DIRECT PRODUCTS OF MODULES(1)

BY
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1. **Introduction.** It is a well-known and basic result of homological algebra that the direct product of an arbitrary family of injective modules over any ring is again injective [3, p. 8]. Such is not the case for projective modules, as is evidenced, for example, by a result of Baer [7, p. 48] which states that the direct product of a countably infinite number of copies of the ring of rational integers is not a free abelian group. It is thus natural to ask for the precise ideal-theoretic conditions which are forced upon a ring by the requirement that its projective modules be preserved by direct products in the manner just described. In this paper we shall present a solution to this problem, as well as an answer to the corresponding question for flat modules.

We then exhibit several applications of these results. First, we derive information concerning semi-hereditary rings which, when applied to integral domains, yields immediately characterizations of Prüfer rings due to Hattori [5] (see also [6]). The second application also concerns integral domains. Let us call a torsion module over an integral domain R a UT-module if it is a direct summand of every R-module of which it is the torsion submodule. We prove that, if every R-module of bounded order is a UT-module, then R must be a Dedekind ring (the converse is well-known; see [8, p. 334]).

As a final application, we obtain a partial solution to the following question of Köthe [9]: For which rings R is it true that every left R-module is a direct sum of cyclic modules? We prove that, if R has the weaker property that every left R-module is a direct sum of finitely generated modules, then R satisfies the minimum condition on left ideals and every indecomposable injective left R-module has finite length. This generalizes a result of Kaplansky and Cohen [4].

Our investigations along these lines were motivated to some extent by the interesting observation of Bass [2] that left Noetherian rings are characterized by the property that their injective left modules are preserved by direct sums. We have, with his kind permission, included this in our paper, as we shall need it in the proof of another result.

Throughout this note we shall assume without further comment that all

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rings possess a unit which acts as the identity on all modules. If $\{A_\alpha\}$ is a family of left or right modules over a ring R, where α traces an index set J, then the direct product of this family will be denoted by $\prod_{\alpha\in J} A_\alpha$ [3, p. 4].

2. **Direct products of flat modules.** We shall begin with some preliminary results on flat modules which will be needed in the proof of the main theorem of this section.

PROPOSITION 2.1. *Let R be a ring, and $0\to K\to F\to A\to 0$ be an exact sequence of right R-modules, where F is free with basis $\{x_\alpha\}$. If $u=x_{\alpha_1}a_1+\cdots+x_{\alpha_r}a_r$ is an element of F, define I_u to be the left ideal in R generated by $a_1,\cdots,a_r$. Then the following statements are equivalent:*

(a) *A is flat.*

(b) *If $u\in K$, then $u\in KI_u$.*

Proof. (a)$\Rightarrow$(b): If A is flat and $u\in K$, then $u\in K\cap FI_u=KI_u$, by [3, p. 123, Exercise 5].

(b)$\Rightarrow$(a): Let I be any left ideal in R, and $u\in K\cap FI$. Then clearly $I_u\subseteq I$, and so $u\in KI_u\subseteq KI$. This is true for all $u\in K\cap FI$, and so it follows that $K\cap FI=KI$. Hence, applying [3, p. 123, Exercise 5], we obtain that A is flat, completing the proof.

We shall need a characterization of flat modules which is due to Villamayor. Since at the time of writing this result has not appeared in the literature, we shall exhibit it here, with proof.

PROPOSITION 2.2. *Let R be a ring, and $0\to K\to F\to A\to 0$ be an exact sequence of right R-modules, where F is free. Then the following statements are equivalent:*

(a) *A is flat.*

(b) *Given any $u\in K$, there exists a homomorphism $\theta\colon F\to K$ such that $\theta(u)=u$.*

(c) *Given any $u_1,\cdots,u_n$ in K, there exists a homomorphism $\theta\colon F\to K$ such that $\theta(u_i)=u_i$ for $i=1,\cdots,n$.*

Proof. (a)$\Rightarrow$(b): Let $u\in K$. Define I_u as in Proposition 2.1; i.e., $u=x_{\alpha_1}a_1+\cdots+x_{\alpha_r}a_r$ (where $\{x_\alpha\}$ is a basis of F) and $I_u=Ra_1+\cdots+Ra_r$. Since A is flat, we have from Proposition 2.1 that $u\in KI_u$, and so $u=v_1a_1+\cdots+v_ra_r$, with $v_i\in K$. Define the homomorphism $\theta\colon F\to K$ by $\theta(x_{\alpha_i})=v_i$ for $i=1,\cdots,r$, and $\theta(x_\alpha)=0$ if $\alpha\neq\alpha_1,\cdots,\alpha_r$. Then clearly θ has the desired property.

