

ON UNIFORMIZATION OF SETS IN TOPOLOGICAL SPACES

BY
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1. Introduction. Let $Q \subset X \times Y$ and P be the projection of Q onto X . A uniformization of Q is a function f on P to Y such that $f \subset Q$, i.e., for every $x \in P$, $(x, f(x)) \in Q$. Clearly with the aid of the axiom of choice, such a function can always be found. The problem, in general, is to find an f that satisfies, in addition, certain conditions. Many authors have considered different aspects of this problem and we mention a few in the bibliography (see also [5, p. 55 and on]).

In case X and Y are Euclidean spaces one important result, due to Luzin [3] and Sierpinski [6] is the following: if Q is Borel, then f can be taken to be the complement of an analytic set in $X \times Y$. Kondô [2] has extended this result to the case where Q is the complement of an analytic set, thereby deriving important properties of projective sets of class 2.

In this paper, we take X and Y to be topological spaces satisfying certain conditions. If Q is compact, then we obtain a uniformization f that is a Borel function (Theorem 4.2). If Q is analytic, then f is a measurable function for a large class of measures on X (Theorem 4.5).

2. Notation and basic definitions.

2.1. A non-negative integer n contains all smaller non-negative integers, i.e., $m \in n$ iff $m < n$ and m is a non-negative integer. Thus, 0 denotes both the empty set and the smallest non-negative integer.

2.2. ω denotes the set of all non-negative integers.

2.3. f is a function on A to B iff $f \subset A \times B$ and, for every $x \in A$, there exists a unique y ($=f(x)$) such that $(x, y) \in f$.

2.4. A is Borel H iff A belongs to the smallest family containing H and closed under countable unions and differences of two sets.

2.5. $K(X)$ denotes the family of all compact sets in X .

2.6. A is analytic in X iff, for some Hausdorff space X' , A is the continuous image of a $K_{\sigma\delta}(X')$.

2.7. $\mathfrak{A}(X)$ denotes the family of all analytic sets in X .

2.8. If $A \subset X \times Y$, then πA is the projection of A onto X .

2.9. Y satisfies condition (I) iff Y is Hausdorff, regular, has a base whose power is at most that of the first noncountable cardinal, and every family of open sets in Y has a countable subfamily with the same cover.

3. Preliminary lemmas.

3.1. *If, for every $n \in \omega$, A_n is compact, $A_{n+1} \subset A_n$, $\bigcap_{n \in \omega} A_n \subset B$, f is a continuous function on B to a Hausdorff space, then*

Presented to the Society, January 29, 1960; received by the editors October 26, 1959.

$$f\left(\bigcap_{n \in \omega} A_n\right) = \bigcap_{n \in \omega} f(B \cap A_n).$$

Proof. Suppose $y \in \bigcap_{n \in \omega} f(B \cap A_n)$. Then, for every $n \in \omega$, $f^{-1}\{y\} \cap B \cap A_n \neq \emptyset$. Since f is continuous on B , we have $f^{-1}\{y\} = B \cap C$ for some closed set C . Thus, for every $n \in \omega$, $C \cap A_n \neq \emptyset$ and hence

$$f^{-1}\{y\} \cap \bigcap_{n \in \omega} A_n = \bigcap_{n \in \omega} (C \cap A_n) \neq \emptyset$$

i.e., $y \in f(\bigcap_{n \in \omega} A_n)$.

The converse is obvious.

3.2. Suppose, for every $i \in I$, $A_i \subset X \times Y$ and, for every $x \in X$, the family $\bigcup_{i \in I} \{\pi^{-1}\{x\} \cap A_i\}$ forms a nest of compact sets in the product topology. Then

$$\bigcap_{i \in I} \pi A_i = \pi \bigcap_{i \in I} A_i.$$

Proof. Let $x \in \bigcap_{i \in I} \pi A_i$. Then, for every $i \in I$, $\pi^{-1}\{x\} \cap A_i \neq \emptyset$ and since the intersection of a nest of nonvoid compact sets is nonvoid, we have

$$\bigcap_{i \in I} (\pi^{-1}\{x\} \cap A_i) \neq \emptyset,$$

i.e.,

$$x \in \pi \bigcap_{i \in I} A_i.$$

The converse is obvious.

