ON THE SPECTRAL THEORY OF SINGULAR INTEGRAL OPERATORS[1],[2]

BY

WALTER KOPPELMAN

1. **Introduction.** Because of its numerous applications, the subject of singular integral equations of Cauchy type has attracted the attention of many authors. The bulk of the theory, as it now stands, has been developed by various Russian mathematicians (see [4; 5] and the references given there). Some interesting refinements in special instances have been obtained by Tricomi [9] and Söhngen [7].

The absence of a unified L^2 theory for singular integral equations involving open line segments has led us to the investigation of the spectral theory of bounded, self-adjoint operators of the form

$$Mx(\lambda) \equiv f(\lambda)x(\lambda) + \frac{1}{\pi i}\int_a^b \frac{g(\lambda, \mu)}{\mu - \lambda} x(\mu)d\mu \begin{pmatrix} \text{with } f(\lambda) \text{ real} \\ \text{and } g(\lambda, \mu) = \bar{g}(\mu, \lambda) \end{pmatrix},$$

acting on functions $x(\lambda)$ belonging to the class $L^2(a, b)$. (The integral which appears here is to be interpreted as a Cauchy principal value.) Needless to say, we have not attempted to give a complete theory but we have restricted ourselves to a setting which permits a unified treatment.

Initial investigations in this subject matter were carried out in a previous paper [3], where a simple singular integral operator was used to furnish an example involving perturbations of continuous spectra. Because of the connection between the present paper and this preceding work, we have gone into considerable detail to obtain spectral representations for a special subclass of the set of operators introduced above.

The operators on which we shall focus our attention have the form

$$(1.1) \qquad Lx(\lambda) \equiv f(\lambda)x(\lambda) + \frac{1}{\pi i}\int_a^b \frac{(k(\lambda)k(\mu))^{1/2}}{\mu - \lambda} x(\mu)d\mu.$$

The domain of L is to be the complex Hilbert space $L^2(a, b)$ of square integrable functions $x(\lambda)$ on the open interval (a, b). We shall assume that $f(\lambda)$ and $k(\lambda)$ are real, continuously differentiable functions on the interval $a \leqq \lambda \leqq b$, such that the functions $f'(\lambda)+k'(\lambda)$ have only isolated zeros and where $k(\lambda)>0$ almost everywhere on the interval $a \leqq \lambda \leqq b$[3]. When referring

Presented to the Society, October 25, 1958; received by the editors September 14, 1959.

(1) This research was carried out under the auspices of the Atomic Energy Commission, while the author was a Visiting Mathematician at Brookhaven National Laboratory.

(2) A preliminary announcement of these results appeared in [11].

(3) The techniques of this paper also apply to the case where $k(\lambda)$ is negative almost everywhere.

to the operator (1.1) or L in the subsequent sections of this paper, we shall tacitly assume that all of the hypotheses which we have just listed, are satisfied.

As we have said, our aim here will be to obtain a spectral representation of the operators L, i.e., an isometry U of $L^2(a, b)$ onto a direct sum of other L^2 spaces, such that ULU^{-1} is just a simple multiplication operator. Again our investigations will be far from complete, especially because of the excessive calculations involved. We hope, however, that our special considerations are a reflection of the general situation.

2. **The dominant singular integral operator.** In conjunction with the operator (1.1) we shall also consider the dominant singular integral operator

$$\tilde{L}x(\lambda) \equiv f(\lambda)x(\lambda) + \frac{k(\lambda)}{\pi i}\int_a^b \frac{x(\mu)}{\mu - \lambda}\,d\mu, \tag{2.1}$$

which differs from the operator L by the completely continuous operator

$$\frac{(k(\lambda))^{1/2}}{\pi i}\int_a^b \frac{(k(\lambda))^{1/2} - (k(\mu))^{1/2}}{\mu - \lambda}\,x(\mu)d\mu.$$

Singular integral equations involving operators of the form (2.1) were already considered by Carleman [1]. His results subsequently played a substantial role in the modern development of the theory of singular integral equations [4; 5]. We shall use Carleman's method here to obtain certain basic results about the spectrum of the operator $\tilde{L}$. First, however, we shall state several known facts concerning integrals of Cauchy type. For the proofs, the reader should consult [6; 8].

Definition 2.1. A function $F(z)$ which is regular analytic in the upper half-plane Im $z>0$ will be said to be of class $\overline{H}_2$ if

$$\int_{-\infty}^{\infty} | F(\lambda + i\eta) |^2 d\lambda \leqq K$$

for all $\eta>0$.

For functions which are regular analytic in the lower half-plane Im $z<0$ a class H_2 is defined in an analogous manner. We now have

Lemma 2.1. *A function $F(z)$ of class $\overline{H}_2$ possesses mean-square limits*

$$F(\lambda + i0) = \underset{\eta\to 0;\ \eta>0}{\text{l.i.m.}} F(\lambda + i\eta).$$

Furthermore, if $0<\text{Im } z<\eta$, *then*

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(\lambda + i0)}{\lambda - z}\,d\lambda - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(\lambda + i\eta)}{\lambda + i\eta - z}\,d\lambda.$$

A similar result holds for functions of class $\boldsymbol{H}_2$. From this lemma we obtain the following

COROLLARY. *If $F(z)$ belongs to $\overline{\boldsymbol{H}}_2$, $G(z)$ belongs to $\boldsymbol{H}_2$, and $F(\lambda+i0) = G(\lambda-i0)$, then $F(z)$ and $G(z)$ are analytic continuations of each other.*

LEMMA 2.2. *If $x(\lambda)$ is of class $L^2(-\infty, \infty)$, then the function*

$$X(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{x(\mu)}{\mu - z}\, d\mu$$

is of class $\overline{\boldsymbol{H}}_2$ for $\operatorname{Im} z>0$ *and of class $\boldsymbol{H}_2$ for* $\operatorname{Im} z<0$. *Furthermore, the limits $X(\lambda \pm i0)$ satisfy the Plemelj relations*

$$X(\lambda + i0) - X(\lambda - i0) = x(\lambda), \qquad X(\lambda + i0) + X(\lambda - i0) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{x(\mu)}{\mu - \lambda}\, d\mu.$$

We are now ready to prove

THEOREM 2.1. *Let ξ be a real number which does not lie in the closed interval $[\min_{a\leq\lambda\leq b} \{f(\lambda)-k(\lambda)\}, \max_{a\leq\lambda\leq b} \{f(\lambda)+k(\lambda)\}]$. Then ξ is not a point eigenvalue of the operator $\tilde{L}$.*

Proof. Suppose that $x(\lambda)$ belongs to $L^2(a, b)$ and that

$$\{f(\lambda) - \xi\}x(\lambda) + \frac{k(\lambda)}{\pi i}\int_a^b \frac{x(\mu)}{\mu - \lambda}\, d\mu = 0. \tag{2.2}$$

We now apply Carleman's method. The function

$$X(z) = \frac{1}{2\pi i}\int_a^b \frac{x(\mu)}{\mu - z}\, d\mu \tag{2.3}$$

is analytic in the complex z-plane cut along the real axis from a to b, is of class $\overline{\boldsymbol{H}}_2$ for Im $z>0$, of class $\boldsymbol{H}_2$ for Im $z<0$ and satisfies

$$X(\infty) = 0 \tag{2.4}$$

and

$$\{f(\lambda) - \xi + k(\lambda)\}X(\lambda + i0) - \{f(\lambda) - \xi - k(\lambda)\}X(\lambda - i0) = 0, \qquad a < \lambda < b. \tag{2.5}$$

(Relation (2.5) follows from the Plemelj formulae.) We now observe that the function

$$E(\xi, z) = \exp \frac{1}{2\pi i}\int_a^b \log\frac{f(\mu) - \xi - k(\mu)}{f(\mu) - \xi + k(\mu)}\, \frac{d\mu}{\mu - z} \tag{2.6}$$

is a solution of the barrier equation (2.5). Of course

$$E(\xi, \infty) = 1. \tag{2.7}$$

One can show that $E(\xi, \lambda \pm i0)$ is Hölder continuous on any subinterval (a', b') of (a, b) and that $E(\xi, z)$ is both *bounded* and *bounded away from zero* [5]. We also have

$$\frac{X(\lambda + i0)}{E(\xi, \lambda + i0)} - \frac{X(\lambda - i0)}{E(\xi, \lambda - i0)} = 0, \qquad a < \lambda < b. \tag{2.8}$$

With the aid of the corollary to Lemma 2.1, we can now conclude that $X(z)/E(\xi, z)$ is regular in the entire plane. In addition, (2.4) and (2.7) show that $X(z)/E(\xi, z)$ vanishes at ∞. Hence $X(z)/E(\xi, z) \equiv 0$, so that $X(z) \equiv 0$. From the Plemelj formulae we then see that $x(\lambda) = 0$ almost everywhere on the interval (a, b).

THEOREM 2.2. *Let ξ satisfy the hypothesis of Theorem 2.1. Then ξ belongs to the resolvent set of the operator $\tilde{L}$.*

Proof. Consider the integral equation

$$\{f(\lambda) - \xi\} x(\lambda) + \frac{k(\lambda)}{\pi i} \int_a^b \frac{x(\mu)}{\mu - \lambda} d\mu = h(\lambda), \qquad h(\lambda) \in L^2(a, b), \tag{2.9}$$

for functions $x(\lambda)$ of the class $L^2(a, b)$. If $x(\lambda)$ is a solution of (2.9), then we again introduce the function

$$X(z) = \frac{1}{2\pi i} \int_a^b \frac{x(\mu)}{\mu - z} d\mu.$$

$X(z)$ satisfies the barrier equation

$$\{f(\lambda) - \xi + k(\lambda)\} X(\lambda + i0) - \{f(\lambda) - \xi - k(\lambda)\} X(\lambda - i0) = h(\lambda), \qquad a < \lambda < b. \tag{2.10}$$

Using the solution (2.7) of the homogeneous barrier equation, we obtain

$$\frac{X(\lambda + i0)}{E(\xi, \lambda + i0)} - \frac{X(\lambda - i0)}{E(\xi, \lambda - i0)} = \frac{h(\lambda)}{\{f(\lambda) - \xi + k(\lambda)\} E(\xi, \lambda + i0)}, \qquad a < \lambda < b. \tag{2.11}$$

The function

$$G(z) = \frac{1}{2\pi i} \int_a^b \frac{h(\mu)}{\{f(\mu) - \xi + k(\mu)\} E(\xi, \mu + i0)} \frac{d\mu}{\mu - z} \tag{2.12}$$

vanishes at ∞ and satisfies

$$G(\lambda + i0) - G(\lambda - i0) = \frac{h(\lambda)}{\{f(\lambda) - \xi + k(\lambda)\} E(\xi, \lambda + i0)} \cdot$$

Hence $X(z)/E(\xi, z)$ differs from $G(z)$ by a function $K(z)$ which vanishes at ∞ and which satisfies $K(\lambda+i0)-K(\lambda-i0)=0$, $a<\lambda<b$. Of course $K(z)$ is of class $\overline{H}_2$ for $\operatorname{Im} z>0$ and of class H_2 for $\operatorname{Im} z<0$. Hence $K(z)$ is analytic in the entire plane and vanishes at ∞. Thus $K(z)\equiv 0$ and $X(z)=E(\xi, z)G(z)$.

Let us therefore consider the function $E(\xi, z)G(z)$. It satisfies (2.10) and is of class $\overline{H}_2$ for $\operatorname{Im} z>0$, H_2 for $\operatorname{Im} z<0$. Furthermore $E(\xi, z)G(z)$ vanishes at ∞. By Lemma 2.1, the function

$$y(\lambda) = E(\xi, \lambda + i0)G(\lambda + i0) - E(\xi, \lambda - i0)G(\lambda - i0)$$

belongs to the class $L^2(a, b)$. Form

$$Y(z) = \frac{1}{2\pi i}\int_a^b \frac{y(\mu)}{\mu - z}\, d\mu.$$

We then have

$$Y(\lambda + i0) - E(\xi, \lambda + i0)G(\lambda + i0) = Y(\lambda - i0) - E(\xi, \lambda - i0)G(\lambda - i0).$$

In addition $Y(z)-E(\xi, z)G(z)$ belongs to the classes $\overline{H}_2$, H_2 and vanishes at ∞. Thus $Y(z)-E(\xi, z)G(z)\equiv 0$ or $Y(z)\equiv E(\xi, z)G(z)$. Of course then

$$E(\xi, \lambda + i0)G(\lambda + i0) + E(\xi, \lambda - i0)G(\lambda - i0) = \frac{1}{\pi i}\int_a^b \frac{y(\mu)}{\mu - \lambda}\, d\mu,$$

and therefore $y(\lambda)$ is a solution of the integral equation (2.9). From Theorem 2.1 we know that this is the only solution of class $L^2(a, b)$. Therefore $\tilde{L}-\xi\cdot 1$ is a 1-1 mapping of $L^2(a, b)$ onto itself. By the closed graph theorem then, the operator $\tilde{L}-\xi\cdot 1$ has a bounded inverse [10]. This completes the proof of the theorem.

