A NEW CLASS OF CONTINUED FRACTION EXPANSIONS FOR THE RATIOS OF HEINE FUNCTIONS III[1]

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1. **Introduction.** The study of new continued fraction expansions for the ratios of contiguous Heine functions, which was begun in [1][2] and [2], is here continued. In [1] the continued fractions were of the type

$$1 + \frac{d_1 z}{f_1 z + 1} + \frac{d_2 z}{f_2 z + 1} + \cdots$$

(1.1)
$$= \frac{\phi(a, b, c, q, z)}{\phi(a, b+1, c+1, q, z)}, \qquad \begin{array}{l} |z| < |q^{-b}|, \ |q| < 1, \\ |z| < |q^{-c+1-a}|, \ |q| > 1, \end{array}$$

$$d_p = -\frac{q^b(1 - q^{a+p-1})(1 - q^{c+p-1-b})}{(1 - q^{c+p-1})(1 - q^{c+p})}, \qquad f_p = \frac{q^b(1 - q^{a+p-1-b})}{1 - q^{c+p}},$$

$$p = 1, 2, \cdots.$$

In [2], the continued fraction expansions for the ratios of continguous Heine functions were of the type

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots = \frac{\phi(a, b, c, q, z)}{\phi(a+1, b+1, c+1, q, z)}, \qquad |q| < 1,$$

(1.2)
$$a_{p+1} = \frac{(1 - q^{a+p+1})(1 - q^{b+p+1})(q^{c+p}z - q^{a+b+2p+1}z^2)}{(1 - q^{c+p+1})(1 - q^{c+p})},$$

$$b_p = 1 - \frac{(q^{a+p} + q^{b+p} - q^{a+b+2p} - q^{a+b+2p+1})z}{1 - q^{c+p}}, \qquad p = 0, 1, \cdots,$$

(formula (2.1)) and of the type

$$\frac{1}{1 - q^{a+b-c-1}z}\left[b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots\right] = \frac{\phi(a, b, c, q, z)}{\phi(a, b, c+1, q, z)}, \qquad |q| > 1,$$

(1.3)
$$a_{p+1} = \frac{q^{a+b-c-p-2}z(1 - q^{c-a+p+1})(1 - q^{c-b+p+1})(1 - q^{a+b-c-p-2}z)}{(1 - q^{c+p+1})(1 - q^{c+p})},$$

$$b_p = 1 - \frac{(q^{a+b-c-p-1} + q^{a+b-c-p-2} - q^{a-1} - q^{b-1})z}{1 - q^{c+p}}, \qquad p = 0, 1, \cdots,$$

(formula (3.1)).

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(2) Numbers in brackets refer to the bibliography at the end of the paper.

In §2 of this paper a standard type of continued fraction expansion which converges to the ratio of contiguous Heine functions is found. In §3, the equivalence of the continued fractions (1.2) and (1.3) is shown, and a functional relation between them is developed. In §4, further functional relations between contiguous Heine functions are considered, and in §5 formulas for the denominators of the approximants of the continued fractions in §1 are found.

The Heine function is the infinite series

$$\phi(a, b, c, q, z) = 1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})} z^2 + \cdots \tag{1.4}$$

with radius of convergence equal to 1 if $|q| < 1$ and equal to $|q^{c+1-a-b}|$ if $|q| > 1$. In (1.4), it is understood that $|q| \neq 1$, $c \neq -p \pm 2n\pi i/u$, where $q = e^u$, $n, p = 0, 1, 2, \cdots$. Except for these conditions, a, b, c, and q are arbitrary. If a or b is equal to $-p \pm 2n\pi i/u$, $\phi(a, b, c, q, z)$ reduces to a polynomial.

In any of the expansions in this paper, for certain values of a, b, or c, a partial numerator of a continued fraction may be zero. In this case, the continued fraction is finite, and its value can be computed therefrom. Furthermore, it is understood throughout that in each case those values of a, b, or c are excluded which make indeterminate the function to which the continued fraction in question converges.

