exact sequence

$$(1)\to F/F_1\to E/F_1\to H\to (1).$$

The above isomorphism of $K\times L$ with F induces an isomorphism of L with F/F_1 $(\cong(K\times L)/K)$. Also, it is clear that the extension E of F by H induces, by the isomorphism of $K\times L$ with F, H-operators coordinate-wise on $K\times L$ by f and g respectively. Hence upon identifying L with F/F_1 by the above isomorphism, we see that the extension E/F_1 induces the H-kernel structure (L, g). That is, (L, g) is extendible.

Definition 3.14. Let C be an abelian Lie group, H an analytic group, and let d be a continuous homomorphism of H into $A(C)$. We call the triple

(C, H, d) an admissible kernel system if there exist analytic H-kernels with center C and inducing d.

We recall that d must necessarily be continuous in order to be induced by an analytic kernel structure. Also, it is clear that if C is connected, such a triple (C, H, d) is admissible; for then (C, d) is itself an analytic H-kernel with center C and induces d.

We now consider only those analytic H-kernels associated with a fixed admissible kernel system (C, H, d).

Definition 3.15. We call two analytic H-kernels (K, f) and (L, g) equivalent if the composite $(K, f) \times_C (L^-, g^-)$ is extendible.

It follows easily from the preceding theorems and from the fact that our process of composing kernels is, up to isomorphism, both associative and commutative that the above notion of equivalence is indeed a relation that is reflexive, symmetric, and transitive. We use this equivalence relation to partition the class of all analytic H-kernels into a set of equivalence classes. It is almost immediate that our method of composing two kernels induces a product on this set of equivalence classes which gives rise to an abelian group structure on this set. We denote this abelian group by $\mathrm{Ker}(C, H, d)$. The identity element of this group is the class consisting of all extendible H-kernels with center C and inducing d.

The homomorphism d of H into $A(C)$ followed by the restriction homomorphism of $A(C)$ into $A(C_0)$ yields an analytic homomorphism of H into $A(C_0)$. Denote by d' the Lie algebra homomorphism of H' into $D(C')$ induced by this analytic homomorphism. The abelian group of equivalence classes of finite dimensional Lie algebra H'-kernels with centers C' and inducing d', which is constructed in a manner similar to the above, is denoted by $\mathrm{Ker}(C', H', d')$. It is an immediate consequence of Theorem 3.10 and of the fact that extendible analytic H-kernels induce extendible Lie algebra kernels that the map which sends an analytic H-kernel (K, f) to its associated Lie algebra H'-kernel (K', f') induces a homomorphism of $\mathrm{Ker}(C, H, d)$ into $\mathrm{Ker}(C', H', d')$.

Theorem 3.16. *If H is simply connected, the homomorphism of* $\mathrm{Ker}(C, H, d)$ *into* $\mathrm{Ker}(C', H', d')$ *is an isomorphism into.*

Proof. We must show, for H simply connected, that if (K, f) is an analytic H-kernel with extendible associated Lie algebra kernel (K', f'), then (K, f) is extendible. Let E be a simply connected analytic group such that (E', p') is an extension of (K', f'). Since E is simply connected, there is an analytic homomorphism p of E onto H which induces p'. Because H is simply connected, the kernel of p is connected and therefore, by Corollary 3.3, is simply connected. Hence the kernel of p is a simply connected analytic group with Lie algebra K'. Consequently, the kernel of p is a simply connected covering group of K and we denote it by K^*. Now we have the two exact sequences

