

ON BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF ORDINARY, NONLINEAR, SECOND ORDER DIFFERENTIAL EQUATIONS⁽¹⁾

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This paper treats various problems connected with systems of differential equations of the form

$$(1) \quad x'' = f(t, x, x')$$

for a vector x . The first part (§§1-5) deals with a priori bounds for $|x'|$ for a solution $x=x(t)$. The next part (§§8-9) gives existence theorems for non-singular boundary value problems

$$(2) \quad x(0) = x_0 \quad \text{and} \quad x(T) = x_T.$$

§§10-11 give existence theorems for solutions of singular boundary value problems, that is, solutions which exist for all $t \geq 0$ and satisfy

$$(3) \quad x(0) = x_0 \quad \text{and} \quad |x| \leq R \quad \text{for} \quad t \geq 0 \quad \text{or} \quad x \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

In §§12-13, there are obtained uniqueness and continuity theorems for the solutions satisfying (2) or (3).

The results are applied in §§14-15 to obtain existence theorems for periodic and almost periodic solutions. This application was suggested by a lecture of G. Seifert.

Finally, §§16-17 deal with existence of solutions of

$$(4) \quad x'' = X(t, x, x', z), \quad z' = Z(t, x, x', z)$$

satisfying

$$(5) \quad x(0) = x_0, \quad x(T) = x_T \quad \text{and} \quad z(0) = z_0,$$

where x and z are vectors (not necessarily of the same dimension).

1. Below, if x, f are vectors, $|x|$ denotes the Euclidean length of x and $x \cdot f$ the scalar product of x and f .

In §§1-5, there will be obtained a priori bounds for the first derivatives of n -vector functions $x(t)$ subject to second order differential inequalities. In this direction, our results are new only for $n > 1$. For reference and comparison, the following is stated for the case $n = 1$.

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LEMMA 1 (NAGUMO [8]). Let $\phi(s)$, where $0 \leq s < \infty$, be a positive continuous function satisfying

$$(1.1) \quad \int^{\infty} s ds / \phi(s) = \infty.$$

Let $R > 0$ and $T \geq S > 0$. Then there exists a constant M , depending only on ϕ and R, S , with the following property: If $x = x(t)$ is a real-valued function of class C^2 for $0 \leq t \leq T$ satisfying

$$(1.2) \quad |x| \leq R, \quad |x''| \leq \phi(|x'|),$$

then $|x'| \leq M$ for $0 \leq t \leq T$.

For example, M can be chosen to be the solution of the equation

$$(1.3) \quad \int_{2R/S}^M s ds / \phi(s) = 2R.$$

(Actually, the condition (1.1) on ϕ can be relaxed to the assumption that equation (1.3) have a solution M .)

Functions $\phi(s)$ satisfying the conditions of Lemma 1 will be called *Nagumo functions*. (For example, $\phi(s) = \gamma s^2 + C > 0$, where γ, C are constants, is a Nagumo function.)

2. An example of Heinz [5] given in connection with partial differential inequalities shows that Lemma 1 is false if $x = x(t)$ is a vector-valued function. His example is the binary vector $x(t) = (\cos pt, \sin pt)$ which satisfies $|x| = 1$, $|x'| = |p|$ and $|x''| = |x'|^2$. Thus (1.2) holds with $R = 1$ and $\phi(s) = s^2 + 1$, but there is no a priori bound for arbitrary p .

Heinz's arguments suggest the consideration of auxiliary inequalities, different in form from (1.2), say

$$(2.1) \quad |x| \leq R, \quad |x''| \leq \rho'',$$

where $\rho(t)$ is a (scalar) function of class C^2 on $0 \leq t \leq T$.

The desired a priori bounds for $|x'|$, in case of a vector x , are given by the following lemma and its consequences.

LEMMA 2. Let $\phi(s)$, $0 \leq s < \infty$, be a Nagumo function. Let $\rho = \rho(t)$ be a (scalar) function of class C^2 and $0 \leq \rho(t) \leq K_1$ on $0 \leq t \leq T$. Let $R > 0$, $T \geq S > 0$. Then there exists a constant M , depending only on $\phi(s)$, K_1 , R and S , with the following property: If $x = x(t)$ is a (vector-valued) function of class C^2 on $0 \leq t \leq T$ satisfying (1.2) and (2.1), then $|x'| \leq M$ for $0 \leq t \leq T$.

Heinz's example above shows that condition (2.1) in $x(t)$ cannot be omitted. It will be seen below that condition (1.2) cannot be omitted either. If, however, (1.2) is omitted (and (2.1) retained), one obtains a priori bounds for $|x'|$ on all subintervals $\mu \leq t \leq T - \mu$ of $0 \leq t \leq T$, where $0 < \mu < T$; cf. §3.

REMARK. It will be shown (§4) that condition (1.2) on $x(t)$ can be omitted in Lemma 2 if $\rho(t)$ satisfies

$$(2.2) \quad |\rho'| \leq \theta |x'| + C_1$$

for some constants θ, C_1 with $0 < \theta < 1$.

If one chooses ϕ, ρ to be $\gamma s^2 + C, \alpha |x|^2 + K$, respectively, then Lemma 2 implies the following:

LEMMA 3. *Let $\alpha, \gamma, R, S, T, C, K$ be non-negative constants and $T \geq S > 0$. Then there exists a constant $M = M(\alpha, \gamma, R, S, C, K)$ with the following property: If $x = x(t)$ is of class C^2 on $0 \leq t \leq T$ satisfying*

$$(2.3) \quad |x| \leq R, \quad |x''| \leq \gamma |x'|^2 + C,$$

$$(2.4) \quad |x| \leq R, \quad |x''| \leq \alpha r'' + K, \quad \text{where } r = |x|^2,$$

then $|x'| \leq M$ on $0 \leq t \leq T$.

Heinz's example of the binary vector $x = (\cos pt, \sin pt)$ shows that condition (2.4) cannot be omitted. It is easy to give an example of a family of (scalar) functions $x(t)$ satisfying inequalities of the form (2.4) but not of the form (2.3) and for which there is no a priori bound for $|x'|$. To this end, let $\epsilon, p > 0$ and let $x(t) = x(t; p, \epsilon)$ be the scalar function which is $1 + \epsilon p^4(t - 1/p)^4$ or 1 according as $0 \leq t \leq 1/p$ or $t > 1/p$. Then x' is $4\epsilon p^4(t - 1/p)^3$ or 0 and x'' is $12\epsilon p^4(t - 1/p)^2$ or 0 according as $0 \leq t \leq 1/p$ or $t > 1/p$. Since $1 \leq x \leq 1 + \epsilon$ and $x'' \geq 0$, it is clear that (2.4) holds with $R = 1 + \epsilon, \alpha = 1/2$ and $K = 0$. As $x'(0) = -4\epsilon p$, there is no a priori bound for all $p > 0$ (and $\epsilon > 0$ fixed).