2. **A type of continued fraction expansion for the ratio of contiguous Heine functions.** In this section a type of continued fraction expansion is identified with one which converges to the ratio of contiguous Heine functions. It is analogous to that in Perron [6, Theorem 6.5, p. 288], for the ratio of contiguous Gauss functions.

On multiplication with an arbitrary number δ ($\neq 0$) and with $1-q^c$, and by an equivalence transformation, relation (1.2) becomes

$$\begin{aligned} &b_0' + \frac{a_1'}{b_1'} + \frac{a_2'}{b_2'} + \cdots = \frac{\delta(1-q^c)\phi(a, b, c, q, z)}{\phi(a+1, b+1, c+1, q, z)}, \qquad |q| < 1, \\ &a_{p+1}' = \delta^2(1-q^{a+p+1})(1-q^{b+p+1})(q^{c+p}z - q^{a+b+2p+1}z^2), \\ &b_p' = \delta[(1-q^{c+p}) - (q^{a+p} + q^{b+p} - q^{a+b+2p} - q^{a+b+2p+1})z], \\ &\qquad p = 0, 1, \cdots. \end{aligned} \tag{2.1}$$

This expansion will now be identified with the continued fraction

$$(2.2)\qquad \begin{aligned} &c_0 + \frac{c_1}{d_1 +} \frac{c_2}{d_2 +} \cdots, \qquad \text{with elements of the form} \\ &c_{p+1} = Aq^{4(p+1)} + Bq^{3(p+1)} + Cq^{2(p+1)} + Dq^{(p+1)}, \\ &d_p = Eq^{2p} + Fq^p + G, \quad |q| < 1, \qquad p = 0, 1, \cdots, \end{aligned}$$

and the value of (2.2) will thus be obtained. The expansions (2.1) and (2.2) are identical if

$$(2.3)\qquad \begin{aligned} &\text{(i)} && A = -\delta^2 q^{2a+2b-1} z^2, \\ &\text{(ii)} && B = \delta^2(q^{a+b+c-1}z + q^{2a+b-1}z^2 + q^{a+2b-1}z^2), \\ &\text{(iii)} && C = \delta^2(-q^{a+c-1}z - q^{b+c-1}z - q^{a+b-1}z^2), \\ &\text{(iv)} && D = \delta^2 q^{c-1} z, \\ &\text{(v)} && E = \delta(q^{a+b}z + q^{a+b+1}z), \\ &\text{(vi)} && F = \delta(-q^c - q^a z - q^b z), \\ &\text{(vii)} && G = \delta \neq 0. \end{aligned}$$

Here there are seven equations in five unknowns a, b, c, z, and δ, for which a solution will be found from (2.3). This is possible when the values of B and E depend on A and F, as follows:

$$(2.4)(\text{i})\qquad \pm F(-Aq)^{1/2} = -Bq = \frac{EF}{1+q}, \qquad (A \neq 0).$$

The latter condition is necessary since, if A were zero, z would be zero, and the continued fraction would not exist. From (2.3)(i), (iii), (iv), (vi), and (vii), q^c must be a root of the equation

$$(2.4)(\text{ii})\qquad G^3x^3 + FG^2x^2 - CqGx \mp Dq(-Aq)^{1/2} = 0.$$

It must be noted, however, that there exists a solution of (2.3) only if a root q^c can be found from (2.4)(ii) such that $c \neq -p \pm 2n\pi i/u$, $q = e^u$, n, $p = 0, 1, \cdots$, as noted in §1. Furthermore, from (2.3)(vii),

$$(2.4)(\text{iii})\qquad \delta = G \neq 0.$$

From (2.3)(iv),

$$(2.4)(\text{iv})\qquad z = \frac{D}{\delta^2 q^{c-1}}, \qquad (D \neq 0).$$

Finally, from (2.3)(i), (iv), (vi), and (vii), q^a and q^b must be the roots of the quadratic equation

$$(2.4)(\text{v})\qquad Dx^2 + Gq^{c-1}(F + Gq^c)x \pm Gq^{c-1}(-Aq)^{1/2} = 0.$$

It is easy to verify that the above solution for a, b, c, z, and δ satisfies equations (2.3). The following theorem has thus been proved.

