AREA AS THE INTEGRAL OF LENGTHS OF CONTOURS(1)

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Introduction. Let Q be the unit square in E^2 and $T:Q\to E^k$ a continuous mapping of Q into Euclidean k-space. Let $T=1\circ m$ be a monotone light factorization of T with monotone factor m, light factor 1, and middle space M. For a, $b\in M$ let $\Gamma(a, b, T)=\{f: [0, 1]\to M: f$ is continuous and $f(0)=a$, $f(1)=b\}$. The geodesic distance, $\mathbf{G}(a, b, T)$, between 1(a) and 1(b) on the surface T is defined by $\mathbf{G}(a, b, T)=\inf_{f\in\Gamma(a,b,T)}$ length $1\circ f$.

For a fixed point p of Q, let $D(t, G, T)=\{q\in Q: \mathbf{G}(m(p), m(q), T)>t\}$. Let $L(D(t, G, T))$ denote the length of T restricted to the boundary of $D(t, G, T)$ as defined by Cesari in [2](2) and $A(T)$ denote the Lebesgue area of T. This paper investigates the relationship between $A(T)$ and $\int_0^\infty L(D(t, G, T))dt$.

For nondegenerate mappings T of finite area, Theorem III asserts that if p is a point of Q such that $\mathbf{G}(m(p), b, T)$ is not infinite for every $b\in M$ different from $m(p)$ then equality holds. If T is a continuous mapping of finite area and $F=\{b\in M: G(m(p), b, T)<\infty\}$ then there is a monotone retraction $r_F: M\to\overline{F}$ of M onto the closure of F. If T_F is defined by $T_F=1\circ r_F\circ m$, Theorems IV and V assert that $A(T_F)=\int_0^\infty L(D(t, G, T_F))=\int_0^\infty L(D(t, G, T))dt$.

The paper consists of four parts. In Part I geodesic and μ_j-geodesic distance are defined and the relationship between them is discussed. These concepts have been used previously by Silverman in [10] and [11]. Some properties of the length of a mapping restricted to the boundary of an open set as defined by Cesari in [2] are stated and for a connected open set an inequality between this definition and the definition given by Federer in [6] is given.

In Part II the inequality $A(T)\geqq\int_0^\infty L(D(t, G, T))dt$ for nondegenerate mappings T is proved. The use of μ_j-geodesic distances makes it possible to give a proof of this inequality which is very similar to the proof of the Cavalieri inequality given by Cesari in [2].

Part III is concerned with proving Theorem III. Two examples are given to show the hypothesis of Theorem III are necessary. Morrey's representation theorem and a recent result found independently by Federer [7] and L. C. Young [13] are used in the proof of Theorem III.

In Part IV the cyclic additivity of $L(D(t, G, T))$ is discussed. Theorems

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IV and V are proved using Theorem III, and the cyclic additivity of Lebesgue area and $L(D(t, G, T))$.

Part I. Let f: $[0, 1] \to Y$ be a continuous mapping of the unit interval into a metric space Y with metric d. Let $t_0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_j$ be a collection of $j+1$ points of $[0, 1]$. Let $I_k = \{t: t_{k-1} \leqq t \leqq t_k\}$ for $k=1, 2, 3, \cdots, j$. The following notion of length is a slight modification of a length used in [10].

DEFINITION I. The μ_j-length of f is defined:

$$\mu_j\text{-length}\, f = \sup_{\substack{\text{all subdivisions} \\ \text{as above by } j+1 \text{ points}}} \sum_{k=1}^{j} \text{diameter}\, f(I_k).$$

The following properties of μ_j-length follow from the definition:

(a) μ_j-length $f \leqq \mu_{j+1}$ length f.

(b) $|\mu_j\text{-length } f - \mu_j\text{-length } g| \leqq 2j \sup_{t\in[0,1]} d(f(t), g(t))$.

(c) μ_j-length $f \leqq j$ diameter $f([0, 1])$.

(d) Let f, g be functions from $[0, 1]$ into Y such that $f(1) = g(0)$. Let $f \# g$ be defined by

$$f \# g(t) = \begin{Bmatrix} f(2t) \text{ if } 0 \leqq t \leqq \dfrac{1}{2} \\ \\ g(2t-1) \text{ if } \dfrac{1}{2} \leqq t \leqq 1 \end{Bmatrix}.$$

Then μ_j-length $f \# g \leqq \mu_j$-length $f + \mu_j$-length g.

Let T: $Q \to E^k$ be a continuous mapping of the unit square Q in E^2 into E^k. Let $T = 1 \circ m$, m: $Q \to M$, 1: $M \to E^k$ be a monotone light factorization of T with monotone factor m, light factor 1 and middle space M. The middle space M will be understood to be metrized by the metric $d(a, b)$ defined as follows: let a and b be points of M, and let C denote the class of continua K in Q that meet $m^{-1}(a)$ and $m^{-1}(b)$, then $d(a, b) = \inf_{K\in C}$ diameter $T(K)$. For $a, b \in M$ let $\Gamma(a, b, T) = \{f\colon [0, 1] \to M\colon f(0) = a, f(1) = b\}$.

DEFINITION II. The μ_j-geodesic distance between $T(p)$ and $T(q)$ on T, $\mu(j, p, q, T)$, is defined:

$$\mu(j, p, q, T) = \inf_{f\in\Gamma(m(p), m(q), T)} \mu_j\text{-length } 1 \circ f.$$

The function $\mu(j, p, q, T)$ has the following properties:

(a) $\mu(j, p, q, T) \leqq \mu(j+1, p, q, T)$.

(b) $\mu(j, p, q, T) \leqq j \inf_{f\in\Gamma(m(p), m(q), T)}$ diameter $1 \circ f$ $([0, 1])$.

(c) $\mu(j, p, \cdot, T)$ is continuous.

(d) Let T and T' be continuous mappings. Let $T = 1 \circ m$ and $T' = 1' \circ m'$ be monotone light factorizations of T and T' with middle spaces

M and M'. If there is a homeomorphism $H: M \to M'$ such that $m' = H \circ m$ then

$$|\mu(j, p, q, T) - \mu(j, p, q, T')| \leqq 2j|T - T'|.$$

(e) If T is quasi linear and q and q' belong to the same triangle on which T is linear then

$$|\mu(j, p, q, T) - \mu(j, p, q', T)| \leqq |T(q) - T(q')|.$$

(f) If T is Lipschitzian with Lipschitz constant K then

$$|\mu(j, p, q, T) - \mu(j, p, q', T)| \leqq K|q - q'|.$$

For each mapping T two geodesic distance functions will be used, one defined on the middle space M, and the other defined on the parameter domain Q. The geodesic function on the middle space, $\mathbf{G}(a, b, T)$, is defined by $\mathbf{G}(a, b, T) = \inf_{f \in \Gamma(a,b,T)}$ length $1 \circ f$. The geodesic distance function defined on Q, $G(p, q, T)$, is defined by

$$G(p, q, T) = \inf_{f \in \Gamma(m(p), m(q), T)} \text{length } 1 \circ f = \mathbf{G}(m(p), m(q), T).$$

Let $\mu(p, q, T) = \lim_{j \to \infty} \mu(j, p, q, T)$. (Note that the limit exists since $\mu(j+1, p, q, T) \geqq \mu(j, p, q, T)$.)

LEMMA I.1. $G(p, q, T) = \mu(p, q, T)$.

Notice that for a collection of curves it is not true that $\inf_\alpha$ length $f_\alpha = \lim_{j \to \infty} \inf_\alpha \mu_j$-length f_α. Let S_k be the circle in E^2 of diameter $1/k$ and center $(0, 1/2k)$. Let f_k be the function which traverses S_k k times starting from $(0, 0)$. Then μ_j-length $f_k \leqq j/k$, so $\inf_k \mu_j$-length $f_k = 0$ for every j but $\inf_k$ length $f_k = \pi$.

Proof of Lemma I.1. $G(p, q, T) \geqq \mu(j, p, q, T)$ for every j hence $G(p, q, T) \geqq \mu(p, q, T)$. To show the other inequality it may be assumed that $\mu(p, q, T) < \infty$. Let $f_j \in \Gamma(m(p), m(q), T)$ be such that μ_j-length $1 \circ f_j - \mu(j, p, q, T) < 1/j$. Since μ_k-length $1 \circ f_k \geqq \mu_j$-length $1 \circ f_k$ for $k \geqq j$, μ_j-length $1 \circ f_k < \mu(p, q, T) + 1$ for $k \geqq j$.

Let $\{f_\alpha\}$ be a collection of functions $f_\alpha: [0, 1] \to Y$ where Y is a compact metric space. According to [8] a necessary and sufficient condition that there exist a collection $\{f'_\alpha\}$ which is relatively compact in the topology of uniform convergence and such that f_α and f'_α are Frechet equivalent is that the $\{f_\alpha\}$ be equally divisible i.e. for each $\epsilon > 0$ there is an integer n such that for each function f_α the interval $[0, 1]$ can be subdivided into n subintervals on which the oscillation of f_α is less than ϵ. (The same n must work for all the f_α's.)

Let ϵ be a positive number and N an integer greater than $(\mu(p, q, T) + 1)\epsilon^{-1}$. For each integer $k > N$ the interval $[0, 1]$ can be divided into fewer than N subintervals such that oscillation $1 \circ f_k$ is $\leqq \epsilon$ on each interval in the following manner:

Let $t_1 = \sup\{t\colon \text{diameter } 1 \circ f_k([s\colon 0 \leqq s \leqq t]) \leqq \epsilon\}$.

Let $t_2 = \sup\{t\colon \text{diameter } 1 \circ f_k([s\colon t_1 \leqq s \leqq t]) \leqq \epsilon\}$, etc.

There are at most $N-1$ of these t_i's which are less than one. Suppose not. Then

$$\text{diameter } 1 \circ f_k([t_i \leqq s \leqq t_{i+1}]) = \epsilon \text{ if } t_{i+1} < 1$$

and

$$\mu_N\text{-length } 1 \circ f_k \geqq N\epsilon > \mu(p, q, T) + 1 > \mu_N\text{-length } 1 \circ f_k,$$

a contradiction.

