OSCILLATION CRITERIA FOR FOURTH-ORDER LINEAR DIFFERENTIAL EQUATIONS(1)

BY

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1. **Introduction.** The problem of obtaining oscillation criteria for self-adjoint linear differential equations of second and fourth order has recently been studied from the point of view of an eigenvalue problem closely related to the differential equation under consideration [4; 5]. A variety of conditions for oscillation or nonoscillation of such equations may be obtained by use of this technique.

The oscillation problem for a particular class of self-adjoint differential equations of fourth order will be discussed here. An eigenvalue problem slightly different from the one considered in [4] forms the basis, in §2, for subsequent oscillation criteria developed in §3. Finally, a variety of comparison theorems for both fourth and second order equations are derived in §4, by use of the basic eigenvalue criteria for oscillation obtained in §2.

2. **The associated eigenvalue problem.** In this section we are concerned with the problem of establishing oscillation criteria for the differential equation

$$(r(x)y''(x))'' = p(x)y(x). \tag{2.1}$$

We shall restrict our considerations to the case in which $r(x)$ is a function that is positive and is of class C'' for x in $(0, \infty)$, while $p(x)$ is a function that is positive and is of class C for x in $(0, \infty)$. It may be remarked that most of the results will remain valid in the somewhat more general case in which $p(x)$ has isolated zeros.

Our basic technique consists in relating this problem to a certain eigenvalue problem for an appropriate differential system associated with (2.1).

It has been shown [4, p. 334] that if no solution of (2.1) is oscillatory, that is, if no solution has an infinite number of zeros in $(0, \infty)$, then there exists a positive number a, such that no solution of (2.1) has more than three zeros in (a, ∞). Following [4] we call Equation (2.1) nonoscillatory in (a, ∞) if any solution of (2.1) vanishes at most three times in this interval. Equation (2.1) will be called nonoscillatory, without the interval being mentioned, if

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there exists a positive number a such that (2.1) is nonoscillatory in (a, ∞). On the other hand, Equation (2.1) will be called oscillatory if at least one of its solutions has an infinite number of zeros in $(0, \infty)$.

We introduce the following notation. Let $u(x)$ and λ_b be, respectively, the first eigenfunction and eigenvalue of the problem

$$(2.2)\qquad \begin{aligned} &(r(x)u''(x))'' = \lambda p(x)u(x),\\ &u(a) = u'(a) = r(b)u''(b) = (r(x)u''(x))_b' = 0, \end{aligned}$$

where $0 < a < b < \infty$. Since $\lambda_b > 0$, it is an easy consequence of Lemma 2.1 of [4] that $u''(a) \neq 0$, so we may normalize $u(x)$ by the condition $r(a)u''(a) = 1$. Let $y(x) = y(x, B)$ and μ_B be, respectively, the first eigenfunction and eigenvalue of the problem

$$(2.3)\qquad \begin{aligned} &(r(x)y''(x))'' = \mu p(x)y(x),\\ &y(a) = y'(a) = y(B) = y'(B) = 0, \end{aligned}$$

where $0 < a < B < \infty$. Since $y''(a) \neq 0$ [4, p. 327], we may normalize $y(x)$ by the condition $r(a)y''(a) = 1$.

In terms of the notation given above, our main result is contained in the following theorem.

THEOREM 2.1. *Let $\int_a^\infty dx/r(x) = \infty$; then the equation $(r(x)u''(x))'' = p(x)u(x)$ is nonoscillatory in (a, ∞) if, and only if, $\lambda_b > 1$, for any $b > a$.*

We remark that the main difficulty is encountered in showing that $\lambda_b > 1$ is a necessary condition. Since, by classical results [2], $\lambda_b < \mu_b$, this result is stronger than the necessary condition $\mu_b > 1$ derived in [4]. As will become apparent later on, the condition $\lambda_b > 1$ is a considerably more effective instrument for obtaining oscillation criteria for Equation (2.1) than the weaker condition $\mu_b > 1$.

The proof of Theorem 2.1 depends on the following two lemmas.

LEMMA 2.1. *The first eigenfunction $u(x)$ of Problem (2.2) is a positive, convex, increasing function in (a, b), for any $b > a$.*

It is well known that the first eigenfunction of (2.2) minimizes the Rayleigh quotient

$$(2.4)\qquad R[z] = \int_a^b r(z'')^2 dx \Big/ \int_a^b p(z)^2 dx$$

within the family of functions Z which are of class D'' in $[a, b]$ and satisfy the conditions $z(a) = z'(a) = 0$ [1]. Suppose $u(x)$ has a zero in (a, b). Then the equation $u'(x) = 0$, with x in (a, b), has at least one root. Denote by x_1 the largest such root; we first assume that $u_1 = u(x_1) > 0$ and $u(x_1) > u(x)$ for x in (x_1, b), by replacing, if necessary, $u(x)$ by $-u(x)$. We define $z(x)$ by

$$z(x) = u(x), \qquad \text{for } x \text{ in } [a, x_1],$$
$$z(x) = 2u_1 - u(x), \qquad \text{for } x \text{ in } [x_1, b].$$

It is readily verified that $z(x)$ is in Z. But $|z(x)| > |u(x)|$ for x in (x_1, b), since $|z(x)| = 2u_1 - u(x) > |u(x)|$. Thus

$$\int_a^b r(z'')^2 dx = \int_a^b r(u'')^2 dx \quad \text{and} \quad \int_a^b p(z)^2 dx > \int_a^b p(u)^2 dx,$$

so $R[z] < R[u]$, a contradiction.