THEOREM 2.1. *The infinite continued fraction* (2.2), *where* A, D, $G\neq 0$, *is equal to the quotient* $\delta(1-q^c)\phi(a, b, c, q, z):\phi(a+1, b+1, c+1, q, z)$ *if* $|q|<1$, *provided* $\pm F(-Aq)^{1/2}=-Bq=EF/(1+q)$. *Here,* q^c *is a root of the equation* $G^3x^3+FG^2x^2-CqGx\mp Dq(-Aq)^{1/2}=0$, $\delta=G$, $z=D/\delta^2q^{c-1}$, *and* q^a *and* q^b *are the roots of the equation* $Dx^2+Gq^{c-1}(F+Gq^c)x\pm Gq^{c-1}(-Aq)^{1/2}=0$.

As an illustration, let formulas (2.4) be applied to the continued fraction

$$
\begin{aligned}
&c_0+\frac{c_1}{d_1+}\frac{c_2}{d_2+}\cdots,\\
(2.5)\qquad &c_{p+1} = -q^{2a'+4p+5}+q^{2a'+3p+3}+q^{2p+3}-q^{p+1},\\
&d_p = -(1+q)q^{a'+1+2p}+q^{a'+p}+1,\\
&\qquad |q|<1,\ p=0,1,\cdots.
\end{aligned}
$$

Here, the minus sign holds in (2.4)(i). From (2.4)(ii), $q^c=q$, from (2.4)(iii), $\delta=1$, from (2.4)(iv), $z=-1$, and from (2.4)(v), $q^a=q^{a'}$, $q^b=q$. Consequently, (2.5) has the value

$$
(2.5')\qquad (1-q)\cdot\frac{\phi(a',1,1,q,-1)}{\phi(a'+1,2,2,q,-1)}=(1-q)(1+q^{a'}),\qquad |q|<1,
$$

since $\phi(a',1,1,q,-1)=\prod_{p=0}^{\infty}(1+q^{a'+p})/(1+q^p)$, (cf. [5, p. 105])(3).

As a second illustration, consider the continued fraction

$$
\begin{aligned}
&K = c_0+\frac{c_1}{d_1+}\frac{c_2}{d_2+}\cdots,\\
(2.6)\qquad &a_{p+1} = q^{a'+b'+p}z(1-q^{c'-a'+p+1})(1-q^{c'-b'+p+1})(1-q^{p+1}z),\\
&b_p = (1-q^{c'+p})-(q^{b'+p}+q^{a'+p}-q^{c'+2p+1}-q^{c'+2p})z,\\
&\qquad |q|<1,\ p=0,1,\cdots.
\end{aligned}
$$

Equations (2.3) for the expansion (2.6) are

$$
\begin{aligned}
A &= -q^{2c'-1}z^2,\\
B &= q^{c'+b'-1}z^2+q^{c'+a'-1}z^2+q^{2c'-1}z,\\
C &= -q^{c'+b'-1}z-q^{c'+a'-1}z-q^{a'+b'-1}z^2,\\
D &= q^{a'+b'-1}z,\\
E &= (q^{c'+1}+q^{c'})z,\\
F &= -q^{c'}-q^{b'}z-q^{a'}z,\\
G &= 1.
\end{aligned}
$$

(3) It is remarked that the second root of (2.4)(ii) gives the value $q^c=-q^{a'}$ from which it is found that $z=q^{1-a'}$, $q^a=q^{a'}$, $q^b=-q^{a'}$, and the value of (2.5) is again (2.5′). Likewise, the third root of (2.4)(ii), $q^c=-q$ gives $z=1$, $q^a=-q^{a'}$, $q^b=q$, and (2.5) again has the value (2.5′).