Note that if $\gamma R < 1$, then (2.3) implies (2.4) with

$$(2.5) \quad \alpha = \gamma/2(1 - \gamma R) \quad \text{and} \quad K = C/(1 - \gamma R).$$

For since

$$(2.6) \quad r'' = 2(x \cdot x'' + |x'|^2),$$

(2.3) shows that $r'' \geq 2(1 - \gamma R)|x'|^2 - 2CR$ and another application of (2.3) gives $r'' \geq 2(1 - \gamma R)(|x''| - C)/\gamma - 2CR$. This inequality is equivalent to (2.4)–(2.5). Conversely, if $2R\alpha < 1$, then (2.4) implies (2.3) with

$$(2.7) \quad \gamma = 2\alpha/(1 - 2R\alpha) \quad \text{and} \quad C = K/(1 - 2R\alpha).$$

It can also be remarked that if (2.3) holds and, in addition,

$$(2.8) \quad x \cdot x'' \geq 0,$$

then (2.4) holds with $\alpha = \gamma/2$ and $K = C$.

In view of the remark concerning (2.5), Lemma 3 has the following consequence:

COROLLARY 1. *Let γ, R, S, T, C be non-negative constants subject to $\gamma R < 1$*

and $T \geq S > 0$. Then the analogue of Lemma 3 holds with an $M = M(\gamma, R, S, C)$ if condition (2.4) on $x(t)$ is omitted.

In view of Heinz's example $x = (\cos pt, \sin pt)$, $\gamma R < 1$ cannot be relaxed to $\gamma R \leq 1$ in this assertion. (Heinz's results on partial differential inequalities involve the condition $\gamma R < 1/2$.)

The remark concerning (2.7) and Lemma 3 (or the remark concerning (2.2) and Lemma 2) imply

COROLLARY 2. Let α, R, S, T, K be non-negative constants subject to $2R\alpha < 1$ and $T \geq S > 0$. Then the analogue of Lemma 3 holds with an $M = M(\alpha, R, S, K)$ if condition (2.3) on $x(t)$ is omitted.

For the family of functions $x = x(t; p, \epsilon)$, mentioned after Lemma 3, $2R\alpha = 1 + \epsilon$. This shows that 1 in the inequality $2R\alpha < 1$ cannot be replaced by a larger constant.

3. A priori bound on $[\mu, T - \mu]$. It will first be shown that the inequality (2.1) implies an a priori bound for $|x'|$ on $\mu \leq t \leq T - \mu$, where $0 < \mu < T$.

Let $0 < \mu < T$ and $0 \leq t \leq T - \mu$. The relations

$$(3.1) \quad x(t + \mu) - x(t) - \mu x'(t) = \int_t^{t+\mu} (t + \mu - s)x''(s)ds,$$

$t + \mu - s \geq 0$, and (2.1) imply that

$$\mu |x'(t)| \leq 2R + \rho(t + \mu) - \rho(t) - \mu \rho'(t).$$

Hence

$$(3.2) \quad |x'(t)| \leq (2R + K_1)/\mu - \rho'(t) \quad \text{for } 0 \leq t \leq T - \mu,$$

where $0 \leq \rho \leq K_1$ for $|x| \leq R$.

Replacing (3.1), for $\mu \leq t \leq T$, by

$$x(t) - x(t - \mu) - \mu x'(t) = - \int_{t-\mu}^t (t - \mu - s)x''(s)ds$$

leads to

$$(3.3) \quad |x'(t)| \leq (2R + K_1)/\mu + \rho'(t) \quad \text{for } \mu \leq t \leq T.$$

Adding (3.2) and (3.3) gives

$$(3.4) \quad |x'(t)| \leq (2R + K_1)/\mu \quad \text{for } \mu \leq t \leq T - \mu.$$

4. On (2.2). If, in addition to (2.1), the inequality (2.2) holds, then the choice $\mu = T/2$ in (3.2) and (3.3) shows that, for $0 \leq t \leq T$,

$$(4.1) \quad |x'(t)| \leq (M_1 + C_1)/(1 - \theta), \quad \text{where } M_1 = 2(2R + K_1)/T.$$

Let M denote the value of $(M_1 + C_1)/(1 - \theta)$, when $T = S$. Since $x(t)$ is given

on an interval of length $T \geq S$, it follows by applying the inequality $|x'| \leq M$ on every subinterval of length S , that M is the desired a priori bound.

5. Proof of Lemma 2. Let $\mu = T/2$ in (3.2) and (3.3) and let $M_1 = M_1(T)$ be the constant defined in (4.1). Then (1.2) and (3.2)–(3.3) imply that

$$(5.1) \quad |x' \cdot x''| / \phi(|x'|) \leq |x'| \leq M_1 \pm \rho',$$

where $\pm$ is required according as $t \geq T/2$ or $t \leq T/2$.

Defined $\Phi(s)$ by

$$(5.2) \quad \Phi(s) = \int_0^s u du / \phi(u).$$

Then

$$(5.3) \quad |\Phi(|x'(t)|) - \Phi(|x'(T/2)|)| = \left| \int x' \cdot x'' dt / \phi(|x'|) \right|,$$

where the integral is taken over the t -interval with endpoints t and $T/2$. It follows, therefore, from (5.1) that the expression on the left of (5.3) is majorized by $2^{-1}M_1T + K_1$. Since $|x'(T/2)| \leq M_1$ by the case $\mu = T/2$ of (3.4), it follows from (5.3) that

$$(5.4) \quad |x'(t)| \leq M_2 \quad \text{on } 0 \leq t \leq T,$$

where $M_2 \equiv M_2(T)$ is defined by

$$M_2 = \Phi^{-1} \left(\frac{1}{2} M_1 T + K_1 + \Phi(M_1) \right),$$

in terms of the inverse function Φ^{-1} of the increasing function Φ . Clearly $M = M_2(S)$ is the desired a priori bound.