$$(0) \to K' \to E' \overset{p'}{\to} H' \to (0)$$

$$(1) \to K^* \to E \overset{p}{\to} H \to (1).$$

Also, we notice that each of these sequences induces the other sequence. By restricting inner automorphisms of E to K^*, we obtain an analytic homomorphism of E onto an analytic subgroup $A_1(K^*)$ of $A_0(K^*)$. This homomorphism induces the homomorphism of E' onto a subalgebra $D_1(K')$ of $D(K')$ obtained by restricting inner derivations of E' to K'. Clearly $D_1(K')$ is the Lie algebra of $A_1(K^*)$. Since (K, f) is an analytic H-kernel, there is an analytic homomorphism f^* of H into A/I and as we saw in the proof of Theorem 3.12, there is an analytic subgroup A_1 of A, containing I, such that f^* maps H onto A_1/I. Let $A_1(K)$ be the analytic subgroup of $A_0(K)$ which is the image of A_1 under the covering homomorphism. Since (E', p') is an extension of (K', f'), the Lie algebra of A_1, and therefore of $A_1(K)$, is $D_1(K')$. Hence, regarding $A(K)$ as a closed subgroup of $A(K^*)$, we see that $A_1(K^*)$ and $A_1(K)$ have the same Lie algebra $D_1(K')$. Therefore $A_1(K^*) = A_1(K)$. In particular, the elements of $A_1(K^*)$ map the fundamental group N of K onto itself. Consequently, N is a normal subgroup of E. Since N is certainly closed in E, we may factor by N to obtain the exact sequence

$$(1) \to K \to E/N \to H \to (1).$$

That this extension of K by H gives the kernel structure (K, f) follows immediately from the fact that this sequence induces the Lie algebra sequence of the extension (E', p'). Hence we see that (K', f') extendible implies that (K, f) is extendible; and we have thereby proved the theorem.

We devote the remainder of this section to show that, for H simply connected, the isomorphism of $\mathrm{Ker}(C, H, d)$ is onto $\mathrm{Ker}(C', H', d')$. We shall require the following well known result which we outline below. The necessary proofs are essentially contained in [1] in the proof that the full matrix algebra may be interpreted as the Lie algebra of the general linear group $GL(n, R)$.

Let A be a finite dimensional associative algebra over the field R of real numbers and form the associative algebra $R+A$ with multiplication defined by $(r+a)(r'+a') = rr' + (ra' + r'a + aa')$. We regard $R+A$ as topologized by the natural vector space topology and then the group of all invertible elements in $R+A$ of the form $1+a$, with a in A, is an open subset of $1+A$. It will follow that the component G of the identity element in this open set of invertible elements is an analytic group and A together with the commutation product $[a_1, a_2] = a_1a_2 - a_2a_1$ is the Lie algebra of G.

The identification of G' with A may be invariantly described as follows. If λ is a linear mapping of A into R and if $\bar{\lambda}$ denotes the restriction to G of the mapping $1+a \to \lambda(a)$ of $1+A$ into R, then $\bar{\lambda}$ is analytic around 1 in G since G is an open subset of $1+A$; also, an element x in G' is completely determined by its effect at 1 in G on the set of all such $\bar{\lambda}$. It follows that there exists a unique element a_x in A such that $x_1(\bar{\lambda}) = \lambda(a_x)$ for all such λ and the mapping $x \rightarrow a_x$ establishes a natural isomorphism of G' with A.

Hochschild [7] has constructed a Lie algebra which is the underlying algebra of a Lie algebra kernel in every class (with prescribed operator Lie algebra H and prescribed H-module as the center of the kernels); that is, a representative of each kernel class can be obtained by defining suitable operations on this standard algebra. The actual construction involved constructing an associative algebra and then taking this associative algebra with the commutation product as the standard Lie algebra. This associative algebra is constructed in several steps. First, $V = C + N$ is constructed, where N is a nilpotent ideal in V and $C = Re + Rf$, as a vector space over R, has multiplication defined by $e^2 = e$, $f^2 = f$, $ef = f$, and $fe = e$. The center of V is (0). Hochschild has shown [8] that in this construction we may actually choose N to be finite dimensional over R to obtain representatives of every class containing a finite dimensional algebra kernel. So in our case we may assume that N is a finite dimensional nilpotent ideal of $V = C + N$. If the center of the Lie algebra to be constructed is M, the associative algebra constructed is $L = M + V$ with multiplication defined by $ML = LM = 0$. In our case, we take Euclidean n-space, for some n, as M. Then L with the commutation product is the standard Lie algebra with center M.