From the solution equations (2.4), one finds that the values of δ, q^c, z, q^a, and q^b are 1, $q^{c'}$, $q^{a'+b'-c'}z$, $q^{c'-a'}$, and $q^{c'-b'}$, respectively. Consequently, the value of K is

$$\begin{aligned} K &= \frac{(1-q^{c'})\,\phi(c'-a', c'-b', c', q, q^{a'+b'-c'}z)}{\phi(c'-a'+1, c'-b'+1, c'+1, q, q^{a'+b'-c'-1}(qz))} \\ &= \frac{(1-q^{c'})\,\phi(a', b', c', q, z)}{\phi(a', b', c'+1, q, qz)} \cdot \frac{\phi(a'+b'-c'-1, 1, 1, q, qz)}{\phi(a'+b'-c', 1, 1, q, z)} \\ &= (1-z)\cdot\frac{(1-q^{c'})\,\phi(a', b', c', q, z)}{\phi(a', b', c'+1, q, qz)}, \qquad |q| < 1, \end{aligned} \tag{2.6'}$$

since

$$\phi(a', b', c', q, z) = \phi(a'+b'-c', 1, 1, q, z)\cdot\phi(c'-a', c'-b', c', q, q^{a'+b'-c'}z)$$

(Heine [5, p. 119]), and $\phi(a', 1, 1, q, z) = \prod_{p=0}^{\infty}(1-q^{a'+p}z)/(1-q^pz)$ (Heine [5, p. 105]). This checks with formula (3.1) of [2], when the primes are omitted from a, b, c. From this is obtained formula (3.4) of [2], namely,

$$\frac{\phi(a, b, c, q, z)}{\phi(a, b, c+1, q, z)} = \frac{1}{1-q^c} + \frac{q^c(1-z)(1-q^c)}{K}, \qquad |q| < 1. \tag{2.6''}$$

Now, consider the case $|q| > 1$. On multiplication with an arbitrary number δ ($\neq 0$) and with $(1-q^{a+b-c-1}z)(1-q^c)$, and by an equivalence transformation, relation (1.3) becomes

$$\begin{aligned} b_0'' + \frac{a_1''}{b_1''} + \frac{a_2''}{b_2''} + \cdots &= \frac{\delta(1-q^{a+b-c-1}z)(1-q^c)\phi(a, b, c, q, z)}{\phi(a, b, c+1, q, z)}, \\ &\qquad |q| > 1, \\ a_{p+1}'' &= \delta^2 q^{a+b-c-1}z(1-q^{c-a+p+1})(1-q^{c-b+p+1})(q^p - q^{a+b-c-2}z), \\ b_p'' &= \delta[(q^p - q^{c+2p}) - (q^{a+b-c-1} + q^{a+b-c-2} - q^{a+p-1} - q^{b+p-1})z], \\ &\qquad p = 0, 1, \cdots. \end{aligned} \tag{2.7}$$

This expansion will now be identified with the continued fraction

$$\begin{aligned} &e_0 + \frac{e_1}{f_1} + \frac{e_2}{f_2} + \cdots, \qquad \text{with elements of the form} \\ &e_{p+1} = A + Bq^{(p+1)} + Cq^{2(p+1)} + Dq^{3(p+1)}, \\ &f_p = E + Fq^p + Gq^{2p}, \qquad |q| > 1,\ p = 0, 1, \cdots, \end{aligned} \tag{2.8}$$

and the value of (2.7) will thus be obtained. The expressions (2.7) and (2.8) are identical if

$$
(2.9)\qquad
\begin{aligned}
&\text{(i)} && A = -\delta^2 q^{2a+2b-2c-3} z^2,\\
&\text{(ii)} && B = \delta^2 q^{a+b-c-1} z (q^{-1} + q^{b-2} z + q^{a-2} z),\\
&\text{(iii)} && C = \delta^2 q^{a+b-c-1} z (-q^{c-a-1} - q^{c-b-1} - q^{c-2} z),\\
&\text{(iv)} && D = \delta^2 q^{c-2} z,\\
&\text{(v)} && E = -\delta q^{a+b-c-1} z (1 + q^{-1}),\\
&\text{(vi)} && F = \delta (1 + q^{a-1} z + q^{b-1} z),\\
&\text{(vii)} && G = -\delta q^c.
\end{aligned}
$$

One can solve these equations in a similar way as equations (2.3), and thus obtain the following theorem.

