SEPARATION THEOREMS FOR SOME PLANE-LIKE SPACES

BY

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In this paper, certain separation theorems, which hold in spaces satisfying R. L. Moore's Axioms 0 and 1–5 of [2], are established for a class of spaces in which there do not necessarily exist either arcs or simple closed curves. The spaces considered satisfy axioms suggested by those of Moore.

Consider a nondegenerate complete metric space S which satisfies the following axioms from [2]:

AXIOM 2. S is locally connected.

AXIOM 3. S is connected and has no cut point.

AXIOM 4. If J is a simple closed curve in S, then $S-J$ has exactly two components and J is the boundary of each of them.

AXIOM 5. If x and y are distinct points and U is an open set containing x, then there is a simple closed curve J in U such that J separates x from y.

It will be shown that even with a weakening of these conditions on S, the following separation theorems hold: (1) If neither of the compact point sets H and K separates the two points x and y, and either H and K are disjoint or $H\cap K$ is connected, then $H\cup K$ does not separate x and y. (2) If the common part of the two compact continua H and K is not connected, then $H\cup K$ separates S.

Principally, the conditions above are weakened by the omission of the stipulation that S be complete. It is assumed instead that S is *locally compactly connected* (a point set K in S is compactly connected if and only if each two points of K belong to a compact continuum in K). As an example in §4 shows, in a locally compactly connected metric space, there does not necessarily exist an arc. However, there exist *chains* of compact continua, and sets which are the unions of links of such chains serve as substitutes for arcs. The sets which serve as substitutes for simple closed curves are the unions of the links of *closed chains* of compact continua. Moore's Axioms 4 and 5 are replaced by analogous axioms for closed chains.

This problem was suggested to the writer by R. L. Moore, and the development presented here is along lines similar to that in [2] which leads to the separation theorems mentioned above.

1. Consequences of Axioms 1 and 2.

AXIOM 1. S is a nondegenerate connected metric space with no cut point.

The diameter of a bounded point set M and the distance between two point sets H and K are denoted by diam M and dist (H, K), respectively.

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The point set X is *compactly connected* if and only if each two points of X belong to a compact continuum contained in X, and X is *locally compactly connected* if and only if for each point p of X and each open set U containing p, there is a compactly connected open set V containing p and contained in U.

AXIOM 2. S is locally compactly connected.

COROLLARY 1.1. *S is locally connected and each connected open set is compactly connected.*

If x and y are two distinct points, then f is a *chain of open sets from x to y* if and only if f is a finite collection $\{f_1, f_2, \cdots, f_n\}$ of open sets such that (1) $x \in f_i$ if and only if $i=1$, (2) $y \in f_i$ if and only if $i=n$, and (3) if i and j are positive integers, $i<j\leqq n$, then f_i and f_j intersect if and only if $j=i+1$. It is well known [2, Chapter I, Theorem 77] that if M is a connected point set, $\mathfrak{U}$ is a collection of open sets covering M, and x and y are points of M, then there is a chain of open sets of $\mathfrak{U}$ from x to y.

THEOREM 1.2. *Suppose that U is a connected open set, B is its boundary, $x \in B$, there is an open set V containing x such that $\overline{V} \cap B$ is compact, W is an open set containing x, and $y \in U$. Then there exist a point z of $B \cap W$ and a compact continuum K containing y and z and such that $K-\{z\} \subset U$.*

Proof. Let T be the component of $W \cap V$ containing x; $U \cup T$ is a connected open set. Let $\mathfrak{A}_1$ be the set of all connected open sets N such that (1) $N \subset U \cup T$, (2) diam $N<1$, (3) if N intersects B, then $\overline{N} \subset T$, and (4) if $y \in N$, then $N \subset U$. There exists a chain $\{U_{11}, U_{12}, \cdots, U_{1n_1}\}$ of open sets of $\mathfrak{A}_1$ from y to x. Let j_1 be the least positive integer i such that U_{1i} intersects B. Let R_1 denote U_{1j_1}, let y_1 be a point of $R_1 \cap U_{1(j_1-1)}$, and let x_1 be a point of $B \cap R_1$. Let D_1 be $\bigcup\{U_{1i}: i<j_1\}$; D_1 is a connected open set contained in U. Let $\mathfrak{A}_2$ be the set of all connected open sets N such that $\overline{N} \subset R_1$, diam $N<1/2$, and if $y_1 \in N$, then $N \subset U$. There is a chain $\{U_{21}, U_{22}, \cdots, U_{2n_2}\}$ of connected open sets of $\mathfrak{A}_2$ from y_1 to x_1. Let j_2 be the least positive integer i such that U_{2i} intersects B. Let R_2 denote U_{2j_2}, let y_2 be a point of $R_2 \cap U_{2(j_2-1)}$, and let x_2 be a point of $B \cap R_2$. Let D_2 be $\bigcup\{U_{2i}: i<j_2\}$. Let this process be continued.