Since it is possible to find a number N' so that for each of the finite number of functions $1 \circ f_1, \cdots, 1 \circ f_N$ the interval $[0, 1]$ may be subdivided into N' intervals on which the corresponding functions have oscillation less than ϵ, it's seen that the collection $\{1 \circ f_k\}_{k=1}^{\infty}$ is equally divisible.

From the definition of the metric in M, diameter $1 \circ f_k([t_i \leqq s \leqq t_{i+1}]) \geqq$ diameter $f_k([t_i \leqq s \leqq t_{i+1}])$. Hence the $\{f_k\}$ are equally divisible. Let $\{f_k^*\}$ be a collection of functions such that f_k and f_k^* are Frechet equivalent and the collection $\{f_k^*\}$ is relatively compact. By selecting a subsequence and relabeling, it may be assumed that the f_k^* converge to a function f. It's seen that f_k^* and $f \in \Gamma(m(p), m(q), T)$ and

$$\mu_j\text{-length } 1 \circ f_k = \mu_j\text{-length } 1 \circ f_k^*.$$

If $k \geqq j$,

$$\mu_j\text{-length } 1 \circ f_k^* \leqq \mu_k\text{-length } 1 \circ f_k^* \leqq \mu(p, q, T) + 1/k.$$

By property (b) of μ_j-lengths

$$\lim_{k \to \infty} \mu_j\text{-length } 1 \circ f_k^* = \mu_j\text{-length } 1 \circ f \leqq \mu(p, q, T),$$

$$\lim_{j \to \infty} \mu_j\text{-length } 1 \circ f = \text{length } 1 \circ f \leqq \mu(p, q, T).$$

Hence $G(p, q, T) \leqq \mu(p, q, T)$.

COROLLARY I (OF THE PROOF). *For $a, b, \in M$ there is an $f \in \Gamma(a, b, T)$ such that length* $1 \circ f = G(a, b, T)$.

COROLLARY II. *$G(p, \cdot, T)$ and $G(m(p), \cdot, T)$ are lower semicontinuous.*

The functions $\mu(j, p, \cdot, T)$ are an increasing sequence of continuous functions such that $\lim_{j\to\infty} \mu(j, p, q, T) = G(p, q, T)$ so $G(p, \cdot, T)$ is lower semicontinuous. $\{b \in M\colon G(m(p), b, T) \leqq t\} = m(\{q \in Q\colon G(p, q, T) \leqq t\})$ and hence is a closed set for each t. Therefore $G(m(p), \cdot, T)$ is lower semicontinuous.

Let Q be the unit square with the relative topology as a subset of E^2. For subsets of Q open, closed, boundary, etc., will be understood to be with respect to this topology. For a set A, A^* will generally be used to denote its boundary. For the square Q it will be necessary to speak of its boundary both using the convention mentioned above and as a subset of E^2, Q^* will be used for the relative boundary and $Q^{\#}$ for the boundary as a subset of E^2.

For a continuous mapping $T: Q \rightarrow E^k$ and an open subset D of Q, let $L(T, D^*)$ denote the length of T restricted to the boundary of D as defined by Cesari in [2].

The following statements about $L(T, D^*)$ are proved (although not stated in quite this manner) in [1; 2] or [3] or follow directly from the definition of length.

LEMMA I.2. *Let $T: Q \rightarrow E^k$ be a continuous mapping and D an open subset of Q.*

Let $(D)_\rho = \{q \in Q: \text{dist}(q, Q-D) \geqq \rho\}$. If $\epsilon > 0$ then there is a $\rho > 0$ such that if T' and D' are a continuous mapping and an open subset of Q such that $|T-T'| < \rho$ and $(D)_\rho \subset D' \subset D$ then $L(T', D'^*) \geqq L(T, D^*) - \epsilon$.

LEMMA I.3. *If D is an open subset of Q and T and T' are continuous mappings of Q into E^k such that $T|D^* = T'|D^*$ then $L(T, D^*) = L(T', D^*)$.*

LEMMA I.4. *Let T and T' be Frechet equivalent mappings of Q into E^k. By [9] or [5] there exist monotone light factorizations $T = 1 \circ m$ and $T' = 1 \circ m'$ of T and T' with a common light factor 1 and a common middle space M. Let D be an open subset of M and let $D(T)$ and $D(T')$ be $m^{-1}(D)$ and $(m')^{-1}(D)$ respectively. Then $L(T, D(T)^*) = L(T', D(T')^*)$.*

LEMMA I.5. *Let α be a connected open subset of Q and $T: Q \rightarrow E^k$ a continuous mapping. Let γ be a component of α^* and $A(\gamma)$ be the associated open set with γ as its boundary defined in [2]. If $L(T, A(\gamma)^*) < \infty$ there is a set B which can be taken to be a circle or a closed interval and a function $f: B \rightarrow \gamma$ such that:*

(a) *$T \circ f: B \rightarrow E^k$ is continuous.*

(b) *The continua of constancy of T which contain a point of $f(B)$ cover γ.*

(c) *$L(T, A(\gamma)^*) =$ length $T \circ f$.*

In Part III essential use will be made of a length defined by Federer in [6]. The following lemma gives a comparison of $L(T, D^*)$ and this length.

LEMMA I.6. *Let $T: Q \rightarrow E^k$ be a continuous mapping. Let α be a connected open subset of Q. Let $T = 1 \circ m$ be a monotone light factorization of T with middlespace M. Let W denote the set of points at which $m(\alpha^*)$ is of positive dimension. Let $N(x, 1, W)$ equal the number, possibly infinite, of points a of W such that $1(a) = x$. Let H_k^1 denote 1-dimensional Hausdorff measure in E^k. Then $L(T, \alpha^*) \geqq \int_{E^k} N(x, 1, W) dH_k^1$.*

It is no loss of generality to assume $L(T, \alpha^*) < \infty$. Let γ be a component of α^*. Since $L(T, \alpha^*) < \infty$, $L(T, A(\gamma)^*) < \infty$ and Lemma I.5 may be applied. Let f be the function described in Lemma I.5. By the theorem relating lengths and Hausdorff measure for curves

$$L(T, A(\gamma)^*) = \text{length } T \circ f = \int_{E^k} N(x, T\circ f, B)\,dH_k^1.$$

By (b) of Lemma I.5,

$$N(x, 1, m(\gamma)) = N(x, 1, m\circ f(B)) \leqq N(x, T\circ f, B),$$

hence

$$L(T, A(\gamma)^*) \geqq \int_{E^k} N(x, 1, m(\gamma))\,dH_k^1.$$

Let $E=\{\gamma\in\alpha^*: L(T, A(\gamma)^*)\neq 0\}$. By 20.2(i) of [2] $L(T, A(\gamma)^*)=0$ if and only if T is constant on γ. Since each point of E belongs to a γ on which T is not constant, $m(E)\subset W$. Let S be the family of all countable collections $\{C_i\}$ of disjoint nondegenerate continua contained in $m(\alpha^*)$. By Theorem 3.3 of [6]

$$\sup_{\{C_i\}\in S} \sum_{i=1}^{\infty} \text{diameter } 1(C_i) = \int_{E^k} N(x, 1, W)\,dH_k^1.$$

Let $\{C_i\}$ be a countable collection of disjoint nondegenerate continua of $m(\alpha^*)$. Since m is monotone, each C_i must be contained in some $m(\gamma)$ where $\gamma\in E$ hence

$$\sum_{i=1}^{\infty} \text{diameter } 1(C_i) \leqq \sum_{i=1}^{\infty}\int_{E^k} N(x, 1, C_i)\,dH_k^1 \leqq \int_{E^k} N(x, 1, m(E))\,dH_k^1$$

so $\int_{E^k}N(x, 1, m(E))dH_k^1=\int_{E^k}N(x, 1, W)dH_k^1$. Now

$$\begin{aligned} L(T, \alpha^*) &= \sum_{\gamma\ni L(T,A(\gamma)^*)\neq 0} L(T, A(\gamma)^*) \\ &\geqq \sum_{\gamma\ni L(T,A(\gamma)^*)\neq 0} \int_{E^k} N(x, 1, m(\gamma))\,dH_k^1 \\ &\geqq \int_{E^k} N(x, 1, m(E))\,dH_k^1 = \int_{E^k} N(x, 1, W)\,dH_k^1. \end{aligned}$$

Part II. Let $D(t,\mu_j, T)=\{q\in Q:\mu(j, p, q, T)>t\}$. Since $\mu(j,p,\cdot, T)$ is continuous $D(t, \mu_j, T)$ is an open set. Denote $L(T, D(t, \mu_j, T)^*)$ more briefly by $L(D(t, \mu_j, T))$.

LEMMA II.1. *Let T_n, T be continuous mappings such that T_n converges uniformly to T and all the T_n's and T are light or all the T_n's and T have $Q^\#$ as their only nondegenerate continuum of constancy. Then*

$$\liminf_{\tau\to t^+}\ \liminf_{n\to\infty} L(D(\tau, \mu_j, T_n)) \geqq L(D(t, \mu_j, T)).$$

Let $T_n = 1_n \circ m_n$, $T = 1 \circ m$ be monotone light factorizations of T and T_n with middle spaces M and M_n. If T and T_n are light m and m_n are homeomorphisms. In the other case m and m_n define homeomorphisms of the quotient space $Q/Q^{\#}$ with M and M_n. Hence property (d) of the function $\mu(j, p, q, T)$ may be used to show $\mu(j, p, \cdot, T_n)$ converges uniformly to $\mu(j, p, \cdot, T)$.

Let ϵ be a positive number and ρ be the corresponding number for T, $D(t, \mu_j, T)$ described in Lemma I.2. Let $t' = \min_{q \in (D(t,\mu_j,T))_\rho} \mu(j, p, q, T)$. Then $t' > t$. Let τ be any number $t < \tau < t'$ and let $\sigma = \min(\tau - t, t' - \tau)$. Let N be an integer so that if $n > N$, $|T_n - T| < \rho$ and $|\mu(j, p, \cdot, T_n) - \mu(j, p, \cdot, T)| < \sigma$.

If $q \in (D(t, \mu_j, T))_\rho$ and $n > N$

$$\mu(j, p, q, T_n) > \mu(j, p, q, T) - \sigma \geqq t' - \sigma \geqq \tau$$

so $D(\tau, \mu_j, T_n) \supset (D(t, \mu_j, T))_\rho$.