(b)$\Rightarrow$(a): Given $u=x_{\alpha_1}a_1+\cdots+x_{\alpha_r}a_r$ in K, let $\theta\colon F\to K$ be a homomorphism such that $\theta(u)=u$. Then $u=\theta(x_{\alpha_1})a_1+\cdots+\theta(x_{\alpha_r})a_r$, and is thus in KI_u. It then follows from Proposition 2.1 that A is flat.

(b)$\Rightarrow$(c): Let $u_1,\cdots,u_n\in K$. If $n=1$, then the existence of θ follows from (b). Proceed by induction on n; assume that $n>1$ and (c) holds for $k<n$. Let $\theta_n\colon F\to K$ be a homomorphism such that $\theta_n(u_n)=u_n$. Let $v_i=u_i-\theta_n(u_i)$ for $i=1,\cdots,n-1$; then, by the induction assumption, there exists

a homomorphism $\theta': F \to K$ such that $\theta'(v_i) = v_i$ for all $i = 1, \cdots, n-1$. Define the homomorphism $\theta: F \to K$ by $\theta = 1 - (1-\theta')(1-\theta_n)$; then it is easy to check that θ has the desired properties.

(c)$\Rightarrow$(b): Obvious. This completes the proof.

A module A over a ring R will be called *finitely related* if there exists an exact sequence $0 \to K \to F \to A \to 0$ of R-modules, where F is free and both F and K are finitely generated. If A is finitely related, then any exact sequence of the form $0 \to K \to F \to A \to 0$ will have the property that K is finitely generated whenever F is free and finitely generated; this follows easily from a result of Schanuel [11, p. 369].

We then have the following immediate corollary to Proposition 2.2.

COROLLARY. *If R is any ring, then a finitely related flat R-module is projective.*

The next proposition is a generalization of a part of Exercise 6 of [3, p. 123].

PROPOSITION 2.3. *Let R be a ring, and A be a right R-module. Then the following statements are equivalent:*

(a) *A is flat.*

(b) *If $a_1\lambda_1 + \cdots + a_r\lambda_r = 0$, where $a_k \in A$ and $\lambda_k \in R$, then there exist $b_1, \cdots, b_n \in A$ and $\{\mu_{ik}\} \subseteq R$ $(i = 1, \cdots, n;\ k = 1, \cdots, r)$ such that $a_k = \sum_{i=1}^n b_i\mu_{ik}$ and $\sum_{k=1}^r \mu_{ik}\lambda_k = 0$.*

(c) *If $\sum_{k=1}^r a_k\lambda_{kj} = 0$, where $a_k \in A$ and $\lambda_{kj} \in R$ $(j = 1, \cdots, s)$ then there exist $b_1, \cdots, b_n \in A$ and $\{\mu_{ik}\} \subseteq R$ $(i = 1, \cdots, n;\ k = 1, \cdots, r)$ such that $a_k = \sum_{i=1}^n b_i\mu_{ik}$ and $\sum_{k=1}^r \mu_{ik}\lambda_{kj} = 0$.*

Proof. (a)$\Rightarrow$(c): Let $f: F \to A$ be an epimorphism, where F is a free right R-module, and let $K = \ker(f)$. Select $x_1, \cdots, x_r$ in F such that $f(x_k) = a_k$, and set $u_j = \sum_{k=1}^r x_k\lambda_{kj}$ for $j = 1, \cdots, s$. Then $f(u_j) = \sum_{k=1}^r a_k\lambda_{kj} = 0$. Since A is flat, we have from Proposition 2.2 that there exists a homomorphism $\theta: F \to K$ such that $\theta(u_j) = u_j$. Write $x_k - \theta(x_k) = \sum_{i=1}^n z_i\mu_{ik}$, where $z_1, \cdots, z_n$ are part of a basis of F. Set $b_i = f(z_i)$; then $a_k = f(x_k) = f\{x_k - \theta(x_k)\} = \sum_{i=1}^n b_i\mu_{ik}$. Finally, $0 = u_j - \theta(u_j) = \sum_{k=1}^r \{x_k - \theta(x_k)\}\lambda_{kj} = \sum_{k=1}^r(\sum_{i=1}^n z_i\mu_{ik})\lambda_{kj} = \sum_{i=1}^n z_i(\sum_{k=1}^r \mu_{ik}\lambda_{kj})$. Since $z_1, \cdots, z_n$ is part of a basis of F, it then follows that $\sum_{k=1}^r \mu_{ik}\lambda_{kj} = 0$.

(c)$\Rightarrow$(b): Trivial.

