LOWER BOUNDS FOR THE DEGREE OF APPROXIMATION(1)

BY

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1. **Introduction.** The problem which is discussed in this paper may be described as follows. Let X be a Banach space of real or complex valued functions f defined on a compact metric space A. Let $G=\{g_1, \cdots, g_n, \cdots\}$, $g_n \in X$. Then

$$(1) \qquad E_n^G(f) = \inf_{a_i} \left\| f - \sum_{i=1}^n a_i g_i \right\|$$

is the degree of approximation of f by G, and

$$(2) \qquad E_n^G(\mathfrak{A}) = \sup_{f \in \mathfrak{A}} E_n^G(f)$$

is the degree of approximation of a class $\mathfrak{A} \subset X$. Finally,

$$(3) \qquad D_n(\mathfrak{A}) = \inf_G E_n^G(\mathfrak{A})$$

is the optimal degree of approximation of $\mathfrak{A}$. In this paper we shall give simple methods which permit to find the order of magnitude of $D_n(\mathfrak{A})$ for several important classes $\mathfrak{A}$: for some classes of analytic functions (§6); for the unit ball $\Lambda^{p+\alpha}$ of the space $C^{p+\alpha}$ of functions with continuous derivatives of order p, which satisfy a Lipschitz condition with exponent α, $0<\alpha\leqq 1$ (§5). We also consider approximation of continuous functions (§§2, 3) and give results about condensation of singularities (Theorems 2, 7).

The main content of this paper consists of results which show that standard means (trigonometric approximation, series of orthogonal polynomials) give the best possible approximation, at least up to a bounded factor. Since estimates of $D_n(\mathfrak{A})$ from above follow from classical results [1; 5], we are interested in estimating $D_n(\mathfrak{A})$ from below. Clearly, results of this type are the better, the smaller the norm used in the definition (1). This is why most of our theorems are for the L^1-norm. Kolmogorov [2] (see also [6]) discussed $D_n(\mathfrak{A})$ in the L^2-norm, and gave an asymptotic formula for it when $\mathfrak{A}$ is the class of functions f on an interval with $f^{(p)}$ bounded in the L^2-norm. Originally the present paper was written to improve (for linear approximation) the

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estimates of $D_n(\mathfrak{A})$ given by Vituškin [7]. After the research announcement [4] had been accepted for publication, a book of Vituškin [8] on the theory of tabulation appeared with improved estimates. We shall compare his results with our Theorems 3, 5, 7. Vituškin considers only approximation of functions f of a variable x in the uniform norm, $\|f\| = \sup_x |f(x)|$. His results are generally of the type $n \geq C\phi(\epsilon)$, where $\epsilon = D_n(\mathfrak{A})$, $\phi(\epsilon)$ is a known decreasing function of ϵ, and C is an unknown bounded function of n. This leads to the same lower bound as that given in our Theorem 5 for the L^1-norm, but his lower estimates for the cases treated by Theorems 3 and 7 are not as good as ours. It should be noted, however, that he considers also approximation by expressions more general than linear forms $\sum_{i=1}^n a_i g_i(x)$. He replaces these by polynomials of degree k in the a_i, $i = 1, \cdots, n$, with coefficients which are given functions of x, or by a quotient of two such polynomials. His results may be roughly described by saying that in his previous inequality, n may now be replaced by $n \log ((K+1)/\epsilon)$, K being the maximum of the degrees of the denominator and the numerator. Vituškin's methods are quite different from ours. Vituškin uses the metric entropy of Kolmogorov (compare [3]), while we prefer a direct approach.

2. **Uniform approximation.** Let A be a compact metric space containing infinitely many points with the distance function ρ, and let $\mathfrak{A}$ be a set of continuous real- or complex-valued functions on A. Throughout this paper, $\|f\|$ shall denote the uniform norm $\|f\| = \sup_{x \in A} |f(x)|$ of f.