If $q \in D(\tau, \mu_j, T_n)$ and $n > N$

$$\mu(j, p, q, T) > \mu(j, p, q, T_n) - \sigma > \tau - \sigma \geqq t$$

so $D(\tau, \mu_j, T_n) \subset D(t, \mu_j, T)$.

Hence by Lemma I.2 for $n > N$, $L(D(\tau, \mu_j, T_n)) \geqq L(D(t, \mu_j, T)) - \epsilon$. Taking limits

$$\liminf_{\tau \to t^+} \liminf_{n \to \infty} L(D(\tau, \mu_j, T_n)) \geqq L(D(t, \mu_j, T)) - \epsilon.$$

Since ϵ was arbitrary the conclusion of the lemma follows.

LEMMA II.2. *Let $T: Q \to E^k$ be a continuous mapping. Let $D(t, G, T) = \{q \in Q: G(p, q, T) > t\}$. Then* $\liminf_{j \to \infty} L(D(t, \mu_j, T)) \geqq L(D(t, G, T))$.

Let ϵ be a positive number and ρ be the corresponding number for T, $(D(t, G, T))$ described in Lemma I.2. Since $\mu(j, p, q, T) \leqq G(p, q, T)$, $D(t, \mu_j, T) \subset D(t, G, T)$. Since $\lim_{j \to \infty} \mu(j, p, q, T) = G(p, q, T)$ the sets $D(t, \mu_j, T)$ cover $D(t, G, T)$. Since $(D(t, G, T))_\rho$ is compact there are a finite number of sets $D(t, \mu_j, T)$ which cover $(D(t, G, T))_\rho$. Let J be the largest of the integers j corresponding to these sets $D(t, \mu_j, T)$. Since $\mu(j+1, p, q, T) \geqq \mu(j, p, q, T)$. $(D(t, G, T))_\rho \subset D(t, \mu_j, T)$ for $j > J$.

By Lemma I.2, $L(D(t, \mu_j, T)) \geqq L(D(t, G, T)) - \epsilon$ for $j > J$. Hence $\liminf_{j \to \infty} L(D(t, \mu_j, T)) \geqq L(D(t, G, T)) - \epsilon$ and since ϵ was arbitrary the conclusion of the lemma follows.

LEMMA II.3. *Let $T: Q \to E^k$ be a quasi linear mapping then $A(T) \geqq \int_0^\infty L(D(t, \mu_j, T))dt$.*

Let $\Omega = \{\tilde{\Delta}_i\}$ be a subdivision of Q into triangles on which T is linear. Let M be a Lipschitz constant for T. Let $\tilde{\Delta}_i$ be a triangle so that $T(\tilde{\Delta}_i)$ is a nondegenerate triangle in E^k. Then $T|\tilde{\Delta}_i$ has an inverse $(T|\tilde{\Delta}_i)^{-1}: T(\tilde{\Delta}_i) \to \tilde{\Delta}_i$ which is linear and hence Lipschitzian on $T(\tilde{\Delta}_i)$. Let N be the maximum of the Lipschitz constants of the $(T|\tilde{\Delta}_i)^{-1}$.

Let S_n denote a strip of width $2(2/n)^{1/2}$ about the edges of the triangles of Ω. Let C_p denote the square of center q and sides $1/n$ about q. Let $\mu^n(q) = 1/n^2\int_{C_q}\mu(j, p, r, T)dL_2(r)$. The functions $\mu^n(q)$ are continuously differentiable, converge uniformly to $\mu(j, p, \cdot, T)$ and using property (f) of the function $\mu(j, p, q, T)$ it is seen that

$$|\mu^n(q') - \mu^n(q)| \leqq M|q' - q|.$$

By 20.4(ii) of [2] there is a quasi linear function $\phi^n: Q \to E^1$ such that $0 < \mu(j, p, q, T) - \phi^n(q) < \epsilon_n$, where ϵ_n is a sequence of numbers tending to 0, and $|\text{grad}\, \phi^n(q) - \text{grad}\, \mu^n(q)| < 1/Nn$ at each interior point of a triangle of linearity of ϕ^n.

Let $\Omega^n = \{\Delta_i\}$ be a triangulation of Q so that ϕ^n and T are linear on each triangle of Ω^n, each Δ_i is contained in a $\tilde{\Delta}_i$ of Ω, and Δ_i is contained either in S_n or $Q - S_n$.

Let Δ_i be a triangle of Ω^n such that $T(\Delta_i)$ is a nondegenerate triangle. Let $D(T|\Delta_i)^{-1}(x)$ be the differential of $(T|\Delta_i)^{-1}$ at a point x of $T(\Delta_i)$.

$$\begin{aligned}
&\big||\text{grad}\, \phi^n \circ (T|\Delta_i)^{-1}(x)| - |\text{grad}\, \mu^n \circ (T|\Delta_i)^{-1}(x)|\big| \\
&\qquad = \big||\text{grad}\, \phi^n[D(T|\Delta_i)^{-1}(x)]| - |\text{grad}\, \mu^n[D(T|\Delta_i)^{-1}(x)]|\big| \\
&\qquad \leqq |\text{grad}\, \phi^n[D(T|\Delta_i)^{-1}(x)] - \text{grad}\, \mu^n[D(T|\Delta_i)^{-1}(x)]| \\
&\qquad \leqq |\text{grad}\, \phi^n((T|\Delta_i)^{-1}(x)) - \text{grad}\, \mu^n((T|\Delta_i)^{-1}(x))||D(T|\Delta_i)^{-1}(x)| \\
&\qquad \leqq (1/Nn)N = 1/n.
\end{aligned}$$

Hence

$$|\text{grad}\, \phi^n \circ (T|\Delta_i)^{-1}(x)| \leqq |\text{grad}\, \mu^n \circ (T|\Delta_i)^{-1}(x)| + 1/n.$$

If q, q' belong to a triangle of Ω^n contained in $Q - S_n$ then

$$\begin{aligned}
&|\mu^n(q) - \mu^n(q')| \\
&\qquad \leqq 1/n^2\int_{C_q} |\mu(j, p, r, T) - \mu(j, p, r + q - q', T)|\, dL_2(r) \\
&\qquad \leqq 1/n^2\int_{C_q} \mu(j, r, r + q - q', T)dL_2(r) \\
&\qquad = 1/n^2\int_{C_q} |T(r) - T(r + q - q')|\, dL_2(r) \\
&\qquad = 1/n^2\int_{C_q} |T(q - q')|\, dL_2(r) = |T(q) - T(q')|\, 1/n^2\int_{C_q} dL_2(r) \\
&\qquad = |T(q) - T(q')|.
\end{aligned}$$

Hence if Δ_i is a triangle of Ω^n contained in $Q - S_n$ so that $T(\Delta_i)$ is a nondegenerate triangle, $|\text{grad}\, \mu^n \circ (T|\Delta_i)^{-1}| \leqq 1$. This implies that

$$|\text{grad } \phi^n \circ (T|\Delta_i)^{-1}(x)| \leqq 1 + 1/n.$$

If Δ_i is any triangle of Ω^n for which $T(\Delta_i)$ is nondegenerate and $x, y \in T(\Delta_i)$

$$|\mu^n \circ (T|\Delta_i)^{-1}(x) - \mu^n \circ (T|\Delta_i)(y)|$$
$$\leqq M|(T|\Delta_i)^{-1}(x) - (T|\Delta_i)^{-1}(y)| \leqq MN|x - y|.$$

Hence $|\text{grad } \mu^n \circ (T|\Delta_i)^{-1}(x)| \leqq MN$ which implies $|\text{grad } \phi^n \circ (T|\Delta_i)^{-1}(x)| \leqq MN + 1/n$.

Let $D(t, \phi^n) = \{q \in Q: \phi^n(q) > t\}$. By Lemma 20.4(i) of [2]

$$(1 + 1/n)A(T|\Delta_i) \geqq \int_0^\infty L(T|\Delta_i, [D(t, \phi^n) \cap \Delta_i]^*)dt$$

if Δ_i is a triangle contained in $Q - S_n$ so that $T(\Delta_i)$ is a nondegenerate triangle and

$$(MN + 1/n)A(T|\Delta_i) \geqq \int_0^\infty L(T|\Delta_i, [D(t, \phi^n) \cap \Delta_i]^*)dt$$

if Δ_i is a triangle contained in S_n so that $T(\Delta_i)$ is a nondegenerate.

Since

$$L(T, D(t, \phi^n)^*) = \sum_{\{\Delta_i : T(\Delta_i) \text{ is nondegenerate}\}} L(T|\Delta_i, [D(t, \phi^n) \cap \Delta_i]^*),$$

$$(1 + 1/n) \sum_{\Delta_i \subset Q - S_n} A(T|\Delta_i) + (MN + 1/n) \sum_{\Delta_i \subset S_n} A(T|\Delta_i)$$
$$\geqq \int_0^\infty L(T, D(t, \phi^n))dt.$$

Since ϕ^n converges uniformly to $\mu(j, p, q, T)$ and $\phi^n \leqq \mu(j, p, q, T)$, Lemma I.2 implies $\liminf_{n\to\infty} L(T, D(t, \phi^n)) \geqq L(D(t, \mu_j, T))$. By taking limits, applying Fatou's lemma, and noting that $\liminf_{n\to\infty} \sum_{\Delta_i \subset S_n} A(T|\Delta_i) = 0$, it's concluded that $A(T) \geqq \int_0^\infty L(D(t, \mu_j, T))dt$.

LEMMA II.4. *Let T be a light mapping or a mapping whose only nondegenerate continuum of constancy is $Q^\#$; then $A(T) \geqq \int_0^\infty L(D(t, \mu_j, T))dt$.*

By [9] if T is light T may be approximated by a sequence T_n of light quasi linear mappings such that $A(T_n)$ converges to $A(T)$ or if $Q^\#$ is the only continuum of constancy of T, T may be approximated by a sequence T_n of quasi linear mappings with the same property such that $A(T_n)$ converges to $A(T)$.