(b)$\Rightarrow$(a): Let $f: F \to A$ be an epimorphism, where F is a free right R-module with basis $\{x_\alpha\}$. Let $K = \ker(f)$, and suppose $u = x_{\alpha_1}\lambda_1 + \cdots + x_{\alpha_r}\lambda_r$ is in K. Let $a_k = f(x_{\alpha_k})$; then $a_1\lambda_1 + \cdots + a_r\lambda_r = f(u) = 0$. By hypothesis, there exist $b_1, \cdots, b_n$ in A and $\{\mu_{ik}\} \subseteq R$ $(i \le n,\ k \le r)$ such that $a_k = \sum_{i=1}^n b_i\mu_{ik}$ and $\sum_{k=1}^r \mu_{ik}\lambda_k = 0$. Select $z_i \in F$ such that $f(z_i) = b_i$. Define a homomorphism $\theta: F \to F$ by $\theta(x_{\alpha_k}) = x_{\alpha_k} - \sum_{i=1}^n z_i\mu_{ik}$ for $k \leqq r$ and $\theta(x_\alpha) = 0$ for $\alpha \ne \alpha_1, \cdots, \alpha_r$. Then $f\{\theta(x_{\alpha_k})\} = a_k - \sum_{i=1}^n b_i\mu_{ik} = 0$, and so $\theta(F) \subseteq K$. Finally, we have that

$$\theta(u) = u - \sum_{k=1}^{r}\left(\sum_{i=1}^{n} z_i\mu_{ik}\right)\lambda_k = u - \sum_{i=1}^{n} z_i\left(\sum_{k=1}^{r} \mu_{ik}\lambda_k\right) = u.$$

It then follows from Proposition 2.2 that A is flat, completing the proof.

We shall now state and prove the principal result of this section.

THEOREM 2.1. *For any ring R the following statements are equivalent:*

(a) *The direct product of any family of flat right R-modules is flat.*

(b) *The direct product of any family of copies of R is flat as a right R-module.*

(c) *Any finitely generated submodule of a free left R-module is finitely related.*

(d) *Any finitely generated left ideal in R is finitely related.*

Proof. (a)$\Rightarrow$(b): Trivial, since R is a flat right R-module.

(b)$\Rightarrow$(c): Let G be a free left R-module, and L be a finitely generated submodule of G. Clearly we may assume that G is finitely generated; hence, for some integer $s>0$, we may identify G with the left R-module of all s-tuples $(\lambda_1, \cdots, \lambda_s)$ of elements of R. Let $u_1, \cdots, u_r$ generate L, where $u_k = (\lambda_{k1}, \cdots, \lambda_{ks})$. Let F be a free left R-module with basis $x_1, \cdots, x_r$, and define an epimorphism $f: F\rightarrow L$ by $f(x_k)=u_k$. Set $K=\ker(f)$, and for each $\alpha\in K$ let $R^{(\alpha)}$ be a copy of R. Define $A=\prod_{\alpha\in K} R^{(\alpha)}$, which we shall view as a right R-module. If $\alpha=a_1(\alpha)x_1+\cdots+a_r(\alpha)x_r$ is in K, then $a_1(\alpha)u_1+\cdots+a_r(\alpha)u_r=f(\alpha)=0$, and so $\sum_{k=1}^{r} a_k(\alpha)\lambda_{kj}=0$ for all $j\leqq s$. Thus, setting $a_k=\{a_k(\alpha)\}\in A$ for $k=1, \cdots, r$, we get that $\sum_{k=1}^{r} a_k\lambda_{kj}=0$ for $j\leqq s$.

By hypothesis, A is a flat right R-module; hence there exist $b_1, \cdots, b_n\in A$ and $\{\mu_{ik}\}\subseteq R$ $(i\leqq n,\ k\leqq r)$ satisfying the conditions of Proposition 2.3(c). Set $z_i=\sum_{k=1}^{r}\mu_{ik}x_k\in F$ for $i\leqq n$; then $f(z_i)=\sum_{k=1}^{r}\mu_{ik}u_k=0$, since $\sum_{k=1}^{r}\mu_{ik}\lambda_{kj}=0$ for all $j\leqq s$. Hence $z_1, \cdots, z_n\in K$. Write $b_i=\{b_i(\alpha)\}$, where $b_i(\alpha)\in R$; then, since $a_k=\sum_{i=1}^{n} b_i\mu_{ik}$ for all $k\leq r$, we obtain easily that $a_k(\alpha)=\sum_{i=1}^{n} b_i(\alpha)\mu_{ik}$ for $k\leqq r$ and $\alpha\in K$. It then follows that $\alpha=\sum_{k=1}^{r} a_k(\alpha)x_k=\sum_{i=1}^{n} b_i(\alpha)z_i$. Hence $z_1, \cdots, z_n$ generate K, completing the proof that L is finitely related.