There exist a sequence $D_1, D_2, D_3, \cdots$ and a sequence $R_1, R_2, R_3, \cdots$ such that $\overline{R}_1 \subset W \cap V$ and if n is a positive integer, then (1) D_n is a connected open set contained in U and intersecting D_{n+1}, (2) R_n is an open set such that R_n intersects B, both D_{n+1} and $\overline{R}_{n+1}$ are subsets of R_n, and diam $R_n<1/n$. Since $\overline{V} \cap B$ is compact, it follows that there is one and only one point z such that for each positive integer n, $z \in \overline{R}_n \cap B$. Clearly $z \in W \cap B$. It follows with the aid of Corollary 1.1 that there exists a sequence $K_1, K_2, K_3, \cdots$ such that (1) K_1 is a compact continuum containing y, intersecting D_2, and contained in D_1, and (2) if n is a positive integer, then K_{n+1} is a compact continuum intersecting both K_n and D_{n+2} and contained in D_{n+1}. Let K be $\bigcup_{n=1}^{\infty} K_n \cup \{z\}$; then K is a compact continuum such that $z \in K$, $y \in K$, and $K-\{z\} \subset U$.

Corollary 1.3. *If U is a connected open set, M is a compact continuum intersecting U, and $z \in U-M$, then there is a compact continuum K such that $z \in K$, $K \subset U$, and K has one and only one point in common with M.*

The statement that g is a *chain* means that g is a finite collection $\{g_1, g_2, \cdots, g_n\}$ of compact continua such that (1) if i and j are positive integers, $i<j \leqq n$, then g_i and g_j intersect if and only if $j=i+1$ and if g_i and g_j intersect, then they have one and only one point in common, and (2) n is large. The sets $g_1, g_2, \cdots$, and g_n are the *links* of the chain g, and g_1 and g_n are the *endlinks* of g.

Lemma 1.4. *If x and y are distinct points of a connected open set U and ϵ is a positive number, then there is a chain g such that* (1) *x belongs to one endlink of g and y belongs to the other endlink of g, and* (2) *each link of g is a subset of U and has diameter less than ϵ.*

Lemma 1.4 follows easily with the aid of Corollary 1.3.

2. **Consequences of Axioms** 1–3. The statement that α is a *closed chain* means that α is a finite set $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of compact continua such that (1) if i and j are positive integers, $i<j \leqq n$, then α_i and α_j intersect if and only if either $j=i+1$ or $i=1$ and $j=n$, and if g_i and g_j intersect, then they have one and only one point in common, and (2) n is large. The sets $\alpha_1, \alpha_2, \cdots$, and α_n are the *links* of the closed chain α.

If M is a closed proper subset of S, then by a *complementary domain* of M is meant a component of $S-M$.

If $\mathfrak{A}$ is a collection of sets, then $\mathfrak{A}^*$ denotes the union of the sets of the collection $\mathfrak{A}$.

Axiom 3. If α is a closed chain, then there exist two and only two complementary domains of α^*, each of which has a boundary point in each link of α, and every other complementary domain of α^* has for its boundary a subset of some one link of α.

If α is a closed chain, then by a *principal complementary domain* of α^* is meant a complementary domain of α^* which has a boundary point in each link of α. If α is a closed chain, then α^* has two and only two principal complementary domains.

Lemma 2.1. *Suppose that α is a closed chain $\{\alpha_1, \alpha_2, \cdots, \alpha_r\}$, D is a principal complementary domain of α^*, n is a positive integer less than r such that if $g=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, then both g and $\alpha-g$ are chains, and $[(\alpha-g)^*-(\alpha_1 \cup \alpha_n)]$ is connected, and h is a chain $\{h_1, h_2, \cdots, h_m\}$ such that* (1) *if $1 \leqq i \leqq m$ and $1 \leqq j \leqq r$, then h_i and α_j intersect if and only if either $(i, j)=(1, 1)$ or $(i, j)=(m, n)$ and if h_i and α_j intersect, then they have only one point in common, and* (2) *for $1 \leqq i \leqq m$, $(h_i-\alpha^*) \subset D$. Then there is one and only one principal complementary domain of $(g \cup h)^*$ which is a subset of D.*

Proof. Let E be the principal complementary domain of α^* distinct from D. Since $(g\cup h)^*$ and E are disjoint, E is a subset of a component U of $S-(g\cup h)^*$; it is easily seen that U is a principal complementary domain of $(g\cup h)^*$. With the aid of the hypothesis, it follows that $[(\bigcup_{i=n+1}^{r} \alpha_i)-(\alpha_1\cup\alpha_n)]\subset U$. Then if I is the principal complementary domain of $(g\cup h)^*$ distinct from U, I and α^* are disjoint; since some link of h is a subset of D, it follows that $I\subset D$. Clearly U is not a subset of D.