On the other hand if $0 < u(x_1) < u(x)$ for x in (x_1, b) then $u''(x_1) \geqq 0$. If $u''(x_1) = 0$ then, since $r(b)u''(b) = (r(x)u''(x))_b' = 0$, the curve ru'' must be concave in an interval, say I, contained in (x_1, b). In I $(ru'')'' < 0$. But (2.4) shows that $\lambda_b > 0$. Hence $(ru'')'' = \lambda_b pu > 0$ for x in (x_1, b), which is a contradiction. If $u''(x_1) > 0$ then there exists at least one root, say x_0, of the equation $u'(x) = 0$, which lies to the left of x_1, since $u(a) = 0$. Hence the equation $u''(x) = 0$ has at least one root between x_0 and x_1 and a repetition of the argument used in the case $u''(x_1) = 0$ again leads to a contradiction.

Finally, if $u(x_1) = 0$ we may assume without loss of generality that $u(x) > 0$ for x in (x_1, b) and then repeat the argument of the previous paragraph to reach contradictions. This proves $u(x)$ is positive in (a, b).

To show that $u(x)$ is convex in (a, b), we observe that $r(b)u''(b) = (r(x)u''(x))_b' = 0$. Hence if at some point in (a, b) we have $u''(x) < 0$ the curve $\xi = ru''$ either has at least one maximum in (a, b) or it must rise toward the point $x = b$ through negative values. The first case is impossible because $(ru'')'' = \lambda_b pu > 0$. In the second case the curve $\xi = ru''$ has a horizontal tangent at $x = b$ so $(ru'')'' < 0$ for x slightly to the left of b, which is also impossible. This proves $u(x)$ is convex.

Finally, $u(a) = u'(a) = 0$, so $u(x)$ must increase with x. This proves the lemma.

We next remark that $u''(x, B)$ vanishes precisely twice in $[a, \infty)$. For it is known [4, p. 338] that the first eigenfunction of Problem (2.3) does not change sign in $[a, B]$. We assume $y \geqq 0$ in $[a, B]$. Moreover if there exists a point $x_0 > B$ such that $y > 0$ in (B, x_0), and $y(x_0, B) = 0$ then there exist roots, x_1 and x_2, of the equation $y'' = 0$ such that $x_1 < B < x_2 < x_0$. Hence y'' has at least one relative maximum between x_1 and x_2 and there must exist an interval I contained in (x_1, x_2) such that $(ry'')'' < 0$ in I, an impossibility, since $y = 0$ at most once in (x_1, x_2). The remaining case in which y changes sign at B means $x_2 = B$ with contradictions being reached as before. Hence we may assume $y \geqq 0$ in $[a, \infty)$. Then $y''(a)$ and $y''(B)$ must both be positive. For if not, ry'' has at least one relative maximum in (a, B). Therefore there exists an interval contained in (a, B) such that $(ry'')'' \leqq 0$ there. But $(ry'')'' = \mu_B py > 0$ in (a, B), a contradiction. Since $y(x)$ has one relative maximum in (a, B) there exists an interval in (a, B) such that $y''(x) < 0$ in this interval.

But ry'' is convex in (a, ∞) so the equation $y''(x)=0$ has exactly two roots in (a, ∞). Next we have the other lemma.

LEMMA 2.2. *Assume that* $\int_a^\infty dx/r(x)=\infty$. *Denote by* $\alpha=\alpha(B)$ *and* $\beta=\beta(B)$ *the first and second zero, respectively, of the equation* $y''(x, B)=0$ *where* $y(x, B)$ *is the first eigenfunction of Problem* (2.3). *Then for any real number* $L>0$ *there exists a value* $B>a$, *such that* $\alpha(B)>L$.

To prove Lemma 2.2 we assume that $\beta=\beta(B)$ is bounded above as $B\to\infty$ and show that this leads to a contradiction. A similar assumption concerning $\alpha=\alpha(B)$ leads to another contradiction, and thus to the desired conclusion.

Suppose there exists a finite real number β_0 such that $\beta=\beta(B)<\beta_0$ where B may assume any value such that $B>a$. Then there exists a constant M, independent of B, such that $y(x)<M$ when x is in $[a, B]$. For the continuity of $r(x)$ in $[a, \beta_0]$ guarantees the existence of a positive minimum for $r(x)$ in this interval, say $r_{\min}$; then $r_{\min}\, y''(x)\leqq r(x)y''(x)\leqq 1$ so $y''(x)\leqq 1/r_{\min}$ and thus (for $i=0, 1$),

$$y^{(i)}(x) \leqq \frac{1}{r_{\min}}\,\frac{(x-a)^{2-i}}{(2-i)!}$$

for x in $[a, \alpha(B)]$. But $y(x)$ has double zeros at $x=a$ and $x=B$ and a single positive maximum between a and B. Thus for x in $[\alpha(B), \beta(B)]$ the curve $y=y(x)$ lies below the tangent to $y(x)$ at $x=\alpha(B)$. Hence

$$y(x) \leqq \frac{\alpha(B)-a}{r_{\min}}(x-\alpha(B)) + \frac{1}{r_{\min}}\,\frac{(\alpha(B)-a)^2}{2!} \tag{2.5}$$

for x in $[\alpha(B), \beta(B)]$. Since the maximum for $y(x)$ in $[a, B]$ is assumed somewhere in $(\alpha(B), \beta(B))$ we have from (2.5)

$$y(x) \leqq \frac{\alpha(B)-a}{r_{\min}}(\beta(B)-\alpha(B)) + \frac{1}{r_{\min}}\,\frac{(\alpha(B)-a)^2}{2} \leqq \frac{3}{2}\,\frac{\beta_0^2}{r_{\min}} = M$$

for x in $[a, B]$, because $0<a<\alpha(B)<\beta(B)<\beta_0$.