Let $\phi(\tau) = \liminf_{n\to\infty} L(D(\tau, \mu_j, T_n))$. By Lemma II.3 and Fatou's lemma

$$A(T) = \lim_{n\to\infty} A(T_n) \geqq \liminf_{n\to\infty} \int_0^\infty L(D(\tau, \mu_j, T_n))d\tau \geqq \int_0^\infty \phi(\tau)d\tau.$$

Let $h=\tau-t$ and define $\phi(\tau)=0$ for $\tau<0$. Then $\int_0^\infty\phi(\tau)d\tau=\int_0^\infty\phi(t+h)dt$. By Lemma II.1 and Fatou's lemma

$$\liminf_{h\to 0^+}\int_0^\infty \phi(t+h)dt \geqq \int_0^\infty \liminf_{h\to 0^+}\phi(t+h)dt \geqq \int_0^\infty L(D, T, \mu_j, T)dt.$$

Hence

$$A(T) \geqq \int_0^\infty L(D(t, \mu_j, T))dt.$$

LEMMA II.5. *Let T be as in Lemma* II.4. *Then* $A(T)\geqq\int_0^\infty L(D(t, G, T))dt$.

By Lemma II.2, Lemma II.4, and Fatou's lemma

$$\begin{aligned} A(T) &\geqq \liminf_{j\to\infty}\int_0^\infty L(D(t, \mu_j, T)) \\ &\geqq \int_0^\infty \liminf_{j\to\infty} L(D(t, \mu_j, T))dt \geqq \int_0^\infty L(D(t, G, T))dt. \end{aligned}$$

THEOREM I. *Let T be an open or closed nondegenerate mapping, then* $A(T)\geqq\int_0^\infty L(D(t, G, T))dt$.

Since T is open or closed nondegenerate T is Frechet equivalent to a mapping T' of the type described in Lemma II.4. Let p' be a point of Q such that $m(p)=m'(p')$ and let $D(t, G, T')=\{q\in Q: G(m'(p'), m'(q), T')>t\}$. Then $m(D(t, G, T))=m'(D(t, G, T'))$ and Lemma II.5 can be applied to give,

$$A(T) = A(T') \geqq \int_0^\infty L(D(t, G, T'))dt = \int_0^\infty L(D(t, G, T))dt.$$

Part III. The following example shows that $\int_0^\infty L(D(t, G, T))dt$ need not equal $A(T)$ if $A(T)$ is infinite. Let (u, v) be coordinate variables in Q. Let ϕ: $[0, 1]\to E^1$ be a nowhere differentiable real valued function defined on the unit interval. Let T: $Q\to E^2$ be defined by $T(u, v)=(\phi(u), \phi(v))$.

ϕ is a light mapping, for if ϕ were constant on some nondegenerate connected set, the set would contain an interval on which ϕ was constant and ϕ would be differentiable on the interior of the interval. This implies T is a light mapping. If I: $Q\to Q$ is the identity mapping of Q onto Q, $T=T\circ I$ is a montone light factorization of T with middle space Q.

Let p, q be distinct points of Q and $f\in\Gamma(p, q, T)$. Let (x_1, x_2) be coordinates in E_2 and f_i: $[0, 1]\to[0, 1]$ be the component mappings of f. For $i=1, 2$ let π_i: $E^2\to E^1$ be defined by $\pi_i(x_1, x_2)=x_i$. π_i is Lipschitzian with Lipschitz constant 1. Since p and q are distinct at least one of $f_i[0, 1]$ covers a nondegenerate interval $[c, d]$. Now $\pi_i\circ T\circ f(t)=\phi(f_i(t))$ and

$$\begin{aligned}\text{length } T\circ f &= \int_{E^2} N(x, T\circ f, [0,1])dH_2^1 \\ &\geqq \int_{E^1} N(x_i, \pi_i\circ T\circ f, [0,1])dH_1^1 \\ &\geqq \int_{E^1} N(x_i, \phi, [c,d])dH_1^1 = \infty\end{aligned}$$

since if the last expression was finite ϕ would be of bounded variation on $[c, d]$ and hence differentiable almost everywhere on $[c, d]$.

This implies $G(p, q, T) = \infty$ if $p \neq q$ so $D(t, G, T)$ is the complement of the point p for every $t > 0$ and hence $L(D(t, G, T)) = 0$ for $t > 0$. Therefore $\int_0^\infty L(D(t, G, T))dt = 0$.

There are many ways of showing $A(T) = \infty$. One is that if this were not so Theorem II proved below would be contradicted.

The following is an example of a mapping of finite area, such that there is a single point at an infinite geodesic distance from every other point. Let C be the closed unit disc in E^2. Let (ρ, θ) be polar coordinates in E^2 and (r, ω, z) be cylindrical coordinates in E^3. Define F: $C \rightarrow E^3$ by $r = \rho$, $\omega = \theta$, $z = \rho \sin 1/\rho$. Let H be any homeomorphism of Q onto C and let $T = F \circ H$. Then point which maps into the center of C is at an infinite geodesic distance from every other point of Q.

Most of the remainder of this part will be concerned with proving the following equality corresponding to the inequality of Theorem I.

THEOREM III. *Let T be an open or closed nondegenerate mapping of Q into E^k for which $A(T) < \infty$. Let $1 \circ m$ be a monotone light factorization of T with middle space M. Let p be a point of Q such that $\mathbf{G}(m(p), b, T)$ is finite for some $b \in M$ different from $m(p)$. Then $\int_0^\infty L(D(t, G, T))dt = A(T)$.*

If T is a closed nondegenerate mapping T is Frechet equivalent to a mapping T' which has $Q^\#$ as it's only nondegenerate continua of constancy [9]. By [9] or [5] there is a monotone mapping m': $Q \rightarrow M$ so that $1 \circ m'$ is a monotone light factorization of T' with the same light factor 1 and middle space M as in the monotone light factorization of T and such that $m(Q^\#)$ and $m'(Q^\#)$ are the same single point of M.

Let Q' be a square contained in $Q - Q^\#$. Then $T' | Q'$ is an open nondegenerate mapping, $m' | Q'$ is a homeomorphism and $1 \circ (m' | Q')$ is a monotone light factorization of $T' | Q'$ with middle space $m'(Q') \subset M$.

In order to handle the cases of open and closed nondegenerate mappings simultaneously let $T' = T$ and $Q' = Q$ if T is open nondegenerate. Then if T is either an open or closed nondegenerate mapping of finite area $T' | Q'$ is an open nondegenerate mapping of finite area. Morrey's theorem may be applied to obtain an almost conformal mapping T'': $Q \rightarrow E^k$ such that T'' and $T' | Q'$

are Frechet equivalent. There will be a monotone mapping $m'': Q \to m'(Q') \subset M$ such that $1 \circ m''$ is a monotone light factorization of T'' with the same light factor as T and middle space $m'(Q')$ contained in the middle space M of T.

Let (u, v) be coordinate variables in Q and $x_1, \cdots, x_k$ be the component mappings of T''. Since T'' is almost conformal it has the following properties: (a) T'' is BVT and ACT (of bounded variation and absolutely continuous in the sense of Tonelli). Hence T'' has partial derivatives almost everywhere in Q.

(b)
$$\text{If } E(q) = \sum_{i=1}^{k} x_{iu}^2 = \left|\frac{\partial T''}{\partial u}\right|^2, \qquad G(q) = \sum_{i=1}^{k} x_{iv}^2 = \left|\frac{\partial T''}{\partial u}\right|^2,$$
$$F(q) = \sum_{i=1}^{k} x_{iu}x_{iv} = \frac{\partial T''}{\partial u}\cdot\frac{\partial T''}{\partial v},$$

and $J(q)$ is the Jacobian matrix of the partial derivatives then $E(q) = G(q)$, and $F(q) = 0$ almost everywhere in Q and

$$A(T'') = \int_Q |J(q)| \, dL_2 = \int_Q (E(q)G(q) - F(q)^2)^{1/2} dL_2 = \int_Q E(q) dL_2.$$

Since T'' is ACT, for almost every parallel to the u-axis and for almost every parallel to the v-axis T'' restricted to this parallel is absolutely continuous as a function of one variable.

Lemma III.1. *Let $d \in M$, $e \in m'(Q')$. Let $f \in \Gamma(d, e, T)$ be such that length $1 \circ f = \mathbf{G}(d, e, T)$. Then there is a continuum $K \subset (m'')^{-1}(f([0, 1]))$ which joins $(m'')^{-1}(d)$ and $(m'')^{-1}(e)$ or $(m'')^{-1}(e)$ and $Q^{\#}$. If T is open nondegenerate K joins $(m'')^{-1}(d)$ and $(m'')^{-1}(e)$.*

If $f[0, 1] \subset m'(Q')$ let $K = (m'')^{-1}(f([0, 1]))$. Note that if T is open nondegenerate $f([0, 1])$ is always contained in $m'(Q')$ since $m'(Q') = M$ in this case. If $f([0, 1])$ is not contained in $m'(Q')$ let $\tau = \sup\{t : f(t) \in M - m'(Q')\}$. Let $m'(Q')^{\#}$ denote the boundary of $m'(Q')$. Then $f([\tau, 1])$ meets $m(Q')^{\#}$. Since m'' is monotone and $m'(Q')$ is a two cell $m''(Q^{\#}) \subset m'(Q')^{\#}$ [9]. Therefore $K = (m'')^{-1} f([\tau, 1])$ joins $(m'')^{-1}(e)$ and $Q^{\#}$.

Theorem II. *Let $T: Q \to E^k$ be an open or closed nondegenerate mapping of finite area. Let $I = \{a \in M : \mathbf{G}(a, b, T) = \infty$ for every $b \in M \ni b \neq a\}$. Then $\mathbf{G}(a, b, T) < \infty$ if $a, b \in M - I$ and $(M - I)^+ = M$.*(3)

If a, b are points of $M - I$ there are points c, d of M such that $a \neq c$, $b \neq d$, $\mathbf{G}(a, c, T) < \infty$, and $\mathbf{G}(b, d, T) < \infty$. If T is closed nondegenerate at least one point in each of the pairs a, c and b, d is not equal to $m(Q^{\#})$ say for

(3) The notation $(M-I)^+$ is used in place of a bar over $M-I$.

definiteness that a and d are different from $m(Q^{\#})$. Let T' and m' be the mappings described previously and let Q' be a square contained in $Q-Q^{\#}$ such that $(m')^{-1}(a)$ and $(m')^{-1}(d)$ are contained in $Q'-Q'^{\#}$. If T is open nondegenerate let $Q'=Q$. Let T'' and m'' be the mappings described previously for the above square Q'.