DEFINITION. *The characteristic (at the origin) of a compact family* $\mathfrak{A}$, $\chi_n(\mathfrak{A})$, *is the largest number* $\delta \geq 0$ *with the following property. There exist* $n+1$ *points of* A, $x_0, x_1, \cdots, x_n$ *such that for each distribution of real or complex signs* ϵ_k, $k = 0, \cdots, n$, *there exists a function* $f \in \mathfrak{A}$ *with*

$$\operatorname{sign} f(x_k) = \epsilon_k, \qquad k = 0, \cdots, n, \tag{4}$$

$$|f(x_k)| \geqq \delta, \qquad k = 0, \cdots, n. \tag{5}$$

It is clear from the compactness of A and $\mathfrak{A}$ that a largest δ exists.

THEOREM 1. *For each compact family* $\mathfrak{A}$,

$$D_n(\mathfrak{A}) \geqq \chi_n(\mathfrak{A}). \tag{6}$$

Proof. Let $\delta = \chi_n(\mathfrak{A})$, and let $x_0, \cdots, x_n$ be the corresponding points of A. Consider arbitrary continuous functions $g_1, \cdots, g_n$ defined on A. We can select constants c_k, $k = 0, \cdots, n$ with $\sum_{k=0}^n |c_k| = 1$ so that

$$\sum_{k=0}^n c_k g_i(x_k) = 0, \qquad i = 1, \cdots, n.$$

Put $L(f) = \sum_{k=0}^n c_k f(x_k)$. Then for arbitrary a_i,

$$\left\| f - \sum_1^n a_i g_i \right\| \geqq L\left(f - \sum_1^n a_i g_i\right) = L(f).$$

We select $f\in\mathfrak{A}$ to satisfy (4) and (5) with $\epsilon_k=\text{sign } c_k$ and $\delta=\chi_n(\mathfrak{A})$. Then

$$E_n^G(f) \geqq L(f) \geqq \chi_n(\mathfrak{A}). \tag{7}$$

This proves (6), since G is arbitrary.

Sometimes one can show that relations such as (7) hold for some $f\in\mathfrak{A}$ and an infinity of n. In this direction we have:

THEOREM 2. *Let the family $\mathfrak{A}$ contain 0 and be compact and convex. Then for each sequence $\epsilon_n\to 0$ and each G there is a function $f_0\in\mathfrak{A}$ such that for infinitely many n,*

$$E_n^G(f_0) \geqq \epsilon_n\chi_n(\mathfrak{A}). \tag{8}$$

Proof. This is a special case of the following lemma. Let A be a convex and closed subset of a Banach space X and let

$$\|f\| = E_0(f) \geqq E_1(f) \geqq \cdots \geqq E_n(f) \geqq \cdots$$

be a sequence of semi-norms on X. Let $\chi_n=\sup_{f\in\mathfrak{A}} E_n(f)<+\infty$, $n=0, 1, \cdots$. Then for each sequence $\epsilon_n\to 0$, $\epsilon_n>0$, there exists an $f_0\in\mathfrak{A}$ with

$$E_n(f_0) \geqq \epsilon_n\chi_n \tag{9}$$

for infinitely many n.

Assume on the contrary that for each $f\in\mathfrak{A}$

$$E_n(f) < \epsilon_n\chi_n \tag{10}$$

for all sufficiently large n. We select $f_n\in\mathfrak{A}$ such that $E_n(f_n)\geqq\chi_n/2$. By induction we construct a sequence $b_k>0$, $k=1, 2, \cdots$, $\sum_1^\infty b_k\leqq 1$ and a sequence of natural numbers $n_1<n_2<\cdots<n_k<\cdots$ as follows. If the b_i, n_i, $i\leqq k-1$ are known, $\sum_1^{k-1} b_i<1$, we first take $b_k>0$ so small that $b_k<b_{k-1}/2$, $\sum_1^k b_i<1$, and

$$b_k < \frac{1}{2}\epsilon_{n_{k-1}}\chi_{n_{k-1}}. \tag{11}$$

We then take n_k so large that

$$b\epsilon_{n_k} < b_k, \tag{12}$$

$$E_{n_k}\left(\sum_1^{k-1} b_i f_{n_i}\right) < \epsilon_{n_k}\chi_{n_k}. \tag{13}$$