LEMMA 2.2. *Suppose that α, g, h, and D satisfy the hypothesis of Lemma 2.1, $g^*-(\alpha_1\cup\alpha_n)$ is connected, and k is a chain $\{k_1, k_2, \cdots, k_j\}$ such that k_1 intersects a link of g but does not intersect an endlink of g, k_j intersects a link of $\alpha-g$ but does not intersect an endlink of g, and if $1<i<j$, $k_i\subset D$. Then h^* and k^* intersect.*

Proof. Suppose that the notation is as in the proof of Lemma 2.1, and let f be the chain whose links are those of $\alpha-g$ together with α_1 and α_n. Let D_1 and D_2 be the principal complementary domains of $(g\cup h)^*$ and $(f\cup h)^*$, respectively, which are subsets of D. It follows, with the aid of results established in the proof of Lemma 2.1, that D_1 and D_2 are disjoint.

Suppose that h^* and k^* are disjoint. With the aid of Theorem 1.2, it follows that there exists a chain s, where $s=\{s_1, s_2, \cdots, s_t\}$, such that (1) s_1 has one and only one point q_1 in common with g^*, $s_1-\{q_1\}$ is a connected subset of D, and s_1 intersects neither α_1 nor α_n, (2) s_t has one and only one point q_t in common with f^*, $s_t-\{q_t\}$ is a connected subset of D, and s_t intersects neither α_1 nor α_n, (3) if $1<i<t$, then $s_i\subset D$, and (4) h^* and s^* are disjoint. It may be shown that s^* intersects neither D_1 nor D_2. There exist two chains u and v such that (1) u has the same relation to g, h, and D_1 as s has to g, f, and D, and (2) v has the same relation to f, h, and D_2 as s has to g, f, and D. There exists a closed chain β such that (1) each of s, u, and v is a subset of β, (2) each link of β is either a link of one of s, u, and v, or a subset of $\alpha^*\cup h^*$, (3) $\alpha^*-\beta^*$ has at most two components, and if there are two, α_1 and α_n are contained in different components of $\alpha^*-\beta^*$, and (4) $h^*-\beta^*$ has at most two components, and if there are two, one intersects α_1 and the other intersects α_n. If E is the principal complementary domain of α^* distinct from D, then E and β^* are disjoint, and E is a subset of a component U of $S-\beta^*$. Let I be a principal complementary domain of β^* such that I is distinct from U. Now $E\cup(\alpha^*-\beta^*)$ is connected and hence is contained in U; it follows that I and α^* are disjoint. It is easily seen that I and h^* are disjoint. Since some link of β is a subset of D_1 and some link of β is a subset of D_2, then I intersects both D_1 and D_2. But this involves a contradiction, and Lemma 2.2 is proved.

If x and y are distinct points and α is a closed chain, then α^* *Separates* x and y if and only if x and y belong to different principal complementary domains of α^*. If x is a point, M is a point set, and α is a closed chain, then

α^* *Separates* x and M if and only if x and M are contained in different principal complementary domains of α^*, and a similar definition holds for two point sets.

Theorem 2.3. *Suppose that x and y are distinct points, U is an open set, α is a closed chain such that α^* is a subset of U and separates the points x and y, V is a connected open set such that $x \in V$, and $\overline{V}$ and α^* are disjoint, and ϵ is a positive number. Then there exists a closed chain γ such that γ^* is a subset of U and Separates y and $\overline{V}$, and each link of γ has diameter less than ϵ.*