4. Uniformization of sets.

4.1. THEOREM. Suppose X is any space; H is any family of subsets of X ; Y satisfies condition (I); $Q \subset X \times Y$; for every $x \in X$, $\pi^{-1}\{x\} \cap Q$ is compact; for every closed set B in Y , $\pi((X \times B) \cap Q) \in H$. Then there exists a function f on πQ such that $f \subset Q$ and, for any open set V in Y , $f^{-1}(V)$ is Borel H .

Proof. Let Ω be the first noncountable ordinal and $\bigcup_{i \in \Omega} \{U_i\}$ be a base for Y . Let, for $\alpha \subset X \times Y$ and $\beta \subset X \times Y$,

$$P(\alpha, \beta) = (\alpha \cap \beta) \cup [((X - \pi(\alpha \cap \beta)) \times Y) \cap \beta].$$

Next, by recursion for every $i \in \Omega$, we define A_i as follows:

$$\begin{aligned} A_0 &= Q, \\ A_{i+1} &= P(X \times \overline{U_i}, A_i), \\ A_i &= \bigcap_{j < i} A_j \quad \text{if } i \text{ is a limit ordinal,} \end{aligned}$$

where $\overline{U_i}$ denotes the closure of U_i . Finally, we set

$$f = \bigcap_{i \in \Omega} A_i$$

and check the following points.

- (i) $P(\alpha, \beta) \subset \beta$ and $\pi P(\alpha, \beta) = \pi\beta$.
- (ii) If $i \in \Omega$, $j \in \Omega$, $j < i$, then $A_i \subset A_j$.
- (iii) If $i \in \Omega$ and $x \in X$, then $\pi^{-1}\{x\} \cap A_i$ is compact.
- (iv) If B' is closed in Y , $B = X \times B'$ and $I \subset \Omega$, then

$$\pi\left(B \cap \bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} \pi(B \cap A_i)$$

Proof. Apply 3.2.

- (v) If $i \in \Omega$ and $i < j \in \Omega$ then

$$\pi((X \times \overline{U}_i) \cap A_i) = \pi((X \times \overline{U}_i) \cap A_j) = \pi((X \times \overline{U}_i) \cap f)$$

Proof. For the first equality, we use induction on j . Let

$$B = \pi((X \times \overline{U}_i) \cap A_i)$$

then

$$(B \times Y) \cap A_{i+1} \subset B \times \overline{U}_i.$$

Hence, in view of (ii), if $i < j \in \Omega$, we have

$$(B \times Y) \cap A_j \subset (B \times \overline{U}_i) \cap A_j$$

and

$$(B \times \overline{U}_j) \cap A_j \subset (B \times \overline{U}_i \cap \overline{U}_j) \cap A_j.$$

Now, if the first equality holds for a given such j , then

$$\begin{aligned} \pi((X \times \overline{U}_i) \cap A_{j+1}) &= \pi((X \times (\overline{U}_i \cap \overline{U}_j)) \cap A_j) \cup (\pi((X \times \overline{U}_i) \cap A_j) - \pi((X \times \overline{U}_j) \cap A_j)) \\ &= \pi((B \times (\overline{U}_i \cap \overline{U}_j)) \cap A_j) \cup (\pi((B \times \overline{U}_i) \cap A_j) \\ &\quad - \pi((B \times (\overline{U}_i \cap \overline{U}_j)) \cap A_j)) \\ &= \pi((B \times \overline{U}_i) \cap A_j) \\ &= B \end{aligned}$$

so that the equality holds for $j+1$. If j is a limit ordinal and the equality holds for all j' such that $i < j' < j$, then by (iv) it also holds for j . The second equality follows from the first and (iv).

- (vi) $\pi f = \pi Q$.

Proof. Apply (i), (iv) and induction or take $U_0 = Y$ and apply (v).

- (vii) f is a function.