Let $f_0=\sum_1^\infty b_k f_{n_k}$. Then, assuming without loss of generality $\chi_0=1$, we have

$$E_{n_k}(f_0) \geqq E_{n_k}(b_k f_{n_k}) - E_{n_k}\left(\sum_1^{k-1} b_i f_{n_i}\right) - E_{n_k}\left(\sum_{k+1}^\infty b_i f_{n_i}\right),$$

$$E_{n_k}\left(\sum_{k+1}^\infty b_i f_{n_i}\right) \leqq \sum_{k+1}^\infty b_i E_0(f_{n_i}) \leqq \sum_{k+1}^\infty b_i \leqq 2b_{k+1} < \epsilon_{n_k}\chi_{n_k}$$

by (11), hence by (12) and (13)

$$E_{n_k}(f_0) \geqq \frac{1}{2} b_k \chi_{n_k} - 2\epsilon_{n_k}\chi_{n_k} \geqq \epsilon_{n_k}\chi_{n_k}, \qquad k = 1, 2, \cdots$$

This is a contradiction with our assumption and proves the lemma.

In §5 we shall see that sometimes ϵ_n may be dropped in the statement of Theorem 2.

3. **Approximation of continuous functions.** We consider a continuous increasing function $\omega(h)$ defined for $h \geqq 0$, vanishing at zero and subadditive: $\omega(h_1+h_2) \leqq \omega(h_1)+\omega(h_2)$. We denote by $\Lambda^\omega = \Lambda^\omega(A)$ the set of all continuous real functions on a compact metric space A with infinitely many points which satisfy $|f(x)| \leqq 1$, $x \in A$ and

$$(14) \qquad |f(x) - f(x')| \leqq \omega(\rho(x, x')).$$

THEOREM 3. *Let $\sigma_n = \sigma_n(A)$ be the largest number $\sigma > 0$ such that A contains $n+1$ points at mutual distances $\geqq 2\sigma$. Then for all large n,*

$$(15) \qquad D_n(\Lambda^\omega) \geqq \frac{1}{2}\omega(\sigma_n);$$

this result is essentially the best possible.

Proof. Let $x_0, x_1, \cdots, x_n$ be points of A at mutual distances $\geqq 2\sigma$, $\sigma = \sigma_n$. Let $\epsilon_k = \pm 1$, $k = 0, \cdots, n$. Put

$$(16) \qquad f(x) = \begin{cases} \frac{1}{2}\epsilon_k \omega(\sigma - \rho(x, x_k)) \text{ if } \rho(x, x_k) \leqq \sigma, & k = 0, \cdots, n, \\ 0 \text{ elsewhere.} \end{cases}$$

To estimate $f(x) - f(x')$ we may assume that $|f(x)| \geqq |f(x')|$ and that x belongs to the ball $\rho(x, x_k) \leqq \sigma$. If also x' belongs to this ball,

$$\begin{aligned} |f(x) - f(x')| &= \frac{1}{2}\{\omega(\sigma - \rho(x, x_k)) - \omega(\sigma - \rho(x', x_k))\} \\ &\leqq \frac{1}{2}\omega(\rho(x', x_k) - \rho(x, x_k)) \leqq \frac{1}{2}\omega(\rho(x, x')); \end{aligned}$$

and if x' does not belong to the ball,

$$|f(x) - f(x')| \leqq 2|f(x)| = \omega(\sigma - \rho(x, x_k)) \leqq \omega(\rho(x, x')).$$

Also, for large n, $|f(x)| \leqq 2^{-1}\omega(\sigma_n) \leqq 1$, $x \in A$. Thus $f \in \Lambda^\omega$. From the definition of χ_n we see that $\chi_n(\Lambda^\omega) \geqq 2^{-1}\omega(\sigma_n)$, and Theorem 1 gives (15).