From the result of the last paragraph and the formula

$$(r(x)y''(x))' = (ry'')'_a + \mu_B\int_a^x p(t)y(t)dt$$

we have, if $\gamma=\gamma(B)$ be taken as that point lying in $[\alpha(B), \beta(B)]$ where $(ry'')'$ vanishes

$$0 = (r(x)y''(x))'_\gamma = (ry'')'_a + \mu_B\int_a^\gamma p(t)y(t)dt,$$

so

$$0 \leqq (ry'')_a' + \mu_B M \int_a^\gamma p(t)dt.$$

Hence

$$-\mu_B M \int_a^\gamma p(t)dt \leqq (ry'')_a'. \tag{2.6}$$

As $B\to\infty$, $(ry'')_a'$ is therefore bounded below by a constant, say N, which is independent of B. Indeed, $\gamma(B)$ is bounded above by β_0, and $p(t)$ is continuous in $[a, \gamma(B)]$; moreover, μ_B decreases as B increases.

Obviously, as $B\to\infty$ either $(ry'')_\gamma$ is bounded above by a negative value, or there exists a sequence $\{B_\nu\}$, $\nu=1, 2, \cdots$, where $B_\nu\to\infty$ as $\nu\to\infty$ such that $(ry'')_{x=\gamma(B_\nu)}\to 0$ as $\nu\to\infty$. In the first case, we have $(ry'')_\gamma \leqq -\epsilon$, $\epsilon>0$, for any $B>a$. But then $r(x)y''(x) \geqq (\epsilon/(\beta_0-a))(x-\beta_0)$ for x in $[\beta_0, \infty)$. If we set $\xi=ry''$, ξ is a convex function of x, and it follows that $\xi(x)\geqq \xi'(\beta_0)(x-\beta_0)$ for x in $[\beta_0, \infty)$ and therefore

$$\xi'(\beta_0) \geqq \frac{0-(-\epsilon)}{\beta(B)-\gamma(B)} \geqq \frac{\epsilon}{\beta_0-\alpha}\cdot$$

Thus

$$y''(x) \geqq \frac{\epsilon}{\beta_0 - a}\cdot\frac{(x-\beta_0)}{r(x)} \tag{2.7}$$

for x in $[\beta_0, \infty)$. Integrating (2.7) from β_0 to B (B chosen large enough to ensure $B>\beta_0$) we have

$$-y'(\beta_0) \geqq \frac{\epsilon}{\beta_0 - a}\int_{\beta_0}^B \frac{(t-\beta_0)}{r(t)}\,dt \to \infty \tag{2.8}$$

as $B\to\infty$ since $\int_a^\infty (dx/r(x)) = \infty$. But (2.8) is impossible. For we may write, for any B sufficiently large

$$N(x-a) + 1 \leqq r(x)y''(x) \tag{2.9}$$

for x in $[\alpha(B), \beta(B)]$, where N is the constant introduced above. The continuity of $r(x)$ in $[\alpha(B), \beta_0]$ guarantees the existence of a positive minimum for $r(x)$ in that interval, say $r_{\min}$. Hence

$$\frac{N(\beta_0 - a) + 1}{r_{\min}} \leqq y''(x) \tag{2.10}$$

for x in $[\alpha(B), \beta(B)]$. Denoting the left-hand side of (2.10) by $-P$, which is independent of B, we get, integrating from $\alpha(B)$ to $\beta(B)$,

$$-P(\beta(B) - \alpha(B)) \leqq y'(\beta(B)) - y'(\alpha(B)). \tag{2.11}$$

Since $y'(\alpha(B))>0$, we have

$$-P(\beta(B)-\alpha(B)) \leqq y'(\beta(B)),$$

and therefore $-Q=-P(\beta_0-a)\leqq y'(\beta(B))$ where Q is independent of B. Since $y'(\beta(B))\leqq y'(\beta_0)$, ($y''(x)$ is positive for x in $(\beta(B), \beta_0)$) we have $-Q\leqq y'(\beta_0)$ and $-y'(\beta_0)\not\to\infty$ as inequality (2.8) requires.

Thus for the case that $(ry'')_{x=\gamma(B)}$ is always bounded above by a negative value as $B\to\infty$ we may conclude that the existence of a β_0 with the above properties is impossible.

In the second case, we must have $-\epsilon < r(x)y''(x)_{x=\gamma(B_\nu)} < 0$ for x in $(\alpha(B_\nu), \beta(B_\nu))$. But we know that, for any sufficiently large B

$$N(x-a)+1 \leqq r(x)y''(x)$$

for x in $(a, \alpha(B))$, by inequality (2.6) above, where N is independent of B. From the last inequality we get

$$y''(x) \geqq \frac{N(x-a)+1}{r(x)} > 0$$

for x in $(a, a-1/N)$. We recall that $-\infty<N<0$, with N independent of B. Thus

$$y'(a-1/N) \geqq \int_a^{a-1/N} \frac{N(t-a)+1}{r(t)}\,dt = \delta > 0.$$

But $\alpha(B)\geqq a-1/N$ for any B, so $y'(\alpha(B))\geqq y'(a-1/N)\geqq\delta$ with δ independent of B. But the hypothesis that $-\epsilon<r(x)y''(x)_{x=\gamma(B_\nu)}<0$, coupled with the fact that $r(x)$ is continuous for x in $[a, \beta_0]$ means that, upon choosing ϵ sufficiently small $-\delta/2<\int_{\alpha(B_\nu)}^{\beta(B_\nu)}y''(t)dt<0$. Combining this last inequality with $\delta\leqq y'(\alpha(B_\nu))$ gives $0<\delta/2<y'(\beta(B_\nu))$, an impossibility, since $y'(x)$, evaluated at the second zero of the equation $y''(x)=0$ must be negative. Thus for the case that $(ry'')_{x=\gamma(B_\nu)}\to 0$ as $\nu\to\infty$, we may conclude that the existence of a β_0, with the above properties, is impossible. Combining these two results shows that corresponding to any number $N>0$, there exists a B_N such that if $B>B_N$, $\beta(B)>N$.