We now wish to find a simply connected analytic group G with connected center and with L as Lie algebra. If we can find a simply connected analytic group $G(V)$ with center (1) and with Lie algebra V, then we can take $G = G(M) \times G(V)$, where $G(M) = R^n$ with $n = \dim(M)$.

In $R + V$ let $G(V)$ denote the component of the identity element in the group consisting of all invertible elements of form $1 + v$, with v in V. We know that $G(V)$ is an analytic group and has V as its Lie algebra. We will show that $G(V)$ has center (1) and is simply connected.

For this purpose, let us remark that, as in the case of matrices, we may define for each v in V

$$\exp v = 1 + v + \frac{v^2}{2!} + \cdots + \frac{v^n}{n!} + \cdots .$$

Convergence can be established by choosing a basis for V over R and showing that this series converges coordinate-wise. As usual, for fixed v in V, the map of R into $1 + V$ which sends t to $\exp tv$ is a homomorphism of the additive group of R into $G(V)$.

Suppose now that $1 + v_0$ is in the center of $G(V)$. Let v denote any element of V. Then $(1 + v_0) \exp tv = (\exp tv)(1 + v_0)$ or $v_0 \exp tv = (\exp tv) v_0$. Hence, for all real t,

$$v_0 + v_0 v t + \cdots = v_0 + v v_0 t + \cdots$$

and we must have $v_0v=vv_0$ for each v in V. That is, v_0 is in the center of V; and since the center of V is (0), the center of $G(V)$ is (1).

In order to show $G(V)$ is simply connected, we first remark that $G(V) = G(V)\cap(1+Re+Rf)+N$. For if $u=1+ae+bf$, with a and b real numbers, and if x belongs to N, then computing modulo N we see that $u+x$ invertible implies u is invertible. Conversely, if u is invertible and if u^{-1} denotes the element such that $u^{-1}u=1$, then $u^{-1}(u+x)=1+u^{-1}x$ and $u^{-1}x$ is in N. Hence to show $u+x$ is invertible we need only show that $1+y$, with y in N, is invertible; and its inverse is, in fact, the finite sum $1-y+y^2-y^3+\cdots$.

We now formulate the exact conditions under which $1+ae+bf$ is invertible. The product $(1+ae+bf)(1+a'e+b'f)$ is easily computed to be $1+(a'+a'a+a'b+a)e+(b'+b'a+b'b+b)f$. Hence an element $1+ae+bf$ is invertible if and only if there exist real numbers a' and b' satisfying $a'(1+a+b) = -a$ and $b'(1+a+b)=-b$. Therefore the element $1+ae+bf$ is invertible if and only if $1+a+b\neq 0$. Hence the correspondence between the elements $1+ae+bf$ and pairs (a, b) induces a homeomorphism between $G(V) \cap(1+Re+Rf)$ and the points in the Euclidean plane lying above the line $1+x+y=0$. Consequently $G(V)\cap(1+Re+Rf)$ is simply connected and since N is simply connected, we have proved that $G(V)=G(V)\cap(1+Re+Rf)+N$ is simply connected.

Taking $G(M)=R^n$, we have that $G=G(M)\times G(V)$ is a simply connected analytic group with center $G(M)$ and Lie algebra L, the standard Lie algebra constructed by Hochschild. We are now in a position to prove the following theorem.

THEOREM 3.17. *Let H be a simply connected analytic group, Z an abelian Lie algebra, and e an algebra homomorphism of H' into $D(Z)$. Let C be an abelian analytic group with Lie algebra Z and also suppose that the Lie algebra $(A(C))'$ contains $e(H')$. Then there exists an analytic group K with center C such that a representative of each element of* Ker(Z, H', e) *is obtained as the induced Lie algebra kernel of a suitable analytic H-kernel structure (K, f) on K.*