This inequality cannot be improved in the following sense. For a given n, let $\sigma_n' = \sigma' > 0$ be chosen so that there exist n open balls $U_1, \cdots, U_n$ with centers $x_1, \cdots, x_n$ and radii σ' which cover A. Then there exists a system of real continuous functions $G = \{g_1, \cdots, g_n\}$ with

$$E_n^G(\Lambda^\omega) \leqq \omega(\sigma_n'). \tag{17}$$

A reasonably chosen σ_n' is usually close to σ_n; for example, one always has $\sigma_{n+1} < \sigma_n'$ [3, p. 8]. To prove (17), consider a decomposition of unity $1 = g_1(x) + \cdots + g_n(x)$, $x \in A$, where $g_k(x)$ are continuous functions on A with $0 \leqq g_k(x) \leqq 1$, $x \in A$, $g_k(x) = 0$ for $x \notin U_k$, $k = 1, \cdots, n$. If $f \in \Lambda^\omega$ and $P = \sum_1^n f(x_k) g_k$, then

$$|f(x) - P(x)| \leqq \sum_{k=1}^{n} |f(x) - f(x_k)| g_k(x) \leqq \omega(\sigma').$$

4. **Approximation in the L^1 norm.** Let A be a compact metric space and μ a positive regular Borel measure on A. Let $\mathfrak{A}$ be a subset of $L^1(A; \mu)$. We prove a theorem similar to Theorem 1 of §2. For this purpose we need the following:

Definition. *For two natural numbers n, p, $\delta_n^p(\mathfrak{A})$ denotes the supremum of numbers $\delta \geqq 0$ with the following property. There exist disjoint measurable subsets A_k, $k = 1, \cdots, n+p$ of A such that for any p different sets A_{k_j}, $j = 1, \cdots, p$ and any distribution of real or complex signs ϵ_k, $k = 1, \cdots, n+p$, there exists a function $f_0 \in \mathfrak{A}$ with*

$$\operatorname{sign} \int_{A_k} f_0 d\mu = \epsilon_k, \qquad k = 1, \cdots, n+p, \tag{18}$$

$$\left| \int_{A_{kj}} f_0 d\mu \right| \geqq \delta, \qquad j = 1, \cdots, p. \tag{19}$$

Theorem 4. *For each subset $\mathfrak{A} \subset L^1(A; \mu)$ one has in the L^1-norm*

$$D_n(\mathfrak{A}) \geqq p\delta_n^p(\mathfrak{A}). \tag{20}$$

Proof. Let g_i, $i = 1, \cdots, n$ be arbitrary functions from $L^1(A; \mu)$. The equations

$$\sum_{k=1}^{n+p} c_k \int_{A_k} g_i d\mu = 0, \qquad i = 1, \cdots, n \tag{21}$$

have at least p linearly independent solutions $c = \{c_k\}_{k=1}^{n+p}$. We use the following:

Lemma. *The vector space X spanned by p linearily independent vectors $c = \{c_k\}_{k=1}^{n+p}$ contains at least one vector $c = \{c_k\}$ for which*

$$|c_k| = \|c\| = \sup_{l=1,\ldots,n+p} |c_l| = 1 \tag{22}$$

holds for at least p values of k.

Using this we select constants c_k satisfying (21) and (22) and put for $f \in L^1$,

$$L(f) = \sum_{k=1}^{n+p} c_k \int_{A_k} f d\mu.$$

This is a continuous linear functional of norm one and $L(g_i)=0, i=1, \cdots, n$. As in §2 we see that $E_n^G(f)\geqq L(f)$. Hence the theorem will be proved if we can find an $f_0\in\mathfrak{A}$ with $L(f_0)\geqq p\delta$ for each $\delta<\delta_n^p(\mathfrak{A})$. If we define the signs ϵ_k by $\epsilon_k=\text{sign } c_k$ and take for $k_1, \cdots, k_p$ some p values of k satisfying (22), the inequality $L(f_0)\geqq p\delta$ follows from the definition of δ_n^p for a properly chosen f_0.