Since our original operator L differs from $\tilde{L}$ by a completely continuous operator, we obtain the following

COROLLARY. *The limit points of the spectrum of the operator L lie in the closed interval* $[\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\}]$.

3. **Relations for the function** $E(l, z)$. In order to obtain more specific information about the spectrum of the operator L, we shall attempt to obtain a spectral representation for L. Our technique is based upon various relations which the fundamental solution

$$E(l, z) = \exp\left\{\frac{1}{2\pi i}\int_a^b \log\frac{f(\mu) - l - k(\mu)}{f(\mu) - l + k(\mu)}\, \frac{d\mu}{\mu - z}\right\}$$

of the Hilbert problem $\{f(\lambda)-l+k(\lambda)\}X(\lambda+i0)-\{f(\lambda)-l-k(\lambda)\}X(\lambda-i0) =0$ satisfies and which will be developed here.

Of course we must have

$$(3.1)\quad \{f(\lambda) - l + k(\lambda)\}E(l, \lambda + i0) - \{f(\lambda) - l - k(\lambda)\}E(l, \lambda - i0) = 0, \qquad a < \lambda < b.$$

If we set $H(l, \lambda) = E(l, \lambda + i0) - E(l, \lambda - i0)$, then (3.1) may also be written in the form

$$(3.2)\quad \{f(\lambda) - l\}H(l, \lambda) = -k(\lambda)\{E(l, \lambda + i0) + E(l, \lambda - i0)\}, \quad a < \lambda < b.$$

From (3.2) one obtains

$$(3.3)\quad \begin{aligned} &(l_2 - l_1)H(l_1, \lambda)\overline{H}(\bar{l}_2, \lambda) \\ &\quad = 2k(\lambda)\{E(l_1, \lambda + i0)\overline{E}(\bar{l}_2, \lambda - i0) - E(l_1, \lambda - i0)\overline{E}(\bar{l}_2, \lambda + i0)\}. \end{aligned}$$

We shall now derive analogous barrier relations which the function $E(l, z)$ satisfies with respect to l. Since

$$E(l, z) = \exp\left\{\frac{1}{2\pi i}\int_a^b \log\left|\frac{f(\mu) - l - k(\mu)}{f(\mu) - l + k(\mu)}\right| \frac{d\mu}{\mu - z} + \frac{1}{2\pi}\int_a^b \arg\frac{f(\mu) - l - k(\mu)}{f(\mu) - l + k(\mu)} \frac{d\mu}{\mu - z}\right\},$$

we have

$$\begin{aligned} \frac{E(\xi + i0, z)}{E(\xi - i0, z)} &= \exp\left\{\frac{1}{2\pi}\int_a^b \left(\arg\frac{f(\mu - \xi - i0 - k(\mu)}{f(\mu) - \xi - i0 + k(\mu)} - \arg\frac{f(\mu) - \xi + i0 - k(\mu)}{f(\mu) - \xi + i0 + k(\mu)}\right)\frac{d\mu}{\mu - z}\right\} \\ &= \exp\left\{\frac{1}{\pi}\int_a^b \arg\frac{f(\mu) - \xi - i0 - k(\mu)}{f(\mu) - \xi - i0 + k(\mu)} \frac{d\mu}{\mu - z}\right\}, \\ &\qquad \min\{f(\lambda) - k(\lambda)\} < \xi < \max\{f(\lambda) + k(\lambda)\}. \end{aligned}$$

From this expression it is not difficult to see that

$$(3.4)\quad \begin{aligned} &c(\xi, z)E(\xi + i0, z) - g(\xi, z)E(\xi - i0, z) = 0, \\ &\qquad \min\{f(\lambda) - k(\lambda)\} < \xi < \max\{f(\lambda) + k(\lambda)\}, \end{aligned}$$

where, for each fixed ξ, $c(\xi, z)$ and $g(\xi, z)$ are polynomials in z with real coefficients and of the same degree.

For convenience in subsequent computations we shall express the quotient $E(\xi+i0, z)/E(\xi-i0, z)$ in another form. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the zeros of $f'(\lambda)+k'(\lambda)$ and let $\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_m$ be the zeros of $f'(\lambda)-k'(\lambda)$. Assume that

$\lambda_0 = a \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_n \leqq b = \lambda_{n+1}$ and $\tilde{\lambda}_0 = a \leqq \tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots < \tilde{\lambda}_m \leqq b = \tilde{\lambda}_{m+1}$. Through integration by parts one can show that

$$\begin{aligned}(3.5)\quad &\int_a^b \log \frac{f(\mu) - l - k(\mu)}{f(\mu) - l + k(\mu)} \frac{d\mu}{\mu - z} \\ &= \log (a - z) \int_{f(a)-k(a)}^{f(a)+k(a)} \frac{dv}{v - l} + \sum_{j=0}^{n} \int_{f(\lambda_j)+k(\lambda_j)}^{f(\lambda_{j+1})+k(\lambda_{j+1})} \log \{ g_j(v) - z \} \frac{dv}{v - l} \\ &\quad - \sum_{s=0}^{m} \int_{f(\tilde{\lambda}_s)-k(\tilde{\lambda}_s)}^{f(\tilde{\lambda}_{s+1})+k(\tilde{\lambda}_{s+1})} \log \{ h_s(v) - z \} \frac{dv}{v - l} - \log(b - z) \int_{f(b)-k(b)}^{f(b)+k(b)} \frac{dv}{v - l},\end{aligned}$$

with $v = f(g_j(v)) + k(g_j(v))$, $\lambda_j \leqq g_j(v) \leqq \lambda_{j+1}$ and $v = f(h_s(v)) - k(h_s(v))$, $\tilde{\lambda}_s \leqq h_s(v) \leqq \tilde{\lambda}_{s+1}$. The right-hand side of (3.5) is simply(4)

$$\int_\alpha^\beta \log \frac{g(v, z)}{c(v, z)} \frac{dv}{v - l} \cdot$$

If we set $F(\xi, z) = E(\xi + i0, z) - E(\xi - i0, z)$, then (3.4) may also be written in the form

$$(3.6)\quad \frac{c(\xi, z) + g(\xi, z)}{c(\xi, z) - g(\xi, z)} F(\xi, z) = - \{ E(\xi + i0, z) + E(\xi - i0, z) \}.$$

From (3.6) we obtain

$$\begin{aligned}(3.7)\quad &\frac{P(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} F(\xi, z_1)\overline{F}(\xi, \bar{z}_2) \\ &\qquad = \{ E(\xi + i0, z_1)\overline{E}(\xi - i0, \bar{z}_2) - E(\xi - i0, z_1)\overline{E}(\xi + i0, \bar{z}_2) \},\end{aligned}$$

with $P(\xi, z_1, z_2) = c(\xi, z_2)g(\xi, z_1) - c(\xi, z_1)g(\xi, z_2)$ and $Q(\xi, z) = c(\xi, z) - g(\xi, z)$. Since $P(\xi, z_1, z_2)$ vanishes for $z_1 = z_2$, we must have $P(\xi, z_1, z_2) = (z_2 - z_1)\tilde{P}(\xi, z_1, z_2)$, where $\tilde{P}(\xi, z_1, z_2)$ is a polynomial in z_1 and z_2 with real coefficients. Thus

$$\begin{aligned}(3.8)\quad &\frac{(z_2 - z_1)\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} F(\xi, z_1)\overline{F}(\xi, \bar{z}_2) \\ &\qquad = \{ E(\xi + i0, z_1)\overline{E}(\xi - i0, \bar{z}_2) - E(\xi - i0, z_1)\overline{E}(\xi + i0, \bar{z}_2) \}, \\ &\qquad\qquad \min \{ f(\lambda) - k(\lambda) \} < \xi < \max \{ f(\lambda) + k(\lambda) \}.\end{aligned}$$

Next we turn our considerations to some relations which certain integrals involving $H(l, \lambda)$ and $F(\xi, z)$ satisfy. If we integrate $E(l, z)$ over a contour in the z-plane which passes around a cut from a to b along the real axis in a clockwise direction, then the calculus of residues yields

(4) Here we have set $\alpha = \min\{f(\lambda) - k(\lambda)\}$ and $\beta = \max\{f(\lambda) + k(\lambda)\}$.

$$\frac{1}{2\pi i}\oint E(l,z)dz = \frac{1}{2\pi i}\int_a^b \log\frac{f(\mu)-l-k(\mu)}{f(\mu)-l+k(\mu)}\,d\mu.$$

Hence

$$\frac{1}{2\pi i}\int_a^b H(l,\lambda)d\lambda = \frac{1}{2\pi i}\int_a^b \log\frac{f(\mu)-l-k(\mu)}{f(\mu)-l+k(\mu)}\,d\mu. \tag{3.9}$$

By integration by parts one can show that

$$\begin{aligned}\int_a^b \log\frac{f(\mu)-l-k(\mu)}{f(\mu)-l+k(\mu)}\,d\mu \\ = a\int_{f(a)-k(a)}^{f(a)+k(a)} \frac{dv}{v-l} + \sum_{j=0}^{n}\int_{f(\lambda_k)+k(\lambda_j)}^{f(\lambda_{j+1})+k(\lambda_{j+1})} \frac{g_j(v)}{v-l}\,dv \\ - \sum_{s=0}^{m}\int_{f(\tilde\lambda_s)-k(\tilde\lambda_s)}^{f(\tilde\lambda_{s+1})-k(\tilde\lambda_{s+1})} \frac{h_s(v)}{v-l}\,dv - b\int_{f(b)-k(b)}^{f(b)+k(b)} \frac{dv}{v-l}.\end{aligned} \tag{3.10}$$

Hence if we define the function $A(\xi)$ by the equation

$$A(\xi) = -\frac{1}{2\pi i}\lim_{\eta\to 0;\ n>0}\int_a^b \{H(\xi+i\eta,\lambda) - H(\xi-i\eta,\lambda)\}d\lambda, \tag{3.11}$$

then the right-hand side of (3.10) is just

$$-\int_\alpha^\beta \frac{A(v)}{v-l}\,dv.$$

Now we use a similar procedure on the function $E(l, z)$ with respect to the l-variable. We integrate $E(l, z)$ over a contour in the l-plane which passes around a cut from min $\{f(\lambda)-k(\lambda)\}$ to max$\{f(\lambda)+k(\lambda)\}$ along the real axis in a clockwise direction. Using the calculus of residues we then obtain

$$\frac{1}{2\pi i}\oint E(l,z)dl = -\frac{1}{\pi i}\int_a^b \frac{k(\mu)}{\mu-z}\,d\mu$$

and hence

$$\frac{1}{2\pi i}\int_\alpha^\beta F(\xi,z)d\xi = -\frac{1}{\pi i}\int_a^b \frac{k(\mu)}{\mu-z}\,d\mu. \tag{3.12}$$

Using the same type of contours we see that

$$\frac{1}{2\pi i}\oint \frac{E(l,z)}{z-w}\,dz = E(l,w) - 1$$

and therefore

$$\frac{1}{2\pi i}\int_a^b \frac{H(l,\lambda)}{\lambda - w}\,d\lambda = E(l,w) - 1. \tag{3.13}$$

Similarly

$$\frac{1}{2\pi i}\oint \frac{E(l,z)}{l-w}\,dl = E(w,z) - 1$$

and

$$\frac{1}{2\pi i}\int_\alpha^\beta \frac{F(\xi,z)}{\xi - w}\,d\xi = E(w,z) - 1. \tag{3.14}$$

$$\frac{1}{2\pi i}\oint E(l_1,z)\overline{E}(\bar{l}_2,\bar{z})dz$$
$$= \frac{1}{2\pi i}\int_a^b \{E(l_1,\lambda+i0)\overline{E}(\bar{l}_2,\lambda - i0) - E(l_1,\lambda - i0)\overline{E}(\bar{l}_2,\lambda + i0)\}d\lambda$$
$$= \frac{1}{2\pi i}\int_\alpha^\beta A(v)\left\{\frac{1}{v-l_2} - \frac{1}{v-l_1}\right\}dv.$$