6. Below there will also be needed the following:

LEMMA 4 (SCORZA-DRAGONI [10]). *Let $g(t, x, x')$ be a continuous and bounded (vector-valued) function for $0 \leq t \leq T$ and arbitrary (x, x') . Then, for arbitrary x_0 and x_T , the system of differential equations*

$$(6.1) \quad x'' = g(t, x, x')$$

has at least one solution $x = x(t)$ satisfying

$$(6.2) \quad x(0) = x_0 \quad \text{and} \quad x(T) = x_T.$$

It has been pointed out by Bass [2] that this lemma is easily derived from Schauder's fixed point theorem if one considers (6.1) as an inhomogeneous form of the linear homogeneous equation $x'' = 0$.

7. In the remainder of the paper, the function $\rho(t)$ in Lemma 2 will be taken to be $\rho(t) = \alpha|x|^2 + K$; $\phi(s)$ will be a Nagumo function that is, a function ϕ satisfying the conditions of Lemma 1.

In order to be able to apply Lemma 4 below, it will be convenient to have the following remark: Let $f(t, x, x')$ be a continuous (vector-valued) function on a set

$$(7.1) \quad D(R, T): \quad 0 \leq t \leq T, \quad |x| \leq R, \quad x' \text{ arbitrary},$$

and let f have one or more of the following properties:

$$(7.2) \quad x \cdot f + |x'|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad |x| > 0,$$

$$(7.3) \quad x \cdot f + |x'|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad |x| = R,$$

$$(7.4) \quad |f| \leq 2\alpha(x \cdot f + |x'|^2) + K,$$

$$(7.5) \quad |f| \leq \phi(|x'|).$$

Let $M > 0$. Then there exists a continuous, bounded function $g(t, x, x')$ defined for $0 \leq t \leq T$ and arbitrary (x, x') with the corresponding set of properties among the following:

$$(7.2') \quad x \cdot g + |x'|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad |x| > 0,$$

$$(7.3') \quad x \cdot g + |x'|^2 > 0 \quad \text{when} \quad x \cdot x' = 0 \quad \text{and} \quad |x| \geq R,$$

$$(7.4') \quad |g| \leq 2\alpha(x \cdot g + |x'|^2) + K,$$

$$(7.5') \quad |g| \leq \phi(|x'|),$$

and, at the same time, satisfying

$$(7.6) \quad g(t, x, x') = f(t, x, x') \quad \text{for} \quad 0 \leq t \leq T, \quad |x| \leq R, \quad |x'| \leq M.$$

In fact, one obtains such a g as follows: Let $\delta(s)$, where $0 \leq s < \infty$, be a scalar continuous function which satisfies

$$\delta = 1, \quad 0 < \delta < 1, \quad \delta = 0 \quad \text{according as} \quad s \leq M, \quad M < s < 2M, \quad s \geq 2M.$$

Put

$$g(t, x, x') = \delta(|x'|)f(t, x, x') \quad \text{on} \quad D(T, R),$$

$$g(t, x, x') = (R/|x|)g(t, Rx/|x|, x') \quad \text{for} \quad |x| > R.$$

On $D(T, R)$, the relation

$$x \cdot g + |x'|^2 = \delta(|x'|)(x \cdot f + |x'|^2) + (1 - \delta(|x'|))|x'|^2$$

makes it clear that g has the desired properties on $D(T, R)$. Furthermore, the validity of the properties for $|x| = R$ implies their validity for $|x| > R$.

Note that (7.5), (7.4), respectively, imply that a solution $x = x(t)$ of $x'' = f(t, x, x')$ satisfies (1.2), (2.1) with $\rho = \alpha|x|^2 + K$.

8. The next desired result in the following theorem dealing with the existence of solutions of nonlinear, nonsingular, boundary value problems (under conditions more general than those in Lemma 4) for a system

$$(8.1) \quad x'' = f(t, x, x').$$

THEOREM 1. Let $f(t, x, x')$ be a continuous function on $D(T, R)$ in (7.1) satisfying

$$(8.2) \quad x \cdot f + |x'|^2 \geq 0 \quad \text{if} \quad x \cdot x' = 0 \quad \text{and} \quad |x| = R.$$

In addition, let f satisfy (7.4) and (7.5), where α, K are non-negative constants and $\phi(s)$ is a Nagumo function. Let $|x_0|, |x_T| \leq R$. Then the system (8.1) has at least one solution $x = x(t)$ satisfying $x(0) = x_0, x(T) = x_T$.

In the case x is scalar, condition (7.4) can be omitted; [8].

In Theorem 1 and the assertions below, (7.5) can be omitted if $2R\alpha < 1$. Also, (7.4) can be omitted, if (7.5) is replaced by

$$(8.3) \quad |f| \leq \gamma |x'|^2 + C,$$

where γ, C are non-negative constants and $\gamma R < 1$. Cf. the remarks concerning (2.5) and (2.7) above.

Proof of Theorem 1. The proof will be given first for the case that f satisfies (7.3) instead of (8.2).

Let M be the constant (with $T = S$) occurring in Lemma 2 (where $\rho = \alpha |x|^2 + K$). Let $g(t, x, x')$ be a continuous bounded function for $0 \leq t \leq T$ and arbitrary (x, x') satisfying (7.3'), (7.6) and, correspondingly, (7.4') (7.5'). By Lemma 4, (6.1) has a solution $x = x(t)$ satisfying the boundary conditions (6.2).

Condition (7.3') means that $r = |x(t)|^2$ satisfies $r'' > 0$ if $r' = 0$ and $r \geq R^2$; cf. (2.6). Hence $r(t)$ cannot have a maximum value $\geq R^2$ in the interval $0 < t < T$. Since $r(0) = |x_0|^2, r(T) = |x_T|^2$ satisfy $r(0), r(T) \leq R^2$, it follows that $r(t) \leq R^2$ (that is, $|x| \leq R$) on $0 \leq t \leq T$. In view of (7.4')–(7.5'), Lemma 2 is applicable to $x(t)$. Hence, $|x'(t)| \leq M$ for $0 \leq t \leq T$.

By virtue of (7.6), it follows that $x = x(t)$ is a solution of (8.1). Hence Theorem 1 is proved provided that (7.3), rather than (8.1), is assumed.

In order to remove this proviso, note that if $\epsilon > 0$, the function $f(t, x, x') + \epsilon x$ satisfies the conditions of Theorem 1 as well as condition (7.3). It is only necessary to replace K, ϕ in (7.4), (7.5) by $K + \epsilon R, \phi + \epsilon R$, respectively. Hence, by what has been proved,

$$x'' = f(t, x, x') + \epsilon x$$

has a solution $x = x_\epsilon(t)$ satisfying the boundary conditions. It is clear that $|x_\epsilon(t)| \leq R$ and that, for a suitable N independent of ϵ (< 1), $|x'_\epsilon(t)| \leq N$ for $0 \leq t \leq T$. Ascoli's selection theorem shows that there exists a sequence $\epsilon_1, \epsilon_2, \dots$ such that $0 < \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and that $x(t) = \lim x_\epsilon(t)$, as $\epsilon = \epsilon_n \rightarrow 0$, exists and is a solution of (8.1) and (6.2). This completes the proof of Theorem 1.