Let K_1 and K_2 be continua as described in Lemma III.1 with $d=a$, $e=c$ for K_1 and $d=b$, $e=d$ for K_2. If T is closed nondegenerate $m'|Q-Q^{\#}$ is a homeomorphism and hence since m'' is monotone $m'(Q'^{\#})=m'(Q')^{\#}=m''(Q^{\#})$. Then since $(m')^{-1}(a)$ and $(m')^{-1}(d)$ are contained in $Q'-Q'^{\#}$, $(m'')^{-1}(a)$ and $(m'')^{-1}(d)$ are contained in $Q-Q^{\#}$. Since K_1 joins $(m'')^{-1}(a)$ and $(m'')^{-1}(c)$ or $(m'')^{-1}(a)$ and $Q^{\#}$ and a similar statement holds for K_2, K_1 and K_2 are nondegenerate if T is closed nondegenerate. If T is open nondegenerate K_1 joins $(m'')^{-1}(a)$ and $(m'')^{-1}(c)$ and K_2 joins $(m'')^{-1}(b)$ and $(m'')^{-1}(d)$; hence K_1 and K_2 are nondegenerate in either case.

Let S be the union of all parallels to the coordinate axes on which T is absolutely continuous as a function of one variable. Since S contains almost every parallel to each axis, S separates any pair of points of Q. Hence $K_1\cap S$ and $K_2\cap S$ are not empty. Let $r\in K_1\cap S$, $s\in K_2\cap S$; then $\mathbf{G}(a, m''(r), T)<\infty$ and $\mathbf{G}(b, m''(s), T)<\infty$.

If P is a parallel to say the u-axis contained in S such that $T''|P$ is absolutely continuous then length $T''|P=\int_0^1|\partial T''/\partial u|\,du<\infty$. Since a similar relationship holds for each parallel contained in S, $\mathbf{G}(m''(r), m''(s), T)<\infty$ for $r, s\in S$. Hence $\mathbf{G}(a, b, T)\leqq \mathbf{G}(a, m''(r), T)+\mathbf{G}(m''(r), m''(s), T)+\mathbf{G}(m''(s), b, T)<\infty$ for $a, b\in M-I$.

Since S is dense in Q, $m''(S)$ is dense in $m''(Q)$. If T is closed nondegenerate let $\{Q_n'\}$ be a sequence of squares contained in $Q-Q^{\#}$ such that $\bigcup_{n=1}^{\infty} Q_n'=Q-Q^{\#}$. If S_n are the corresponding sets of parallels described above $\bigcup_{n=1}^{\infty} m_n''(S_n)\subset M-I$ and $(\bigcup_{n=1}^{\infty} m_n''(S_n))^{+}=M$. If T is open nondegenerate $m''(Q)=M$; hence in either case $(M-I)^{+}=M$.

It follows from Theorem II that if T is an open or closed nondegenerate mapping of finite area there always is a point p as described in the hypothesis of Theorem III. From now on unless stated otherwise the hypothesis of Theorem III will be assumed and T', m', Q', T'', and m'' will be as defined previously.

A mapping which may be discontinuous but satisfies all the other conditions for an ACT mapping is said to be ACE (absolutely continuous in the sense of Evans).

LEMMA III.2. *Let $G(q, T'')=\mathbf{G}(m(p), m''(q), T)$. Then $G(q, T'')$ is* ACE.

Let Q_{u_0} denote a subset of S of the type $Q_{u_0}=\{(u, v)\in Q: u=u_0\}$. Let $v_1\leqq v_2\leqq \cdots \leqq v_n$ be points of $[0, 1]$ and $q_i=(u_0, v_i)$ for $i=1, 2, \cdots, n$ be the corresponding points of Q_{u_0}. Since T'' is absolutely continuous as a function of v on Q_{u_0}

$$G(m''(q_{i+1}), m''(q_i), T) \leqq \int_{v_i}^{v_{i+1}} \left| \frac{\partial T''}{\partial v} \right| dv.$$

If I is the set described in Theorem II, both $m(p)$ and $m''(S)$ are contained in $M-I$. Hence by Theorem II $G(m(p), m''(q), T) < \infty$ if $q \in S$.

$$\sum_{i=1}^{n-1} | G(q_{i+1}, T'') - G(q_i, T'') |$$

$$= \sum_{i=1}^{n-1} | G(m(p), m''(q_{i+1}), T) - G(m(p), m''(q_i), T) |$$

$$\leqq \sum_{i=1}^{n-1} G(m''(q_{i+1}), m''(q_i) T) \leqq \sum_{i=1}^{n-1} \int_{v_i}^{v_{i+1}} \left| \frac{\partial T''}{\partial v} \right| dv.$$

Since similar considerations hold for analogous sets Q_{v_0}, it follows that $G(q, T'')$ is ACE.

Since T'' is almost conformal and $G(g, T'')$ is ACE there is a measurable set B contained in Q with the following properties:

(a) The partial derivatives of T'' and $G(q, T'')$ exist and $E(q) = G(q)$, $F(q) = 0$ at every point of B.

(b) $B = \bigcup_{i+1}^{\infty} B_i$ where the B_i are disjoint measurable sets on which T'' and $G(q, T'')$ are continuously differentiable. (A function $f: E^m \to E^n$ is said to be continuously differentiable on a set $H \subset E^m$ if its restriction to H may be extended to E^m so that the resulting function is continuously differentiable.)

(c) At every point of B, T'' and $G(q, T'')$ have regular approximate differentials, and the same set A of boundaries of squares with sides parallel to the coordinate axes on which the limits are taken may be used for both functions.

(d) For each i let $\phi_i: E^2 \to E^1$ and $\psi_i: E^2 \to E^k$ be continuously differentiable functions such that $\phi_i | B_i = G(q, T'') | B_i$ and $\psi_i | B_i = T'' | B_i$. Let $D\phi_i(q)$ and $D\psi_i(q)$ denote the differentials of ϕ_i and ψ_i at a point q. Let $DT''(q)$ and $DG(g, T'')$ denote the regular approximate differentials of T'' and $G(q, T'')$ at a point q. Then the linear transformation associated with the Jacobian matrix $J(q)$ of the partial derivatives of T'', $D\psi_i(q)$, and $DT''(q)$ are equal at every point of B_i and the linear transformation associated with grad $G(q, T'')$, $D\phi_i(q)$, and $DG(q, T'')$ are equal at every point of B_i.

(e) B is contained in the set S of parallels to the axes on which T'' is absolutely continuous as a function of one variable.

(f) No point of $Q^\#$ is in B.

(g) $|J(q)| \neq 0$ at every point of B.

(h) $L_2((Q-B) \cap \{q \in Q: |J(q)| \neq 0\}) = 0$.

The set B may be obtained as follows: Since T'' is almost conformal and

$G(g,T'')$ is ACE the statement of (a) holds almost everywhere in Q. By Theorem I of [14] the existence of the partial derivatives of T'' and $G(g, T'')$ almost everywhere implies that Q may be written $Q=\bigcup_{i=0}^{\infty} P_i$ where $L_2(P_0)=0$ and the sets P_i for $i\geqq 1$ are a disjoint sequence of closed sets such that $T''|P_i$ and $G(q, T'')|P_i$ are continuously differentiable. By 26.2 (i) and (ii) of [2] the existence of the partial derivatives of T'' and $G(q, T'')$ almost everywhere in Q imply the statement of (c) holds at almost every point of Q.

Theorem 26.2 (i) of [2] implies the equality of the linear transformation associated with $J(q)$ and $DT''(q)$ and the equality of the linear transformation associated with grad $G(g, T'')$ and $DG(q, T'')$ at almost every point of Q. It is seen that $D\phi_i(q)=DG(q, T'')$ and $D\psi_i(q)=DT''(q)$ at every point of density of P_i where the quantities on the right of the equal sign exist, hence almost everywhere in P_i.

Hence by discarding several sets of measure 0 and the set of points where $|J(q)|=0$ from the sets P_i for $i\geqq 1$ sets B_i are obtained such that $B=\bigcup_{i=1}^{\infty} B_i$ has all the desired properties.

LEMMA III.3. $|\text{Grad } G(q, T'')|=E(q)^{1/2}$ *at every point of* B.

Consider the regular approximate differential $DT''(q)$ of T'' at a point q of B. Let $w=(a, b)$ be a point of E^2. From conditions (a) and (d) it is seen that

$$|DT''(q)(w)| = \left|\begin{pmatrix} x_{1u} & x_{1v} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ x_{ku} & x_{kv}\end{pmatrix}\binom{a}{b}\right| = \left(\sum_{i=1}^{k}(x_{iu}a + x_{iv}b)^2\right)^{1/2}$$

$$= (a^2E(q) + b^2G(q) + 2abF(q))^{1/2} = |w|\, E(q)^{1/2}.$$

Let $f\in\Gamma(m(p), m''(q), T)$ be such that length $1\circ f=\mathbf{G}(m(p), m''(q), T)$. Let K be the corresponding continuum described in Lemma III.1 with $d=m(p)$ and $e=m''(q)$. Let $\{q_n\}$ be a sequence of points approaching q so that $\{q_n\}\subset A\cap K$ where A is the set described in property (c) of the set B. Since $q_n\in K$,

$$|G(q, T'') - G(q_n, T'')| = |\mathbf{G}(m(p), m''(q), T) - \mathbf{G}(m(p), m''(q_n), T)|$$
$$= \mathbf{G}(m''(q), m''(q_n), T).$$

Hence

$$\lim_{q_n\to q}\frac{|G(q, T'') - G(q_n, T'')|}{|q - q_n|} \geqq \lim_{q_n\to q}\frac{|T''(q_n) - T''(q)|}{|q_n - q|}$$
$$= \lim_{q_n\to q}\left|DT''(q)\left(\frac{q_n - q}{|q_n - q|}\right)\right| = E(q)^{1/2}.$$

If $q=(u_0, v_0)$ let q_n^* be a sequence of points approaching q which belong to $Q_{u_0}\cap A$. Then

$$| G(q, T'') - G(q_n^*, T'') | = | G(m(p), m''(q), T) - G(m(p), m''(q_n^*), T) | \leqq G(m''(q), m''(q_n^*), T).$$

By condition (e) of the set B, $B\subset S$. Hence

$$\lim_{q_n^*\to q} \frac{| G(q, T'') - G(q_n^*, T'') |}{| q - q_n^* |} \leqq \lim_{q_n^*\to q} \frac{G(m''(q), m''(q_n^*), T)}{| q - q_n^* |} \leqq \lim_{q_n\to q} \frac{\left| \int_{v_n}^{v_0} E(q)^{1/2} dv \right|}{| v_0 - v_n |} = E(q)^{1/2}.$$

Lemma III.4. *Let $C_t=\{q\in Q: G(q, T'')=t\}$. Using the preceding definitions and conventions*

$$A(T'') = \int_0^\infty \left[\int_{E^k} N(x, T'', B\cap C_t) dH_k^1 \right] dt.$$

The following theorem is a special case of a more general theorem recently found independently by Federer [7] and L. C. Young [13].