Using (3.3) we then have

$$\frac{l_2 - l_1}{2\pi i}\int_a^b \frac{H(l_1,\lambda)\overline{H}(\bar{l}_2,\lambda)}{2k(\lambda)}\,d\lambda = \frac{1}{2\pi i}\int_\alpha^\beta A(v)\left\{\frac{1}{v-l_2} - \frac{1}{v-l_1}\right\}dv. \tag{3.15}$$

$$\frac{1}{2\pi i}\oint E(l,z_1)\overline{E}(\bar{l},\bar{z}_2)dl$$
$$= \frac{1}{2\pi i}\int_\alpha^\beta \{E(\xi+i0,z_1)\overline{E}(\xi - i0,\bar{z}_2) - E(\xi - i0,z_1)\overline{E}(\xi + i0,\bar{z}_2)\}d\xi$$
$$= \frac{1}{\pi i}\int_a^b k(\mu)\left\{\frac{1}{\mu - z_2} - \frac{1}{\mu - z_1}\right\}d\mu.$$

From (3.8), we obtain

$$\frac{z_2 - z_1}{2\pi i}\int_\alpha^\beta \frac{\tilde{P}(\xi,z_1,z_2)}{Q(\xi,z_1)Q(\xi,z_2)}F(\xi,z_1)\overline{F}(\xi,\bar{z}_2)d\xi = \frac{1}{\pi i}\int_a^b k(\mu)\left\{\frac{1}{\mu - z_2} - \frac{1}{\mu - z_1}\right\}d\mu. \tag{3.16}$$

4. **Properties of the function $A(\xi)$.** In the preceding section we defined the function $A(\xi)$ through the relation (3.11). From (3.9) it is not difficult to see that

$$
\begin{aligned}
A(\xi) &= -\frac{1}{2\pi}\int_a^b \left\{ \arg\frac{f(\mu)-\xi-i0-k(\mu)}{f(\mu)-\xi+i0+k(\mu)} - \arg\frac{f(\mu)-\xi+i0-k(\mu)}{f(\mu)-\xi+i0+k(\mu)} \right\} d\mu \\
&= -\frac{1}{\pi}\int_a^b \arg\frac{f(\mu)-\xi-i0-k(\mu)}{f(\mu)-\xi-i0+k(\mu)}\, d\mu,
\end{aligned}
\tag{4.1}
$$

for all but a finite number of values ξ.

Formula (4.1) immediately gives us the following lemma:

LEMMA 4.1. *$A(\xi)$ is non-negative for all but a finite number of values ξ.*

A more thorough investigation will show that we can strengthen the statement of Lemma 4.1. In fact we obtain

LEMMA 4.2. *$A(\xi)>0$ for almost all ξ such that* $\min\{f(\lambda)-k(\lambda)\}<\xi<\max\{f(\lambda)+k(\lambda)\}$.

Proof. Suppose, first, that ξ lies in the given interval but that ξ does not lie in the ranges of the functions $f(\lambda)\pm k(\lambda)$. Then $\xi>\max\{f(\lambda)-k(\lambda)\}$ and $\xi<\min\{f(\lambda)+k(\lambda)\}$. Therefore

$$
\frac{f(\lambda)-\xi-k(\lambda)}{f(\lambda)-\xi+k(\lambda)}<0, \qquad a\leqq\lambda\leqq b.
$$

From (4.1) we see that $A(\xi)>0$.

Assume, next, that ξ lies in the *interior* of the range of at least one of the functions $f(\lambda)\pm k(\lambda)$, say $f(\lambda)-k(\lambda)$. Suppose, furthermore, that $\xi=f(\Lambda)-k(\Lambda)$, where $k(\Lambda)>0$ and $f'(\Lambda)-k'(\Lambda)\neq 0$. Then there exists an interval (μ_1,μ_2) such that $\mu_1<\Lambda<\mu_2$, $k(\lambda)\geqq m>0$ for $\mu_1<\lambda<\mu_2$ and $f'(\lambda)-k'(\lambda)\neq 0$ for $\mu_1<\lambda<\mu_2$.

Thus there exists a Λ_1, $\mu_1<\Lambda_1<\mu_2$, such that $\xi>f(\Lambda_1)-k(\Lambda_1)>\xi-m$. We shall now show that $f(\Lambda_1)+k(\Lambda_1)>\xi$. In fact $f(\Lambda_1)+k(\Lambda_1)=\{f(\Lambda_1)-k(\Lambda_1)\}+2k(\Lambda_1)>\{\xi-m\}+2m=\xi+m>\xi$. Hence

$$
\arg\left\{\frac{f(\Lambda_1)-\xi-i0-k(\Lambda_1)}{f(\Lambda_1)-\xi-i0+k(\Lambda_1)}\right\}<0.
$$

Formula (4.1) now shows that $A(\xi)>0$. A similar argument holds when ξ lies in the interior of the range of the function $f(\lambda)+k(\lambda)$ with $\xi=f(\Lambda)+k(\Lambda)$, $k(\Lambda)>0$ and $f'(\Lambda)+k'(\Lambda)\neq 0$.

That we have omitted only a set of values ξ of measure zero, now follows immediately from our hypothesis about the zeros of the functions $f'(\lambda)\pm k'(\lambda)$ and $k(\lambda)$, and the absolute continuity of the functions $f(\lambda)\pm k(\lambda)$.

Finally we shall prove the formula

$$
\int_\alpha^\beta A(v)dv=\int_a^b 2k(\mu)d\mu. \tag{4.2}
$$

By integration by parts we have

$$\int_a^b 2k(\mu)d\mu = 2\mu k(\mu)\Big|_a^b - 2\int_a^b \mu k'(\mu)d\mu$$

$$= -\mu\{f(\mu)-k(\mu)\}\Big|_a^b + \mu\{f(\mu)+k(\mu)\}\Big|_a^b + \int_a^b \mu\{f'(\mu)-k'(\mu)\}d\mu$$

$$- \int_a^b \mu\{f'(\mu)+k'(\mu)\}d\mu$$

$$= \sum_{s=0}^{m}\int_{f(\tilde\lambda_s)-k(\tilde\lambda_s)}^{f(\tilde\lambda_{s+1})-k(\tilde\lambda_{s+1})} h_s(v)dv + b\int_{f(b)-k(b)}^{f(b)+k(b)} dv - a\int_{f(a)-k(a)}^{f(a)+k(a)} dv$$

$$- \sum_{j=0}^{n}\int_{f(\lambda_j)+k(\lambda_j)}^{f(\lambda_{j+1})+k(\lambda_{j+1})} g_j(v)dv,$$

where we have used the notation which occurs in (3.5) and (3.10). Through the remark after formula (3.11), we see that our last expression is just

$$\int_\alpha^\beta A(v)dv.$$

This proves (4.2).

5. Spectral transformations for the operator L. In this section we shall prove the following theorem:

THEOREM 5.1. *The transformations*

$$(5.1)\qquad \hat{R}h(\lambda) = -\frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_a^b \frac{\overline{H}(\xi+i\eta,\lambda)-\overline{H}(\xi-i\eta,\lambda)}{(2k(\lambda)A(\xi))^{1/2}}h(\lambda)d\lambda$$

and

$$(5.2)\qquad \hat{S}g(\xi) = \frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_\alpha^\beta \frac{F(\xi,\lambda+i\eta)-F(\xi,\lambda-i\eta)}{(2k(\lambda)A(\xi))^{1/2}}g(\xi)d\xi$$

generate isometries of a subspace $\overline{\mathfrak{L}}_\lambda$ *of* $L^2(a, b)$ *onto*

$$L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$$

and S *of* $L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$ *onto* $\overline{\mathfrak{L}}_\lambda$, *such that* $RS=1$ *and* $SR=1$. *Furthermore, the subspace* $\overline{\mathfrak{L}}_\lambda$ *is invariant under the operation* L. *If* $h(\lambda)$ *belongs to* $\overline{\mathfrak{L}}_\lambda$ *and if* $g(\xi)=Rh(\lambda)$, *then* $\xi g(\xi)=RLh(\lambda)$.

Proof. Let us denote the linear manifold generated by the elements $\{(2k(\lambda))^{1/2}, H(w,\lambda)/(2k(\lambda))^{1/2}\}$, where w is any complex number in the exterior of the closed interval $[\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\}]$, by $\mathfrak{L}_\lambda$. From (3.11) we then obtain

$$(A(\xi))^{1/2} = -\frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_a^b \frac{\overline{H}(\xi+i\eta,\lambda)-\overline{H}(\xi-i\eta,\lambda)}{(2k(\lambda)A(\xi))^{1/2}}\{-(2k(\lambda))^{1/2}\}d\lambda$$

and thus

$$\hat{R}\{-(2k(\lambda))^{1/2}\} = (A(\xi))^{1/2}. \tag{5.3}$$

From (3.15) we have

$$-\frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_a^b \frac{\overline{H}(\xi+i\eta,\lambda)-\overline{H}(\xi-i\eta,\lambda)}{(2k(\lambda)A(\xi))^{1/2}}\,\frac{H(w,\lambda)}{(2k(\lambda))^{1/2}}d\lambda = \frac{(A(\xi))^{1/2}}{\xi-w}$$

and hence

$$\hat{R}\left\{\frac{H(w,\lambda)}{(2k(\lambda))^{1/2}}\right\} = \frac{(A(\xi))^{1/2}}{\xi-w}. \tag{5.4}$$

Thus the transformation $\hat{R}$ maps the linear manifold $\mathfrak{L}_\lambda$ onto the linear manifold $\mathfrak{L}_\xi$ which is generated by the functions $\{(A\xi))^{1/2},\ (A(\xi))^{1/2}/(\xi-w)\}$.

From (3.12) we see that

$$\frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_\alpha^\beta \frac{F(\xi,\lambda+i\eta)-F(\xi,\lambda-i\eta)}{(2k(\lambda)A(\xi))^{1/2}}(A(\xi))^{1/2}d\xi = -(2k(\lambda))^{1/2}.$$

Thus

$$\hat{S}\{(A(\xi))^{1/2}\} = -(2k(\lambda))^{1/2}. \tag{5.5}$$

From (3.14) we obtain

$$\frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_\alpha^\beta \frac{F(\xi,\lambda+i\eta)-F(\xi,\lambda-i\eta)}{(2k(\lambda)A(\xi))^{1/2}}\,\frac{(A(\xi))^{1/2}}{\xi-w}d\xi = \frac{H(w,\lambda)}{(2k(\lambda))^{1/2}}.$$

Therefore

$$\hat{S}\left\{\frac{(A(\xi))^{1/2}}{\xi-w}\right\} = \frac{H(w,\lambda)}{(2k(\lambda))^{1/2}}. \tag{5.6}$$

Thus if we restrict our considerations to the manifolds $\mathfrak{L}_\lambda$ and $\mathfrak{L}_\xi$, we have $\hat{R}\hat{S}=1$ and $\hat{S}\hat{R}=1$. Now formula (4.2) states that

$$\int_\alpha^\beta A(v)dv = \int_a^b 2k(\mu)d\mu.$$

From (3.9), (3.10) and (3.11) we obtain

$$\int_a^b \frac{H(w,\lambda)}{(2k(\lambda))^{1/2}}\left\{-(2k(\lambda))^{1/2}\right\}d\lambda = \int_\alpha^\beta \frac{(A(\xi))^{1/2}}{\xi-w}\left\{(A(\xi))^{1/2}\right\}d\xi. \tag{5.7}$$

Using (3.15), we conclude that

$$\int_a^b \frac{H(w_1, \lambda)}{(2k(\lambda))^{1/2}} \cdot \frac{\bar{H}(w_2, \lambda)}{(2k(\lambda))^{1/2}} d\lambda = \int_\alpha^\beta \frac{(A(\xi))^{1/2}}{\xi - w_1} \cdot \frac{(A(\xi))^{1/2}}{\xi - \bar{w}_2} d\xi. \tag{5.8}$$

From (4.2), (5.7), and (5.8) we see that when $\hat{R}$ and $\hat{S}$ are restricted to $\mathfrak{L}_\lambda$ and $\mathfrak{L}_\xi$ respectively, they are isometries. Hence $\hat{R}$ and $\hat{S}$ have bounded extensions R and S to the closures $\bar{\mathfrak{L}}_\lambda$ and $\bar{\mathfrak{L}}_\xi$ of the linear manifolds $\mathfrak{L}_\lambda$ and $\mathfrak{L}_\xi$. R and S are also isometries such that $RS=1$ and $SR=1$.