Suppose, finally, that there exists a finite real number α_0 such that $\alpha=\alpha(B)<\alpha_0$ where B may assume any value such that $B>a$. We take, in particular, $\alpha_0=1+\text{lub}_{B>a}(\alpha(B))$. We show first, under this assumption, that $(r(\alpha_0)y''(\alpha_0, B))\to 0$ as $B\to\infty$. For suppose $r(\alpha_0)y''(\alpha_0)\leqq -\epsilon$, $\epsilon>0$. Note that as $B\to\infty$ $r(\alpha_0)y''(\alpha_0)$ must eventually be negative, for if not α_0 would serve as a finite upper bound for $\beta(B)$, an impossibility as we have just seen. From the last inequality we have at once

$$r(x)y''(x) \leqq -\frac{\epsilon}{2} \qquad \text{for } x \text{ in } \left(\alpha_0, \frac{\alpha_0+\beta(B)}{2}\right)$$

since $(ry'')'' \geqq 0$ and $ry'' < 0$ in that interval. Thus for sufficiently large B

$$(2.13)\quad y(x) = y(x, B) \leqq -\frac{\epsilon}{2}\int_{\alpha_0}^{x}\left[\int_{\alpha_0}^{t}\frac{dt}{r(t)}\right]dt + y'(\alpha_0)(x - \alpha_0) + y(\alpha_0)$$

for x in $(\alpha_0, (\alpha_0+\beta(B))/2)$. But if we set $r_{\min} = \min_{x\in[a,\alpha_0]} r(x)$ then for x in $(a, \alpha(B))$ $y''(x) \leqq 1/r_{\min}$ so $y'(\alpha_0) \leqq 1/r_{\min}\,(\alpha_0 - a)$ and $y(\alpha_0) \leqq (1/r_{\min})(\alpha_0 - a)^2/2$. That is, for any sufficiently large B, $y'(\alpha_0)$ and $y(\alpha_0)$ are bounded independently of B. Since $\beta(B)\to\infty$ as $B\to\infty$ and $\int_a^\infty dx/r(x) = \infty$ we may thus conclude from inequality (2.13) that, for all sufficiently large B, $y((\alpha_0+\beta(B))/2, B) < 0$; evidently the first term on the right-hand side of (2.13) will eventually be the dominant one. But this is absurd, since $y(x) \geqq 0$ for x in (a, ∞). We next show, still under the assumption of the existence of a finite upper bound α_0, that $(r(x)y''(x))_a'$ is bounded below for all sufficiently large B. Suppose the assertion is false. We know that

$$y(x) \leqq \frac{1}{r_{\min}}\,\frac{(\alpha_0 - a)^2}{2}, \qquad \text{for } x \text{ in } (a, \alpha_0),$$

and any sufficiently large B. Since $p(x)$ is continuous in $[a, \alpha_0]$ we have

$$0 \leqq \int_a^{\alpha_0} p(t)y(t)dt \leqq \frac{M(\alpha_0 - a)^3}{2} = P,$$

where $M = \max_{x\in[a,\alpha_0]} p(x)$ and, by construction, P is independent of B. But from the equation

$$(r(x)y''(x))' = (r(x)y''(x))_a' + \mu_B\int_a^x p(t)y(t)dt$$

we get

$$(r(x)y''(x))' \leqq (r(x)y''(x))_a' + \mu_B P$$

for x in (a, α_0). Since μ_B decreases as $B\to\infty$ we may, for sufficiently large B, replace $\mu_B P$ by a constant, say Q, independent of B. By our assumption there exists a sequence of B's $\{\overline{B}_\nu\}$ say, with $\nu = 1, 2, \cdots$, and with $\overline{B}_\nu\to\infty$ as $\nu\to\infty$ such that

$$(r(x)y''(x))_a' \leqq \left(\frac{-2}{\alpha_0 - a} - Q\right).$$

Hence $(r(x)y''(x))' \leqq -2/(\alpha_0 - a)$ for x in (a, α_0) whence $r(x)y''(x) \leqq -1$. But this is contradictory for all ν sufficiently large, because $r(x)y''(x, B)_{\alpha_0}\to 0$ as $B\to\infty$ as shown above.