Proof. Let L be the standard Lie algebra with center Z constructed by Hochschild. For any given element in Ker(Z, H', e), we know there exists a Lie algebra homomorphism r of H' into $D(L)/J(L)$ so that (L, r) is a representative of the given element. Let G be the simply connected analytic group with Lie algebra L and with center $G(Z)=R^n$, where $n=\dim(Z)$. Notice that, since G is simply connected, $A(G)$ is isomorphic with $A(L)$ and therefore $(A(G))'$ is isomorphic with the full derivation algebra $D(L)$. Let A be a simply connected covering group of $A_0(G)$ and let I be the closed normal analytic subgroup of A with Lie algebra $J(L)$. Since H is simply connected and since $A'=D(L)$, the Lie algebra homomorphism r induces an analytic homomorphism g^* of H into A/I such that $r=(g^*)'$. Let g be the homomorphism of H into $A(G)/I(G)$ obtained by following g^* with the natural homomorphism of A/I into $A(G)/I(G)$, which is induced by the covering homomorphism. Then (G, g) is an analytic H-kernel with (L, r) as its associated Lie algebra kernel. If C were to be simply connected, we would now be finished by taking G for the K in the theorem. The remainder of the proof is devoted to showing that we may reduce the center of G to any C satisfying the conditions of the theorem.

The image of H in A/I under g^* is, as we have seen before, an analytic subgroup of the form A_1/I, where A_1 is an analytic subgroup of A. Let $A_1(G)$ be the image in $A(G)$ of A_1 under the covering homomorphism. Let $A_1(G(Z))$ be the analytic subgroup of $A_0(G(Z))$ which is the image of $A_1(G)$ under the restriction homomorphism of $A(G)$ into $A(G(Z))$. Since $A_1(G(Z))$ is also the image of H under g^* followed by the "restriction" of A/I to $G(Z)$, the Lie algebra of $A_1(G(Z))$ is precisely $e(H')$. $G(Z)$ is a simply connected covering group of C since they have the same Lie algebra Z. Let N in $G(Z)$ be the fundamental group of C. Regarding $A(C)$ as a closed subgroup of $A(G(Z))$, we see that $A_1(G(Z))$ is contained in $A(C)$, since $A_1(G(Z))$ has Lie algebra $e(H')$ which is contained in $(A(C))'$ by assumption. In particular, N is mapped onto itself by the elements of $A_1(G(Z))$. Since $A_1(G(Z))$ is the restriction of $A_1(G)$ to the center $G(Z)$, every element of $A_1(G)$ maps N onto itself. The center of G/N is $G(Z)/N$ since the center of a covering group is always mapped by the covering homomorphism onto the center of the group it covers. Let K be the analytic group obtained from G/N by identifying its center with C. Since N is mapped onto itself by the elements of $A_1(G)$, $A_1(G)$ is a subgroup of $A(K)$ where $A(K)$ is regarded as a closed subgroup of $A(G)$. Let A_1^* be a simply connected covering group of A_1 and let I^* be the closed normal analytic subgroup with Lie algebra $J(L)$. Let B be a simply connected covering group of $A_0(K)$ and let B_I be the closed normal analytic subgroup of B with Lie algebra $J(L)$—recall that K' is identified with L. We lift the covering homomorphism of A_1^* onto $A_1(G)$ to an analytic homomorphism of A_1^* into B. This homomorphism maps I^* onto B_I and therefore induces an analytic homomorphism of A_1^*/I^* into B/B_I, which in turn induces the identity map on $r(H')=(A_1^*/I^*)'$. We lift g^* to obtain an analytic homomorphism of H into A_1^*/I^*—which we again denote by g^*. Let f^* be this g^* followed by the above homomorphism of A_1^*/I^* into B/B_I. Then $(f^*)'=r$. Hence by letting f be f^* followed by the natural homomorphism of B/B_I onto $A_0(K)/I(K)$, we see that (K, f) is an analytic H-kernel with center C and with (L, r) as associated Lie algebra H'-kernel.

Remark. The condition in the preceding theorem that $(A(C))'$ contain $e(H')$ is easily seen to be a necessary condition.