Proof. Suppose that $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Let D and E be the principal complementary domains of α^* containing x and y, respectively. Suppose that r and t are positive integers, $r < t < n$, such that r, $(t-r)$, and $(n-t)$ are large. Let A and B denote $\alpha_r \cup \alpha_{r+1} \cup \alpha_{r+2}$ and $\alpha_t \cup \alpha_{t+1} \cup \alpha_{t+2}$, respectively. With the aid of Theorem 1.2 and Axiom 3, it follows that there exists a chain h, where $h = \{h_1, h_2, \cdots, h_m\}$, such that (1) h_1 has one and only one point a_0 in common with A and does not intersect any link of α distinct from α_r, α_{r+1}, and α_{r+2}, (2) h_m has one and only one point a_m in common with B and does not intersect any link of α distinct from α_t, α_{t+1}, and α_{t+2}, (3) if, for $1 \leqq i < m$, a_i is the point common to h_i and h_{i+1}, then, for $1 \leqq i \leqq m$, h_i is irreducible between a_{i-1} and a_i, and (4) $h^* - \{a_0, a_m\} \subset D$. There exists a chain k, where $k = \{k_1, k_2, \cdots, k_j\}$, such that (1) k_1 has one and only one point b_0 in common with A and does not intersect any link of α distinct from α_r, α_{r+1}, and α_{r+2}, (2) k_j has one and only one point b_m in common with B and does not intersect any link of α distinct from α_t, α_{t+1}, and α_{t+2}, (3) if, for $1 \leqq i < j$, b_i is the point common to k_i and k_{i+1}, then for $1 \leqq i \leqq j$, k_i is irreducible between b_{i-1} and b_i, and (4) $k^* - \{b_0, b_j\} \subset E$. Since $h^* - \{a_0, a_m\} \subset D$, there exists a compact continuum H lying in D containing x, and intersecting h^*. Similarly, there is a compact continuum K lying in E, containing y, and intersecting k^*. There exists a connected open set R such that $R \subset U$, $\alpha_1 \subset R$, $R \cap \alpha^* \subset \alpha_2 \cup \alpha_1 \cup \alpha_n$, and R intersects no one of h^*, k^*, A, B, H, K, and $\overline{V}$. With the aid of Theorem 1.2, it follows that there exists a chain g_1, where $g_1 = \{g_{11}, g_{12}, \cdots, g_{1n_1}\}$, such that (1) each link of g_1 is a subset of R and has diameter less than ϵ, (2) g_{11} has one and only one point x_1 in common with $(\alpha - \{\alpha_1\})^*$ and $x_1 \in \alpha_2$, (3) g_{1n_1} has one and only one point x_2 in common with $(\alpha - \{\alpha_1\})^*$ and $x_2 \in \alpha_n$, and (4) if $1 < i < n_1$, then g_{1i} and $(\alpha - \{\alpha_1\})^*$ are disjoint. Let γ_1 be the closed chain $(\alpha - \{\alpha_1\}) \cup g_1$. Since $(h^* - \{a_0, a_m\}) \cup H$ is a connected set not intersecting γ_1^*, containing x, and having limit points both in A and in B, then by Axiom 3, x belongs to some principal complementary domain of γ_1^*. By a similar argument, y belongs to a principal complementary domain of γ_1^*.

Suppose that x and y are not Separated by γ_1^*. Then let β be the closed chain whose links are those of h and those of k, together with A and B, except that if $b_0 = a_0$, omit A as a link, and if $b_j = a_m$, omit B as a link. By

Axiom 3, there is a principal complementary domain I of β^* which contains each link α_i of α for $1 \leqq i < (r-1)$ and $(t+1) < i \leqq n$. Clearly then $g_1^* \subset I$. Since x and y belong to the same principal complementary domain of γ_1^*, it may be seen that there is a chain s, where $s = \{s_1, s_2, \cdots, s_q\}$, such that (1) for some positive integer p_1, $1 < p_1 < m$, s_1 intersects h_{p_1}, (2) for some positive integer p_2, $1 < p_2 < j$, s_q intersects k_{p_2}, (3) if $1 < i < q$, then s_i and β^* are disjoint, and (4) s^* and γ_1^* are disjoint. Since s^* intersects α^*, then s^* intersects α_1 and hence for $1 < i < q$, $s_i \subset I$. But this contradicts Lemma 2.2, and therefore γ_1^* Separates x and y.

Now γ_1^* Separates y and $\overline{V}$ and is a subset of U, and γ_1 has at most $(n-1)$ links, each of which has diameter greater than or equal to ϵ. This process may be repeated, using the closed chain γ_1 and the link α_2 of γ_1 in place of the closed chain α and the link α_1 of α. There results a closed chain γ_2, at most $(n-2)$ links of which have diameter greater than or equal to ϵ, and such that γ_2^* is a subset of U and Separates y and $\overline{V}$. Let this process continue; it terminates after a finite number of steps with a closed chain γ such that each link of γ has diameter less than ϵ, $\gamma^* \subset U$, and γ^* Separates y and $\overline{V}$.

Suppose that p is a point and α is a closed chain. Then α is *bounded with respect to* p if and only if p belongs to some principal complementary domain of α^*. If α is bounded with respect to p, then the *interior, with respect to* p, *of* α^* is the principal complementary domain of α^* which does not contain p, and the *exterior, with respect to* p, *of* α^* is the principal complementary domain of α^* which contains p.

Throughout the remainder of this section, let ω be some definite point, and let the terms "bounded," "interior," and "exterior" mean "bounded with respect to ω," "interior with respect to ω," and "exterior with respect to ω," respectively. If α is a bounded closed chain, int α^* and ext α^* denote the interior and exterior, respectively, of α^*.

LEMMA 2.4. *Suppose that U_1 and U_2 are open sets, α_1 and α_2 are bounded closed chains such that $\alpha_1^* \subset U_1$ and $\alpha_2^* \subset U_2$, V_1 and V_2 are connected open sets such that $\omega \in V_1 \cap V_2$, $\overline{V}_1 \subset \text{ext } \alpha_1^*$, and $\overline{V}_2 \subset \text{ext } \alpha_2^*$, and $p \in (\text{int } \alpha_1^*) \cap (\text{int } \alpha_2^*)$. Then there exist two bounded closed chains γ_1 and γ_2 such that $\gamma_1^* \subset U_1$, $\gamma_2^* \subset U_2$, $\overline{V}_1 \subset \text{ext } \gamma_1^*$, $\overline{V}_2 \subset \text{ext } \gamma_2^*$, $p \in (\text{int } \gamma_1^*) \cap (\text{int } \gamma_2^*)$, and if x is a link of γ_1 intersecting $\overline{V}_2$, then there exists a bounded closed chain σ_x such that* (1) *each link of σ_x is either a link of γ_1, or a link of γ_2 which lies, except possibly for one point, in the interior of γ_1^*, and* (2) *p belongs to the interior of σ_x^* and $x \subset (\text{ext } \sigma_x^*)$.*