We may thus write $r(x)y''(x) \geqq N(x-a)+1$ for x in (a, ∞), where $N<0$ is used to stand for a lower bound to $(r(x)y''(x, B))_a'$, and the only requirement on B is that it be sufficiently large. Since $r(x)$ is positive and continuous for x in $[a, \infty)$ we have

$$y''(x) \geqq \frac{N(x-a)+1}{r(x)} .$$

Thus

$$y'(a-1/N) \geqq \int_a^{a-1/N} \frac{N(t-a)+1}{r(t)} dt = \delta' > 0$$

$$y(a-1/N) \geqq \int_a^{a-1/N} (a-1/N-t) \frac{(N(t-a)+1)}{r(t)} dt = \delta > 0$$

where δ and δ' are independent of B for any sufficiently large B. Now $r(x)y''(x) = 0$ at $x = \alpha(B)$ and $x = \beta(B)$, $r(x)y''(x)_{\alpha_0} \to 0$ as $B \to \infty$, and $(r(x)y''(x))'' \geqq 0$ for x in $[a, \infty)$. Hence $r(x)y''(x) \to 0$ for x in $[\alpha(B), \alpha_0+1]$ as $B \to \infty$. But this means $y''(x) \to 0$ for x in $[\alpha(B), \alpha_0+1]$ as $B \to \infty$. We know, consequently, that

$$y'(\alpha_0+1) = y'(a-1/N) + \int_{a-1/N}^{\alpha_0+1} y''(t)dt \geqq \frac{\delta'}{2} > 0$$

since the integral may be made to give as small a negative contribution as desired by choosing B sufficiently large. Hence $y(x) \geqq \delta > 0$ for any sufficiently large B, with δ independent of B, for x in $[\alpha_0, \alpha_0+1]$. But from the equation

$$r(\alpha_0+1)y''(\alpha_0+1) = r(\alpha_0)y''(\alpha_0) + (r(x)y''(x))'_{\alpha_0} + \mu_B \int_{\alpha_0}^{\alpha_0+1} p(t)y(t)dt$$

we note that as $B \to \infty$ every term except the integral tends to zero, since we know [4, p. 347] that $\mu_B > 1$ for any $B > a$. This leads to $0(\epsilon) = 0(1)$ which is absurd.

It follows that the existence of an α_0, with the above properties is impossible. Hence there exists a $B > a$ such that, for any real number $L > 0$, $\alpha(B) > L$. This proves the second lemma.

We now prove Theorem 2.1. For the proof of the necessity we note that

$$\begin{aligned}(\lambda_b - \mu_B) \int_a^b p(t)y(t, B)u(t)dt &= \int_a^b (y(ru'')'' - u(ry'')'')dt \\ &= (y(ru'')' - y'(ru'') + (ry'')u' - (ry'')'u \big|_a^b \\ &= ((ry'')u' - (ry'')'u \big|_b.\end{aligned}$$

For any $b > a$ we may choose B such that $\alpha(B) > b$, by Lemma 2.2. Then $(ry'')_b > 0$ while $(ry'')'_b < 0$. For we know [4, p. 347] that $\mu_B > 1$ for any $B > a$, since $(ru'')''=pu$ is nonoscillatory in (a, ∞). Thus $(ry'')''=\mu_B py>0$ in $(a, \alpha(B))$ and ry'' is a positive, convex, decreasing function in $(a, \alpha(B))$. Since $u(b)$ and $u'(b)$ are positive by Lemma 2.1 we conclude that

$$(\lambda_b - \mu_B)\int_a^b puydt > 0.$$

Lemma 2.1 shows that this integral is positive since we know [4, p. 338] that $y(x, B)>0$ in (a, B). Thus $(\lambda_b-\mu_B)>0$ for some B; since $\mu_B>1$ the result follows.

The proof of the sufficiency is simple. We note that if $(ru'')''=pu$ had a solution with four zeros in (a, ∞) then there would also exist a solution $u(x)$ with the properties $u(c)=u'(c)=u(d)=u'(d)=0$ where $a<c<d$ (cf. [4, p. 332]). Therefore $\mu_d\leqq 1$; but $\mu_d\geqq\lambda_d$ since μ_d is the minimum of the Rayleigh quotient (2.4) under more restrictive conditions (see, for example, [1] or [2]). By hypothesis $\lambda_b>1$ for any b, so $1\geqq\mu_d\geqq\lambda_d>1$, a contradiction. This completes the proof of the sufficiency and of Theorem 2.1.

3. **Oscillation criteria.** In this section we apply Theorem 2.1 to derive necessary conditions for the nonoscillation of solutions of equation (2.1). We have the following general theorem.

THEOREM 3.1. *Let $v(t)$ be a function of class D'' in $[a, b]$ such that $v(a)=v'(a)=0$. If equation* (2.1) *is nonoscillatory in (a, ∞) and $\int_a^\infty (dx/r(x))=\infty$ then*

$$\int_a^b p(v)^2dt < \int_a^b r(v'')^2dt. \tag{3.1}$$

Indeed, by the minimizing property of the Rayleigh quotient we have

$$1 < \lambda_b \leqq \int_a^b r(v'')^2dt \Big/ \int_a^b p(v)^2dt.$$

We consider some applications of Theorem 3.1. Let $0<a<b$ and choose a number x such that $a<x<b$; define $v(t)$ by

$$\begin{aligned} v(t) &= c_1(t-a)^{\beta/2}, && \text{for } t \text{ in } [a, x], \\ v(t) &= c_2(t-a)^{\alpha/2} - 1, && \text{for } t \text{ in } [x, b], \end{aligned}$$

where

$$c_1 = \frac{\alpha}{(\beta-\alpha)(x-a)^{\beta/2}} \qquad c_2 = \frac{\beta}{(\beta-\alpha)(x-a)^{\alpha/2}}.$$