THEOREM 2.2. *The infinite continued fraction* (2.8), *where* $A, D, G \neq 0$, *is equal to the quotient* $\delta(1-q^{a+b-c-1}z)(1-q^c)\phi(a, b, c, q, z) : \phi(a, b, c+1, q, z)$ *if* $|q| > 1$, *provided*

$$(2.10)(\mathrm{i}) \qquad \pm F(-Aq)^{1/2} = Bq = -EF/(1 + q^{-1}).$$

Here, q^c *is a root of the equation*

$$(2.10)(\mathrm{ii}) \qquad \pm Dq(-Aq)^{1/2} x^3 - CGqx^2 + FG^2 x + G^3 = 0,$$

$$(2.10)(\mathrm{iii}) \qquad \delta = -G/q^c,$$

$$(2.10)(\mathrm{iv}) \qquad z = D/\delta^2 q^{c-2},$$

and q^a *and* q^b *are the roots of the equation*

$$(2.10)(\mathrm{v}) \qquad Dx^2 + G(Fq^{-1} + Gq^{-c-1}) \mp Gq^{c-1}(-Aq)^{1/2} = 0.$$

3. **Equivalence of the continued fraction expansions** (1.2) **and** (1.3). Let a, b, c, q, z be replaced by $c'-a'$, $c'-b'$, c', $1/q'$, z'/q', respectively, in (1.2), which is valid for $|q| < 1$. The resulting continued fraction is the same as that in (1.3), that is,

$$
(3.1)\qquad
\begin{aligned}
\frac{\phi(a, b, c, z, z)}{\phi(a+1, b+1, c+1, q, z)} &= \frac{\phi(c'-a', c'-b', c', 1/q', z'/q')}{\phi(c'-a'+1, c'-b'+1, c'+1, 1/q', z'/q')}\\
&= (1 - (q')^{a'+b'-c'-1} z') \cdot \frac{\phi(a', b', c', q', z')}{\phi(a', b', c'+1, q', z')}, \qquad |q| < 1,\ |q'| > 1.
\end{aligned}
$$

Similarly, if one writes (1.3) with primes, and then in (1.3) replaces a', b', c', q', z' with $c-a$, $c-b$, c, $1/q$, z/q, respectively, one obtains the continued fraction in (1.2), that is,

$$
(3.2)\qquad
\begin{aligned}
(1 - (q')^{a'+b'-c'-1} z') \cdot \frac{\phi(a', b', c', q', z')}{\phi(a', b', c'+1, q', z')} &= \frac{\phi(c-a, c-b, c, 1/q, z/q)}{\phi(c-a, c-b, c+1, 1/q, z/q)}\\
&= \frac{\phi(a, b, c, q, z)}{\phi(a+1, b+1, c+1, q, z)}, \qquad |q| < 1,\ |q'| > 1.
\end{aligned}
$$

4. **Other functional relations between the ratios of Heine functions.** In (3.1) and (3.2) certain functional relations between the ratios of Heine functions were given. These held since expressions of different form give the same continued fraction. Heine [4, p. 296] showed that, for his continued fraction (cf. (1.4) of [1])([4]),

$$\frac{\phi(a, b, c, q, z)}{\phi(a, b+1, c+1, q, z)} = \frac{\phi(c-b, c-a, c, q, q^{a+b-c}z)}{\phi(c-b, c-a+1, c+1, q, q^{a+b-c}z)}, \qquad |q| \neq 1. \tag{4.1}$$

This same formula can be derived from the continued fraction (1.1).