Proof. Let W_1 be an open set such that $\alpha_1^* \subset W_1$, $\overline{W}_1 \subset U_1$, and $\overline{W}_1$ and $\overline{V}_1$ are disjoint. With the aid of Theorem 2.3, it follows that there exists a closed chain β_1 such that (1) $\beta_1^* \subset W_1$ and $p \in \text{int } \beta_1^*$, (2) each link of β_1 which intersects $\overline{V}_2$ belongs to a chain of links of β_1, no one of which intersects α_2^*, and (3) each link of β_1 is irreducible between the two points it has in common with other links of β_1. By taking unions of links of β_1, if necessary, there results a closed chain μ_1 such that $\mu_1^*\subset U_1$, $p\in\text{int}\,\mu_1^*$, μ_1^* and $\overline{V}_1$ are disjoint, and α_2^* contains no point common to two links of μ_1. Let F be an open set such that $\alpha_2^*\subset F$ and $\overline{F}$ contains no point common to two links of μ_1. As in the proof of Theorem 2.3, it may be shown that there is a closed chain γ_2 such that (1) $\gamma_2^*\subset F\cap U_2$, γ_2^* and $\overline{V}_2$ are disjoint, and $p\in\text{int}\,\gamma_2^*$, (2) each link of γ_2 is irreducible between the two points it has in common with other links of γ_2, (3) no link of γ_2 intersects two distinct links of μ_1, and (4) the links of γ_2 have small diameters. Let γ_1 be the closed chain such that y is a link of γ_1 if and only if for some link z of μ_1, y is the union of z and all the links of γ_2 which intersect z.

Suppose that x is a link of γ_1 intersecting $\overline{V}_2$. It may be shown, principally with the aid of Lemma 2.1 or of a modification of it, that there exist a chain h_x of links of γ_2 and a closed chain σ_x whose links are those of h_x together with certain links of γ_1 such that σ_x satisfies the conditions of the conclusion of Lemma 2.4.

THEOREM 2.5. *Suppose that U_1 and U_2 are open sets, α_1 and α_2 are bounded closed chains such that $\alpha_1^*\subset U_1$ and $\alpha_2^*\subset U_2$, V_1 and V_2 are connected open sets such that $\omega\in V_1\cap V_2$, $\overline{V}_1\subset\text{ext}\,\alpha_1^*$, and $\overline{V}_2\subset\text{ext}\,\alpha_2^*$, and $p\in(\text{int}\,\alpha_1^*)\cap(\text{int}\,\alpha_2^*)$. Then there exists a bounded closed chain β such that $\beta^*\subset U_1\cup U_2$, $p\in\text{int}\,\beta^*$, and $\overline{V}_1\cup\overline{V}_2\subset\text{ext}\,\beta^*$.*

Proof. Let γ_1 and γ_2 be bounded closed chains satisfying the conditions of Lemma 2.4 relative to U_1, U_2, V_1, V_2, and p. Let $x_1, x_2, \cdots$, and x_n be the links of γ_1 which intersect $\overline{V}_2$. By Lemma 2.4, there is a bounded closed chain μ_1, each link of which either is a link of γ_1 or is a link of γ_2 which, except possibly for one point, lies in the interior of γ_1^*, and such that $p\in\text{int}\,\mu_1^*$ and $x_1\subset\text{ext}\,\mu_1^*$; note that $\text{int}\,\mu_1^*\subset\text{int}\,\gamma_1^*$. Now if for some positive integer m, $m\leqq n$, x_m intersects μ_1^*, then x_m is a link of μ_1. Let n_2 be the least positive integer i such that x_i intersects μ_1^*. Again by Lemma 2.4, there is a bounded closed chain μ_2, each link of which either is a link of μ_1 or is a link of γ_2 which, except possibly for one point, lies in the interior of μ_1^*, and such that $p\in\text{int}\,\mu_2^*$, $x_{n_2}\subset\text{ext}\,\mu_2^*$, and $\text{int}\,\mu_2^*\subset\text{int}\,\mu_1^*$. Neither x_1 nor x_2 is a link of μ_2. Let this process be continued; there results a bounded closed chain β such that $p\in\text{int}\,\beta^*$, $\beta^*\subset U_1\cup U_2$, and no one of $x_1, x_2, \cdots$, and x_n intersects β^*. It follows that $\overline{V}_1\cup\overline{V}_2\subset\text{ext}\,\beta^*$.