It is easily confirmed that $v(t)$ satisfies the hypotheses of Theorem 3.1 if $\beta>2$. We have then, by (3.1),

$$\frac{\alpha^2}{(\beta-\alpha)^2(x-a)^\beta}\int_a^x (t-a)^\beta p(t)dt$$
$$+\frac{\beta^2}{(\beta-\alpha)^2(x-a)^\alpha}\int_x^b\left((t-a)^{\alpha/2}-\frac{(\beta-\alpha)}{\beta}(x-a)^{\alpha/2}\right)^2 p(t)dt$$
$$<\frac{\alpha^2}{(\beta-\alpha)^2(x-a)^\beta}\cdot\left(\frac{\beta}{2}\right)^2\left(\frac{\beta}{2}-1\right)^2\frac{(x-a)^{\beta-3}}{\beta-3}$$
$$+\frac{\beta^2}{(\beta-\alpha)^2(x-a)^\alpha}\cdot\left(\frac{\alpha}{2}\right)^2\left(\frac{\alpha}{2}-1\right)^2\left(\frac{(b-a)^{\alpha-3}}{\alpha-3}-\frac{(x-a)^{\alpha-3}}{\alpha-3}\right).$$

These formulae are valid provided $\alpha\neq 3$ and $\beta>3$. In particular, if $\alpha-3<0$ then $(b-a)^{\alpha-3}\to 0$ as $b\to\infty$ and these results may be put into the following form.

THEOREM 3.2. *If $y^{\text{iv}}=py$ is nonoscillatory in (a,∞), then*

$$\alpha^2(x-a)^{3-\beta}\int_a^x (t-a)^\beta p(t)dt$$
$$+\beta^2(x-a)^{3-\alpha}\int_x^\infty\left((t-a)^{\alpha/2}-\frac{(\beta-\alpha)}{\beta}(x-a)^{\alpha/2}\right)^2 p(t)dt$$
$$\leqq\left(\frac{\alpha\beta}{4}\right)^2(\beta-\alpha)\left(1+\frac{1}{(\beta-3)(3-\alpha)}\right)$$

where $a<x$ and $\alpha<3<\beta$.

We next apply Theorem 3.1 to the function $v(t)$ defined by

$$v(t)=f(t)\quad\text{with}\quad f(a)=f'(a)=0,\qquad \text{for } t \text{ in } [a,x],$$
$$v(t)=f'(x)(t-x)+f(x),\qquad \text{for } t\geqq x,$$

where $f(t)$ is a function of class C'' in $[a,x]$. The function $v(t)$ belongs to D'' in $[a,\infty)$ and (3.1) yields

$$\int_a^x p(t)f^2(t)dt+\int_x^b p(t)((f'(x))^2(t-x)^2+2f(x)f'(x)(t-x)+(f(x))^2)dt$$
$$<\int_a^x (f''(t))^2dt.$$

If $f'(x)f(x)>0$ we have, by letting $b\to\infty$

$$\int_x^\infty (t-x)^2p(t)dt\leqq\int_a^x\frac{(f''(t))^2dt}{(f'(x))^2},\tag{3.2}$$

$$\int_x^\infty (t-x)p(t)dt\leqq\int_a^x\frac{(f''(t))^2dt}{2f'(x)f(x)},\tag{3.3}$$

$$\int_x^\infty p(t)dt \leqq \int_a^x \frac{(f''(t))^2 dt}{(f(x))^2}. \tag{3.4}$$

In order to derive as precise estimates as possible we seek to minimize the integral on the right-hand side of (3.2) where each member f of the class of competing functions must have the properties $f(a)=f'(a)=0$. The situation with respect to the inequalities (3.3) and (3.4) is analogous.

Since the Euler equation associated with the integrals on the right-hand side of the inequalities is $f^{\text{iv}}=0$, we choose $f(t)$ to be a cubic polynomial with a double zero at $t=a$. Since $f(t)$ will still contain two arbitrary constants we set $f(x)=\alpha, f'(x)=\beta$, and then minimize the right-hand sides of (3.2) through (3.4) with respect to the parameters α and β, subject only to the restriction that $\alpha\beta>0$.

Omitting the details of calculation we have the following result.

THEOREM 3.3. *If $y^{\text{iv}}=py$ is nonoscillatory in (a, ∞), then*

$$\int_x^\infty (t-x)^2 p(t)dt \leqq 1/(x-a),$$

$$\int_x^\infty (t-x)p(t)dt \leqq (12-6(3)^{1/2})/3^{1/2}(x-a)^2$$

and

$$\int_x^\infty p(t)dt \leqq 3/(x-a)^3.$$

As a final example of the use of Theorem 3.1 we outline the proof of the following result.

THEOREM 3.4. *If $y^{\text{iv}}=py$ is nonoscillatory in (a, ∞), if $p'(x)$ exists and is integrable, and if there exists a constant m such that*

$$0 < p(x) \leqq m^{4/3},\ p'(x) \leqq m^4/p(x)\int_a^x p(s)ds,\ (p'(x))^2 \leqq m^4 p(x) \Big/ \left(\int_a^x p(s)ds\right)^2$$

then there exists a universal constant c_0 such that the inequality

$$\int_a^\infty p(x)dx \leqq cm \qquad \textit{holds for } c = c_0$$

but not in general for $c=c_0-\epsilon$. c_0 satisfies the inequalities

$$1/(2(3)^{1/2}) = .289 \leqq c_0 \leqq (1+2^{3/2})^{7/4}/3^{3/4}(1+2^{1/2})^{1/2} = 2.96.$$

The lower bound follows from consideration of the equation $y^{\text{iv}}=(9/16)(1/x^4)y$ which is nonoscillatory in (a, ∞), $a>0$. To obtain the upper bound we set

$$v(t) = \left(\int_a^t p(s)ds\right)^\nu, \qquad 3/2 < \nu, \qquad \text{for } t \text{ in } [a, x],$$

$$v(t) = \nu\left(\int_a^x p(s)ds\right)^{\nu-1} \cdot (p(x)(t - x)) + \left(\int_a^x p(s)ds\right)^\nu, \qquad \text{for } t \text{ in } [x, b].$$

The proof may be completed by an argument analogous to one given by Nehari [5] and the details are therefore omitted.