Proof of the lemma. Let j, $1\leq j\leq p$ denote the largest integer for which X contains a vector $x=\{x_k\}$ with

$$|x_k| = \|x\| = \sup_{l=1,\ldots,n+p} |x_l| = 1 \tag{23}$$

holding for j different values of k. We shall show that the assumption $j<p$ leads to a contradiction. Let x be a vector in X for which (23) holds for $k=k_1, \cdots, k_j$; and consider the $j'\leqq j$ dimensional space Y spanned by the vectors $d=\{c_{k_i}\}_{i=1}^j$. We can find j' of the vectors c for which the corresponding d span Y. Subtracting proper linear combinations of these c from other vectors c, we obtain vectors c', not all zero, with coordinates $c'_{k_i} i=0, =1, \cdots, j$. For one of such c' and the x chosen above put $y=x+\lambda c'$, with real λ. The coordinates of y change continuously with λ, and $|y_{k_i}|=|x_{k_i}|=1$. It is easy to see that one can find a value of λ for which y will satisfy (23) for more than j values of k. This is a contradiction.

As an application of Theorem 4, we can obtain for the class $\mathfrak{A}$ of analytic functions of §6, for the L^1-norm and the *Lebesgue measure*, $D_n(\mathfrak{A})\geqq Cn^{-1}\rho^{-n}$. We omit the details.

5. L^1 **approximation of smooth functions.** Let A_0 be a fixed parallelepiped of the q-dimensional euclidean space, and let A be a subset of A_0, which is a union of finitely many parallelepipeds. A function $f(x)$, $x=(x^{(1)}, \cdots, x^{(q)})$, defined on A_0, belongs to $C^{p+\alpha}(A_0)$, where $p=0, 1, \cdots, 0<\alpha\leqq 1$, if it has continuous partial derivatives of order p on A_0, each of which satisfies a Lipschitz condition

$$|f^{(p)}(x) - f^{(p)}(x')| \leqq M\rho(x, x')^\alpha. \tag{24}$$

If M_0 is the smallest possible value of M in (24), we put $\|f\|_{p+\alpha}=\max(\|f\|, M_0)$. Finally, $\Lambda^{p+\alpha}$ will be the set of all functions f on A_0 with $\|f\|_{p+\alpha}\leqq 1$.

The following theorem states essentially that $D_n(\Lambda^{p+\alpha})\geqq\text{const. } n^{-(p+\alpha)/q}$; details will be needed for a "gliding hump" argument in Theorem 6. In this section, μ will be the Lebesgue measure.

THEOREM 5. *There exists a constant $B>0$ depending only on $p+\alpha$, q, $m>0$ such that for each A with $\mu A\geqq m$ and each $\epsilon>0$, for each sufficiently large n, $n\geqq n(A, \epsilon, p+\alpha, q)$, and each system $G=\{g_1, \cdots, g_n, \cdots\}$ of integrable functions on A_0, there exists a function $f_0\in\Lambda^{p+\alpha}$ defined on A_0 and vanishing outside of A such that for its degree of approximation in the L^1-norm on A_0,*

(25) $$E_n^G(f_0) \geqq Bn^{-(p+\alpha)/q};$$

moreover for all partial derivatives $f_0^{(k)}$ of orders $k=0, 1, \cdots, p$,

(26) $$\|f_0^{(k)}\| < \epsilon.$$

This f_0 belongs to $C^{p+1}(A')$, where A' is obtained from A by removing a finite number of open parallelepipeds of arbitrary small total measure.