THEOREM 2.6. *Suppose that U_1 and U_2 are open sets, α_1 and α_2 are bounded closed chains such that $\alpha_1^*\subset U_1$ and $\alpha_2^*\subset U_2$, and V_1 and V_2 are intersecting connected open sets such that $\overline{V}_1\subset\text{int}\,\alpha_1^*$ and $\overline{V}_2\subset\text{int}\,\alpha_2^*$. Then there exists a bounded closed chain γ such that $\gamma^*\subset U_1\cup U_2$ and $\overline{V}_1\cup\overline{V}_2\subset\text{int}\,\gamma^*$.*

Proof. Let q be a point of $V_1\cap V_2$, and apply Theorem 2.5, regarding "interior" and "exterior" as "interior with respect to q" and "exterior with respect to q," respectively.

3. Consequences of Axioms 1–4.

AXIOM 4. If x and y are distinct points and U is an open set containing x, then there exists a closed chain α such that $\alpha^* \subset U$ and α^* Separates x and y.

THEOREM 3.1. *Suppose that H and K are disjoint compact point sets, $x \in H$, and $y \in K$. Then there is a closed chain γ such that γ^* Separates x and y and γ^* does not intersect $H \cup K$.*

Proof. Let D be the component of $S-K$ which contains x. If $p \in D$, then by Axiom 4, there is a closed chain α_p such that α_p^* is a subset of D and Separates p and y. There is a connected open set V_p such that $p \in V_p$, $\overline{V}_p \subset D$, and $\overline{V}_p$ and α_p^* are disjoint. Let $\mathcal{V}$ be $\{V_p : p \in D\}$; since $D \cap H$ is compact, there is a finite subset $\{p_1, p_2, \cdots, p_n\}$ of $D \cap H$ such that $\{V_{p_1}, V_{p_2}, \cdots, V_{p_n}\}$ covers $D \cap H$. Let i be a positive integer, $i \leqq n$. Since D is connected, there is a chain $\{U_{i1}, U_{i2}, \cdots, U_{in_i}\}$ of open sets of $\mathcal{V}$ such that $x \in U_{i1}$ and $U_{in_i} = V_{p_i}$. For each positive integer j, $j \leqq n_i$, there is a point p_{ij} of D such that $U_{ij} = V_{p_{ij}}$, and let V_{ij} and α_{ij} denote $V_{p_{ij}}$ and $\alpha_{p_{ij}}$, respectively. By repeated application of Theorem 2.6 to the chains $\alpha_{i1}, \alpha_{i2}, \cdots$, and α_{in_i}, it follows that there exists a closed chain β_i such that $\beta_i^* \subset D$, β_i^* Separates x and y, and $\bigcup_{j=1}^{n_i} \overline{V}_{ij}$ and β_i^* are disjoint. For each positive integer i, $i \leqq n$, let W_i denote $\bigcup_{j=1}^{n_i} V_{ij}$. Since for each positive integer i, $i \leqq n$, $x \in W_i$, it follows with the aid of Theorem 2.6 that there exists a closed chain γ such that $\gamma^* \subset D$, γ^* Separates x and y, and $\bigcup_{i=1}^{n} \overline{W}_i$ and γ^* are disjoint. Since $H \cap D \subset \bigcup_{i=1}^{n} W_i$, then γ^* and H are disjoint, and since $\gamma^* \subset D$, then K and γ^* are disjoint.

THEOREM 3.2. *If x and y are distinct points, H and K are disjoint compact point sets, and neither H nor K separates x from y, then $H \cup K$ does not separate x from y.*

Proof. There exist compact continua A and B, each containing both x and y, such that A and H are disjoint, and B and K are disjoint. Suppose that B intersects H, for otherwise the conclusion clearly holds. Suppose that $p \in B \cap H$; since H and $K \cup A$ are disjoint compact sets, then by Theorem 3.1, there is a closed chain γ_p such that γ_p^* Separates p and x, and γ_p^* and $H \cup K \cup A$ are disjoint. Let V_p be a connected open set containing p and such that $\overline{V}_p$ and γ_p^* are disjoint. Now $\{V_p : p \in B \cap H\}$ covers the compact set $B \cap H$ and there is a finite subset $\{p_1, p_2, \cdots, p_n\}$ of $B \cap H$ such that $\{V_{p_1}, V_{p_2}, \cdots, V_{p_n}\}$ covers $B \cap H$. Let W denote $\bigcup_{i=1}^{n} V_{p_i}$. W has only finitely many components and it follows, with the aid of Theorem 2.6, that there is a finite set $\{\beta_1, \beta_2, \cdots, \beta_m\}$ of closed chains such that (1) if U is a component of W, then for some positive integer j, $j \leqq m$, β_j^* Separates x and $\overline{U}$, and (2) for each positive integer j, $j \leqq m$, $H \cup K \cup A$ and β_j^* are disjoint. Let $K_1, K_2, \cdots, K_r$ be the components of $\bigcup_{i=1}^{m} \beta_i^*$, and let K_0, K_{r+1}, and K_{r+2} denote $\{x\}$, $\{y\}$, and H, respectively. Let δ be the minimum of $\{\operatorname{dist}(K_i, K_j) : 0 \leqq i \leqq r+2,\ 0 \leqq j \leqq r+2, \textit{ and } i \neq j\}$.