Let $F: Q\to E^1$ be a continuously differentiable real valued function on Q and $h: Q\to E^1$ a nonnegative Borel measurable real valued function on Q. Let $A_t=\{q\in Q: F(q)=t\}$ and K be measurable subset of Q. Then

$$\int_K h | \operatorname{grad} F | dL_2 = \int_{-\infty}^\infty \left[\int_{A_t\cap K} h\, dH_2^1 \right] dt.$$

Proof of Lemma III.4. Let ϕ_i be the function described in property (d) of the set B. It follows from property (d) that $B_i\cap\{q\in Q: \phi_i(q)=t\}=B_i\cap C_t$ and that grad $G(q, T'')=$ grad $\phi_i(q)$ on B_i. Applying the above theorem with $F=\phi_i$, $h=E^{1/2}$, and $K=B_i$ it follows using Lemma III.3 that

$$\int_0^\infty \left[\int_{C_t\cap B_i} E(q)^{1/2} dH_2^1 \right] dt = \int_{B_i} E(q)^{1/2} | \operatorname{grad} \phi_i(q) | dL_2$$
$$= \int_{B_i} E(q)^{1/2} | \operatorname{grad} G(q, T) | dL_2 = \int_{B_i} E(q) dL_2 = \int_{B_i} | J(q) | dL_2.$$

Since $|\operatorname{grad} \phi(q)| = E(q)^{1/2} \neq 0$ at every point of B_i and $C_t\cap B_i =B_i\cap\{q\in Q: \phi_i(q)=t\}$ the classical implicit function theorem implies that $C_t\cap B_i$ is the disjoint union of countably many sets C_i each of which lies on a differentiable arc γ_i.

Let $u(q)$ denote a unit tangent vector to γ_i at the point q. Let ψ_i be the function described in property (d) of the set B and let $D\psi_i(q)$ denote the differential of ψ_i at q. The directional derivative of ψ_i in the direction of the unit tangent vector $u(q)$ at the point q is given by $D\psi_i(q)(u(q))$. By a classical theorem on Hausdorff measures

$$\int_{C_i} D\psi(q)(u(q))dH_2^1 = \int_{E^k} N(x, \psi, C_i)dH_k^1.$$

Since by property (d) of the set B, $D\psi_i(q) = DT''(q)$ at each point q of B_i and $T''|B_i = \psi|B_i$

$$\int_{C_i} DT''(q)(u(q))dH_2^1 = \int_{E^k} N(x, T'', C_i)dH_k^1.$$

By the argument used in Lemma III.2 $DT''(q)(u(q)) = E(q)^{1/2}|u(q)| = E(q)^{1/2}$. Hence since the sets C_i are disjoint

$$\int_{C_t \cap B_i} E(q)^{1/2} dH_2^1 = \int_{E^k} N(x, T'', C_t \cap B_i)dH_k^1.$$

Hence

$$\int_0^\infty \left[\int_{E^k} N(x, T'', B_i \cap C_t)dH_k^1\right] dt = \int_0^\infty \left[\int_{C_t \cap B_i} E(q)^{1/2} dH_2^1\right] dt$$
$$= \int_{B_i} |J(q)|\, dL_2.$$

Now

$$\begin{aligned} A(T'') &= \int_Q |J(q)|\, dL_2 = \int_B |J(q)|\, dL_2 \\ &= \sum_{i=1}^\infty \int_{B_i} |J(q)|\, dL_2 = \sum_{i=1}^\infty \int_0^\infty \left[\int_{E^k} N(x, T'', B_i \cap C_t)dH_k^1\right] dt \\ &= \int_0^\infty \left[\int_{E^k} N(x, T'', B \cap C_t)dH_k^1\right] dt. \end{aligned}$$

Lemma III.5. *No point of B is contained in a nondegenerate continuum of constancy of T''.*

Suppose q were a point of B belonging to such a continuum C. Let A be the set described in property (c) of the set B. Let q_n be a sequence of points belonging to $C \cap A$ such that $q_n \neq q$ and q_n converges to q. There is a point v on the unit circle and subsequence $\{q_n'\}$ of q_n such that $(q_n' - q)/|q_n' - q|$ converges to v. Now

$$\lim_{n\to\infty}\left|\frac{T''(q_n')-T''(q)-DT''(q)(q_n'-q)}{|q_n'-q|}\right| = 0.$$

Since $T''(q_n')=T''(q)$ and $DT''(q)$ is continuous,

$$DT''(q)(v) = \lim_{n\to\infty} DT''(q)\left(\frac{q_n'-q}{|q_n'-q|}\right) = 0$$

contradicting $|J(q)|\neq 0$.

The following can be easily shown to hold by using the usual argument for relative maximum and minimum of functions.

LEMMA III.6. *Let $F: G\to E^1$ be a function from an open subset G of E^n into the real numbers. Let x be a point of G at which the partial derivatives of F exist. If $F(x)=t$ and there is a neighborhood of x on which $F(x)\geqq t(F(x)\leqq t)$ or there are arbitrarily small neighborhoods of x on whose boundaries $F(x)\geqq t(F(x)\leqq t)$ then* grad $F(x)=0$.

LEMMA III.7. *Let $D(t, G, T'')=\{q\in Q: G(q, T'')>t\}$. Then $C_t\cap B \subset D(t, G, T'')^*$.*

If q_0 were a point of $C_t\cap B$ not in $D(t, G, T'')^*$, the interior of $Q-D(t, G, T'')$ would be a neighborhood of q_0 on which $G(q, T'')\leqq t$ and hence by Lemma III.6 grad $G(q_0, T'')=0$. But since $|$gradient $G(q, T'')|$ $=E(q)^{1/2}$ and $E(q)=|J(q)|\neq 0$ at every point of B this is a contradiction.

LEMMA III.8. *The point p does not belong to B.*

Since no point of $Q^\#$ belongs to B this follows from Lemma III.6 by an argument entirely analogous to that of Lemma III.7.

The proofs of the following two lemmas are elementary.

LEMMA III.9. *Let X be a locally connected unicoherent Hausdorff space. Let A and B be disjoint, nonempty, closed, connected subsets of X. Let K be the component of $X-A$ containing B. Then $X-K$ is connected and hence by unicoherence $K^*=\overline{K}\cap[X-K]$ is a connected subset of A^*.*

LEMMA III.10. *Let X and Y be compact Hausdorff spaces and $f: X\to Y$ be a continuous mapping of X onto Y. Let U be an open subset of Y and $V=f^{-1}(U)$. Then $f(V^*)=U^*$.*

LEMMA III.11. *Let $T: Q\to E^k$ be a continuous mapping. Let $\mathbf{D}(t, \mathbf{G}, T)$ $=\{a\in M: \mathbf{G}(m(p), a, T)>t\}$. Let $\{\beta_i\}$ be the collection of components of $\mathbf{D}(t, \mathbf{G}, T)$. Then β_i^* is a connected set. If T satisfies the hypothesis of Theorem* III *and t is greater than* 0, *β_i^* is a nondegenerate connected set.*

$M-\mathbf{D}(t, \mathbf{G}, T)$ is closed and connected, for if $c\in M-\mathbf{D}(t, \mathbf{G}, T)$ there is a function $f\in\Gamma(m(p), c, T)$ for which length $1\circ f=\mathbf{G}(m(p), c, T)$ and hence

$f([0, 1])$ is a connected set contained in $M-\mathbf{D}(t, \mathbf{G}, T)$ joining $m(p)$ and c. Let e be a point of β_i. Applying Lemma III.9 with $A=M-\mathbf{D}(t, \mathbf{G}, T)$, $B=\{e\}$ it is seen that β_i^* is a connected set.

If T satisfies the hypothesis of Theorem III, M is a 2-sphere or a 2-cell. Since β_i is open β_i^* will be a single point if and only if $M-\beta_i$ is the single point $m(p)$. There is a point $b\in M$ different from $m(p)$ for which $\mathbf{G}(m(p), b, T) < \infty$. Let $f\in\Gamma(m(p), b, T)$ be such that length $1 \circ f=\mathbf{G}(m(p), b, T)$. Let $\tau=\sup\{s: \mathbf{G}(m(p), f(s), T)\leqq t\}$. Since $t>0$, $f(0, \tau)$ is a subset of $M-\beta_i$ which is not a single point. Therefore β_i^* is a nondegenerate set.

LEMMA III.12. *For $t>0$ let $\mathbf{D}(t, \mathbf{G}, T)=\{a\in M: \mathbf{G}(m(p), a, T)>t\}$ and $\{\beta_i\}$ be the collection of components of $\mathbf{D}(t, \mathbf{G}, T)$. Then*

$$L(D(t, G, T)) \geqq \sum_{i=1}^{\infty}\int_{E^k} N(x, 1, \beta_i^*)dH_k^1.$$

Let $\{\alpha_i\}$ be the collection of components of $D(t, G, T)$. Since m is monotone and $D(t, G, T)=m^{-1}(\mathbf{D}(t, \mathbf{G}, T))$ each α_i is of the form $m^{-1}(\beta_j)$ for some j. For convenience it will be supposed that the indices have been chosen such that $\alpha_i=m^{-1}(\beta_i)$. By the additivity of the length

$$L(D(t, G, T)) = \sum_{i=1}^{\infty} L(T, \alpha_i^*).$$

Let $W_i=\{b\in m(\alpha_i^*)$: dimension $(m(\alpha_i^*), b)>0\}$. By Lemma I.6 $L(T, \alpha_i^*) \geqq\int_{E^k}N(x, 1, W_i)dH_k^1$. By Lemma III.10 $m(\alpha_i^*)=\beta_i^*$. Since by Lemma III.11 β_i^* is a nondegenerate connected set, $\beta_i^*=W_i$. Hence the conclusion of the lemma follows.