Proof. Suppose $(x, y) \in f$ and $(x, y') \in f$ and $y \neq y'$. Let $y \in U_i$, $y' \in U_j$, $\overline{U}_i \cap \overline{U}_j = \emptyset$, $i < j$. Then by (v) we have

$$x \in \pi((X \times \overline{U}_i) \cap A_i) \quad \text{and} \quad x \in \pi((X \times \overline{U}_j) \cap A_j),$$

but

$$A_j \subset A_{i+1} = P(X \times \overline{U}_i, A_i) \subset ((X \times \overline{U}_i) \cap A_i) \cup ((X - \{x\}) \times Y).$$

Hence

$$(X \times \overline{U}_j) \cap A_j \subset (X - \{x\}) \times Y,$$

i.e.

$$x \notin \pi((X \times \overline{U}_j) \cap A_j)$$

in contradiction to the above.

(viii) If $i \in \Omega$ and B' is closed in Y and $B = X \times B'$ then $\pi(B \cap A_i)$ is Borel H .

Proof. We use induction on i . It is clearly true for $i=0$. Suppose it is true for j and $i=j+1$. Then:

$$\begin{aligned} \pi(B \cap A_i) &= \pi(B \cap P(X \times \overline{U}_j, A_j)) \\ &= \pi(B \cap (X \times \overline{U}_j) \cap A_j) \\ &\quad \cup \pi(B \cap ((X - \pi((X \times \overline{U}_j) \cap A_j)) \times Y) \cap A_j) \\ &= \pi(B \cap (X \times \overline{U}_j) \cap A_j) \cup (\pi(B \cap A_j) - \pi((X \times \overline{U}_j) \cap A_j)) \end{aligned}$$

so that $\pi(B \cap A_i)$ is Borel H .

If i is a limit ordinal and the result is true for all $j < i$, by (iv), we have

$$\pi(B \cap A_i) = \pi\left(\bigcap_{j < i} (B \cap A_j)\right) = \bigcap_{j < i} \pi(B \cap A_j)$$

and the last set is Borel H since the set of all $j < i$ is countable.

(ix) If V is open in Y , then $f^{-1}(V)$ is Borel H .

Proof. Let

$$I = \{i \in \Omega: \overline{U}_i \subset V\}.$$

Then there exists a countable $I' \subset I$ such that

$$\bigcup_{i \in I'} U_i = \bigcup_{i \in I} U_i = V.$$

Hence

$$\bigcup_{i \in I'} \overline{U}_i = V.$$

Then

$$\begin{aligned}
 f^{-1}(V) &= \pi((X \times V) \cap f) = \bigcup_{i \in I'} \pi((X \times \bar{U}_i) \cap f) \\
 &= (by(v)) = \bigcup_{i \in I'} \pi((X \times \bar{U}_i) \cap A_i)
 \end{aligned}$$

so that $f^{-1}(V)$ is Borel H , in view of (viii).

4.2. COROLLARY. Suppose X is any topological space; Y satisfies condition (I); Q is compact in $X \times Y$. Then there exists a function f on πQ such that $f \subset Q$ and, for every open set V in Y , $f^{-1}(V)$ is Borel $K(X)$. (If in addition, Q is metrizable then f can be chosen to be of Baire first class.)

4.3. THEOREM. Suppose X and Y are Hausdorff and Q is analytic in $X \times Y$. Then there exists Q' such that: $Q' \subset Q$; $\pi Q' = \pi Q$; for every $x \in X$, $\pi^{-1}\{x\} \cap Q'$ is compact; for every closed set B in $X \times Y$, $\pi(B \cap Q')$ is Borel $\mathfrak{A}(X)$.

Proof. Since Q is analytic in $X \times Y$, there exists a Hausdorff space Z such that Q is the continuous image of a $K_{\sigma\delta}(Z)$. For every $n \in \omega$ and $i \in \omega$, let $d(n, i)$ be compact in Z ,

$$D = \bigcap_{n \in \omega} \bigcup_{i \in \omega} d(n, i),$$

f be a continuous function on D , and $Q = f(D)$. Next, let S_n be the set of all $(n+1)$ -tuples $k = (k_0, \dots, k_n)$ with $k_i \in \omega$. For every $n \in \omega$ and $k \in S_n$, let

$$\Delta(k) = f\left(D \cap \bigcap_{i \in (n+1)} d(i, k_i)\right),$$

and then define $A(k)$ by recursion on n as follows: if $k \in S_0$, then

$$A(k) = \Delta(k) \cap \left(\left(X - \bigcup_{j \in k} \pi \Delta(j) \right) \times Y \right).$$