If, in (4.1), one replaces a, b, c, q, z by $a, a-c, a-b, q, q^{c+1-a-b}/z$, respectively, one obtains the following functional equation (where the first two elements are interchanged since the ϕ-function is symmetric in these two elements):

$$\frac{\phi(a, a-c, a-b, q, q^{c+1-a-b}/z)}{\phi(a, a-c+1, a-b+1, q, q^{c+1-a-b}/z)} = \frac{\phi(-b, c-b, a-b, q, q/z)}{\phi(1-b, c-b, a-b+1, q, q/z)}. \tag{4.2}$$

It is remarked that results (4.1) and (4.2) can be obtained without the use of continued fractions. Heine [5, p. 119] showed that the following four relations hold for the four solutions of the Heine difference equation (2a), [5, p. 119]:

$$\begin{aligned}
&\text{(i)} \quad \phi(a, b, c, q, z) = \phi(a+b-\mathrm{c}, 1, 1, q, z)\cdot\phi(c-a, c-b, c, q, q^{a+b-c}z);\\
&\text{(ii)} \quad z^{1-c}\phi(a+1-c, b+1-c, 2-c, q, z)\\
&\qquad = z^{1-c}\phi(a+b-c, 1, 1, q, z)\cdot\phi(1-a, 1-b, 2-c, q, q^{a+b-c}z);\\
&\text{(iii)} \quad z^{-a}\phi(a, a+1-c, a+1-b, q, q^{c+1-a-b}/z)\\
&\qquad = z^{-a}\phi(a+b-c, 1, 1, q, q^{c+1-a-b}/z)\\
&\qquad\quad \cdot\phi(1-b, c-b, a+1-b, q, q/z);\\
&\text{(iv)} \quad z^{-b}\phi(b, b+1-c, b+1-a, q, q^{c+1-a-b}/z)\\
&\qquad = z^{-b}\phi(a+b-c, 1, 1, q, q^{c+1-a-b}/z)\\
&\qquad\quad \cdot\phi(1-a, c-a, b+1-a, q, q/z).
\end{aligned} \tag{4.3}$$

The identity (4.1) follows from (4.3)(i), (4.2) from (4.3)(iii).

On the interchange of a and b in (4.2), one obtains an identity analogous to (4.2). This could also be found from (4.3)(iv).

([4]) Perron [6, p. 124] shows an analogous functional relation between the ratios of specialized Gauss functions.

It is further remarked that the identity (4.3)(ii) can be obtained from (4.3)(i) if a, b, c, q, z are replaced by $a+1-c$, $b+1-c$, $2-c$, q, z, respectively.

5. **Recurrence formulas for the denominators of the approximants of continued fraction expansions of the type** (1.1). Let P_n and Q_n be the numerator and denominator, respectively, of the nth approximant of the continued fraction

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_{n-1}}{b_{n-1}} + \frac{a_n}{b_n} + \cdots. \tag{5.1}$$

Then (cf., for example, [4, p. 301]),

$$\frac{Q_n}{Q_{n-1}} = b_n + \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_2}{b_1}, \tag{5.2}$$

$$\frac{P_n/a_1}{P_{n-1}/a_1} = b_n + \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_3}{b_2}. \tag{5.3}$$

These formulas are now applied to the continued fraction expansion (1.1). On the addition of $f_0 z = q^b z(1-q^{a-b-1})/(1-q^c)$ to both sides of (1.1), and by use of the Heine identity

$$\phi(a, b, c, q, z) - \phi(a-1, b+1, c, q, z) = -q^b z \cdot \frac{1-q^{a-b-1}}{1-q^c} \cdot \phi(a, b+1, c+1, q, z),$$

the reciprocal of this expansion becomes

$$\begin{aligned} &\frac{\phi(a, b+1, c+1, q, z)}{\phi(a, b, c, q, z) + q^b z \cdot \dfrac{1-q^{a-b-1}}{1-q^c} \cdot \phi(a, b+1, c+1, q, z)} \\ &= \frac{\phi(a, b+1, c+1, q, z)}{\phi(a-1, b+1, c, q, z)} = \frac{1}{f_0 z + 1} + \frac{d_1 z}{f_1 z + 1} + \frac{d_2 z}{f_2 z + 1} + \cdots. \end{aligned} \tag{5.4}$$