LEMMA III.13. *Let t be a number greater than 0. Let $\{\alpha_i''\}$ be the collection of components of $D(t, G, T'')$. If $L(D(t, G, T))<\infty$ then*

$$C_t \cap B \cap \left[D(t, G, T'')^* - \bigcup_{i=1}^{\infty} \alpha_i''^* \right]$$

is empty.

(Note that the assertion $L(D(t, G, T))<\infty$ has to do with the set $D(t, G, T)$ and the mapping T while the conclusion involves the set $D(t, G, T'')$ associated with the mapping T''.)

Let $\{\gamma_j\}$ denote the collection of components of $m'(Q')\cap \mathbf{D}(t, \mathbf{G}, T)$. Since $(m'')^{-1}(m'(Q')\cap \mathbf{D}(t, \mathbf{G}, T))=D(t, G, T'')$ each γ_j is contained in a component β_i of $\mathbf{D}(t, \mathbf{G}, T)$ and if γ_j^* denotes the boundary of γ_j relative to $m'(Q')$ it is seen that $\gamma_j^*\subset\beta_i^*$.

By Lemma III.10 $m''(\alpha_j^*)=\gamma_j^*\subset\beta_i^*$. By Lemma III.12

$$\sum_{i=1}^{\infty}\int_{E^k} N(x, 1, \beta_i^*) dH_k^1 < \infty.$$

This implies there is an integer j_n so that

$$\int_{E^k} N\left(x, 1, \bigcup_{j=j_n}^{\infty} \gamma_j^*\right) dH_k^1 < 1/n.$$

Suppose $q \in C_t \cap B \cap [D(t, G, T'')^* - \bigcup_{i=1}^{\infty} \alpha_i''^*]$. Consider the mapping $R: E^k \to E^1$ defined by $R(x) = |x - T''(q)|$. R is Lipschitzian with Lipschitz constant 1, hence

$$\int_{E^1} N\left(y, R \circ 1, \bigcup_{j=j_n}^{\infty} \gamma_j^*\right) dH_1^1(y) \leq \int_{E^k} N\left(x, 1, \bigcup_{j=j_n}^{\infty} \gamma_j^*\right) dH_k^1(x) < \frac{1}{n}.$$

For each integer n there is a number r_n, $0 < r_n \leq 1/n$ such that the boundary of the sphere $S(T''(q), r_n)$ of center $T''(q)$ and radius r_n does not meet $1(\bigcup_{j=j_n}^{\infty} \gamma_j^*)$. If there were not $R \circ 1(\bigcup_{j=j_n}^{\infty} \gamma_j^*)$ would cover the interval $0 < r_n \leq 1/n$ contradicting

$$\int_{E^1} N\left(y, R \circ 1, \bigcup_{j=j_n}^{\infty} \gamma_i^*\right) dH_1^1(y) < 1/n.$$

Since $q \in B$ by Lemma III.5 q does not belong to a nondegenerate continuum of constancy of T''. Hence the components G_n of $(T'')^{-1}(S(T''(q), r_n))$ that contain q form a basis of connected neighborhoods of q whose boundaries do not meet $\bigcup_{j=j_n}^{\infty} \alpha_j^*$. If the sets G_n did not form a basis of neighborhoods of q the intersection of their closures would be a nondegenerate continuum of constancy of T'' containing q contradicting Lemma III.5.

Since $q \in B$, $q \notin Q^\#$. Since the sets G_n are a basis of neighborhoods of q it may be assumed by selecting a subsequence that G_n is contained in $S(q, 1/n)$ the sphere of radius $1/n$ about q and that $Q^\#$ and $m''^{-1}(m(p))$ are contained in the same component of the complement of $Q - \overline{G}_n$. Denote this component by K_n. Let $G_n' = Q - \overline{K}_n$. By Lemma III.9 $G_n'^*$ is connected and $G_n'^* \subset G_n^*$. Since $G_n \subset S(q, 1/n)$, $G_n' \subset S(q, 1/n)$ and hence the sets G_n' form a basis of neighborhoods of q with connected boundaries whose boundaries do not meet $\bigcup_{j=j_n}^{\infty} \alpha_j''^*$.

Let $f \in \Gamma(m(p), m''(q), T)$ be such that length $f = \mathbf{G}(m(p), m''(q), T)$. Let K be the continuum described in Lemma III.1 connecting $Q^\#$ and q or q and $(m'')^{-1}(m(p))$. Since $\mathbf{G}(m(p), m''(q), T) = t$ and $m''(K) \subset f([0, 1])$, $K \subset Q - D(t, G, T'')$. Since $G_n'^*$ separates q and $Q^\#$ or $(m'')^{-1}(m(p))$, $G_n'^* \cap K \neq 0$. Hence $G_n'^* \cap [Q - D(t, g, T'')] \neq 0$. This implies, since $G_n'^* \cap \bigcup_{j=j_n}^{\infty} \alpha_j''^* = 0$ and $G_n'^*$ is connected, that $G_n'^* \cap \bigcup_{j=j_n}^{\infty} \alpha_j'' = 0$.

Since $q \in [D(t, G, T'')^* - \bigcup_{j=1}^{\infty} \alpha''^*]$, $q \in Q - \bigcup_{j=1}^{j_n} \bar{\alpha}_j''$. Let

$$H_n = G_n' \cap \left[Q - \bigcup_{j=1}^{j_n} \bar{\alpha}_j'' \right].$$

Since

$$H_n^* \subset [Q - \cup_{j=1}^{j_n} \bar{\alpha}_j'']^+, \quad H_n^* \cap \cup_{j=1}^{j_n} \alpha_j'' = 0.$$

Since $H_n^* \subset G_n'^* \cup [Q - \cup_{j=1}^{j_n} \bar{\alpha}_j'']^*$ and neither of these meets $\cup_{j=j_n}^{\infty} \alpha_j''$, $H_n^* \cap \cup_{j=j_n}^{\infty} \alpha_j'' = 0$.

Thus the sets H_n are a basis of neighborhoods of q on whose boundaries $G(\cdot, T'') \leqq t$, hence by Lemma III.6 grad $G(q, T'') = 0$ contradicting $q \in B$.

Lemma III.14. *$A(T'') \leqq \int_0^\infty L(D(t, G, T))dt$.*

By Lemma III.4 $A(T'') = \int_0^\infty \int_{E^k} N(x, T'', B \cap C_t) dH_k^1 dt$. By Lemma III.12 $L(D(t, G, T)) \geqq \sum_{i=1}^\infty \int_{E^k} N(x, 1, \beta_i^*) dH_k^1$. Using Lemma III.10 it is seen that $m''(\alpha_i''^*) \subset \beta_j^*$ for some j. Therefore

$$L(D(t, G, T)) \geqq \int_{E^k} N\left(x, 1, m''\left(\bigcup_{j=1}^\infty \alpha_j''^*\right)\right) dH_k^1.$$

By Theorem I, $A(T) \geqq \int_0^\infty L(D(t, G, T))dt$, hence $L(D(t, G, T))$ is finite for almost every t. Therefore by Lemma III.13 and Lemma III.7 $B \cap C_t \subset \cup_{j=1}^\infty \alpha_j''^*$ for almost every t. By Lemma III.5 $N(x, T'', B \cap C_t) = N(x, 1, m''(B \cap C_t))$. Hence for almost every t

$$L(D(t, G, T)) \geqq \int_{E^k} N\left(x, 1, m''\left(\bigcup_{j=1}^\infty \alpha_j''^*\right)\right) dH_k^1 \geqq \int_{E^k} N(x, T'', B \cap C_t) dH_k^1.$$

Therefore

$$A(T'') = \int_0^\infty \left[\int_{E^k} N(x, T'', B \cap C_t) dH_k^1 \right] dt \leqq \int_0^\infty L(D(t, G, T))dt.$$

Proof of Theorem III. If T is open nondegenerate Theorem III follows from Lemma III.14 and Theorem I since T'' and T are Frechet equivalent. If T is closed nondegenerate a succession of squares Q_n may be chosen in a manner similar to the choice of Q' so that the Q_n expand outward to Q. Let $T_n' = T' | Q_n$ and T_n'' be defined as T'' was defined. Then $A(T_n') = A(T_n'')$ and $A(T_n')$ converges to $A(T') = A(T)$. Hence using Lemma III.14

$$\lim_{n \to \infty} A(T_n'') = A(T) \leqq \int_0^\infty L(D(t, G, T))dt.$$

Part IV.

Lemma IV.1. *Let $T: Q \to E^k$ be a continuous mapping and $1 \circ m$ a monotone light factorization of T with middle space M. Let β be a component of $D(t, G, T)$. Then either β^* is a single point or β^* belongs to a single proper cyclic element of M.*

Suppose not, then there would be a pair of distinct points a and b of β^* and a point c of M such that c separates a and b. By Lemma III.11 β^* is connected, hence c must belong to β^*. Since β is open c must belong to $M-\beta$. Let A be the component of $M-\{c\}$ containing a and $B=M-A\cup\{c\}$. Since c separates a and b and $c\notin\beta$, $A\cap\beta$ and $B\cap\beta$ are nonempty sets which form a separation of β contradicting that β is connected.

If C is a proper cyclic element of M each boundary point of C separates M and there is a monotone retraction r_C: $M\rightarrow C$ of M onto C which takes the part of M separated from the rest of C by a boundary point of C onto the boundary point [9]. Let T_C be defined by $T_C=1\circ r_C\circ \boldsymbol{m}$; then $1|C$ and $r_C\circ \boldsymbol{m}$ are the light and monotone factors and C the middle space in a monotone light factorization of T_C.