Proof. We use the construction of Theorem 4 to obtain our f_0. Let $F(u)$ denote a fixed increasing continuous function for $0 \leqq u \leqq 1$, given by $F(u) = u^{p+\alpha}$ for $0 \leqq u \leqq 1/2$, with continuous derivatives of all orders up to $p+1$ for $0 < u \leqq 1$, which all vanish at $u=1$. Clearly, $F^{(p)}(u)$ will satisfy on $[0, 1]$ a Lipschitz condition $|F^{(p)}(u) - F^{(p)}(u')| \leqq B_1|u-u'|^\alpha$. (The constants $B_1, B_2, \cdots$ will depend only on $p+\alpha$, q, m.)

For all sufficiently large n, A contains $2n$ points $x_1, \cdots, x_{2n}$ at mutual distances from each other $\geqq 2B_2 n^{-1/q}$, such that the closed balls A_k with center x_k and radius $r = B_2 n^{-1/q}$ are contained in A. Let $\epsilon_k = \pm 1$ be an arbitrary distribution of signs. We put

$$f_1(x) = \begin{cases} \epsilon_k r^{p+\alpha} F(1 - r^{-1}\rho(x, x_k)), & x \in A_k, \qquad k = 1, \cdots, 2n, \\ 0 \text{ elsewhere on } A_0. \end{cases}$$

Elementary computations show that $f_1 \in C^{p+\alpha}$ with $\|f_1\|_{p+\alpha} \leqq B_3$. On the other hand, $|f_1(x)| \geqq B_4 r^{p+\alpha}$ for $\rho(x, x_k) \leqq r/2$, hence

(27) $$\left| \int_{A_k} f_1 d\mu \right| \geqq B_5 r^{p+\alpha+q} \geqq B_6 n^{-(p+\alpha)/q-1} = \delta.$$

If we put $f_0 = f_1/B_3$, then f_0 belongs to $\Lambda^{p+\alpha}$ and satisfies conditions (18) and (19) of §4, with n replaced by $2n$, and p by n, hence $E_n^G(f_0) \geqq n\delta$ with δ given by (27). This proves (25), and the other conditions are also satisfied for each large n.

For $0 < \alpha < 1$ we can prove more.

THEOREM 6. *Let A be a closed q-dimensional parallelepiped and $0 < \alpha < 1$. There exists a constant B such that for each system $G = \{g_n\}$ of integrable functions on A, there exists an $f_0 \in \Lambda^{p+\alpha}$ for which (for the L^1-approximation)*

(28) $$E_n^G(f_0) \geqq Bn^{-(p+\alpha)/q}$$

holds for an infinity of values of n.

Proof. We shall have $B = B_1/6$, where B_1 is the constant of Theorem 5 corresponding to $m = \mu A/2$. Let the sequence G be given. For the construction we shall need two sequences of sets, A_k, A_k' consisting each of finitely many closed parallelepipeds and satisfying

$$A = A_1 \supset A_1' \supset \cdots \supset A_k \supset A_k' \supset \cdots; \mu A_k > m,$$

functions f_k defined on A and vanishing outside of A_k', integers $1 \leq n_1 < \cdots < n_k < \cdots$, and two sequences of numbers $\epsilon_k > 0$, $\delta_k > 0$ with $\sum_1^\infty \epsilon_k < 1$; δ_k will be the distance from A_k' to the boundary of A_k. We put $M_k = \sum_1^k \epsilon_i$, $F_k = \sum_1^k f_i$, and denote by N_k the C^{p+1}-norm of F_k on A_{k+1}.

Let all these elements with subscripts not exceeding $k-1$ be defined; we shall define all elements with subscripts k. We can assume that for all sufficiently large n,

$$E_n(F_{k-1}) < \frac{1}{3} B_1 n^{-\rho}, \qquad \rho = \frac{p+\alpha}{q}, \tag{29}$$

for otherwise Theorem 6 would follow immediately.