Szegö [2] has shown that the linear manifold generated by the elements $\{1/(\xi-w)\}$ is dense in the L^2 space over a finite interval. Formula (4.1) shows that $A(\xi)$ is bounded. From Lemma 4.3 it therefore follows that the set of elements $\{(A(\xi))^{1/2}/(\xi-w)\}$ generates a linear manifold which is dense in the space $L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$. Hence we see that $\bar{\mathfrak{L}}_\xi=L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$. Furthermore, since S is an isometry of $\bar{\mathfrak{L}}_\xi$ onto $\bar{\mathfrak{L}}_\lambda$, the linear manifold generated by the elements $\{H(w, \lambda)/(2k(\lambda))^{1/2}\}$ must be dense in $\bar{\mathfrak{L}}_\lambda$. Using the fact that

$$E(w, z) - 1 = \frac{1}{2\pi i}\int_a^b \frac{H(w, \mu)}{\mu - z} d\mu$$

(see (3.13)) and the Plemelj formulae, we see that

$$\begin{aligned}
L\left\{\frac{H(w, \lambda)}{(2k(\lambda))^{1/2}}\right\} &\equiv f(\lambda)\left\{\frac{H(w, \lambda)}{(2k(\lambda))^{1/2}}\right\} + \frac{1}{\pi i}\int_a^b \frac{(k(\lambda)k(\mu))^{1/2}}{\mu - \lambda} \frac{H(w, \mu)}{(2k(\mu))^{1/2}} d\mu \\
&= \frac{1}{(2k(\lambda))^{1/2}} \left\{f(\lambda)H(w, \lambda) + \frac{k(\lambda)}{\pi i}\int_a^b \frac{H(w, \mu)}{\mu - \lambda} d\mu\right\} \\
&= \frac{1}{(2k(\lambda))^{1/2}} \{f(\lambda)H(w, \lambda) + k(\lambda)\{E(w, \lambda + i0) \\
&\qquad + E(w, \lambda - i0) - 2\}\} \\
&= \frac{1}{(2k(\lambda))^{1/2}} \{f(\lambda)H(w, \lambda) - \{f(\lambda) - w\}H(w, \lambda) - 2k(\lambda)\} \\
&\qquad \text{(from (3.2))} \\
&= -(2k(\lambda))^{1/2} + w\frac{H(w, \lambda)}{(2k(\lambda))^{1/2}}.
\end{aligned}$$

This proves

$$L\left\{\frac{H(w, \lambda)}{(2k(\lambda))^{1/2}}\right\} = -(2k(\lambda))^{1/2} + w\frac{H(w, \lambda)}{(2k(\lambda))^{1/2}}. \tag{5.9}$$

Since the linear manifold generated by $\{H(w, \lambda)/(2k(\lambda))^{1/2}\}$ is dense in $\bar{\mathfrak{L}}_\lambda$

and since L is a bounded operator, (5.9) shows that $\bar{\mathfrak{L}}_\lambda$ is invariant under L. Finally, we observe that

$$RL\left\{\frac{H(w,\lambda)}{(2k(\lambda))^{1/2}}\right\} = R\{-(2k(\lambda))^{1/2}\} + wR\left\{\frac{H(w,\lambda)}{(2k(\lambda))^{1/2}}\right\}$$

$$= (A(\xi))^{1/2} + w\frac{(A(\xi))^{1/2}}{\xi - w} = \xi\frac{(A(\xi))^{1/2}}{\xi - w},$$

while $R\{H(w,\lambda)/(2k(\lambda))^{1/2}\} = (A(\xi))^{1/2}/(\xi - w)$. Since R is an isometry and since the ξ interval is bounded, we can now conclude that if $g(\xi) = Rh(\lambda)$, with $h(\lambda)$ in $\bar{\mathfrak{L}}_\lambda$, then $\xi g(\xi) = RLh(\lambda)$. This completes the proof of the theorem.

6. Further development of the spectral transformations for the operator L. In the investigations of this section we shall restrict ourselves to the following hypothesis:

HYPOTHESIS 6.1.

$$\frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} = \frac{1}{A(\xi)} + \sum_{j=1}^{N} \frac{C_j(\xi, z_1)C_j(\xi, z_2)}{Q(\xi, z_1)Q(\xi, z_2)},$$

where the $C_j(\xi, z)$ are polynomials in z, with real coefficients. (Here $\tilde{P}(\xi, z_1, z_2)$ and $Q(\xi, z)$ are the polynomials which we defined in §3).

LEMMA 6.1. *Under Hypothesis 6.1, there exists an isometry $\mathfrak{R}$ of $L^2(a, b)$ into the direct sum $\mathfrak{H}$ of $N+1$ $L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$ spaces, which coincides with the transformation R of §5 on the space $\bar{\mathfrak{L}}_\lambda$.*

Proof. Let $(f_0(\xi), f_1(\xi), \cdots, f_N(\xi))$ be an element of $\mathfrak{H}$. Consider the following transformation:

$$\hat{\mathfrak{S}}(f_0(\xi), f_1(\xi), \cdots, f_N(\xi)) = \frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_\alpha^\beta \frac{F(\xi,\lambda+i\eta) - F(\xi,\lambda-i\eta)}{(2k(\lambda)A(\xi))^{1/2}} f_0(\xi)d\xi$$

$$+ \sum_{j=1}^{N}\int_\alpha^\beta \left\{\frac{C_j(\xi,\lambda+i\eta)F(\xi,\lambda+i\eta)}{Q(\xi,\lambda+i\eta)(2k(\lambda))^{1/2}} - \frac{C_j(\xi,\lambda-i\eta)F(\xi,\lambda-i\eta)}{Q(\xi,\lambda-i\eta)(2k(\lambda))^{1/2}}\right\} f_j(\xi)d\xi. \tag{6.1}$$

Utilizing (3.16), we see that

$$\frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_\alpha^\beta \left\{\frac{\tilde{P}(\xi,\lambda+i\eta,z)F(\xi,\lambda+i\eta)}{Q(\xi,\lambda+i\eta)(2k(\lambda))^{1/2}} - \frac{\tilde{P}(\xi,\lambda-i\eta,z)F(\xi,\lambda-i\eta)}{Q(\xi,\lambda-i\eta)(2k(\lambda))^{1/2}}\right\}$$

$$\frac{\bar{F}(\xi,\bar{z})}{Q(\xi,z)}d\xi = \frac{(2k(\lambda))^{1/2}}{\lambda - z}.$$

Under Hypothesis 6.1, this last formula just states that

$$\hat{\mathfrak{S}}\left\{\frac{\bar{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}, \frac{C_1(\xi,z)\bar{F}(\xi,\bar{z})}{Q(\xi,z)}, \cdots, \frac{C_N(\xi,z)\bar{F}(\xi,\bar{z})}{Q(\xi,z)}\right\} = \frac{(2k(\lambda))^{1/2}}{\lambda - z}. \tag{6.2}$$

Using (3.16) again, we also have

$$\int_\alpha^\beta \frac{\tilde{P}(\xi, \bar{z}_1, z_2)}{Q(\xi, \bar{z}_1)Q(\xi, z_2)} F(\xi, \bar{z}_1)\bar{F}(\xi, \bar{z}_2)d\xi = \int_a^b \frac{(2k(\mu))^{1/2}}{\mu - \bar{z}_1}\cdot\frac{(2k(\mu))^{1/2}}{\mu - z_2} d\mu.$$

Under Hypothesis 6.1 this equation becomes

$$\int_\alpha^\beta \frac{\bar{F}(\xi, \bar{z}_2)}{(A(\xi))^{1/2}}\cdot\frac{F(\xi, \bar{z}_1)}{(A(\xi))^{1/2}} d\xi + \sum_{j=1}^N \int_\alpha^\beta \frac{C_j(\xi, z_2)\bar{F}(\xi, \bar{z}_2)}{Q(\xi, z_2)}\cdot\frac{\bar{C}_j(\xi, z_1)F(\xi, \bar{z}_1)}{\bar{Q}(\xi, z_1)} d\xi$$
$$(6.3) \qquad = \int_a^b \frac{(2k(\mu))^{1/2}}{\mu - z_2}\cdot\frac{(2k(\mu))^{1/2}}{\mu - \bar{z}_1} d\mu.$$

The left-hand side of (6.3), however, is just the scalar product of the two elements

$$\left(\frac{\bar{F}(\xi, \bar{z}_s)}{(A(\xi))^{1/2}}, \frac{C_1(\xi, z_s)\bar{F}(\xi, \bar{z}_s)}{Q(\xi, z_s)}, \cdots, \frac{C_N(\xi, z_s)\bar{F}(\xi, \bar{z}_s)}{Q(\xi, z_s)}\right), \qquad s = 1, 2.$$

On the linear manifold $(\mathfrak{L}_\xi, 0, \cdots, 0)$, with $\mathfrak{L}_\xi$ as in §5, the transformation $\hat{\mathfrak{S}}$ coincides with the transformation S of §5. We shall now show that if we restrict the domain of the transformation $\hat{\mathfrak{S}}$ to the linear manifold $\mathfrak{M}_\xi$ generated by the elements $(\mathfrak{L}_\xi, 0, \cdots, 0)$ and

$$\left(\frac{\bar{F}(\xi, \bar{z})}{(A(\xi))^{1/2}}, \frac{C_1(\xi, z)\bar{F}(\xi, \bar{z})}{Q(\xi, z)}, \cdots, \frac{C_N(\xi, z)\bar{F}(\xi, \bar{z})}{Q(\xi, z)}\right),$$

where z is any complex number which does not lie in the closed interval $[a, b]$, then $\hat{\mathfrak{S}}$ is an isometry. The image of $\mathfrak{M}_\xi$ is the linear manifold $\mathfrak{M}_\lambda$ generated by the elements $\{(2k(\lambda))^{1/2}, H(w, \lambda)/(2k(\lambda))^{1/2}, (2k(\lambda))^{1/2}/(\lambda - z)\}$. Since the linear manifold generated by the elements $\{(2k(\lambda))^{1/2}/(\lambda - z)\}$ is dense in $L^2(a, b)$, $\mathfrak{M}_\lambda$ is also dense in $L^2(a, b)$.

From (3.12) we have

$$(6.4) \qquad \int_\alpha^\beta (A(\xi))^{1/2} \frac{F(\xi, \bar{z})}{(A(\xi))^{1/2}} d\xi = \int_a^b \left\{-(2k(\mu))^{1/2} \frac{(2k(\mu))^{1/2}}{\mu - \bar{z}}\right\} d\mu.$$

Using (3.13) and (3.14), we see that

$$(6.5) \qquad \int_\alpha^\beta \frac{(A(\xi))^{1/2}}{\xi - w} \frac{F(\xi, \bar{z})}{(A(\xi))^{1/2}} d\xi = \int_a^b \frac{H(w, \mu)}{(2k(\mu))^{1/2}} \frac{(2k(\mu))^{1/2}}{\mu - \bar{z}} d\mu.$$

The relations (4.2), (5.7), (5.8), (6.3), (6.4) and (6.5) show that $\hat{\mathfrak{S}}$ is an isometry when its domain is restricted to $\mathfrak{M}_\xi$. Thus $\hat{\mathfrak{S}}$ is a 1-1 map of $\mathfrak{M}_\xi$ onto $\mathfrak{M}_\lambda$. $\hat{\mathfrak{S}}$, then, defines an inverse map $\hat{\mathfrak{R}}$ whose domain is $\mathfrak{M}_\lambda$. Since $\hat{\mathfrak{R}}$ is bounded on $\mathfrak{M}_\lambda$, we may extend $\hat{\mathfrak{R}}$ to the closure of $\mathfrak{M}_\lambda$, which is just $L^2(a, b)$, as a bounded transformation. We shall call this extension $\mathfrak{R}$. $\mathfrak{R}$ is indeed an isometry of $L^2(a, b)$ into $\mathfrak{K}$. To show that $\mathfrak{R}$ coincides with R on the space $\overline{\mathfrak{L}}_\lambda$, we observe that the image of $(\mathfrak{L}_\xi, 0, \cdots, 0)$ under $\hat{\mathfrak{S}}$ is just $\mathfrak{L}_\lambda$ and that $\hat{\mathfrak{S}}$ coincides with the transformation S for these elements. Thus $\hat{\mathfrak{R}}$ coincides with $\hat{R}$ on $\mathfrak{L}_\lambda$. These transformations have unique bounded extensions to $\overline{\mathfrak{L}}_\lambda$. This completes the proof.