REMARK. In Theorem 1, let (8.2) be strengthened to

$$(8.4) \quad x \cdot f + |x'|^2 \geq 0 \quad \text{if } x \cdot x' = 0,$$

and let

$$(8.5) \quad x_T = 0.$$

Then, for the solution $x = x(t)$ just obtained, $r = |x(t)|^2$ satisfies

$$(8.6) \quad r \geq 0, \quad r' \leq 0.$$

For if (8.4) is first replaced by (7.2), it is seen that $r(t)$ has no maximum on $0 < t < T$. Hence $r(t) \leq \max(r(0), r(T)) = r(0)$. Since $r(T) = 0$, the same argument applies if $0 < t < T$ is replaced by any subinterval $t_0 < t < T$. This gives (8.6) if (7.2) holds. If (8.4) holds, the proof of Theorem 1 shows that $r = |x_\epsilon(t)|^2$ satisfies (8.6). But these inequalities are not lost during the limit process $\epsilon = \epsilon_n \rightarrow 0$.

9. In this section, there will be proved a theorem analogous to Theorem 1, but the assumption (8.2) will be replaced by conditions on the magnitude of $|x_0|$, $|x_T|$ and T .

THEOREM 2. *Let f satisfy the conditions of Theorem 1 except that (8.2) need not hold. Let x_0 , x_T , R and T be such that*

$$(9.1) \quad \beta = \max(|x_0|, |x_T|)$$

satisfies

$$(9.2) \quad \alpha\beta^2 + \beta + KT^2/8 \leq R.$$

Then (8.1) has at least one solution $x = x(t)$ satisfying $x(0) = x_0$, $x(T) = x_T$.

One can obtain the following assertion:

COROLLARY. *Let f be defined and continuous on $D(T, R)$ and satisfy (8.3) for some non-negative constants γ , C such that $\gamma R < 1$. Let β in (9.1) and T satisfy*

$$(9.3) \quad \gamma\beta^2 + 2(1 - \gamma R)\beta + CT^2/4 \leq 2R(1 - \gamma R).$$

Then (8.1) has at least one solution $x = x(t)$ satisfying $x(0) = x_0$, $x(T) = x_T$.

Theorems 1 and 2 can be considered to be the ordinary (vector) analogue of Nagumo's results [9] for a partial (scalar) differential equation.

Proof of Theorem 2. Let M be the constant supplied by Lemma 2 (with $T = S$ and $\rho = \alpha|x|^2 + K$) and let $g(t, x, x')$ be the function supplied by §7.

By Lemma 4, (6.1) has a solution satisfying (6.2). Let $y = y(t)$ be the linear (vector) function satisfying

$$(9.4) \quad y(0) = x_0 \quad \text{and} \quad y(T) = x_T,$$

so that

$$x(t) = y(t) - \int_0^T G(t, s)x''(s)ds,$$

where $TG(t, s)$ is $(T-t)s$ or $t(T-s)$ according as $0 \leq s \leq t \leq T$ or $0 \leq t \leq s \leq T$. By (6.1) and (7.4'), x satisfies the differential inequality in (2.4). Thus $G \geq 0$ implies

$$(9.5) \quad |x(t)| \leq |y(t)| + \int_0^T G(t, s)(\alpha r''(s) + K)ds.$$

In this inequality, r'' can be replaced by $(r-u)''$, where $u=u(t)$ is the linear function determined by

$$(9.6) \quad u(0) = r(0) = |x_0|^2, \quad u(T) = r(T) = |x_T|^2.$$

Thus, by (9.5),

$$|x(t)| \leq |y(t)| + \alpha(u(t) - r(t)) + 2^{-1}K(T-t)t.$$

Since $|y(t)| \leq \max(|x_0|, |x_T|) = \beta$, $u(t) \leq \max(|x_0|^2, |x_T|^2) = \beta^2$ and $r \geq 0$,

$$(9.7) \quad |x(t)| \leq \beta + \alpha\beta^2 + KT^2/8.$$

By condition (9.3), $|x(t)| \leq R$ for $0 \leq t \leq T$. Also, $|x'(t)| \leq M$ by Lemma 2. It follows from (7.6) that $x=x(t)$ is a solution of (8.1). This proves Theorem 2.

10. In some of the theorems to follow, the bounded interval $0 \leq t \leq T$ is replaced by $0 \leq t < \infty$.

THEOREM 3. *Let $f(t, x, x')$ be defined and continuous on*

$$(10.1) \quad D(R): 0 \leq t < \infty, \quad |x| \leq R < \infty, \quad x' \text{ arbitrary.}$$

For every $T > 0$, let f satisfy the conditions of Theorem 1 on $D(T, R)$, where the constants α , K and Nagumo function $\phi(s)$ which occur can depend on T . Then, for every x_0 in the sphere $|x_0| \leq R$, there is at least one solution $x=x(t)$ of (8.1) which satisfies $x(0)=x_0$ and exists for $t \geq 0$.

REMARK. If, in addition (8.4) is assumed in Theorem 3, then $r=|x(t)|^2$ satisfies (8.6). Also, if

$$(10.2) \quad x \cdot f + |x'|^2 \geq 0,$$

then

$$(10.3) \quad r \geq 0, \quad r' \leq 0, \quad r'' \geq 0.$$

For a scalar equation in which f does not depend on x' , this type of theorem goes back to A. Kneser [6]; cf. [7]. For the scalar analogue of Theorem 3 in which the conditions (7.4), (7.5) on $D(T, R)$ are replaced by a Nagumo condition (7.5) alone, see [3]. For an f linear in x and independent of x' ,

see [12]; for the general linear case, [4]. For a nonlinear system, see [2], where f is subject to a majorant linear in $|x'|$,

$$(10.4) \quad |f| \leq \gamma |x'| + C,$$

on $D(T, R)$. In contrast to (10.4), Theorem 3 implies that it is sufficient to require on each $D(T, R)$ an inequality of the form (8.3) if $\gamma R < 1$. There is, of course, no limitation on R if γ can be chosen arbitrarily small, that is, if

$$f(t, x, x')/|x'|^2 \rightarrow 0 \text{ as } |x'| \rightarrow \infty$$

uniformly for bounded t and $|x| \leq R$.