LEMMA IV.2. *Let $s_C=G(m(p), r_C\circ m(p), T)$. Let t be a number greater than or equal to s_C. Let $\boldsymbol{D}(s, \boldsymbol{G}, T_C)=\{a\in C: \boldsymbol{G}(r_C\circ \boldsymbol{m}(p), a, T_C)>s\}$. Let B_C be the union of the components β_i of $\boldsymbol{D}(t, \boldsymbol{G}, T)$ such that $\beta_i\cap C\neq 0$. Then $B_C=r_C^{-1}(\boldsymbol{D}(t-s_C, \boldsymbol{G}, T_C))$.*

If $\boldsymbol{m}(p)\notin C$, $r_C\circ \boldsymbol{m}(p)$ is a boundary point of C and hence separates $\boldsymbol{m}(p)$ and $C-\{r_C\circ \boldsymbol{m}(p)\}$. If $\boldsymbol{m}(p)\in C$, $\boldsymbol{G}(\boldsymbol{m}(p), r_C\circ \boldsymbol{m}(p), T)=0$. Hence if $a\in C$, $\boldsymbol{G}(\boldsymbol{m}(p), a, T)=\boldsymbol{G}(\boldsymbol{m}(p), r_C\circ \boldsymbol{m}(p), T)+\boldsymbol{G}(r_C\circ \boldsymbol{m}(p), a, T)$. Since if $a, b\in C$, $\boldsymbol{G}(a, b, T)=\boldsymbol{G}(a, b, T_C)$, $\boldsymbol{G}(\boldsymbol{m}(p), a, T)=s_C+\boldsymbol{G}(r_C\circ \boldsymbol{m}(p), a, T_C)$ for $a\in C$.

Since each β_i is connected, $\beta_i\cap C\neq 0$, and r_C has the properties described above, $r_C(\beta_i)\subset\beta_i$. Hence if $b\in\beta_i$

$$s_C+\boldsymbol{G}(r_C\circ \boldsymbol{m}(p), r_C(b), T_C) = \boldsymbol{G}(\boldsymbol{m}(p), r_C(b), T_C) > t$$

which implies $\beta_i\subset r_C^{-1}(\boldsymbol{D}(t-s_C, \boldsymbol{G}, T_C))$ for each β_i.

If $a\in r_C^{-1}(\boldsymbol{D}(t-s_C, \boldsymbol{G}, T_C))$, $\boldsymbol{G}(r_C\circ \boldsymbol{m}(p), r_C(a), T)>0$. This implies $\boldsymbol{m}(p)$ and a do not belong to the same component of the complement of C, hence either $a\in C$ or $r_C(a)$ separates $\boldsymbol{m}(p)$ and a in M. Therefore $\boldsymbol{G}(\boldsymbol{m}(p), a, T)\geqq \boldsymbol{G}(\boldsymbol{m}(p), r_C(a), T)=s_C+G(r_C\circ \boldsymbol{m}(p), r_C(a), T_C)>t$. Hence $a\in D(t, G, T)$.

Since r_C is a monotone retraction each component of $r_C^{-1}(\boldsymbol{D}(t-s_C, \boldsymbol{G}, T_C))$ meets C and hence is contained in a component β_i of B_C. Therefore $B_C=r_C^{-1}(\boldsymbol{D}(t-s_C, \boldsymbol{G}, T_C)$.

LEMMA IV.3. *If C is a cyclic element for which $s_C\leqq t$ and there is a component β of $\boldsymbol{D}(t, \boldsymbol{G}, T)$ which meets C, β^* meets C. Conversely if β^* meets C, $s_C\leqq t$.*

Suppose C was a cyclic element for which $s_C\leqq t$ and a component β of $\boldsymbol{D}(t, \boldsymbol{G}, T)$ met C and β^* did not meet C. Since $s_C\leqq t$, $C-\beta$ is not empty. Since β^* separates M and $\beta^*\cap C=0$, $\beta\cap C$ and $C-\beta$ would be a separation of C contradicting that C is connected.

If $b\in\beta^*\cap C$, $\boldsymbol{G}(\boldsymbol{m}(p), b, T)\leqq t$. If $\boldsymbol{m}(p)\in C$, $s_C=\boldsymbol{G}(\boldsymbol{m}(p), r_C\circ \boldsymbol{m}(p), T)=0$.

If $m(p) \notin C$, $r_C \circ m(p)$ separates $m(p)$ and the rest of C hence in either case if $\beta^* \cap C \neq 0$, $s_C \leqq t$.

COROLLARY. *There is only one cyclic element C for which $\beta \cap C \neq 0$ and $s_C \leqq t$.*

By Lemma IV.1 β^* is contained in a single proper cyclic element of M.

LEMMA IV.4. *Let $F = \{a \in M: G(m(p), a, T) < \infty\}$. For a proper cyclic element C of M let*

$$L_C(t) = \begin{cases} L(D(t - s_C, G, T_C)) & \text{if } s_C \leqq t, \\ 0 & \text{if } s_C > t. \end{cases}$$

Then $L(D(t, G, T)) = \sum_{C \subset \bar{F}} L_C(t)$.

For each cyclic element C of M for which $s_C \leqq t$, let B_C be the union of the components of $D(t, G, T)$ which meet C and let $\{\beta\}$ be the collection of components not meeting any cyclic element C for which $s_C \leqq t$. Let $A_C = m^{-1}(B_C)$ and $\{\alpha\} = \{m^{-1}(\beta)\}$. By Lemma IV.3 and its corollary $D(t, G, T)$ is a disjoint union of the sets of $\{\alpha\}$ and the sets A_C. By the additivity properties of the length

$$L(D(t, G, T)) = \sum_{\alpha \in \{\alpha\}} L(T, \alpha^*) + \sum_{\{C: s_C \leq t\}} L(T, A_C^*).$$

By Lemma IV.1 β^* is either a single point or is contained in a proper cyclic element of M. From Lemma IV.3 if β^* meets a proper cyclic element C, $s_C \leqq t$. Since $\beta \cap C = 0$ for each C for which $s_C \leqq t$, $\beta \subset M - C$. Then β^* must be contained in $C \cap (M - C)^+$ which is a single point. Hence β^* is always a single point of M. Therefore Lemma III.10 implies that T is constant on α^*, and hence $L(T, \alpha^*) = 0$ for every $\alpha \in \{\alpha\}$.

By Lemma IV.2 $r_C \circ m^{-1}(D(t - s_C, G, T_C)) = m^{-1}(B_C) = A_C$. Since $m(A_C^*) \subset C$, it follows from Lemma III.10 that $r_C \circ m(A_C^*) = m(A_C^*) = B_C^*$. Hence $T | A_C^* = T_C | A_C^*$ which implies by Lemma I.3 that $L(D(t - s_C, G, T_C)) = L(T, A_C^*)$.

For the mapping T_C let I_C be the set I described in Theorem II. If $C \not\subset \bar{F}$ either s_C is infinite or $r_C \circ m(p)$ belongs to I_C. If this were not so, by Theorem II and the fact that if $b \in C$, $G(m(p), b, T) = s_C + G(r_C \circ m(p), b, T_C)$, F would be dense in C contradicting $C \not\subset \bar{F}$.

If s_C is finite then $r_C \circ m(p) \in I_C$ and hence for every $t > 0$, $D(t, G, T_C)$ consists of the complement of the single point p. Hence $L(D(t, G, T_C)) = 0$ for every t in this case.

Combining the preceding remarks it is seen that

$$L(D(t, G, T)) = \sum_{\{C: s_C \leq t \text{ and } C \subset \bar{F}\}} L(D(t - s_C, G, T_C)) = \sum_{C \subset \bar{F}} L_C(t).$$

THEOREM IV. *Let $T: Q \to E^k$ be a continuous mapping of finite area. Let $T = 1 \circ m$ be a monotone light factorization of T with middle space M. Let C denote the collection of proper cyclic elements of M. Let $D(t, G, T)$, T_C, and F be as defined previously. Then*

$$\int_0^\infty L(D(t, G, T))dt = \sum_{C\subset \overline{F}} A(T_C).$$

From Lemma IV.4, $L(D(t, G, T)) = \sum_{C\subset\overline{F}} L_C(t)$. Integrating and using Lebesgue's convergence theorem

$$\int_0^\infty L(D(t, G, T))dt = \sum_{C\subset\overline{F}} \int_0^\infty L_C(t)dt = \sum_{C\subset\overline{F}} \int_0^\infty L(D(t - s_C, G, T_C)d(t - s_C).$$

Since $C \subset \overline{F}$ the mappings T_C satisfy the conditions of Theorem III hence

$$\int_0^\infty L(D(t, G, T))dt = \sum_{C\subset \overline{F}} A(T_C).$$

If C is a proper cyclic element of M and $b \in C$, $G(m(p), b, T) = s_C + G(r_C \circ m(p), b, T_C)$, hence by Theorem II $F \cap C$ is either empty, the single point $r_C \circ m(p)$, or dense in C. Since F is connected this implies that $\overline{F}$ is either a single point or an A-set in the terminology of [9]. By [9] there is a monotone retraction $r_F: M \to \overline{F}$ retracting M onto $\overline{F}$. Let $T_F = 1 \circ r_F \circ m$. Then $r_F \circ m$ is the monotone factor, 1 the light factor and $\overline{F}$ the middle space in a monotone light factorization of T_F.

THEOREM V. *Let T be as in Theorem* IV *and suppose the hypothesis of Theorem* IV *holds. Let F and T_F be as described above. Then*

$$A(T_F) = \int_0^\infty L(D(t, G, T_F))dt = \int_0^\infty L(D(t, G, T))dt.$$

If $\overline{F}$ is not a single point, $\overline{F}$ is a A-set and each boundary point of $\overline{F}$ separates M. Since $m(p) \in F$ and r_F is a retraction $r_F \circ m(p) = m(p)$. These two statements imply that if $b \in \overline{F}$

$$G(r_F \circ m(p), b, T_F) = G(m(p), b, T).$$

Hence $\{b \in \overline{F}: G(r_F \circ m(p), b, T_F) < \infty\} = F$.

Since $r_C \circ r_F = r_C$ if $C \subset \overline{F}$, by applying Theorem IV to the mapping T_F it is seen that

$$\int_0^\infty L(D(t, G, T_F))dt = \sum_{C\subset \overline{F}} A(T_C).$$

Hence by the cyclic additivity theorem of [9] it is seen that $\int_0^\infty L(D(t, G, T_F))dt = A(T_F)$.

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