There exists a chain f, where $f=\{f_1, f_2, \cdots, f_d\}$, such that (1) $x\in f_1$ and $y\in f_d$, (2) each link of f has diameter less than δ/N, where N is some large positive integer, and (3) K and f^* are disjoint and $f^*\cap H$ is a subset of W. Let s_1 be the least positive integer i such that f_i intersects $\bigcup_{1=q}^r K_q$. Since no link of f intersects distinct components of $\bigcup_{q=1}^r K_q$, there is a positive integer m_1 such that f_{s_1} intersects K_{m_1} and no other component of $\bigcup_{q=1}^r K_q$. Let t_1 be the largest positive integer i such that f_i intersects K_{m_1}. Note that $1<s_1\leqq t_1<d$. Let g_1 denote $\{f_1, f_2, \cdots, f_{s_1}\}$. Let s_2 be the least positive integer i such that $i>t_1$ and f_i intersects $\bigcup_{q=1}^r K_q$. There is a positive integer m_2 such that f_{s_2} intersects K_{m_2}. Let t_2 be the largest positive integer i such that f_i intersects K_{m_2}. Then $t_1<s_2\leqq t_2<d$. Let g_2 denote $\{f_{t_1}, f_{t_1+1}, \cdots, f_{s_2}\}$. Let this process be continued and suppose that u is the largest positive integer i such that f_i intersects $\bigcup_{q=1}^r K_q$, and let g_{u+1} denote $\{f_{t_u}, f_{t_u+1}, \cdots, f_d\}$. It is clear that

$$\left(\bigcup_{i=1}^{u+1} g_i^*\right)\cup\left(\bigcup_{i=1}^{u} K_{m_i}\right)$$

is a continuum containing x and y and intersecting neither H nor K. Therefore $H\cup K$ does not separate x and y.

THEOREM 3.3. *If x and y are distinct points, neither of the compact point sets H and K separates x and y, and $H\cap K$ is connected, then $H\cup K$ does not separate x and y.*

Proof. There exist compact continua A and B, each containing both x and y, such that A and H are disjoint, and K and B are disjoint. Let b be a point of $H\cap K$; by Theorem 3.1, there is a closed chain α such that α^* Separates x and b and α^* does not intersect $A\cup B\cup(H\cap K)$. There exists a connected open set V such that $b\in V$, $H\cap K\subset V$, and $\overline{V}$ and α^* are disjoint. There is, by Theorem 2.3, a closed chain β such that β^* Separates $A\cup B$ and $\overline{V}$, and each link of β has diameter less than $(1/N)$ dist $(H-V, K-V)$, where N is a large positive integer. There exists a finite set $\{f_1, f_2, \cdots, f_j\}$ such that (1) for $1\leqq i\leqq j$, f_i is a chain of links of β and f_i^* intersects at most one of H and K, (2) if $1\leqq i<r\leqq j$, then f_i^* and f_r^* are disjoint, and (3) if a link of β intersects $H\cup K$, then it belongs to some one of $f_1, f_2, \cdots$, and f_j. If $p\in H\cap B$, there is a closed chain γ_p such that γ_p^* Separates p and x, γ_p^* does not intersect either A, K, or $\bigcup_{i=1}^j f_i^*$, and the links of γ_p have small diameters. For each point p of $H\cap B$, let U_p be a connected open set containing p such that $\overline{U}_p$ intersects neither γ_p^* nor β^*. Then there is a finite subset $\{p_1, p_2, \cdots, p_n\}$ of $H\cap B$ such that $\{U_{p_1}, U_{p_2}, \cdots, U_{p_n}\}$ covers $H\cap B$. For each positive integer i, $i\leqq n$, let γ_i be γ_{p_i} and let U_i be U_{p_i}. Then it may be shown, by a process similar to that used to establish Lemma 2.4, that for each positive integer i, $i\leqq n$, there is a chain h_i of links of γ_i, where $h_i=\{h_{i1}, h_{i2}, \cdots, h_{in_i}\}$, and

a chain k_i of links of β, where $k_i=\{k_{i1}, k_{i2}, \cdots, k_{im_i}\}$, such that if μ_i is the chain whose links are those of h_i except for its endlinks, those of k_i except for its endlinks, $h_{i1}\cup k_{i1}$, and $h_{in_i}\cup k_{im_i}$, then μ_i^* intersects neither A nor K, and μ_i^* Separates x and $\overline{U}_i$.