Theorem 3.18. *Let (C, H, d) be an admissible kernel system with H simply connected. Then there exists an analytic group K with center C such that a representative of every element of* Ker(C, H, d) *is obtained by choosing a suitable ana-*

lytic kernel structure on K and every element of $\operatorname{Ker}(C', H', d')$ *is represented by the induced Lie algebra kernel of such an analytic kernel structure on K.*

Proof. Let C_0 be the component of the identity element of C; then $(C_0)' = C'$. Since (C, H, d) is an admissible kernel system, there exists an analytic H-kernel with center C and inducing the given H operators on C. The composition of such a kernel with its canonical inverse yields an extendible H-kernel associated with the system (C, H, d). Let (E, e) be any such extendible H-kernel. Let G be the analytic group, with center C_0, guaranteed by the last theorem (G is the K of Theorem 3.17). Let (G, g) be an analytic H-kernel structure on G. By Theorem 3.10, we know that $(K, f) = (E, e) \times_{C_0} (G, g)$ is an analytic H-kernel with center C, inducing d, and has $(K', f') = (E', e') \times_{C'} (G', g')$ as associated Lie algebra H'-kernel. Since (E', e') is extendible, (K', f') and (G', g') represent the same element of $\operatorname{Ker}(C', H', d')$. So, in view of the last theorem, we see that by varying the analytic kernel structure on G, and therefore on $K = E \times_{C_0} G$, we can obtain analytic H-kernels, with center C and inducing d, with associated Lie algebra kernels representing any element of $\operatorname{Ker}(C', H', d')$. Hence our isomorphism of $\operatorname{Ker}(C, H, d)$ into $\operatorname{Ker}(C', H', d')$ is actually onto $\operatorname{Ker}(C', H', d')$.

This proves Theorem 3.18 and also the following result.

THEOREM 3.19. *Let (C, H, d) be an admissible kernel system with H simply connected. Then the map sending an analytic H-kernel to its associated Lie algebra kernel induces an isomorphism of* $\operatorname{Ker}(C, H, d)$ *onto* $\operatorname{Ker}(C', H', d')$.

4. **Analytic cohomology.** If C is an abelian analytic group and if G is an analytic group which operates on C, then throughout this section we shall denote by $H^n(G, C)$ the nth cohomology group for G in C which is computed from cochains that are also analytic maps. Consequently in this section we reserve the terms cochain and cocycle for analytic maps that are cochains or cocycles in the usual sense.

The following lemma is of fundamental importance for obtaining the results of this section.

LEMMA 4.1. *Let K be an analytic group and let G be a simply connected analytic group. Let f be an analytic map of G into $I(K)$ such that $f(1) = 1$. Then there exists an analytic map g of G into K such that, for all x in G, $f(x)$ is the inner automorphism effected on K by $g(x)$ and further, $g(L) = (1)$ for every analytic subgroup L of G for which $f(L) = (1)$.*

Proof. Let K^* be a simply connected covering group of K and let C be the center of K^*. Under the embedding of $A(K)$ into $A(K^*)$, $I(K)$ and $I(K^*)$ coincide. Hence we may regard f as an analytic map of G into $I(K^*)$. Since K^*/C is isomorphic with $I(K^*)$, K^*/C_0 is a covering group of $I(K^*)$, where C_0 is the component of the identity in C. Therefore, since G is simply connected, we may lift f uniquely to an analytic map f_1 of G into K^*/C_0 such that

$f_1(1)=1$. If L is an analytic subgroup of G such that $f(L)=(1)$, then $f_1(L)=(1)$; for $f_1(L)$ is contained in the discrete kernel of the homomorphism of K^*/C_0 onto $I(K^*)$ and since L is connected and $f_1(1)=1$, we must have $f_1(L)=(1)$. Since C_0 is a closed connected normal subgroup of the simply connected analytic group K^*, we have by Theorem 3.2 that there exists an analytic map of K^*/C_0 into K^* which is inverse to the natural homomorphism of K^* onto K^*/C_0. Clearly we may assume this analytic map is adjusted so that 1 maps to 1. This analytic map following f_1 yields an analytic map f_2 of G into K^*. Also it is clear that $f_2(L)=(1)$ for every analytic subgroup L of G for which $f(L)=(1)$. Let g be f_2 followed by the covering homomorphism of K^* onto K. Then g is an analytic map of G into K that satisfies the conditions of the lemma.