4. **Comparison theorems.** In this section we consider theorems, dealing with differential equations of the type $y^{\text{iv}} = py$, which are obtained by comparing the given equation with one whose oscillatory or nonoscillatory character is known. This comparison procedure will also be applied to obtain theorems for the more general equations of type (2.1). (Cf. [4] and [5] for further comparison theorems pertaining to related second and fourth order equations.) We begin with the following theorem.

THEOREM 4.1. *Let $p(x)$ be positive and continuous for x in $[a, b]$ and let $q(x)$ be continuous for x in $[a, b]$. Denote by λ_1 the first eigenvalue of the differential system*

$$(rv'')'' = \lambda p v,$$
$$v(a) = v'(a) = (rv'')_b = (rv'')_b' = 0,$$

and by μ_1 the first eigenvalue of the differential system

$$(rw'')'' = \mu q w,$$
$$w(a) = w'(a) = (rw'')_b = (rw'')_b' = 0.$$

If we define $\phi(t)$ and $\psi(t)$ by

$$\phi(t) = \int_t^b f(s)p(s)ds \quad \text{and} \quad \psi(t) = \int_t^b f(s)q(s)ds$$

where $f(s)$ is a positive, nonincreasing function of class C' in $[a, b]$, and if $\phi(t) \leqq \psi(t)$ for all t in $[a, b]$ then $\lambda_1 \geqq \mu_1$.

By a direct generalization of a procedure used in [5] we have

$$0 < \int_a^b p(v)^2 dt \leqq \int_a^b q(v)^2 dt$$

and consequently $\lambda_1 \geqq \mu_1$ by use of the Rayleigh quotients associated with the eigenvalues.

By use of the same technique we may prove the following theorem.

THEOREM 4.2. *Let $p(x)$ be positive and continuous for x in $[a, b]$ and let $q(x)$ be continuous for x in $[a, b]$. Denote by λ_1 the first eigenvalue of the differential system*

$$(rv'')'' = \lambda p v,$$
$$v(a) = v'(a) = (rv'')_b = (rv'')_b' = 0$$

and by μ_1 the first eigenvalue of the differential system

$$(rw'')'' = \mu q w,$$
$$w(a) = w'(a) = (rw'')_b = (rw'')_b' = 0.$$

If we define $\phi(t)$ and $\psi(t)$ by

$$\phi(t) = \int_t^b (s - a)^\nu p(s) ds \quad \text{and} \quad \psi(t) = \int_t^b (s - a)^\nu q(x) ds$$

where $\nu \leqq 2$ and if $\phi(t) \leqq \psi(t)$ for all t in $[a, b]$ then $\lambda_1 \geqq \mu_1$.

We have as before that

$$0 < \int_a^b p(v)^2 dt \leqq \int_a^b q(v)^2 dt$$

and consequently $\lambda_1 \geqq \mu_1$ by precisely the same reasoning as used in the conclusion of Theorem 4.1.

Combining Theorem 2.1 and Theorem 4.1 we obtain the following result.

THEOREM 4.3. *Let $u(s)$ be a positive and nonincreasing function of class C' in $[a, \infty)$. If*

$$\int_x^\infty \frac{u(s)}{s^4} ds < \infty \quad \text{and} \quad \operatorname*{Lim\,sup}_{x \to \infty} \left(\int_x^\infty u(s) p(s) ds \Big/ \int_x^\infty \frac{u(s)}{s^4} ds \right) < 9/16,$$

the equation $y^{\mathrm{iv}} = py$ is nonoscillatory. If

$$\operatorname*{Lim\,inf}_{x \to \infty} \left(\int_x^\infty u(s) p(s) ds \Big/ \int_x^\infty \frac{u(s)}{s^4} ds \right) > 9/16,$$

then the equation $y^{\mathrm{iv}} = py$ is oscillatory.

Combining Theorem 2.1 and Theorem 4.2 we obtain the following result.

THEOREM 4.4. *If*

$$\operatorname*{Lim\,sup}_{x \to \infty} \int_x^\infty (s - a)^\nu p(s) ds \Big/ \int_x^\infty \frac{(s - a)^\nu}{s^4} ds < 9/16, \qquad \nu \leqq 2,$$

then the equation $y^{\mathrm{iv}} = py$ is nonoscillatory. If

$$\operatorname*{Lim\,inf}_{x \to \infty} \left(\int_x^\infty (s - a)^\nu p(s) ds \Big/ \int_x^\infty \frac{(s - a)^\nu}{s^4} ds \right) > 9/16, \qquad \nu \leqq 2,$$

then the equation $y^{\mathrm{iv}} = py$ is oscillatory.

The following results, which are analogues of corresponding results of Hille [3] concerning second-order equations, are direct consequences of Theorem 4.3 and Theorem 4.4.

THEOREM 4.5. *If*

$$\operatorname*{Lim\,sup}_{x\to\infty} x^{3-\nu}\int_x^\infty s^\nu p(s)ds < (9/16)(1/(3-\nu)), \qquad \nu \leqq 2,$$

then the equation $y^{\text{iv}} = py$ *is nonoscillatory. If*

$$\operatorname*{Lim\,inf}_{x\to\infty} x^{3-\nu}\int_x^\infty s^\nu p(s)ds > (9/16)(1/(3-\nu)), \qquad \nu \leqq 2,$$

then the equation $y^{\text{iv}} = py$ *is oscillatory.*

The following theorem, like Theorem 4.1, is proved by a generalization of a technique due to Nehari [5] so the result is merely stated.