REMARK. In some of the papers just mentioned, it is assumed that

$$x \cdot f(t, x, x') \geq 0.$$

In this case, the conditions (7.4), (7.5) on f on $D(T, R)$ are satisfied if, for example, (8.3) holds on $D(T, R)$ with *arbitrary* constants $\gamma = \gamma(T) > 0$, $C = C(T) > 0$. (A restriction of the type $\gamma R < 1$ is not needed.)

Proof of Theorem 3. Let $m = 1, 2, \dots$. By Theorem 1, (8.1) has a solution $x = x_m(t)$ on $0 \leq t \leq m$ satisfying $x(0) = x_0$, $x(m) = 0$. Let $m \geq T$. Then, by Lemma 2, there is a constant $M = M_T$ such that $|x'_m(t)| \leq M$ for $0 \leq t \leq T$. Hence the sequences $\{x_m(t)\}$, $\{x'_m(t)\}$, $\{x''_m(t)\}$, where $m \geq T$, are uniformly bounded and equicontinuous on $0 \leq t \leq T$. Theorem 3 follows from Ascoli's selection theorem applied to a sequence of intervals $0 \leq t \leq T$, where $T \rightarrow \infty$.

The assertion concerning (8.6) follows from the Remark at the end of §8 and that concerning (10.3) follows from (2.6).

11. The next theorem gives a sufficient condition for the solutions $x = x(t)$ of (8.1) given by Theorem 3 to satisfy

$$(11.1) \quad x(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

THEOREM 4. Let $f(t, x, x')$ be defined on $D(R)$. For every number m , $0 < m < R$, let there exist a non-negative function $\sigma(t) = \sigma(t, m)$ for large t satisfying

$$(11.2) \quad x \cdot f(t, x, x') \geq \sigma(t) \geq 0 \text{ for large } t, 0 < m \leq |x| \leq R, x' \text{ arbitrary,}$$

$$(11.3) \quad \int_0^\infty t \sigma(t) dt = \infty.$$

Let $x = x(t)$ be a solution of (8.1) on $t \geq 0$. Then (11.1) holds.

This is an analogue of (IV) in [3] dealing with scalar equations.

REMARK. Let $f(t, x, x')$ satisfy the conditions of Theorem 3 and, in addition, let the constants α , K and the function $\phi(s)$ be independent of T . Let $x = x(t)$ be a solution (8.1) satisfying (11.1). Then

$$(11.4) \quad x'(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

For, by Lemma 2, $x'(t)$ is bounded for large t and, by (7.5), $x''(t)$ is bounded

for large t . The relation (11.4) then follows from the simplest Tauberian theorem (Hadamard) which states that $M_1 \leq \text{Const}(M_0 M_2)^{1/2}$ if M_0, M_1, M_2 are the least upper bounds for the moduli of a C^2 function and its first and second derivatives on $T \leq t < \infty$, respectively.

Proof of Theorem 4. Let $r(t) = |x(t)|^2$. Since (11.2) holds for large t , r satisfies $r'' \geq 0$ for large t . Suppose, if possible, that (11.1) fails to hold. Then there exists a constant m such that $0 < m \leq r(t) \leq R$ for large t . Let $\sigma(t)$ be the function belonging to the number m . Then

$$q(t) \equiv 2(x \cdot f(t, x, x') + |x'|^2)/r, \quad \text{where } x = x(t) \quad \text{and} \quad r = |x(t)|^2,$$

satisfies

$$(11.5) \quad q(t) \geq 2\sigma(t)/m \text{ for large } t.$$

Note that $r = r(t)$ satisfies the linear equation

$$(11.6) \quad r'' - q(t)r = 0;$$

cf. (2.6). But the boundedness of $r(t)$ and (11.3), (11.5), (11.6) imply that

$$(11.7) \quad r(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

(Weyl; cf., e.g., [13, pp. 601–602]). This contradicts $r \geq m > 0$ for large t . Hence Theorem 4 is proved.

12. This section deals with the uniqueness of solutions of (8.1) given by Theorems 1–3. In order to obtain a uniqueness criterion, consider the linear system of differential equations

$$(12.1) \quad y'' = A(t)y + B(t)y',$$

where $A(t), B(t)$ are real matrices and y is a vector. It is easily verified that, by virtue of (12.1),

$$y \cdot y'' + |y'|^2 = \left| y' + \frac{1}{2} B^* y \right|^2 + y \cdot A y - B^* y \cdot B^* y / 4,$$

where B^* is the transpose of B . Thus a sufficient condition (cf. [4]) for every solution $y = y(t)$ of (12.1) to satisfy

$$(12.2) \quad |y(t)| \geq 0, \quad (|y(t)|^2)'' \geq 0,$$

is that

$$(12.3) \quad 4A - BB^* \geq 0,$$

where “ $Q \geq 0$ ” for a matrix Q means that “ $y \cdot Q y \geq 0$ (or, equivalently, $y \cdot (Q + Q^*) y \geq 0$) for all real vectors y .” The fact that (12.3) implies (12.2) leads to the following uniqueness theorem:

THEOREM 5. Let $f(t, x, x')$ be defined on $D(T, R)$ [or $D(R)$] and possess

continuous partial derivatives with respect to the components of x and x' . Let $F(t, x, x')$ and $G(t, x, x')$ denote the Jacobian matrices

$$(12.4) \quad F(t, x, x') = (\partial f / \partial x), \quad G(t, x, x') = (\partial f / \partial x'),$$

and suppose that

$$(12.5) \quad 4F - GG^* \geq 0.$$

Then (8.1) has, at most, one solution which satisfies $x(0) = x_0$ and $x(T) = x_T$ [or which satisfies $x(0) = x_0$ and exists for $t \geq 0$].

Of course, if an a priori bound $|x'| \leq M$ is known for the possible solutions involved, then (12.5) is only required for $0 \leq t \leq T$ [or $t \geq 0$], $|x| \leq R$, $|x'| \leq M$.

In the case that x is a scalar and f does not depend on x' , the proof of Theorem 5 will show that the conditions on f can be replaced by the assumption that $f(t, x)$ is nondecreasing in x for fixed t .