THEOREM 4.2. *Let H be a simply connected analytic group and let (K, f) be an analytic H-kernel with center C. Then there exist 3-cocycles of H in C_0 which are associated with (K, f) by the usual method* ([3]; *recall that here a cocycle is also an analytic map*).

Proof. Since (K, f) is an analytic H-kernel and since H is simply connected, there is an analytic map f_0 of H into $A(K)$ which defines the kernel structure (K, f). As we have seen previously, we may assume $f_0(1)=1$. Consider the analytic map of $H\times H$ into $A(K)$ which sends a pair (x, y) to $f_0(x)f_0(y)f_0(xy)^{-1}$. Since f_0 followed by the natural homomorphism of $A(K)$ into $A(K)/I(K)$ is a homomorphism, the image of $H\times H$ under the above map lies in $I(K)$. By Proposition 1 on page 95 of [1] it follows that the above map is an analytic map of $H\times H$ into $I(K)$. Also, we notice that the analytic subgroups $H\times(1)$ and $(1)\times H$ are mapped to (1). By Lemma 4.1, there exists an analytic map g of $H\times H$ into K such that $f_0(x)f_0(y)f_0(xy)^{-1}$ is the inner automorphism effected on K by $g(x, y)$ and further, $g(x, 1)=g(1, y)=1$ for all x and y in H. Denote by $a(k)$ the inner automorphism effected on K by an element k of K. On evaluating the identity $f_0(x)(f_0(y)f_0(z))=(f_0(x)f_0(y))f_0(z)$, where x, y and z belong to H, we obtain

$$a(f_0(x)(g(y, z))g(x, yz)) = a(g(x, y)g(xy, z)).$$

Hence for all x, y, and z in H, $t(x, y, z)=f_0(x)(g(y, z))g(x, yz)g(xy, z)^{-1}g(x, y)^{-1}$ defines an analytic map t of $H\times H\times H$ into C_0. It is also clear that $t(1, y, z)=t(x, 1, z)=t(x, y, 1)=1$—this is immediate from the fact that the similar property is satisfied by f_0 and g. It is well known [3] that this construction yields a 3-cocycle t in the ordinary sense. Hence the theorem is proved.

If f_0 is kept fixed, we claim that the cohomology class of the constructed cocycle t does not depend on the choice of g and further, every 3-cocycle which is cohomologous to t can be obtained by a suitable choice of g. Also, if f_1 is a second analytic map of H into $A(K)$ that defines the kernel structure (K, f) and is such that $f_1(1)=1$, then there exists an analytic map g_1 of $H\times H$ into K, which is related to f_1 as g is related f_0, such that t is the 3-cocycle constructed from f_1 and g_1 in exactly the same way as t is constructed from f_0 and g. These statements are proved exactly as in [3] except that Lemma 4.1 is required to ensure that certain discrete choices made in [3] can be accomplished by analytic maps. It is immediate from this discussion that we have associated with each analytic H-kernel (K, f) a unique element of $H^3(H, C_0)$.

THEOREM 4.3. *Let H be a simply connected analytic group and let (K, f) and (L, g) be two analytic H-kernels with the same center C and inducing identical H operators on C. Then the element of $H^3(H, C_0)$ associated with $(K, f) \times_C (L, g)$ is the product of the elements associated with (K, f) and (L, g).*