We first select A_k, A_k' with $A_{k-1}' \supset A_k \supset A_k'$ and $\mu A_k' > m$ such that F_{k-1} belongs to the class C^{p+1} on A_k, and that the distance δ_k of A_k' to the boundary of A_k satisfies

$$0 < \delta_k < (N_{k-1}^{-1} \epsilon_{k-1})^{1/(1-\alpha)}, \qquad \delta_k < 1. \tag{30}$$

Then we take $n_k > n_{k-1}$ so large that (29) holds with $n = n_k$. Next $\epsilon_k > 0$ is defined with $\sum_1^k \epsilon_i < 1$ and

$$\epsilon_k < \frac{1}{3} B_1 n_k^{-\rho} \mu A, \tag{31}$$

$$\epsilon_k < \frac{1}{2} \epsilon_{k-1} \delta_k^\alpha. \tag{32}$$

Thus $\epsilon_k < \epsilon_{k-1}/2$. Finally we take f_k to be the function f_0 of Theorem 5, with n, A_0, A, ϵ now replaced by n_k, A, A_k' and ϵ_k, respectively. Put

$$f = \sum_1^\infty f_i. \tag{33}$$

As in Theorem 2, we obtain for the degree of approximation by the g_1, $g_2, \cdots$ in the L^1 norm on A,

$$\begin{aligned} E_{n_k}(f) &\geq E_{n_k}(f_k) - E_{n_k}(F_{k-1}) - \sum_{i=k+1}^\infty \|f_i\| \mu A \\ &\geq B_1 n_k^{-\rho} - \frac{1}{3} B_1 n_k^{-\rho} - \sum_{k+1}^\infty \epsilon_i \mu A \\ &\geq \frac{2}{3} B_1 n_k^{-\rho} - \epsilon_k \mu A \geq \frac{1}{3} B_1 n_k^{-\rho}. \end{aligned}$$

We now have to show that $f \in \Lambda^{p+\alpha}$. We have $\|f\| \leq \sum \|f_i\| \leq \sum \epsilon_i < 1$. The derivatives of f of orders $k = 1, \cdots, p$ are obtained by termwise differentiation of (33); in particular, each partial derivative $f^{(p)}$ is by (26) and the choice of ϵ_k equal to $f^{(p)}(x) = \lim_{k\to\infty} F_k^{(p)}(x)$. We show by induction that

$$(34)\qquad \left| F_k^{(p)}(x) - F_k^{(p)}(x') \right| \leqq (1 + M_{k-1})h^\alpha, \qquad h = \rho(x, x').$$

We may assume that at least one of the points x, x' belongs to A_k', for otherwise $F_k = F_{k-1}$ at these points. Let $x \in A_k'$. We consider two cases. If $h \leqq \delta_k$, then $x' \in A_k$ and by (30), $N_{k-1}h^{1-\alpha} \leqq \epsilon_{k-1}$. Therefore

$$\begin{aligned} \left| F_k^{(p)}(x) - F_k^{(p)}(x') \right| &\leqq \left| F_{k-1}^{(p)}(x) - F_{k-1}^{(p)}(x') \right| + \left| f_k^{(p)}(x) - f_k^{(p)}(x') \right| \\ &\leqq N_{k-1}h + h^\alpha \leqq (\epsilon_{k-1} + 1)h^\alpha \\ &\leqq (1 + M_{k-1})h^\alpha. \end{aligned}$$

If on the other hand $h > \delta_k$, then by (32), $\epsilon_{k-1}h^\alpha > 2\epsilon_k$, hence

$$\begin{aligned} \left| F_k^{(p)}(x) - F_k^{(p)}(x') \right| &\leqq (1 + M_{k-2})h^\alpha + 2\left\| f_k^{(p)} \right\| \\ &\leqq (1 + M_{k-2})h^\alpha + 2\epsilon_k \\ &\leqq (1 + M_{k-1})h^\alpha. \end{aligned}$$

If we put $f_0 = f/2$, we see that $f_0 \in \Lambda^{p+\alpha}$ and that $E_{n_k}(f_0) \geqq 6^{-1}B_1 n_k^{-p}$, $k = 1, 2, \cdots$.