LEMMA 6.2. *Under Hypothesis* 6.1, *the transformation $\hat{R}$ defined through formula* (5.1) *generates a bounded transformation $\tilde{R}$ of norm* 1 *of the space $L^2(a, b)$ on the space* $L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$.

Proof. From (3.13) we obtain

$$-\frac{1}{2\pi i}\lim_{\eta\to 0;\ \eta>0}\int_a^b \frac{\overline{H}(\xi+i\eta,\lambda)-\overline{H}(\xi-i\eta,\lambda)}{(2k(\lambda)A(\xi))^{1/2}}\,\frac{(2k(\lambda))^{1/2}}{\lambda-z}\,d\lambda = \frac{\overline{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}\cdot$$

Thus

$$\hat{R}\left\{\frac{(2k(\lambda))^{1/2}}{\lambda-z}\right\} = \frac{\overline{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}\cdot \tag{6.6}$$

Now let $\mathcal{P}$ be the projection of $\mathfrak{K}$ onto its first component. If we restrict our domains to $\mathfrak{M}_\lambda$, then we see that $\hat{R}=\mathcal{P}\hat{\mathfrak{R}}$. Consequently $\hat{R}$ is bounded on $\mathfrak{M}_\lambda$ and has a unique bounded extension $\tilde{R}$ to the closure of $\mathfrak{M}_\lambda$ which is just $L^2(a, b)$.

COROLLARY. $\tilde{R}=\mathcal{P}\mathfrak{R}$, *where $\mathcal{P}$ is the projection of $\mathfrak{K}$ onto its first component.*

Next, we turn to

LEMMA 6.3. *Let P denote the projection of the space $L^2(a, b)$ onto $\overline{\mathfrak{L}}_\lambda$. Then $\tilde{R}=\tilde{R}P$.*

Proof. The map $S\tilde{R}$ maps $L^2(a, b)$ onto $\overline{\mathfrak{L}}_\lambda$. ($\tilde{R}$ agrees with R on the subspace $\overline{\mathfrak{L}}_\lambda$.) Furthermore, $\tilde{R}S=1$. Hence $(S\tilde{R})^2=S\tilde{R}$. Thus $S\tilde{R}$ is just the projection of $L^2(a, b)$ onto $\overline{\mathfrak{L}}_\lambda$, i.e. $S\tilde{R}=P$. Now $\tilde{R}P=\tilde{R}S\tilde{R}=\tilde{R}$ since $\tilde{R}S=1$.[5]

COROLLARY I. *Under Hypothesis* 6.1, *the linear manifold generated by the elements* $\{\overline{F}(\xi,\bar{z})/(A(\xi))^{1/2}\}$ *is dense in* $L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$.

Proof. Since the linear manifold generated by the elements

$$\left\{\frac{(2k(\lambda))^{1/2}}{\lambda-z}\right\}$$

is dense in $L^2(a, b)$, the manifold generated by the elements

[5] From (5.6), (6.5), and (6.6), one readily concludes that $\tilde{R}$ and S are adjoints of each other, so that $S\tilde{R}$ is self-adjoint.

$$\left\{P\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right\}$$

is dense in $\bar{\mathfrak{L}}_\lambda$. Now

$$\tilde{R}P\left\{\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right\} = \tilde{R}\left\{\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right\} = \frac{\bar{F}(\xi, \bar{z})}{(A(\xi))^{1/2}}$$

from (6.6) and Lemma 6.3. But when $\tilde{R}$ is restricted to $\bar{\mathfrak{L}}_\lambda$, it coincides with R which is an isometry of $\bar{\mathfrak{L}}_\lambda$ onto $L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$. Hence the manifold generated by the elements $\{\bar{F}(\xi, \bar{z})/(A(\xi))^{1/2}\}$ is dense in $L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{(f\lambda)+k(\lambda)\})$.

Corollary II. *Under Hypothesis* 6.1, *if* $\tilde{R}h(\lambda)=g(\xi)$, *then* $\tilde{R}Lh(\lambda)=\xi g(\xi)$.

Proof. $\tilde{R}h(\lambda)=\tilde{R}Ph(\lambda)$ by Lemma 6.3. Since $\bar{\mathfrak{L}}_\lambda$ is invariant under the operation L (see Theorem 5.1), we must have $LP=PL$. Now $\tilde{R}Lh(\lambda)=\tilde{R}PLh(\lambda)$ from Lemma 6.3. $\tilde{R}PLh(\lambda)=\tilde{R}LPh(\lambda)=\xi g(\xi)$ by Theorem 5.1. ($\tilde{R}$ agrees with R on $\bar{\mathfrak{L}}_\lambda$.)

Next we turn to the more elaborate

Lemma 6.4. *If* $\mathfrak{R}h(\lambda)=(g_0(\xi), g_1(\xi), \cdots, g_N(\xi))$, *then*

$$\mathfrak{R}Lh(\lambda) = (\xi g_0(\xi), \xi g_1(\xi), \cdots, \xi g_N(\xi)).$$

We outline the proof.

By considering the expansion of the function $E(l, z_1)\bar{E}(\bar{l}, \bar{z}_2)$ in powers of $1/l$ up to the second power, one can evaluate the integral

$$\frac{1}{2\pi i}\oint lE(l, z_1)\bar{E}(\bar{l}, \bar{z}_2)dl \tag{6.7}$$

$$= \frac{1}{2\pi i}\int_\alpha^\beta \xi\{E(\xi+i0, z_1)\bar{E}(\xi-i0, \bar{z}_2) - E(\xi-i0, z_1)\bar{E}(\xi+i0, \bar{z}_2)\}d\xi \tag{6.8}$$

$$= \frac{z_2-z_1}{2\pi i}\int_\alpha^\beta \frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)}\,\xi F(\xi, z_1)\bar{F}(\xi, \bar{z}_2)d\xi. \tag{6.9}$$

(The last equality follows from (3.8).) The computation leads to the formula

$$\frac{1}{2\pi i}\lim_{\eta\to 0;\,\eta>0}\int_\alpha^\beta\left\{\frac{\tilde{P}(\xi, \lambda+i\eta, z)F(\xi, \lambda+i\eta)}{Q(\xi, \lambda+i\eta)(2k(\lambda))^{1/2}} - \frac{\tilde{P}(\xi, \lambda-i\eta, z)F(\xi, \lambda-i\eta)}{Q(\xi, \lambda-i\eta)(2k(\lambda))^{1/2}}\right\}$$

$$\cdot\frac{\xi\bar{F}(\xi, \bar{z})}{Q(\xi, z)}d\xi = f(\lambda)\left\{\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right\} + \frac{1}{\pi i}\int_a^b \frac{(k(\lambda)k(\mu))^{1/2}}{\mu - \lambda}\left\{\frac{(2k(\mu))^{1/2}}{\mu - z}\right\}d\mu,$$

or, under Hypothesis 6.1,

$$\hat{\mathfrak{S}}\left\{\left(\frac{\xi\overline{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}, \frac{\xi C_1(\xi,z)\overline{F}(\xi,\bar{z})}{Q(\xi,z)}, \cdots, \frac{\xi C_N(\xi,z)\overline{F}(\xi,\bar{z})}{Q(\xi,z)}\right)\right\} = L\left\{\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right\}. \tag{6.10}$$

We now indicate how to show that $\hat{\mathfrak{S}}$ is an isometry on the linear manifold $\mathfrak{M}_\xi$ generated by the elements

$$\left\{(\mathfrak{L}_\xi, 0, \cdots, 0), \left(\frac{\overline{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}, \frac{C_1(\xi,z)\overline{F}(\xi,\bar{z})}{Q(\xi,z)}, \cdots, \frac{C_N(\xi,z)\overline{F}(\xi,\bar{z})}{Q(\xi,z)}\right), \left(\frac{\xi\overline{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}, \frac{\xi C_1(\xi,z)\overline{F}(\xi,\bar{z})}{Q(\xi,z)}, \cdots, \frac{\xi C_N(\xi,z)\overline{F}(\xi,\bar{z})}{Q(\xi,z)}\right)\right\}.$$

A lengthy computation beginning with the results obtained in evaluating the integral (6.9) leads to the formula

$$\int_\alpha^\beta \frac{\xi\overline{F}(\xi,\bar{z}_2)}{(A(\xi))^{1/2}} \frac{F(\xi,z_1)}{(A(\xi))^{1/2}} d\xi + \sum_{j=1}^N \int_\alpha^\beta \frac{C_j(\xi,z_2)\overline{F}(\xi,\bar{z}_2)C_j(\xi,\bar{z}_1)F(\xi,z_1)}{Q(\xi,z_2)\qquad \overline{Q}(\xi,\bar{z}_1)} d\xi = \int_a^b \left\{L\frac{(2k(\mu))^{1/2}}{\mu - z_2}\right\}(\lambda)\cdot\frac{(2k(\lambda))^{1/2}}{\lambda - z_1} d\lambda. \tag{6.11}$$

In this last set of computations, the equation

$$\int_a^b \frac{2k(\lambda)}{\lambda - z}\left\{\int_a^b \frac{k(\mu)}{\mu - \lambda} d\mu\right\} d\lambda = \left(\int_a^b \frac{k(\mu)}{\mu - z} d\mu\right)^2, \tag{6.12}$$

which follows from

$$\frac{1}{2\pi i}\oint \frac{1}{\zeta - z}\left(\int_a^b \frac{k(\mu)}{\mu - \zeta} d\mu\right)^2 d\zeta = \left(\int_a^b \frac{k(\mu)}{\mu - z} d\mu\right)^2$$

$$= \frac{1}{2\pi i}\int_a^b \frac{1}{\lambda - z}\left[\left(\pi i k(\lambda) + \int_a^b \frac{k(\mu)}{\mu - \lambda} d\mu\right)^2 - \left(-\pi i k(\lambda) + \int_a^b \frac{k(\mu)}{\mu - \lambda} d\mu\right)^2\right] d\lambda$$

$$= \int_a^b \frac{2k(\lambda)}{\lambda - z}\left\{\int_a^b \frac{k(\mu)}{\mu - \lambda} d\mu\right\} d\lambda,$$

plays a key role.

If one expands $E(l, z_1)\overline{E}(\bar{l}, \bar{z}_2)$ in powers of $1/l$ up to the third power, one can evaluate the integral

$$(6.13)\quad \frac{1}{2\pi i}\oint l^2 E(l, z_1)\bar{E}(\bar{l}, \bar{z}_2)dl$$

$$(6.14)\quad = \frac{1}{2\pi i}\int_\alpha^\beta \xi^2\{E(\xi+i0, z_1)\bar{E}(\xi-i0, \bar{z}_2) - E(\xi-i0, z_1)\bar{E}(\xi+i0, \bar{z}_2)\}d\xi$$

$$(6.15)\quad = \frac{z_2 - z_1}{2\pi i}\int_\alpha^\beta \frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)}\xi F(\xi, z_1)\cdot\xi\bar{F}(\xi, \bar{z}_2)d\xi \qquad \text{(from (3.8))}.$$

An extremely cumbersome computation leads to the equation

$$(6.16)\quad \begin{aligned}&\int_\alpha^\beta \frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)}\xi F(\xi, z_1)\cdot\xi\bar{F}(\xi, \bar{z}_2)d\xi \\ &\qquad = \int_a^b \left\{L\frac{(2k(\mu))^{1/2}}{\mu - z_2}\right\}(\lambda)\cdot\left\{\bar{L}\frac{(2k(\mu))^{1/2}}{\mu - \bar{z}_1}\right\}(\lambda)d\lambda.\end{aligned}$$