Then, by formula (5.2),

$$\frac{Q_{p-1}}{Q_p} = \frac{1}{f_{p-1}z + 1} + \frac{d_{p-1}z}{f_{p-2}z + 1} + \frac{d_{p-2}z}{f_{p-3}z + 1} + \cdots + \frac{d_2 z}{f_1 z + 1} + \frac{d_1 z}{f_0 z + 1}, \quad p = 1, 2, \cdots. \tag{5.5}$$

A comparison of (1.1) and (5.4) with (5.5) shows that (5.5) coincides with the beginning of the continued fraction development of

$$\frac{\phi(-c+b-p+2, a-c, -c-p+2, q, z)}{\phi(-c+b-p+1, a-c, -c-p+1, q, z)}. \tag{5.6}$$

If (5.6) should break off with the terms which are no longer in Q_{p-1}/Q_p, then (5.6) would be identical with (5.5). The continued fraction expansion for (5.6) breaks off with $d_1z/(f_0z+1)$ if $a=1$ or if $c=b+1$. If $a=1$, the ratio of the denominators of the approximants of the expansion (5.4) for

$$\phi(1, b+1, c+1, q, z) \tag{5.7}$$

is

$$\frac{Q_{p-1}}{Q_p} = \frac{\phi(-c+b-p+2, 1-c, -c-p+2, q, z)}{\phi(-c+b-p+1, 1-c, -c-p+1, q, z)} = \frac{\phi(-p+1, -b, -c-p+2, q, q^{b-c+1}z)}{\phi(-p, -b, -c-p+1, q, q^{b-c+1}z)} \tag{5.7'}$$

by the Heine identity (62) [4, p. 296],

$$\frac{\phi(a+1, b, c+1, q, z)}{\phi(a, b, c, q, z)} = \frac{\phi(c-b+1, c-a, c+1, q, q^{a+b-c}z)}{\phi(c-b, c-a, c, q, q^{a+b-c}z)}, \tag{5.8}$$

(since the Heine function is symmetric in the first two elements). The denominators of the approximants of the expansion (5.4) for (5.7) are, therefore,

$$Q_p = \phi(-p, -b, -c-p+1, q, q^{b-c+1}z), \qquad p = 1, 2, \cdots. \tag{5.7''}$$

If $c=b+1$, the denominators of the approximants of the continued fraction (5.4) for

$$\frac{\phi(a, b+1, b+2, q, z)}{\phi(a-1, b+1, b+1, q, z)} = \phi(1, b-a+2, b+2, q, q^{a-1}z) \tag{5.9}$$

(by (5.8)) are

$$Q_p = \phi(-p, a-b-1, -b-p, q, z), \qquad p = 1, 2, \cdots. \tag{5.9'}$$

The same can be concluded from (5.7) with $c=b+1$ and $a=1$.

The following theorem has thus been proved.

THEOREM 5.1. *The denominators of the pth approximants of the continued fraction expansions* (5.4) *for the functions* (5.7) *and* (5.9) *are given by* (5.7″) *and* (5.9′), *respectively.*

One can apply the same technique as used above to the other formulas of [1] of the type (1.1). By (5.3), similar recurrence formulas for the numerators of these expansions can be found. Also, one can apply a similar procedure to obtain formulas for the numerators and denominators of the other continued fraction expansions of §1.

BIBLIOGRAPHY

1. E. Frank, *A new class of continued fraction expansions for the ratios of Heine functions*, Trans. Amer. Math. Soc. vol. 88 (1958) pp. 288–300.

2. ———, *A new class of continued fraction expansions for the ratios of Heine functions* II, Trans. Amer. Math. Soc. vol. 95 (1960) pp. 17–26.

3. E. Heine, *Über die Reihe*

$$1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}z+\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}z^2+\cdots,$$

J. für Math. vol. 32 (1846) pp. 210–212.

4. ———, *Untersuchungen über die Reihe*

$$1+\frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}z+\frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}z^2+\cdots,$$

J. für Math. vol. 34 (1847) pp. 285–328.

5. ———, *Handbuch der Kugelfunctionen*, vol. 1, Berlin, Reimer, 1878.

6. O. Perron, *Die Lehre von den Kettenbrüchen*, vol. II, 3d ed., Stuttgart, Teubner, 1957.

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