Since B is a continuum, there is a chain g, one of whose endlinks contains x and the other of which contains y, such that (1) g^* intersects neither K nor β^*, (2) if a link of g intersects H, then it is a subset of $\bigcup_{i=1}^n U_i$, and (3) the links of g are small. By a process similar to that used in the proof of Theorem 3.2, it may be shown that there exist a finite set $\{s_1, s_2, \cdots, s_r\}$, each element of which is a chain of links of g, and a finite set $\{t_1, t_2, \cdots, t_q\}$, each element of which is a chain of links of some one of $\mu_1, \mu_2, \cdots$, and μ_n, such that

$$\left(\bigcup_{i=1}^{r} s_i^*\right)\cup\left(\bigcup_{i=1}^{q} t_i^*\right)$$

is a continuum containing both x and y and intersecting neither H nor K. Then $H\cup K$ does not separate x and y.

THEOREM 3.4. *If H and K are compact continua and $H\cap K$ is not connected, then $H\cup K$ separates S.*

Proof. Suppose that A and B are closed disjoint point sets such that $H\cap K=A\cup B$. Let x and y be points of A and B, respectively. By Theorems 3.1 and 2.3, there exists a closed chain γ such that (1) γ^* Separates x and y, (2) γ^* does not intersect $H\cap K$, and (3) there exists a finite set $\{g_1, g_2, \cdots, g_n\}$ of chains of links of γ such that (i) $g_1^*, g_2^*, \cdots$, and g_n^* are mutually disjoint, (ii) each link of γ which intersects $H\cup K$ belongs to some chain of $g_1, g_2, \cdots$, and g_n, and (iii) no one of $g_1^*, g_2^*, \cdots$, and g_n^* intersects both H and K, and each of them intersects one of H and K. Let $\{\gamma_{n_1}, \gamma_{n_2}, \cdots, \gamma_{n_j}\}$ be the set of links of γ which belong to no one of $g_1, g_2, \cdots$, and g_n.

Suppose that $H\cup K$ separates no two of $\gamma_{n_1}, \gamma_{n_2}, \cdots$, and γ_{n_j}. Let H_0 be the union of those sets of $g_1^*, g_2^*, \cdots$, and g_n^* which intersect H, and let K_0 be the union of the other sets of $g_1^*, g_2^*, \cdots$, and g_n^*. For each pair s and t of positive integers, $s\leqq j$ and $t\leqq j$, let L_{st} be a compact continuum intersecting both γ_{n_s} and γ_{n_t} and not intersecting $H\cup K$. Let L be

$$\left(\bigcup_{i=1}^{j} \gamma_{n_i}\right)\cup\left(\bigcup\{L_{st}: 1\leqq s\leqq j,\ 1\leqq t\leqq j\}\right).$$

Clearly L, $H_0\cup L$, and $K_0\cup L$ are compact continua, $H_0\cup L$ and K are disjoint, and $K_0\cup L$ and H are disjoint. Therefore neither $H_0\cup L$ nor $K_0\cup L$ separates x from y. However $(H_0\cup L)\cup(K_0\cup L)$ has γ^* as a subset and contains neither x nor y, and hence separates x and y. Since the common part of $H_0\cup L$ and $K_0\cup L$ is the continuum L, this contradicts Theorem 3.3. It follows that $H\cup K$ separates S.

4. **An example.** In this section, there is described an example of a space satisfying Axioms 1–4 of this paper and in which there exists no arc. The example is a subspace of the Cartesian plane.

Suppose that n is a positive integer, and each of i and j is an integer. Let K_{nij} denote the square in the plane whose vertices are the points $(i/2^n, j/2^n)$, $([i+1]/2^n, j/2^n)$, $(i/2^n, [j+1]/2^n)$, and $([i+1]/2^n, [j+1]/2^n)$. Let A_{nij} be a pseudoarc (for a description of a pseudoarc, see [1]) which contains the two points $(i/2^n, j/2^n)$ and $([i+1]/2^n, [j+1]/2^n)$ and, except for these points, lies in the interior of K_{nij}, and let B_{nij} be a pseudoarc which contains the two points $(i/2^n, [j+1]/2^n)$ and $([i+1]/2^n, j/2^n)$ and, except for these points, lies in the interior of K_{nij}. Then let S denote $\bigcup\{A_{nij}\cup B_{nij}$: *$n$ is a positive integer and each of i and j is an integer*$\}$. It is easily seen that the space S satisfies Axioms 1–4 of this paper, and it is of some interest to note that S has dimension 1.

THEOREM 4.1. *S contains no arc.*

Proof. Suppose that S contains an arc A. S is the union of countably many pseudoarcs S_1, S_2, $\cdots$. If n is a positive integer and A and S_n intersect, then $A\cap S_n$ is closed and totally disconnected, for S_n contains no arc. Then the compact continuum A is the union of countably many closed totally disconnected sets. But this is contradictory [2, Chapter I, Theorem 44′]. Hence S contains no arc.

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