Proof of Theorem 5. Note that if there are two such solutions $x = x_1(t)$ and $x = x_2(t)$, the difference $y = x_2 - x_1$ satisfies a linear equation (12.1), where

$$A(t) = \int_0^1 F ds, \quad B(t) = \int_0^1 G ds,$$

and the argument of F, G in these integrals is

$$(12.6) \quad (t, sx_2(t) + (1-s)x_1(t), sx'_2(t) + (1-s)x'_1(t)).$$

For any (constant) vector y , Schwarz's inequality (applied to each component of $B^*(t)y$) gives

$$|B^*(t)y|^2 \leq \int_0^1 |G^*y|^2 ds,$$

where the argument of G^* is (12.6). Hence

$$y \cdot (4A - BB^*)y \geq y \cdot \int_0^1 (4F - GG^*) ds y;$$

that is, (12.3) follows from (12.5). Consequently, $y(t) = x_2(t) - x_1(t)$ satisfies (12.2).

If either $y(0) = y(T) = 0$ or $y(0) = 0$ and $y(t)$ exists and is bounded for $t \geq 0$, then (12.2) implies $y(t) \equiv 0$. Hence Theorem 5 is proved.

13. This section gives a "continuity" theorem for the solutions furnished by Theorems 1-3; namely:

LEMMA 5. Let $f(t, x, x')$ and $f_1(t, x, x'), f_2(t, x, x'), \dots$ be continuous functions defined on $D(T, R)$ [or on $D(R)$] such that

$$(13.1) \quad f_n(t, x, x') \rightarrow f(t, x, x') \quad \text{as } n \rightarrow \infty$$

uniformly on compact subsets of $D(T, R)$ [or of $D(R)$]. On $D(T, R)$ [or on every $D(T, R)$], let f satisfy the conditions of Theorem 1 [with constants α, K and Nagumo function $\phi(s)$ depending on T]. Let $|x_0|, |x_T| \leq R$. Finally, let

$$(13.2) \quad x'' = f_n(t, x, x')$$

possess a solution $x = x_n(t)$ on $0 \leq t \leq T$ satisfying $x(0) = x_0, x(T) = x_T$ [or on $0 \leq t < \infty$ and satisfying $x(0) = x_0$]. Then there exists a sequence of positive integers $n_1 < n_2 < \dots$ such that

$$(13.3) \quad \lim_{k \rightarrow \infty} x_{n_k}(t) = x(t), \quad \text{where } n = n_k,$$

exists uniformly on $0 \leq t \leq T$ [or compact subsets of $t \geq 0$] and is a solution of (8.1) satisfying $x(0) = x_0, x(T) = x_T$ [or $x(0) = x_0$].

In order to see this, consider only the case of the nonsingular boundary value problem $x(0) = x_0, x(T) = x_T$. The considerations in the singular case are similar.

Lemma 5 is an immediate consequence of Lemma 2. In fact, the inequality

$$(13.4) \quad |f_n - f| \leq 1$$

together with the inequalities for f in (7.4) and (7.5) imply that

$$|f_n| \leq 2\alpha(x \cdot f_n + |x'|^2) + K + 1 + 2R\alpha, \quad |f_n| \leq \phi(|x'|) + 1.$$

Let M denote the constant furnished by Lemma 2, where $\rho(t), \phi(s)$ are replaced by $\alpha|x|^2 + K + 1 + 2R\alpha, \phi(s) + 1$, respectively.

In view of assumption (13.1), the inequality (13.4) holds for $0 \leq t \leq T, |x| \leq R, |x'| \leq M$ if n is sufficiently large. It follows from Lemma 2 that $|x'| \leq M$ for $0 \leq t \leq T$ and large n . Hence there exists a sequence of positive integers $n_1 < n_2 < \dots$ such that $\lim_{n \rightarrow \infty} x'_n(0)$ exists as $n = n_k \rightarrow \infty$. Lemma 5 now follows from standard theorems.

14. Existence of periodic or almost periodic solutions is usually proved under conditions which assure that all solutions exist for $t \geq 0$. Recently, Seifert [11] has given an existence theorem for almost periodic solutions in which this is not the case. Theorem 7 of the next section can be considered an analogue of his result for systems (even though the scalar case of Theorem 7, without modification, does not yield Seifert's theorem which involves a differential equation of a rather special form). In this section, there will be obtained a similar theorem for the existence of periodic solutions.

THEOREM 6. Let $f(t, x, x')$ be defined for $-\infty < t < \infty, |x| \leq R, x'$ arbitrary with the properties: (i) f is continuous and periodic in t of period 1 for fixed (x, x') ; (ii) the Jacobian matrices (12.4) exist and are continuous; (iii) f satisfies (8.2); (iv) f satisfies (7.4) and (7.5) with non-negative constants α, K and Nagumo function $\phi(s)$ independent of t, x ; finally, (v) if M is the constant sup-

plied by Lemma 2 (with some fixed $S > 0$ and $\rho(t) = \alpha|x|^2 + K$), then (12.5) holds for $-\infty < t < \infty$, $|x| \leq R$, $|x'| \leq M$.

Then (8.1) has at least one periodic solution of period 1. If $x = x_1(t)$ and $x = x_2(t)$ are bounded solutions for $-\infty < t < \infty$, then $|x_1(t) - x_2(t)| \equiv \text{const}$ (and $\text{const} = 0$ if " ≥ 0 " in (12.5) is replaced by " > 0 "). If $x = x_1(t)$ and $x = x_2(t)$ are bounded solutions for $t \geq 0$ or $t \leq 0$, then $|x_1(t) - x_2(t)| \leq |x_1(0) - x_2(0)|$.

REMARK. When x is a scalar, one can improve Theorem 6 slightly: condition (7.4) can be omitted in (iv); in which case, M in (v) should be replaced by the M supplied by Lemma 1. If, in addition, $f(t, x, x') = f(t, x)$ does not depend on x' , the differentiability assumptions (ii) and (v) can be replaced by the condition that f is nondecreasing in x for fixed t (and uniqueness results if f is increasing in x).

Proof of Theorem 6. By Theorems 3 and 5, the equation (8.2) has a unique solution on $t \geq 0$ satisfying $x(0) = x_0$ for any x_0 in the sphere $|x_0| \leq R$. Let this solution be denoted by $x = x(t, x_0)$. Define a map $x_0 \rightarrow x_1$ of the sphere $|x_0| \leq R$ into itself by putting $x_1 = x(1, x_0)$. It is clear from $|x'(t, x_0)| \leq M$ and from the uniqueness of the solution $x = x(t, x_0)$ that the map $x_0 \rightarrow x_1$ is continuous. Hence, by Brouwer's fixed point theorem, there exists a point $x_0 = x^*$ such that $x(1, x^*) = x^*$.