In the derivation of (6.16), (6.12) and

$$(6.17)\quad \begin{aligned}&\int_a^b \frac{k(\lambda)}{\lambda - z}\left(\frac{1}{\pi i}\int_a^b \frac{k(\mu)}{\mu-\lambda}d\mu\right)^2 d\lambda \\ &\qquad = -\frac{1}{3\pi^2}\left(\int_a^b \frac{k(\mu)}{\mu - z}d\mu\right)^3 - \frac{1}{3}\int_a^b \frac{k^3(\lambda)}{\lambda - z}d\lambda\end{aligned}$$

are utilized. (6.17) is a consequence of the following computation:

$$\begin{aligned}\frac{1}{2\pi i}\oint \frac{1}{\zeta - z}\left(\int_a^b \frac{k(\mu)}{\mu-\zeta}d\mu\right)^3 d\zeta &= \left(\int_a^b \frac{k(\mu)}{\mu - z}d\mu\right)^3 \\ = \frac{1}{2\pi i}\int_a^b \frac{1}{\lambda - z}\Bigg[\left(\pi i k(\lambda) + \int_a^b \frac{k(\mu)}{\mu-\lambda}d\mu\right)^3 \\ &\qquad - \left(-\pi i k(\lambda) + \int_a^b \frac{k(\mu)}{\mu-\lambda}d\mu\right)^3\Bigg]d\lambda \\ = \frac{1}{2\pi i}\int_a^b \frac{1}{\lambda - z}\left[2(\pi i)^3 k^3(\lambda) + 6\pi i k(\lambda)\left(\int_a^b \frac{k(\mu)}{\mu-\lambda}d\mu\right)^2\right]d\lambda.\end{aligned}$$

Now we shall show that

$$(6.18)\quad \int_\alpha^\beta \frac{\xi\bar{F}(\xi, \bar{z})}{(A(\xi))^{1/2}}(A(\xi))^{1/2}d\xi = \int_a^b \left\{L\frac{(2k(\mu))^{1/2}}{\mu - z}\right\}(\lambda)\cdot\{-(2k(\lambda))^{1/2}\}d\lambda.$$

If we expand $E(l, z)$ in powers of $1/l$, we obtain

$$
\begin{aligned}
E(l, z) &= 1 + \frac{1}{l}\,\frac{1}{\pi i}\int_a^b \frac{k(\mu)}{\mu - z}\,d\mu \\
&\quad + \frac{1}{l^2}\left\{\frac{1}{\pi i}\int_a^b \frac{k(\mu)f(\mu)}{\mu - z}\,d\mu + \frac{1}{2}\left(\frac{1}{\pi i}\int_a^b \frac{k(\mu)}{\mu - z}\,d\mu\right)^2\right\} + \cdots .
\end{aligned}
\tag{6.19}
$$

Hence

$$
\frac{1}{2\pi i}\oint lE(l, z)\,dl = -\frac{1}{\pi i}\int_a^b \frac{k(\mu)f(\mu)}{\mu - z}\,d\mu - \frac{1}{2}\left(\frac{1}{\pi i}\int_a^b \frac{k(\mu)}{\mu - z}\,d\mu\right)^2.
$$

From (6.12) we have

$$
\left(\frac{1}{\pi i}\int_a^b \frac{k(\mu)}{\mu - z}\,d\mu\right)^2 = -\frac{2}{\pi^2}\int_a^b \frac{k(\lambda)}{\lambda - z}\left\{\int_a^b \frac{k(\mu)}{\mu - \lambda}\,d\mu\right\}d\lambda,
$$

so that

$$
\begin{aligned}
\frac{1}{2\pi i}\int_\alpha^\beta \xi F(\xi, z)\,d\xi &= -\frac{1}{\pi i}\int_a^b f(\mu)\,\frac{k(\mu)}{\mu - z}\,d\mu \\
&\quad - \frac{1}{\pi^2}\int_a^b \frac{k(\lambda)}{\lambda - z}\left\{\int_a^b \frac{k(\mu)}{\mu - \lambda}\,d\mu\right\}d\lambda + \frac{1}{\pi^2}\int_a^b \frac{k(\lambda)}{\lambda - z}\left\{\int_a^b \frac{k(\mu)}{\mu - z}\,d\mu\right\}d\lambda.
\end{aligned}
\tag{6.20}
$$

However,

$$
\begin{aligned}
&-\frac{1}{\pi^2}\int_a^b \frac{k(\lambda)}{\lambda - z}\left\{\int_a^b k(\mu)\left\{\frac{1}{\mu - \lambda} - \frac{1}{\mu - z}\right\}d\mu\right\}d\lambda \\
&\qquad\qquad = -\frac{1}{\pi^2}\int_a^b k(\lambda)\left\{\int_a^b \frac{k(\mu)\,d\mu}{(\mu - \lambda)(\mu - z)}\right\}d\lambda.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_\alpha^\beta \xi F(\xi, z)\,d\xi &= \int_a^b \left\{-f(\lambda)\,\frac{2k(\lambda)}{\lambda - z} + \frac{k(\lambda)}{\pi i}\int_a^b \left\{\frac{2k(\mu)}{\mu - z}\right\}\frac{d\mu}{\mu - \lambda}\right\}d\lambda \\
&= \int_a^b \left\{f(\lambda)\,\frac{(2k(\lambda))^{1/2}}{\lambda - z}\right. \\
&\quad \left. - \frac{1}{\pi i}\int_a^b \frac{(k(\lambda)k(\mu))^{1/2}}{\mu - \lambda}\left\{\frac{(2k(\mu))^{1/2}}{\mu - z}\right\}d\mu\right\}\left\{-(2k(\lambda))^{1/2}\right\}d\lambda.
\end{aligned}
\tag{6.21}
$$

From (6.21) we easily obtain (6.18).

Finally we shall prove that

$$
(6.22)\quad \begin{aligned}\int_\alpha^\beta \frac{\xi\bar{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}\,\frac{(A(\xi))^{1/2}}{\xi-\bar{w}}\,d\xi &= \int_a^b \left\{L\,\frac{(2k(\mu))^{1/2}}{\mu-z}\right\}(\lambda)\cdot\frac{\bar{H}(w,\lambda)}{(2k(\lambda))^{1/2}}\,d\lambda \\ &= \int_a^b \frac{(2k(\lambda))^{1/2}}{\lambda-z}\cdot\left(\bar{L}\left\{\frac{H(w,\mu)}{(2k(\mu))^{1/2}}\right\}\right)(\lambda)d\lambda.\end{aligned}
$$

Now

$$
\begin{aligned}\int_\alpha^\beta \frac{\xi\bar{F}(\xi,\bar{z})}{\xi-\bar{w}}\,d\xi &= \int_\alpha^\beta \bar{F}(\xi,\bar{z})d\xi + \bar{w}\int_\alpha^\beta \frac{\bar{F}(\xi,\bar{z})}{\xi-\bar{w}}\,d\xi \\ &= \int_a^b \frac{(2k(\lambda))^{1/2}}{\lambda-z}\left\{-(2k(\lambda)^{1/2}\right\}d\lambda + \bar{w}\int_a^b \frac{(2k(\lambda))^{1/2}}{\lambda-z}\,\frac{\bar{H}(w,\lambda)}{(2k(\lambda))^{1/2}}\,d\lambda \\ &\qquad\qquad \text{(from (6.4),(6.5))}, \\ &= \int_a^b \frac{(2k(\lambda))^{1/2}}{\lambda-z}\left\{-(2k(\lambda))^{1/2} + \bar{w}\,\frac{\bar{H}(w,\lambda)}{(2k(\lambda))^{1/2}}\right\}d\lambda \\ &= \int_a^b \frac{(2k(\lambda))^{1/2}}{\lambda-z}\left(\bar{L}\left\{\frac{H(w,\mu)}{(2k(\mu))^{1/2}}\right\}\right)(\lambda)d\lambda \qquad \text{(from (5.9))}.\end{aligned}
$$

Formulae (6.10), (6.11), (6.16), (6.18) and (6.22) show that $\mathcal{S}$ is an isometry on the linear manifold $\mathfrak{N}_\xi$. The image of $\mathfrak{N}_\xi$ is the linear manifold $\mathfrak{N}_\lambda$ generated by the elements

$$
\left\{(2k(\lambda))^{1/2},\ \frac{H(w,\lambda)}{(2k(\lambda))^{1/2}},\ \frac{(2k(\lambda))^{1/2}}{\lambda-z},\ L\,\frac{(2k(\lambda))^{1/2}}{\lambda-z}\right\}.
$$

Of course, $\mathfrak{N}_\xi \supset \mathfrak{M}_\xi$ and $\mathfrak{N}_\lambda \supset \mathfrak{M}_\lambda$. Thus $\hat{\mathcal{S}}$ is a 1-1 map of $\mathfrak{N}_\xi$ onto $\mathfrak{N}_\lambda$. $\hat{\mathcal{S}}$, then defines an inverse map $\hat{\mathcal{R}}$ whose domain is $\mathfrak{N}_\lambda$. $\tilde{\mathcal{R}}$ is a bounded extension of the map $\hat{\mathcal{R}}$ used in the proof of Lemma 6.1. Since $\hat{\mathcal{R}}$ has a unique bounded extension to $L^2(a, b)$, the mapping $\mathcal{R}$ must also be an extension of $\tilde{\mathcal{R}}$, i.e. $\mathcal{R}$ agrees with $\tilde{\mathcal{R}}$ on $\mathfrak{N}_\lambda$. Suppose now that $\mathcal{R}h(\lambda) = (g_0(\xi), g_1(\xi), \cdots, g_N(\xi))$. Then the image element $(g_0(\xi), g_1(\xi), \cdots, g_N(\xi))$ lies in the closure of $\mathfrak{M}_\xi$. In fact, the element must lie in the closure of the linear manifold generated by the elements

$$
\left\{\left(\frac{(A(\xi))^{1/2}}{\xi-w}, 0, \cdots, 0\right), \left(\frac{\bar{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}, \frac{C_1(\xi,z)\bar{F}(\xi,\bar{z})}{Q(\xi,z)}, \cdots, \frac{C_N(\xi,z)\bar{F}(\xi,\bar{z})}{Q(\xi,z)}\right)\right\}.
$$

Since the ξ-interval is bounded, $(\xi g_0(\xi), \xi g_1(\xi), \cdots, \xi g_N(\xi))$ can be approximated by elements from the linear manifold generated by

$$
\left\{\left(\xi\,\frac{(A(\xi))^{1/2}}{\xi-w}, 0, \cdots, 0\right), \left(\frac{\xi\bar{F}(\xi,\bar{z})}{(A(\xi))^{1/2}}, \frac{\xi C_1(\xi,z)\bar{F}(\xi,\bar{z})}{Q(\xi,z)}, \cdots, \frac{\xi C_N(\xi,z)\bar{F}(\xi,\bar{z})}{Q(\xi,z)}\right)\right\}.
$$

Thus $(\xi g_0(\xi), \xi g_1(\xi), \cdots, \xi g_N(\xi))$ lies in the closure of $\mathfrak{N}_\xi$. Since all the operations involved are continuous, we must have $(\xi g_0(\xi), \xi g_1(\xi), \cdots, \xi g_n(\xi)) = \mathcal{R}Lh(\lambda)$. This completes the proof.