6. **L^1-approximation of analytic functions.** Let $\mathfrak{A}$ be the family of the complex valued analytic functions on $[-1, +1]$, which have analytic continuation bounded by one into the ellipse E_ρ with foci -1, $+1$ and the sum of the half axes $\rho > 1$. We consider the measure given by $d\mu = (1 - x^2)^{-1/2}dx$ on $[-1, +1]$.

THEOREM 7. *For the family $\mathfrak{A}$ in the $L^1(\mu)$ norm,*

$$(35)\qquad D_n(\mathfrak{A}) \geqq C\rho^{-n}, \qquad C = 1 - \rho^{-1}.$$

For the uniform approximation, (35) may be derived from Theorem 1. One can take for the function f with properties (4), (5) the interpolating polynomial with values $\epsilon_k\delta$ assumed at the $n+1$ zeros x_k of the Chebyshev polynomial T_{n+1}.

Turning to the proof of Theorem 7, we note that $|T_k(z)| \leqq \rho^k$ in E_ρ [5], hence the functions $t_k = \rho^{-k}T_k$ belong to $\mathfrak{A}$.

Let $g_1, \cdots, g_n$ be arbitrary functions from $L^1(\mu)$; there exist linear combinations h_k of the g_i such that

$$(36)\qquad \| t_k - h_k \|_{L^1} \leqq E_n^G(\mathfrak{A}), \qquad k = 0, 1, \cdots, n.$$

As a set of $n+1$ vectors in an n-dimensional space is always linearily dependent, there exist constants $c_0, \cdots, c_n$ such that

(37) $$\sum_{k=0}^{n} c_k h_k(x) = 0,$$

(38) $$\sum_{k=0}^{n} |c_k| = 1.$$

The function $f=\sum_{k=0}^{n} c_k t_k$ belongs to $\mathfrak{A}$ by (38). By (36) and (37), $\|f\|_{L^1} \leqq E_n^G(\mathfrak{A})$. We also have

$$\|f\|_{L^1} \geqq \left| \int_{-1}^{+1} f(x) T_k(x) d\mu(x) \right| = c_k \rho^{-k}.$$

Hence

$$\rho^k E_n^G(\mathfrak{A}) \geqq |c_k|, \qquad k = 0, 1, \cdots, n.$$

Summing over k we get

$$(\rho^{n+1} - 1) E_n^G(\mathfrak{A}) \geqq \rho^{-1}.$$

From this the inequality (35) follows. Classical estimates in the uniform norm give $D_n(\mathfrak{A}) \leqq C_1 \rho^{-n}$, $C_1 = 2(\rho-1)^{-1}$.

We conclude by remarking that similar estimates can be given for the class $\mathfrak{A}'$ of functions analytic on $|z| \leqq 1$, which have analytic continuation bounded by one into the circle $|z| < \rho$, $\rho > 1$.

References

1. N. I. Achiezer, *Theory of approximation*, New York, Ungar Publishing Co., 1956.

2. A. Kolmogorov, *Über die beste Annäherung von Funktionen einer gegebenen Funktionenklasse*, Ann. of Math. (2) vol. 37 (1936) pp. 107–110.

3. A. N. Kolmogorov and V. N. Tihomirov, *ε-entropy and ε-capacity of sets in spaces of functions*, Uspehi Mat. Nauk vol. 14 no. 2(86) (1959) pp. 3–86.

4. G. G. Lorentz, *Approximation of smooth functions*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 124–125.

5. I. P. Natanson, *Konstruktive Funktionentheorie*, Akademie-Verlag, Berlin, 1955.

6. W. Rudin, *L^2-approximation by partial sums of orthogonal developments*, Duke Math. J. vol. 19 (1952) pp. 1–4.

7. A. G. Vituškin, *On best approximation of differentiable and analytic functions*, Dokl. Akad. Nauk SSSR vol. 119 (1958) pp. 418–420.

8. ———, *Estimation of the difficulty of the tabulation problem*, Gosizdat, Moscow, 1959, 228 pp.

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