The periodicity of f implies that if $x = x(t)$ is a solution of (8.1), then $x = x(t+1)$ is also. In particular, $x(t+1, x_0) \equiv x(t, x_1)$. For the fixed point $x_0 = x^*$, we have $x(t+1, x^*) \equiv x(t, x^*)$, i.e., periodicity of period 1. This gives the existence assertion of Theorem 6.

The "uniqueness" assertions have nothing to do with the periodicity of f . If $x = x_1(t)$, $x_2(t)$ are two solutions of (8.1) for $-\infty < t < \infty$, then the proof of Theorem 5 shows that $r = |x_1 - x_2|^2$ satisfies $r'' \geq 0$ for all t . But the boundedness of r , $0 \leq r \leq 4R^2$, implies therefore that $r(t)$ is a constant. (If " > 0 " holds in (12.5) it is seen that $r'' > 0$ for some t unless $r \equiv 0$.) The stability assertion concerning $|x_1 - x_2|$ for $t \geq 0$ or $t \leq 0$ follows similarly. This completes the proof of Theorem 6.

15. An analogous theorem for almost periodic solutions is the following:

THEOREM 7. Let $f(t, x, x')$ be defined for $-\infty < t < \infty$, $|x| \leq R$, x' arbitrary with the properties (i) $f(t, x, x')$ is uniformly continuous for $-\infty < t < \infty$, $|x| \leq R$, x' bounded and is uniformly almost periodic in t for fixed (x, x') ; (ii) the Jacobian matrices (12.4) exist and are uniformly bounded and uniformly continuous for $-\infty < t < \infty$, $|x| \leq R$, x' bounded; and conditions (iii)-(v) of Theorem 6 hold.

Then (8.1) has at least one uniformly almost periodic solution.

The last parts of Theorem 6 concerning uniqueness on $-\infty < t < \infty$ and stability for $t \geq 0$ or $t \leq 0$ are valid here. Also, the Remark following Theorem 6 on the scalar case is applicable to Theorem 7.

Theorem 7 is a consequence of Theorem 3, the proof of Theorem 5 and

results of Amerio [1]; cf. [1] for references to Favard. By Theorem 3, (8.1) possesses solutions $x = x(t)$ on $0 \leq t \leq \infty$. Also, by Lemma 2, any solution of (8.1) satisfies $|x'(t)| \leq M$ if $x(t)$ exists on an interval of length $\geq S$. In particular, $|x'(t)| \leq M$ on $0 \leq t \leq \infty$. The boundedness of $x(t)$ and $x'(t)$ on $t \geq 0$ for some solution implies, by [1], the existence of a solution $x = x_1(t)$ for $-\infty < t < \infty$.

If $x = x_1(t)$, $x_2(t)$ are two solutions of (8.1) for $-\infty < t < \infty$, then, as in the proof of Theorem 6, $|x_1(t) - x_2(t)| \equiv \text{const.}$

Let $f_1(t, x, x')$ belong to the closure of the set $\{f(t+s, x, x') : -\infty < s < \infty\}$ with respect to the sup norm for $-\infty < t < \infty$, $|x| \leq R$, $|x'| \leq M$. It is clear that f_1 has properties (i)–(v) analogous to those of f (if, in (i) and (ii), “ x' bounded” is replaced by “ $|x'| \leq M$ ”). Thus, if $x = x_1(t)$ and $x = x_2(t)$ are two solutions of $x'' = f_1(t, x, x')$ for $-\infty < t < \infty$, then $|x_1 - x_2| \equiv \text{const.}$

It follows from [1] that if $x = x_1(t)$ is any solution of (8.1) on $-\infty < t < \infty$, then $x_1(t)$ is uniformly almost periodic. This completes the proof of Theorem 7.

16. Let x, z be vectors, not necessarily of the same dimension, and let x' be a vector of the same dimension as x . Let R, Q be positive constants and E the (t, x, x', z) -set:

$$(16.1) \quad E: 0 \leq t \leq T, \quad |x| \leq R, \quad x' \text{ arbitrary}, \quad |z| \leq 2Q.$$

Let X, Z be continuous vector valued functions on E of the same dimension as x, z , respectively. The system of differential equations

$$(16.2) \quad x'' = X(t, x, x', z), \quad z' = Z(t, x, x', z),$$

will now be considered.

The following conditions will be imposed on X :

$$(16.3) \quad x \cdot X(t, x, x', z) + |x'|^2 \geq 0 \text{ if } x \cdot x' = 0 \text{ and } |x| = R;$$

there exist non-negative constants α, K and a Nagumo function $\phi(s)$ such that

$$(16.4) \quad |X| \leq 2\alpha(x \cdot X + |x'|^2) + K, \quad |X| \leq \phi(|x'|);$$

the Jacobian matrices

$$(16.5) \quad F(t, x, x', z) = (\partial X / \partial x), \quad G(t, x, x', z) = (\partial X / \partial x')$$

exist, are continuous and satisfy (12.5) on E .

For Z , it will be supposed that there exist continuous, positive functions $\sigma(t), \tau(s)$ for $0 \leq t \leq T, Q^2 \leq s \leq (2Q)^2$, respectively, satisfying

$$(16.6) \quad 2|z \cdot Z| \leq \sigma(t)\tau(|z|^2) \text{ for } 0 \leq t \leq T, Q \leq |z| \leq 2Q, (x, x') \text{ arbitrary},$$

$$(16.7) \quad \int_0^T \sigma(t) dt < \int_{Q^2}^{(2Q)^2} ds / \tau(s) < \infty,$$

and that the Jacobian matrix $(\partial Z/\partial z)$ exists and is continuous on E .

THEOREM 8. *Let $|x_0|, |x_T| \leq R, |z_0| \leq Q$. The system (16.2) has at least one solution $x=x(t), z=z(t)$ which satisfies*

$$(16.8) \quad x(0) = x_0, \quad x(T) = x_T \quad \text{and} \quad z(0) = z_0.$$

It is clear that the first inequality for X in (16.4) is redundant if the second is of the form $|X| \leq \gamma |x'|^2 + C$ and $\gamma R < 1$. It is also clear that Theorem 8 leads to an analogue of Theorem 3.

The proof of Theorem 8 depends on Lemma 5 and on Schauder's fixed point theorem.