7. Some properties of the polynomials $\tilde{P}(\xi, z_1, z_2)$ and $Q(\xi, z)$. In §3 we defined certain polynomials $c(\xi, z)$ and $g(\xi, z)$ with real coefficients and of the same degree (see (3.4)). Let D_ξ be the degree of $c(\xi, z)$ and $g(\xi, z)$. Let $r_1(\xi), r_2(\xi), \cdots, r_{D_\xi}(\xi)$ be the roots of $c(\xi, z)$ and let $s_1(\xi), s_2(\xi), \cdots, s_{D_\xi}(\xi)$ be the roots of $g(\xi, z)$. Because of the way in which $c(\xi, z)$ and $g(\xi, z)$ were formed, we may assume that

$$s_1(\xi) < r_1(\xi) < s_2(\xi) < r_2(\xi) < \cdots < s_{D_\xi}(\xi) < r_{D_\xi}(\xi). \tag{7.1}$$

Furthermore, $c(\xi, z)$ and $g(\xi, z)$ have leading coefficient 1. Hence

$$c(\xi, z) = \sum_{j=0}^{D_\xi} (-1)^j \sigma_j(r(\xi)) z^{D_\xi - j}, \qquad g(\xi, z) = \sum_{j=0}^{D_\xi} (-1)^j \sigma_j(s(\xi)) z^{D_\xi - j}, \tag{7.2}$$

where σ_j represents the elementary symmetric function of weight j. In formula (3.7) we introduced $P(\xi, z_1, z_2) = c(\xi, z_2)g(\xi, z_1) - c(\xi, z_1)g(\xi, z_2)$ and $Q(\xi, z) = c(\xi, z) - g(\xi, z)$. Thus

$$\begin{aligned} &Q(\xi, z_1)Q(\xi, z_2) \\ &= \sum_{j=0}^{D_\xi} \sum_{k=0}^{D_\xi} (-1)^{j+k} \{\sigma_j(r(\xi)) - \sigma_j(s(\xi))\}\{\sigma_k(r(\xi)) - \sigma_k(s(\xi))\} z_1^{D_\xi - j} z_2^{D_\xi - k} \end{aligned} \tag{7.3}$$

Now

$$c(\xi, z_1)g(\xi, z_2) = \sum_{j=0}^{D_\xi} \sum_{k=0}^{D_\xi} (-1)^{j+k} \sigma_j(r(\xi))\sigma_k(s(\xi)) z_1^{D_\xi - j} z_2^{D_\xi - k}.$$

Hence

$$P(\xi, z_1, z_2) = \sum_{j=0}^{D_\xi} \sum_{k=0}^{D_\xi} (-1)^{j+k} \{\sigma_k(r(\xi))\sigma_j(s(\xi)) - \sigma_j(r(\xi))\sigma_k(s(\xi))\} z_1^{D_\xi - j} z_2^{D_\xi - k}. \tag{7.4}$$

In $P(\xi, z_1, z_2)$ the coefficients of $z_1^{D_\xi - j} z_2^{D_\xi - k}$ and $z_1^{D_\xi - k} z_2^{D_\xi - j}$ differ only by a minus sign. Therefore

$$\begin{aligned} &P(\xi, z_1, z_2) \\ &= \sum_{j<k} (-1)^{j+k} \{\sigma_k(r(\xi))\sigma_j(s(\xi)) - \sigma_j(r(\xi))\sigma_k(s(\xi))\} (z_1^{D_\xi - j} z_2^{D_\xi - k} - z_1^{D_\xi - k} z_2^{D_\xi - j}). \end{aligned} \tag{7.5}$$

Now

$$z_1^{D_\xi - j} z_2^{D_\xi - k} - z_1^{D_\xi - k} z_2^{D_\xi - j} = -(z_2 - z_1) \sum_{\nu=0}^{k-j-1} z_1^{D_\xi - j - 1 - \nu} z_2^{D_\xi - k + \nu}, \qquad j < k.$$

Therefore

$$\tilde{P}(\xi, z_1, z_2)$$
$$(7.6) \qquad = \sum_{j<k} (-1)^{j+k} \{\sigma_j(r(\xi))\sigma_k(s(\xi)) - \sigma_k(r(\xi))\sigma_j(s(\xi))\} \sum_{\nu=0}^{k-j-1} z_1^{D_\xi-j-1-\nu} z_2^{D_\xi-k+\nu}$$

In order to verify the Hypothesis 6.1, we shall have to examine the expressions

$$\frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} - \frac{1}{A(\xi)} = \frac{A(\xi)\tilde{P}(\xi, z_1, z_2) - Q(\xi, z_1)Q(\xi, z_2)}{A(\xi)Q(\xi, z_1)Q(\xi, z_2)}.$$

Using (4.1) and the expression for $E(\xi+i0, z)/E(\xi-i0, z)$ in §3, one can see that

$$(7.7) \qquad A(\xi) = \sigma_1(r(\xi)) - \sigma_1(s(\xi)).$$

From (7.3), (7.6), and (7.7), we readily obtain

LEMMA 7.1. *If the degree D_ξ of the polynomials $c(\xi, z)$ and $g(\xi, z)$ is* 1, *then*

$$\frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} = \frac{1}{A(\xi)}.$$

Next, let us examine the case where $D_\xi = 2$. Here

$$\begin{aligned} \tilde{P}(\xi, z_1, z_2) &= \{\sigma_1(r(\xi)) - \sigma_1(s(\xi))\}z_1z_2 + \{\sigma_2(s(\xi)) - \sigma_2(r(\xi))\}(z_1 + z_2) \\ &\quad + \{\sigma_2(r(\xi))\sigma_1(s(\xi)) - \sigma_2(s(\xi))\sigma_1(r(\xi))\}, \\ Q(\xi, z_1)Q(\xi, z_2) &= \{\sigma_1(r(\xi)) - \sigma_1(s(\xi))\}^2 z_1z_2 \\ &\quad + \{\sigma_1(r(\xi)) - \sigma_1(s(\xi))\}\{\sigma_2(s(\xi)) - \sigma_2(s(r(\xi))\}(z_1 + z_2) \\ &\quad + \{\sigma_2(r(\xi)) - \sigma_2(s(\xi))\}^2. \end{aligned}$$

Using (7.7), we have

$$\begin{aligned} &A(\xi)\tilde{P}(\xi, z_1, z_2) - Q(\xi, z_1)Q(\xi, z_2) \\ &\quad = \{\sigma_1(r(\xi)) - \sigma_1(s(\xi))\}\{\sigma_2(r(\xi))\sigma_1(s(\xi)) - \sigma_2(s(\xi))\sigma_1(r(\xi))\} \\ &\qquad - \{\sigma_2(r(\xi)) - \sigma_2(s(\xi))\}^2 \\ &\quad = \{r_1 + r_2 - s_1 - s_2\}\{r_1r_2(s_1 + s_2) - s_1s_2(r_1 + r_2)\} - \{r_1r_2 - s_1s_2\}^2. \end{aligned}$$

It is easy to verify that this last expression vanishes when $r_1 = s_1$, $s_2 = r_1$, $r_2 = s_2$, or $s_1 = r_2$. From algebraic considerations it then follows that

$$A(\xi)\tilde{P}(\xi, z_1, z_2) - Q(\xi, z_1)Q(\xi, z_2) = (r_1 - s_1)(s_2 - r_1)(r_2 - s_2)(r_2 - s_1).$$

In view of the inequalities (7.1), we must have

$$A(\xi)\tilde{P}(\xi, z_1, z_2) - Q(\xi, z_1)Q(\xi, z_2) > 0.$$

Hence if $D_\xi = 2$,

$$\frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} - \frac{1}{A(\xi)} = \frac{B(\xi)/A(\xi)}{Q(\xi, z_1)Q(\xi, z_2)}$$

where $B(\xi) > 0$. Thus we can write

$$\frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} = \frac{1}{A(\xi)} + \frac{(B(\xi)/A(\xi))^{1/2}}{Q(\xi, z_1)} \frac{(B(\xi)/A(\xi))^{1/2}}{Q(\xi, z_2)}$$

and the Hypothesis 6.1 is satisfied. This proves

Lemma 7.2. *If the degree D_ξ of the polynomials $c(\xi, z)$ and $g(\xi, z)$ is $\leqq 2$, then the Hypothesis* 6.1 *is satisfied.*

8. **Singular integral operators with simple continuous spectra.** If the polynomials $c(\xi, z)$ and $g(\xi, z)$ are of degree $\leqq 1$, then Lemma 7.1 can be used to show that the transformation $\hat{\mathfrak{S}}$ defined through (6.1) is just the transformation $\hat{S}$ defined by (5.2). In this case $\hat{S}$ furnishes an isometry of $\mathfrak{M}_\xi$ onto $\mathfrak{M}_\lambda$. Furthermore if $\hat{R}$ (see (5.1)) is restricted to $\mathfrak{M}_\lambda$, then $\hat{R}\hat{S} = 1$ and $\hat{S}\hat{R} = 1$. Thus the bounded extensions R and S of $\hat{R}$ and $\hat{S}$ to the spaces $L^2(a, b)$ and $L^2(\min \{f(\lambda) - k(\lambda)\}, \max \{f(\lambda) - k(\lambda)\})$ are isometries such that $RS = 1$ and $SR = 1$. The range of S is all of $L^2(a, b)$ and the results of Theorem 5.1 yield a complete spectral representation for our operator. The purpose of this section is to give a criterion which guarantees that the degrees of $c(\xi, z)$ and $g(\xi, z)$ do not exceed 1. We shall prove

Theorem 8.1. *Consider the function*

$$\frac{f(\lambda) - \xi - k(\lambda)}{f(\lambda) - \xi + k(\lambda)}, \qquad a \leqq \lambda \leqq b.$$

Suppose that when $f(a) - k(a) \leqq \xi \leqq f(a) + k(a)$, the function changes signs at most once, and that for all other values of ξ, the function changes signs at most twice. Then the operator L has a simple continuous spectrum from $\min \{f(\lambda) - k(\lambda)\}$ *to* $\max \{f(\lambda) + k(\lambda)\}$ *and the transformations R and S of* §5 *furnish a complete spectral representation for L.*

Proof. In (3.4) we saw that

$$\log \frac{g(\xi, z)}{c(\xi, z)} = \frac{1}{\pi} \int_a^b \arg \frac{f(\mu) - \xi - i0 - k(\mu)}{f(\mu) - \xi - i0 + k(\mu)} \frac{d\mu}{\mu - z}.$$

The integrand is zero unless $(f(\mu) - \xi - k(\mu))/(f(\mu) - \xi + k(\mu))$ is negative. If

$$\frac{f(\mu) - \xi - k(\mu)}{f(\mu) - \xi + k(\mu)} < 0,$$

then we must have $f(\mu) - \xi - k(\mu) < 0$ and $f(\mu) - \xi + k(\mu) > 0$, since $k(\mu) \geqq 0$.

Suppose, first, that

$$\frac{f(\mu) - \xi - k(\mu)}{f(\mu) - \xi + k(\mu)} < 0$$

in some half-open interval $a < \mu \leqq \lambda_1$. Then $f(\mu) - k(\mu) < \xi < f(\mu) + k(\mu)$ for $a < \mu \leqq \lambda_1$ and therefore $f(a) - k(a) \leqq \xi \leqq f(a) + k(a)$. According to the hypothesis of the theorem, the function $(f(\mu) - \xi - k(\mu))/(f(\mu) - \xi + k(\mu))$ changes signs at most once. From this fact it follows that $c(\xi, z)$ and $g(\xi, z)$ are of degree 1 in z.

For all other values of ξ,

$$\frac{f(a) - \xi - k(a)}{f(a) - \xi + k(a)} > 0.$$

According to the hypothesis of the theorem, the function

$$(f(\mu) - \xi - k(\mu))/(f(\mu) - \xi + k(\mu))$$

now changes signs at most twice. From this fact it is easily seen that the degrees of $c(\xi, z)$ and $g(\xi, z)$ do not exceed 1.

9. **Singular integral operators with continuous spectra of multiplicity 2.** In §7 we saw that if $c(\xi, z)$ and $g(\xi, z)$ are of degree $D_\xi \leqq 2$ for $\min\{f(\lambda) - k(\lambda)\} < \xi < \max\{f(\lambda) + k(\lambda)\}$, then

$$\frac{\tilde{P}(\xi, z_1, z_2)}{Q(\xi, z_1)Q(\xi, z_2)} = \frac{1}{A(\xi)} + \frac{(B(\xi)/A(\xi))^{1/2}}{Q(\xi, z_1)} \frac{(B(\xi)/A(\xi))^{1/2}}{Q(\xi, z_2)},$$

where $B(\xi) \geqq 0$. Furthermore, in Lemma 6.4 we proved that if

$$(g_0(\xi), g_1(\xi), \cdots, g_N(\xi))$$

lies in the domain of the transformation $\mathcal{S}$, then $(\xi g_0(\xi), \xi g_1(\xi), \cdots, \xi g_N(\xi))$ also lies in the domain of $\mathcal{S}$. Suppose, now, that z is a *real* number which does not lie in the closed interval $[a, b]$. The element

$$\left(0, \left(\frac{B(\xi)}{A(\xi)}\right)^{1/2} \frac{\overline{F}(\xi, z)}{Q(\xi, z)}\right)$$

lies in the domain of $\mathcal{S}$. Thus if $T(\xi)$ is any polynomial in ξ, then

$$\left(0, T(\xi)\left(\frac{B(\xi)}{A(\xi)}\right)^{1/2} \frac{\overline{F}(\xi, z)}{Q(\xi, z)}\right)$$

lies in the domain of $\mathcal{S}$. Now

$$F(\xi, z) = \left\{\frac{-Q(\xi, z)}{(c(\xi, z)g(\xi, z))^{1/2}}\right\} \exp\left\{\frac{1}{2\pi i}\int_a^b \log\left|\frac{f(\mu) - \xi - k(\mu)}{f(\mu) - \xi + k(\mu)}\right| \frac{d\mu}{\mu - z}\right\}.$$

Since z does not lie in the interval $[a, b]$, then one sees easily from (3.5) that