Proof. Let f_0 be an analytic map of H into $A(K)$ which defines the kernel structure (K, f) and is such that $f_0(1) = 1$. Let p be an analytic map of $H \times H$ into K such that $f_0(x)f_0(y)f_0(xy)^{-1}$ is the inner automorphism effected on K by $p(x, y)$ and $p(x, 1) = p(1, y) = 1$ for all x and y in H. Let g_0 and q be defined similarly for the kernel (L, g). We identify $A(K) \times A(L)$ with the corresponding subgroup of $A(K \times L)$ and consider the analytic maps (f_0, g_0) of H into $A(K \times L)$ and (p, q) of $H \times H$ into $K \times L$. In the proof of Theorem 3.10 we showed that the image of H under (f_0, g_0) is contained in an analytic subgroup E_1 of $A(K \times L)$ whose elements map the subgroup S of $K \times L$, consisting of all pairs (c, c^{-1}) with c in C, into itself. We had an analytic homomorphism of E_1 into $A(K \times_C L)$. Let h_0 be the analytic map of H into $A(K \times_C L)$ obtained by following (f_0, g_0) by this analytic homomorphism on E_1. Then h_0 defines the kernel structure $(K, f) \times_C (L, g)$ and also $h_0(1) = 1$. Let r be the analytic map of $H \times H$ into $K \times_C L$ obtained by following (p, q) by the natural analytic homomorphism of $K \times L$ onto $K \times_C L$. Then r satisfies $r(x, 1) = r(1, y) = 1$ for all x and y in H. Also, simple computations show that $h_0(x)h_0(y)h_0(xy)^{-1}$ is the inner automorphism effected on $K \times_C L$ by $r(x, y)$ and that the 3-cocycle associated with h_0 and r is the product of the 3-cocycle associated with f_0 and p and the 3-cocycle associated with g_0 and q. This proves the theorem.

THEOREM 4.4. *Let (K, f) be an extendible H-kernel with center C and with H simply connected. Then the identity element of $H^3(H, C_0)$ is associated with (K, f).*

Proof. Let (E, p) be an extension of (K, f). By Corollary 3.4, there exists an analytic map f_1 of H into E which is inverse to p. As usual, we may assume $f_1(1) = 1$. Then f_1 followed by the natural homomorphism of E into $A(K)$ yields an analytic map f_0 which defines the kernel structure (K, f) and is such that $f_0(1) = 1$. Define an analytic map g of $H \times H$ into K by $g(x, y) = f_1(x)f_1(y)f_1(xy)^{-1}$. We easily see that f_0 and g are suitable maps for associating a 3-cocycle with (K, f) and by using the relationships of f_0 and g with f_1, we see that the cocycle determined by f_0 and g is identically equal to 1; that is, the identity element of $H^3(H, C_0)$ is associated with (K, f).

THEOREM 4.5. *Let (C, H, d) be an admissible kernel system with H simply connected. The map sending analytic H-kernels to their associated elements in $H^3(H, C_0)$ induces an isomorphism of* Ker(C, H, d) *into* $H^3(H, C_0)$.

Proof. It follows trivially from Theorems 4.3 and 4.4 that this map induces a homomorphism. Hence it remains only to show the following: if (K, f) is associated with the identity element of $H^3(H, C_0)$, then (K, f) is extendible.

Let f_0 be any analytic map of H into $A(K)$ which defines the kernel structure (K, f) and is such that $f_0(1)=1$. Since (K, f) is mapped to the identity element of $H^3(H, C_0)$, we know there exists a map g of $H\times H$ into K which satisfies:

(1) $g(x, 1)=g(1, y)=1$ for all x and y in H,

(2) $f_0(x)f_0(y)f_0(xy)^{-1}$ is the inner automorphism effected on K by $g(x, y)$,

(3) the 3-cocycle constructed from f_0 and g is identically equal to 1.

Let E by the product manifold $K\times H$ and define a group structure on E by

$$(u, x)(v, y) = (u(f_0(x)(v))g(x, y), xy)$$

for all u and v in K and all x and y in H. It is easily verified that E is a group and since all operations are analytic, E is an analytic group. Perhaps it should be remarked that the associative law holds in E in virtue of condition (3). We identify K with the subgroup $K\times(1)$ of E. A simple computation will show that E together with the projection map of E onto H is an extension of the H-kernel (K, f). This proves Theorem 4.5.

We have not succeeded in interpreting $H^3(H, C_0)$ as the group Ker(C, H, d), as did Eilenberg and MacLane in the discrete case, since it remains unknown whether or not the isomorphism of the last theorem is onto $H^3(H, C_0)$.

5. **Admissible kernel systems.** We shall prove the following theorem in easy stages.