17. Proof of Theorem 8. Let H be the Banach space of vector functions $(x(t), z(t))$ on $0 \leq t \leq T$ with the product topology arising from $x(t) \in C^2, z(t) \in C^1$. Let M be the constant furnished by Lemma 2 (with $S=T$ and $\rho = \alpha |x|^2 + K$) and let N be a bound for $|X|, |Z|$ on the set

$$(17.1) \quad E_M: 0 \leq t \leq T, \quad |x| \leq R, \quad |x'| \leq M, \quad |z| \leq 2Q.$$

Let $\omega(\epsilon) = \omega_M(\epsilon)$ be defined by

$$(17.2) \quad \omega(\epsilon) = \max_{J=X,Z} \sup |\Delta J|,$$

where $\Delta J = J(t, x, x', z) - J(t^*, x^*, x'^*, z^*)$ and sup refers to $(t, x, x', z), (t^*, x^*, x'^*, z^*)$ in E_M and subject to $|t - t^*| \leq \epsilon, |x - x^*| \leq M\epsilon, |x' - x'^*| \leq N\epsilon, |z - z^*| \leq N\epsilon$.

Let H_0 be the subset of H consisting of vector functions $(x(t), z(t))$ which, for $0 \leq t \leq T$, satisfy $|x| \leq R, |x'| \leq M, |x''| \leq N, |z| \leq 2Q, |z'| \leq N$ and

$$|j(t) - j(t^*)| \leq \omega(\epsilon) \text{ if } 0 \leq t, t^* \leq T, |t - t^*| \leq \epsilon \text{ and } j = x'', z',$$

and, in addition, satisfy the boundary condition (16.8). Clearly, H_0 is a compact, convex subset of H .

Define a map $L: H_0 \rightarrow H_0$ as follows: if $(\bar{x}(t), \bar{z}(t)) \in H_0$, let $L(\bar{x}(t), \bar{z}(t)) = (x(t), z(t))$, where $z(t)$ is defined as the unique solution of

$$(17.3) \quad z' = Z(t, \bar{x}(t), \bar{z}(t), z), \quad z(0) = z_0,$$

and $x(t)$ is the unique solution of

$$(17.4) \quad x'' = X(t, x, x', z(t)), \quad x(0) = x_0 \quad \text{and} \quad x(T) = x_T.$$

In order to see that L is well defined, note that (17.3) defines $z(t)$ uniquely, at least for small $t \geq 0$, since $(\partial Z/\partial z)$ exists and is continuous. Actually, $z(t)$ is defined for $0 \leq t \leq T$ (and satisfies $|z(t)| < 2Q$). For otherwise, there is a subinterval $t_0 \leq t \leq t_1$ of $0 \leq t \leq T$ on which $Q \leq |z| \leq 2Q$ and $|z(t_0)| = Q, |z(t_1)| = 2Q$. But if $s(t) = |z(t)|^2$, then $s' = 2z \cdot Z(t, \bar{x}(t), \bar{z}(t), z)$ satisfies $|s'| \leq \sigma(t)\tau(s)$. An integration of this differential inequality over the interval $t_0 \leq t \leq t_1$ leads to

$$\int_{t_0}^{t_1} \sigma(t) dt \geq \int_{Q^2}^{(2Q)^2} ds/\tau(s),$$

which contradicts (16.7). Also, (17.4) has a unique solution $x=x(t)$ by the existence Theorem 1 and the uniqueness Theorem 5. Finally, $(x(t), z(t))$ is in H_0 .

The mapping $L: H_0 \rightarrow H_0$ is continuous. In order to see this, it is sufficient to show that if $(\bar{x}_n(t), \bar{z}_n(t))$, $n=1, 2, \dots$, is a sequence of elements of H_0 such that $(\bar{x}_n, \bar{z}_n) \rightarrow (\bar{x}, \bar{z})$ in H , as $n \rightarrow \infty$, and $(x_n, z_n) = L(\bar{x}_n, \bar{z}_n)$, then $(x_n, z_n) \rightarrow (x, z) = L(\bar{x}, \bar{z})$ in H . That $z_n \rightarrow z$ in $C^1(0, T)$ is clear from (17.3) and the uniqueness of the solution of (17.3). That $x_n \rightarrow x$ in $C^2(0, T)$ follows from (17.4) and Lemma 5.

Schauder's fixed point theorem implies that there is a point $(x(t), z(t)) \in H$ which is a fixed point of the map L . The point $(x(t), z(t))$ is a solution of (16.2) satisfying (16.8). This gives Theorem 8.

REFERENCES

1. L. Amerio, *Soluzioni quasi-periodiche, o limitate, di sistemi differenziale non-lineari quasi-periodici, o limitati*, Ann. Mat. Pura Appl. ser. 4 vol. 39 (1955) pp. 97-119.
2. R. W. Bass, *On non-linear repulsive forces*, Contributions to the theory of non-linear oscillations, vol. 4 (1958) pp. 201-211.
3. P. Hartman and A. Wintner, *On the non-increasing solutions of $y''=f(x, y, y')$* , Amer. J. Math. vol. 73 (1951) pp. 390-404.
4. ———, *On disconjugate differential systems*, Canad. J. Math. vol. 8 (1956) pp. 72-81.
5. E. Heinz, *On certain non-linear elliptic differential equations and univalent mappings*, J. Analyse Math. vol. 5 (1956/1957) pp. 197-272.
6. A. Kneser, *Untersuchung und asymptotische Darstellung der Integrale gewisser Differentialgleichungen bei grossen reellen Werthen des Arguments*, I, J. Reine Angew. Math. vol. 116 (1896) pp. 178-212.
7. A. Mambriani, *Su un teorema relativo alle equazioni differenziali ordinarie del 2° ordine*, Atti della Reale Accademia Nazionale dei Lincei ser. 6 vol. 9 (1929) pp. 620-622.
8. M. Nagumo, *Ueber die Differentialgleichung $y''=f(x, y, y')$* , Proceedings of the Physico-Mathematical Society of Japan ser. 3 vol. 19 (1937) pp. 861-866.
9. ———, *On principally linear elliptic differential equations of second order*, Osaka Math. J. vol. 6 (1954) pp. 207-229.
10. G. Scorza-Dragoni, *Sul problema dei valori ai limiti per i sistemi di equazioni differenziali del secondo ordine*, Boll. Un. Mat. Ital. vol. 14 (1935) pp. 225-230.
11. G. Seifert, *Almost periodic solutions for systems of differential equations near points of nonlinear first approximation*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 429-435.
12. A. Wintner, *On linear repulsive forces*, Amer. J. Math. vol. 71 (1949) pp. 362-366.
13. ———, *On almost free linear motions*, Amer. J. Math. vol. 71 (1949) pp. 595-602.

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