THEOREM 4.6. *Let* $p(x)$ *be positive and continuous for* x *in* $[a, b]$, *let* $r(x)$ *be positive and of class* C'' *for* x *in* $[a, b]$, *and let* $r_1(x)$ *be positive and of class* C'' *for* x *in* $[a, b]$. *Denote by* λ_1 *and* μ_1, *respectively, the lowest eigenvalues of the differential systems*

$$(4.1) \qquad \begin{aligned} (rv'')'' &= \lambda pv, \\ v(a) = v'(a) &= (rv'')_b = (rv'')_b' = 0, \end{aligned}$$

$$(4.2) \qquad \begin{aligned} (rw'')'' &= \mu pw, \\ w(a) = w'(a) &= (r_1w'')_b = (r_1w'')_b' = 0. \end{aligned}$$

If we define $\phi(t)$ *and* $\psi(t)$ *by*

$$\phi(t) = \int_a^t r(s)f(s)ds, \qquad \psi(t) = \int_a^t r_1(s)f(s)ds,$$

where $(r_1(s))^2f(s)$ *is nondecreasing in* $[a, b]$, *and if* $\phi(t) \leqq \psi(t)$ *for all* t *in* $[a, b]$ *then* $\lambda_1 \leqq \mu_1$.

Combining Theorem 2.1 and Theorem 4.6 we obtain the following result.

THEOREM 4.7. *If*

$$\int_a^\infty \frac{dx}{r(x)} = \infty \quad \text{and} \quad \operatorname*{Lim\,sup}_{x\to\infty}\left(\int_a^x r(s)f(s)ds \Big/ \int_a^x r_1(x)f(s)ds\right) < 1,$$

where $f(s)$ *is a positive function of class* C' *in* $[a, \infty)$ *such that* $(r_1(s))^2f(s)$ *is nondecreasing in* $[a, \infty)$, *then* $(rv'')'' = pv$ *is oscillatory if* $(r_1w'')'' = pw$ *is oscillatory.*

The techniques used in the proofs of Theorems 4.1, 4.3, 4.6, and 4.7 may also be applied to second-order equations, and they will yield similar results. The following are examples of theorems obtainable in this way.

THEOREM 4.8. *Let $p(x)$ be positive and continuous for x in $[a, b]$ and let $q(x)$ be continuous for x in $[a, b]$. Denote by λ_1 the first eigenvalue of the differential system*

$$v'' + \lambda p v = 0,$$
$$v(a) = v'(b) = 0,$$

and by μ_1 the first eigenvalue of the differential system

$$w'' + \mu q w = 0,$$
$$w(a) = w'(b) = 0.$$

If we define $\phi(t)$ and $\psi(t)$ by

$$\phi(t) = \int_t^b f(s)p(s)ds \quad \text{and} \quad \psi(t) = \int_t^b f(s)q(s)ds,$$

where $f(s)$ is a positive, nonincreasing function of class C' in $[a, b]$ and if $\phi(t) \leqq \psi(t)$ for all t in $[a, b]$ then $\lambda_1 \geqq \mu_1$.

The proof parallels previous ones and details are accordingly omitted.

THEOREM 4.9. *Let $u(s)$ be a positive and nonincreasing function of class C' in $[a, \infty)$. If $\int_x^\infty (u(s)/s^2)ds < \infty$ and* Lim $\sup_{x\to\infty} (\int_x^\infty u(s)p(s)ds / \int_x^\infty u(s)/s^2(ds))$ $< 1/4$ *then the equation $y'' + py = 0$ is nonoscillatory. If*

$$\operatorname*{Lim\,inf}_{x\to\infty} \left(\int_x^\infty u(s)p(s)ds \Big/ \int_x^\infty \frac{u(s)}{s^2}\, ds \right) > \frac{1}{4}$$

then the equation $y'' + py = 0$ is oscillatory.

The proof is similar to previous proofs and details are omitted.

In parallel to Theorems 4.6 and 4.7 we have the following two results. The proofs will be omitted.

THEOREM 4.10. *Let $p(x)$ be positive and continuous for x in $[a, b]$, let $r(x)$ be positive and of class C'' for x in $[a, b]$, and let $r_1(x)$ be positive and of class C'' for x in $[a, b]$. Denote by λ_1 and μ_1 the lowest eigenvalues of the differential systems*

$$(rv')' + \lambda p v = 0, \quad v(a) = v'(b) = 0, \tag{4.3}$$

$$(r_1 w')' + \mu p w = 0, \quad w(a) = w'(b) = 0. \tag{4.4}$$

If we define $\phi(t)$ and $\psi(t)$ by

$$\phi(t) = \int_a^t r(s)f(s)ds, \quad \text{and} \quad \psi(t) = \int_a^t r_1(s)f(s)ds,$$

where $f(s)$ is a positive function of class C' in $[a, b]$ such that $(r_1(s))^2f(s)$ is nondecreasing in $[a, b]$, and if $\phi(t) \leqq \psi(t)$ for all t in $[a, b]$, then $\lambda_1 \leqq \mu_1$.

THEOREM 4.11. *If*

$$\operatorname*{Lim\,sup}_{x \to \infty} \left(\int_a^x f(s)ds \Big/ \int_a^x r_1(s)f(s)ds \right) < 1$$

where $f(s)$ is a positive function of class C' in $[a, \infty)$ such that $(r_1(s))^2f(s)$ is nondecreasing in $[a, \infty)$ then $v''+pv=0$ is oscillatory if $(r_1w')'+pw=0$ is oscillatory.

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