$c(\xi, z)$ and $g(\xi, z)$ do not vanish in the interval $\min\{f(\lambda)-k(\lambda)\}<\xi<\max\{f(\lambda)+k(\lambda)\}$, except for a finite number of points ξ. Now $Q(\xi, z)=c(\xi, z)-g(\xi, z)$. Since

$$\log\frac{g(\xi,z)}{c(\xi,z)}=\frac{1}{\pi}\int_a^b \arg\frac{f(\mu)-\xi-i0-k(\mu)}{f(\mu)-\xi-i0+k(\mu)}\,\frac{d\mu}{\mu-z},$$

the argument used in the proof of Lemma 4.2 shows that $\log(g(\xi, z)/c(\xi, z))\neq 0$ for almost all ξ in the interval $\min\{f(\lambda)-k(\lambda)\}<\xi<\max\{f(\lambda)+k(\lambda)\}$. Thus $Q(\xi, z)\neq 0$ for almost all ξ from this interval and therefore $F(\xi, z)\neq 0$ for almost all ξ from the same interval. From the expression for $\log(g(\xi, z)/c(\xi, z))$ one can also easily establish that $F(\xi, z)$ is bounded. Hence the linear manifold generated by elements of the form

$$\left\{T(\xi)\left(\frac{B(\xi)}{A(\xi)}\right)^{1/2}\frac{\overline{F}(\xi, z)}{Q(\xi, z)}\right\}$$

is dense in $L^2(\Sigma)$, where

$$\Sigma=\{\xi\mid B(\xi)\neq 0\}\cap\{\xi\mid \min\{f(\lambda)-k(\lambda)\}<\xi<\max\{f(\lambda)+k(\lambda)\}\}.$$

The discussion of §6 can be used to show that $\mathcal{R}(1-P)=(1-\mathcal{P})\mathcal{R}$ is an isometry on $\overline{\mathcal{L}}_\lambda^\perp$, the orthogonal complement of $\overline{\mathcal{L}}_\lambda$ in $L^2(a, b)$. Thus the set of elements of $\mathcal{H}$ which lie in the domain of $\mathcal{S}$ and which are of the form $(0, g(\xi))$ is closed. Therefore the closure of the linear manifold generated by the elements of the form

$$\left\{\left(0,\ T(\xi)\left(\frac{B(\xi)}{A(\xi)}\right)^{1/2}\frac{\overline{F}(\xi, z)}{Q(\xi, z)}\right)\right\}$$

lies in the domain of $\mathcal{S}$. But then the domain of $\mathcal{S}$ is just the direct sum of the spaces $L^2(\min\{f(\lambda)-k(\lambda)\}, \max\{f(\lambda)+k(\lambda)\})$ and $L^2(\Sigma)$. Hence the transformations $\mathcal{R}$ and $\mathcal{S}$ of §6 furnish a complete spectral representation for the operator L in this case.

Just as in §8 we can now formulate suitable conditions on the functions $f(\lambda)$ and $k(\lambda)$, so that the degrees of $c(\xi, z)$ and $g(\xi, z)$ do not exceed 2.

THEOREM 9.1. *Consider the function*

$$\frac{f(\lambda)-\xi-k(\lambda)}{f(\lambda)-\xi+k(\lambda)},\qquad a\leqq\lambda\leqq b.$$

Suppose that when $f(a)-k(a)\leqq\xi\leqq f(a)+k(a)$, the function changes signs at most three times, and that for all other values of ξ, the function changes signs at most four times. Then the operator L has a continuous spectrum of multiplicity $\leqq 2$ and the transformations $\mathcal{R}$ and $\mathcal{S}$ of §6 furnish a spectral representation for L.

Proof. We again utilize the expression

$$\log \frac{g(\xi, z)}{c(\xi, z)} = \frac{1}{\pi} \int_a^b \arg \frac{f(\mu) - \xi - i0 - k(\mu)}{f(\mu) - \xi - i0 + k(\mu)} \frac{d\mu}{\mu - z} .$$

If

$$\frac{f(\mu) - \xi - k(\mu)}{f(\mu) - \xi + k(\mu)} < 0$$

in a half-open interval $a < \mu \leqq \lambda_1$, then we again have $f(a) - k(a) \leqq \xi \leqq f(a) + k(a)$. Thus the sign of $(f(\mu) - \xi - k(\mu))/(f(\mu) - \xi + k(\mu))$ changes at most three times, by hypothesis. Hence it follows that the degrees of $c(\xi, z)$ and $g(\xi, z)$ are $\leqq 2$.

For all other values ξ,

$$\frac{f(a) - \xi - k(a)}{f(a) - \xi + k(a)} > 0$$

and the sign of $(f(\mu) - \xi - k(\mu))/(f(\mu) - \xi + k(\mu))$ can change at most four times according to our hypothesis. Thus the degrees of $c(\xi, z)$ and $g(\xi, z)$ are again $\leqq 2$. This completes the proof.

10. **Remarks about the general operator** L. From the corollary to Theorem 2.2 and from Theorem 5.1 we immediately obtain

THEOREM 10.1. *The set of limit points of the spectrum of the operator L is the entire interval*

$$\min\{f(\lambda) - k(\lambda)\} \leqq \xi \leqq \max\{f(\lambda) + k(\lambda)\}.$$

If $k(\lambda) > 0$, $a \leqq \lambda \leqq b$, then the operator L is similar to the dominant singular integral operator $\tilde{L}$ of §2. Hence we obtain

THEOREM 10.2. *If $k(\lambda) > 0$, $a \leqq \lambda \leqq b$, then the spectrum of the operator L is precisely the interval*

$$\min\{f(\lambda) - k(\lambda)\} \leqq \xi \leqq \max\{f(\lambda) + k(\lambda)\}.$$

11. **Remarks about more general self-adjoint singular integral operators.** In this section we shall give a general indication of the reduction of the problem of finding the spectrum of operators of the form

$$Mx(\lambda) \equiv f(\lambda)x(\lambda) + \frac{1}{\pi i} \int_a^b \frac{g(\lambda, \mu)}{\mu - \lambda} x(\mu) d\mu, \tag{11.1}$$

where $f(\lambda)$ and $g(\lambda, \mu)$ are smooth functions, to the problem of determining the spectra of certain multiplication operators, operators of the form (1.1) and completely continuous operators.

First, we observe that the operator M differs from the corresponding dominant singular integral operator

$$\tilde{M}x(\lambda) \equiv f(\lambda)x(\lambda) + \frac{g(\lambda, \lambda)}{\pi i}\int_a^b \frac{x(\mu)}{\mu - \lambda}\, d\mu \tag{11.2}$$

by a completely continuous operator. We now define

$$\begin{aligned} k_1(\lambda) &= g(\lambda, \lambda), \text{ if } \lambda \in \{\lambda \mid g(\lambda, \lambda) \geqq 0\} \\ &= 0, \text{ otherwise} \end{aligned}$$

and

$$\begin{aligned} k_2(\lambda) &= -\, g(\lambda, \lambda), \text{ if } \lambda \in \{\lambda \mid g(\lambda, \lambda) \leqq 0\} \\ &= 0, \text{ otherwise.} \end{aligned}$$

We then consider the operator

$$\hat{M}x(\lambda) = f(\lambda)x(\lambda) + \frac{1}{\pi i}\int_a^b \frac{(k_1(\lambda)k_1(\mu))^{1/2} - (k_2(\lambda)k_2(\mu))^{1/2}}{\mu - \lambda}\, x(\mu)d\mu. \tag{11.3}$$

This operator $\hat{M}$ commutes with the cut-off projection

$$\begin{aligned} \Pi x(\lambda) &= x(\lambda), \ \lambda \in \{\lambda \mid g(\lambda, \lambda) \geqq 0\} \\ &= 0, \text{ otherwise.} \end{aligned} \tag{11.4}$$

We can thus analyze the spectra of the operators

$$\begin{aligned} \hat{M}_1x(\lambda) &\equiv f(\lambda)x(\lambda) + \frac{1}{\pi i}\int_{\{\mu \mid g(\mu,\mu) \geqq 0\}} \frac{(k_1(\lambda)k_1(\mu))^{1/2}}{\mu - \lambda}\, x(\mu)d\mu \\ &\qquad \text{on } L^2(\{\lambda \mid g(\lambda, \lambda) \geqq 0\}), \end{aligned} \tag{11.5}$$

$$\begin{aligned} \hat{M}_2x(\lambda) &\equiv f(\lambda)x(\lambda) - \frac{1}{\pi i}\int_{\{\mu \mid g(\mu,\mu) < 0\}} \frac{(k_2(\lambda)k_2(\mu))^{1/2}}{\mu - \lambda}\, x(\mu)d\mu \\ &\qquad \text{on } L^2(\{\lambda \mid g(\lambda, \lambda) \leqq 0\}) \end{aligned} \tag{11.6}$$

separately. The operator $\hat{M}_2$ is essentially of the form (1.1). The operator $\hat{M}_1$ commutes with the cut-off projection

$$\begin{aligned} \Pi_1x(\lambda) &= x(\lambda), \ \lambda \in \{\lambda \mid k_1(\lambda) > 0\} \\ &= 0, \ \lambda \in \{\lambda \mid k_1(\lambda) = 0\} \cap \{\lambda \mid g(\lambda, \lambda) \geqq 0\}. \end{aligned} \tag{11.7}$$

Hence we can analyze the spectra of the multiplication operator

$$\hat{M}_{11}x(\lambda) \equiv f(\lambda)x(\lambda) \quad \text{on } L^2(\{\lambda \mid k_1(\lambda, \lambda) = 0\} \cap \{\lambda \mid g(\lambda, \lambda) \geqq 0\}) \tag{11.8}$$

and the operator

$$\begin{aligned} \hat{M}_{12}x(\lambda) &\equiv f(\lambda)x(\lambda) + \frac{1}{\pi i}\int_{\{\mu \mid k_1(\mu) > 0\}} \frac{(k_1(\lambda)k_1(\mu))^{1/2}}{\mu - \lambda}\, x(\mu)d\mu \\ &\qquad \text{on } L^2(\{\lambda \mid k_1(\lambda) > 0\}) \end{aligned} \tag{11.9}$$

separately. The operator $\hat{M}_{12}$ is essentially of the form (1.1). Finally we observe that the operator $\hat{M}$ (see (11.3)) differs from $\tilde{M}$ by a completely continuous operator, since

$$(\hat{M} - \tilde{M})x(\lambda) \equiv \Pi(\hat{M} - \tilde{M})x(\lambda) + (1 - \Pi)(\hat{M} - \tilde{M})x(\lambda)$$
$$= \frac{(k_1(\lambda))^{1/2}}{\pi i}\int_a^b \frac{(k_1(\mu))^{1/2} - (k_1(\lambda))^{1/2}}{\mu - \lambda} x(\mu)d\mu$$
$$+ \frac{(k_2(\lambda))^{1/2}}{\pi i}\int_a^b \frac{(k_2(\lambda))^{1/2} - (k_2(\mu))^{1/2}}{\mu - \lambda} x(\mu)d\mu,$$

is the sum of two completely continuous operators.

Bibliography

1. T. Carleman, *Sur la résolution de certaines équations intégrales*, Arkiv för Matematik, Astronomi och Fysik vol. 16 (1922).

2. R. Courant and D. Hilbert, *Methods of mathematical physics*, vol. 1, New York, Interscience, 1953.

3. W. Koppelman and J. D. Pincus, *Perturbation of continuous spectra and singular integral operators*, Report IMM-NYU 242, New York University, 1957.

4. S. G. Michlin, *Singular integral equations*, Amer. Math. Soc. Translation no. 24, 1950.

5. N. I. Muskhelishvili, *Singular integral equations*, (translated from the Russian), Groningen, Noordhoff, 1953.

6. I. I. Priwalow, *Randeigenschaften analytischer Funktionen*, Berlin, Deutscher Verlag der Wissenschaften, 1956.

7. H. Söhngen, *Zur Theorie der endlichen Hilbert-Transformation*, Math. Z. vol. 60 (1954) pp. 31–51.

8. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, London, Oxford University Press, 1937.

9. F. G. Tricomi, *Integral equations*, New York, Interscience, 1957.

10. A. C. Zaanen, *Linear analysis*, New York, Interscience, 1953.

11. W. Koppelman, *On the spectral theory of singular integral operators*, Proc. Nat. Acad. Sci. U.S.A. vol. 44 (1958) pp. 1252–1254.

Brookhaven National Laboratory,
Upton, New York