If $k \in S_{n+1}$, then

$$\begin{aligned}
 A(k) &= A(k_0, \dots, k_n) \cap \Delta(k) \\
 &\cap \left(\left(X - \bigcup_{j \in k_{n+1}} \pi(A(k_0, \dots, k_n) \cap \Delta(k_0, \dots, k_n, j)) \right) \times Y \right).
 \end{aligned}$$

Finally, we set

$$\begin{aligned}
 A'_n &= \bigcup_{k \in S_n} A(k), \\
 Q' &= \bigcap_{n \in \omega} A'_n,
 \end{aligned}$$

and check the following points.

(i) If $n \in \omega$, $k \in S_n$, $k' \in S_n$, $k \neq k'$, then

$$\pi A(k) \cap \pi A(k') = 0.$$

(ii) For every $n \in \omega$ and $k \in S_n$,

$$A(k) \subset \Delta(k) = \bigcup_{j \in \omega} \Delta(k_0, \dots, k_n, j),$$

and

$$Q = \bigcup_{j \in \omega} \Delta(j).$$

(iii) For every $n \in \omega$, $\pi A'_n = \pi Q$.

Proof. We use induction on n . Let $x \in \pi Q$. Then by (ii)

$$x \in \bigcup_{j \in \omega} \pi \Delta(j).$$

Let j_0 be the first integer such that $x \in \pi \Delta(j_0)$. Then,

$$x \in \pi A(j_0) \subset \pi A'_0.$$

Next, suppose for some $k \in S_n$, $x \in \pi A(k)$. Then by (ii)

$$x \in \bigcup_{j \in \omega} \pi (A(k) \cap \Delta(k_0, \dots, k_n, j)).$$

Let j_0 be the first integer such that

$$x \in \pi (A(k) \cap \Delta(k_0, \dots, k_n, j_0)).$$

Then

$$x \in \pi A(k_0, \dots, k_n, j_0) \subset \pi A'_{n+1}.$$

Thus, for every $n \in \omega$, $\pi Q \subset \pi A'_n$. The converse is immediate.

(iv) For every $x \in X$ there exists a unique sequence k such that for every $n \in \omega$:

$$\pi^{-1}\{x\} \cap A'_n = \pi^{-1}\{x\} \cap A(k_0, \dots, k_n) = \pi^{-1}\{x\} \cap \Delta(k_0, \dots, k_n).$$

Proof. Use induction on n and (i).

(v) If B is closed in $X \times Y$ then

$$\pi(B \cap Q') = \bigcap_{n \in \omega} \pi(B \cap A'_n).$$

Proof. Suppose

$$x \in \bigcap_{n \in \omega} \pi(B \cap A'_n).$$

Let k be the sequence given by (iv). Then for every $n \in \omega$:

$$0 \neq \pi^{-1}\{x\} \cap B \cap A'_n = \pi^{-1}\{x\} \cap B \cap \Delta(k_0, \dots, k_n).$$

Since f is continuous on D and $\pi^{-1}\{x\} \cap B$ is closed, there exists a closed set C in Z such that

$$f^{-1}(\pi^{-1}\{x\} \cap B) = D \cap C.$$

Recalling the definition of $\Delta(k_0, \dots, k_n)$ we conclude

$$D \cap C \cap \bigcap_{i \in (n+1)} d(i, k_i) \neq 0.$$

Since the $d(i, k_i)$ are compact, we have

$$\bigcap_{n \in \omega} \left(C \cap \bigcap_{i \in (n+1)} d(i, k_i) \right) \neq 0.$$

Therefore

$$\pi^{-1}\{x\} \cap B \cap f \left(\bigcap_{n \in \omega} \bigcap_{i \in (n+1)} d(i, k_i) \right) \neq 0$$

and, by 3.1,

$$\pi^{-1}\{x\} \cap B \cap \bigcap_{n \in \omega} f \left(D \cap \bigcap_{i \in (n+1)} d(i, k_i) \right) \neq 0$$

i.e.,

$$\bigcap_{n \in \omega} (\pi^{-1}\{x\} \cap B \cap \Delta(k_0, \dots, k_n)) \neq 0$$

Hence, by (iv)

$$\bigcap_{n \in \omega} (\pi^{-1}\{x\} \cap B \cap A'_n) \neq 0$$

i.e.,

$$x \in \pi \bigcap_{n \in \omega} (B \cap A'_n) = \pi(B \cap Q').$$

Thus,

$$\bigcap_{n \in \omega} \pi(B \cap A'_n) \subset \pi(B \cap Q').$$

The converse is immediate.