THEOREM 5.1. *Let d be a continuous homomorphism of an analytic group H into $A(C)$ where C is an abelian Lie group. If C has a finitely generated H-subgroup D such that $C=C_0\times D$, then (C, H, d) is an admissible kernel system.*

Let us first remark that it is a necessary condition that the group of components C/C_0 be finitely generated. More specifically, if C is the center of an analytic group K, then C/C_0 is finitely generated. For the natural homomorphism of K onto K/C_0 maps C onto the group of components of C, which is a discrete central subgroup of K/C_0. This discrete central subgroup is easily shown to be a homomorphic image of the fundamental group of K/C. Hence we need only show that the fundamental group of an analytic group is finitely generated. In [9] it is shown that the manifold of every analytic group can be decomposed into the product manifold of a Euclidean space with the manifold of a compact subgroup of the given analytic group. The result now follows from the classical result that the fundamental group of a compact analytic group is finitely generated.

It should also be remarked that in the decomposition $C = C_0 \times D$, with D a discrete H-subgroup of C, H operates trivially on D. This is an immediate consequence of the more general statement that if G is a totally disconnected locally compact group, then $A(G)$ is also totally disconnected. In order to see this latter statement let x be any element of G and consider the map of $A(G)$ into G that sends an element a in $A(G)$ to $a(x)$. It follows from the discussion immediately following Definition 2.2 that this is, for each x in G, a continuous map of $A(G)$ into G. Since $A_0(G)$ is connected and since G is totally disconnected, the image of $A_0(G)$ must be a single point of G; and since 1 belongs to $A_0(G)$, this image is in fact the point x. Because this is true for each x in G, we must have $A_0(G) = (1)$; that is $A(G)$ is totally disconnected. Hence, since H is connected, the image of H in $A(D)$ must be contained in the component of identity in $A(D)$, which is (1), and therefore H operates trivially on D.

Since D is a finitely generated abelian group, D decomposes into a direct product $D_1 \times \cdots \times D_k$ of cyclic groups. We see from the above discussion that H operates trivially on each D_i and hence each D_i is an H-subgroup of C.

LEMMA 5.2. *Every infinite cyclic group can be embedded as the discrete center of an analytic group.*

Proof. Let G denote the additive group of complex numbers and let R denote the additive group of real numbers. Define an analytic homomorphism p of R into $A(G)$ by: for all x in R and all g in G, let $p(x)(g) = ge^{2\pi ix}$. Let K be the semi-direct product $(G \times R)_p$. The center of K is easily computed to be the set of all pairs $(0, z)$, where z is an integer. That is, the center of K is isomorphic with the additive group of integers.

LEMMA 5.3. *Every finite cyclic group can be embedded as the discrete center of an analytic group.*

Proof. Let n denote the order of a finite cyclic group. Let G denote the additive group of complex numbers and let T denote the one-dimensional torus. We regard T as the unit circle in the complex plane. Define an analytic homomorphism p of T into $A(G)$ by: for all x in T and all g in G, let $p(x)(g) = gx^n$. Let K be the semi-direct product $(G \times T)_p$. Straightforward computation will show that the center of K is precisely all pairs $(0, x)$, where x is an nth root of unity. Hence the center of K is isomorphic with the group of nth roots of unity.

For each D_i in the decomposition of D, let K_i be an analytic group with D_i as center. Then $L = K_1 \times \cdots \times K_k$ is an analytic group with discrete center D. Let g denote the trivial homomorphism of H into $A(L)/I(L)$; that is, $g(H) = (1)$. Then (L, g) is an analytic H-kernel with center D. Now (C_0, d_0) is an analytic H-kernel with center C_0, where d_0 is induced by restricting $d(H)$ to C_0. Hence the direct product $(C_0, d_0) \times (L, g)$ is an analytic H-kernel with center C, inducing d. This proves Theorem 5.1.

We conclude with the remark that Hochschild has proved in [5] that $C = C_0 \times D$ with D an H-subgroup of C if any one of the following conditions holds: (1) H is semi-simple; (2) C_0 is simply connected and H is compact; (3) every element of C/C_0 is of finite order and H is compact.

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University of Illinois,
Urbana, Illinois
University of Washington,
Seattle, Washington