(vi) If B is either closed or analytic in $X \times Y$ then, for every $n \in \omega$ and $k \in S_n$, $\pi(B \cap A(k))$ is Borel $\mathfrak{A}(X)$.

Proof. We use induction on n . We first observe that the intersection of two analytic sets, the intersection of an analytic set with a closed set, and the continuous image of an analytic set are all analytic. Thus, for every $n \in \omega$ and $k \in S_n$, $\pi(B \cap \Delta(k))$ is analytic in X . Now, if $k \in S_0$, since

$$\pi(B \cap A(k)) = \pi(B \cap \Delta(k)) - \bigcup_{j \in k} \pi \Delta(j)$$

we see that $\pi(B \cap A(k))$ is the difference of two analytic sets in X . Next, suppose the result holds for a given $n \in \omega$ and let $k \in S_{n+1}$. Then

$$\pi(B \cap \Delta(k_0, \dots, k_n, j) \cap A(k_0, \dots, k_n))$$

is Borel $\mathfrak{A}(X)$ for every $j \in \omega$, and since

$$\begin{aligned} \pi(B \cap A(k)) &= \pi(B \cap \Delta(k) \cap A(k_0, \dots, k_n)) \\ &= \bigcap_{j \in k_{n+1}} \pi(A(k_0, \dots, k_n) \cap \Delta(k_0, \dots, k_n, j)), \end{aligned}$$

we see that $\pi(B \cap A(k))$ is the difference of two Borel $\mathfrak{A}(X)$ sets.

(vii) For every closed set B in $X \times Y$, $\pi(B \cap Q')$ is Borel $\mathfrak{A}(X)$.

Proof. By (v) we have:

$$\pi(B \cap Q') = \bigcap_{n \in \omega} \pi(B \cap A'_n) = \bigcap_{n \in \omega} \bigcup_{k \in S_n} \pi(B \cap A(k)).$$

Since S_n is countable, the result follows from (vi).

(viii) For every $x \in X$, $\pi^{-1}\{x\} \cap Q'$ is compact.

Proof. Let k be the sequence given by (iv). Then

$$\begin{aligned} \pi^{-1}\{x\} \cap Q' &= \bigcap_{n \in \omega} (\pi^{-1}\{x\} \cap A'_n) = \bigcap_{n \in \omega} (\pi^{-1}\{x\} \cap \Delta(k_0, \dots, k_n)) \\ &= \pi^{-1}\{x\} \cap \bigcap_{n \in \omega} f\left(D \cap \bigcap_{i \in (n+1)} d(i, k_i)\right) = (\text{by 3.1}) \\ &= \pi^{-1}\{x\} \cap f\left(\bigcap_{i \in \omega} d(i, k_i)\right) \end{aligned}$$

so that $\pi^{-1}\{x\} \cap Q'$ is the intersection of a closed set with a compact set.

4.4. COROLLARY. Suppose X is Hausdorff, Y satisfies condition (I), and Q is analytic in $X \times Y$. Then there exists a function f on πQ such that $f \subset Q$ and, for every open set V in Y , $f^{-1}(V)$ is Borel $\mathfrak{A}(X)$.

4.5. COROLLARY. Suppose X is Hausdorff, Y satisfies condition (I), and Q is analytic in $X \times Y$. Then there exists a function f on πQ such that $f \subset Q$ and, for every open set V in Y , $f^{-1}(V)$ is μ -measurable for all Carathéodory outer measures μ on X such that closed sets are μ -measurable.

Proof. In this case (see [7]), the analytic sets in X can be obtained by applying Souslin's operation \mathfrak{A} to the family of closed sets in X . Hence they are μ -measurable whenever closed sets are μ -measurable. (See e.g., S. Saks, *Theory of the integral*, p. 50.)

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