(c)$\Rightarrow$(d): Trivial.

(d)$\Rightarrow$(b): Let $\{R^{(\alpha)}\}$ be any family of copies of R, and $A=\prod_\alpha R^{(\alpha)}$, which we shall view as a right R-module. Suppose that $a_1\lambda_1+\cdots+a_r\lambda_r=0$, where $a_k=\{a_k(\alpha)\}\in A$ and $\lambda_k\in R$, $k\leqq r$. Let I be the left ideal in R generated by $\lambda_1, \cdots, \lambda_r$, F be a free left R-module with basis $x_1, \cdots, x_r$, and $f: F\rightarrow I$ be the epimorphism defined by $f(x_k)=\lambda_k$. Let K be the kernel of f; by hypothesis, K is finitely generated. Let $z_1, \cdots, z_n$ be a set of generators of K, and write $z_i=\sum_{k=1}^{r}\mu_{ik}x_k$. Setting $u(\alpha)=a_1(\alpha)x_1+\cdots+a_r(\alpha)x_r$, we have that $f(u(\alpha))=a_1(\alpha)\lambda_1+\cdots+a_r(\alpha)\lambda_r=0$, and so there exist $b_i(\alpha)\in R$ such that $u(\alpha)=\sum_{i=1}^{n} b_i(\alpha)z_i=\sum_{k=1}^{r}(\sum_{i=1}^{n} b_i(\alpha)\mu_{ik})x_k$. Since $x_1, \cdots, x_r$ is a basis of F, it follows that $a_k(\alpha)=\sum_{i=1}^{n} b_i(\alpha)\mu_{ik}$ for all α, and so $a_k=\sum_{i=1}^{n} b_i\mu_{ik}$ for $k=1, \cdots, r$. Finally, $\sum_{k=1}^{r}\mu_{ik}\lambda_k=f(z_i)=0$ for each $i\leqq n$, since $z_i\in K$. It then follows from Proposition 2.3 that A is flat.

(c)⇒(a): Let $\{A_\alpha\}$ be a family of flat right R-modules, and set $A = \prod_\alpha A_\alpha$. Define a functor V from the category of left R-modules to the category of abelian groups by $V(C) = \prod_\alpha (A_\alpha \otimes_R C)$. It is well-known that V is additive and exact [3, p. 31, Exercise 2]. Define a natural transformation $t: A \otimes_R (\cdot) \to V(\cdot)$ as follows: If C is a left R-module, then $t_C: A \otimes_R C \to V(C)$ is defined by $t_C(\{a_\alpha\} \otimes c) = \{a_\alpha \otimes c\}$, where $c \in C$ and $\{a_\alpha\} \in A$.

Now let $0 \to K \to F \to C \to 0$ be an exact sequence of left R-modules, where C is finitely generated and F is free of finite rank. We then get the following commutative diagram:

$$\begin{array}{ccccccc} & A \otimes_R K & \to & A \otimes_R F & \to & A \otimes_R C & \to 0 \\ & \downarrow t_K & & \downarrow t_F & & \downarrow t_C & \\ 0 \to & V(K) & \to & V(F) & \to & V(C) & \to 0 \end{array}$$

where the rows are exact. That V is additive, $V(R) \approx A$, and F is free of finite rank implies immediately that t_F is an isomorphism. It then follows from routine diagram-chasing that t_C is an epimorphism.

Suppose now that K is also finitely generated; i.e., C is finitely related. Then, replacing C by K in the above argument, we obtain that t_K is an epimorphism. Further diagram-chasing then shows that t_C is an isomorphism. But since K is a finitely generated submodule of a free left R-module, it follows from our hypotheses that K is finitely related; hence t_K is an isomorphism, too. We may then conclude that the sequence $0 \to A \otimes_R K \to A \otimes_R F \to A \otimes_R C \to 0$ is exact, and thus $\mathrm{Tor}_1^R(A, C) = 0$.

Now let C be any finitely generated left R-module, and $0 \to K \to F \to C \to 0$ be an exact sequence, where F is free of finite rank. The family $\{K_\beta\}$ of all finitely generated submodules of K form, in the obvious way, a directed system of which the direct limit is K. Then C is the direct limit of the induced directed system $\{C_\beta\}$, where $C_\beta = F/K_\beta$. We obtain from our previous remarks that $\mathrm{Tor}_1^R(A, C_\beta) = 0$ for all β, since C_β is finitely related. Since the torsion functor preserves direct limits, it then follows that $\mathrm{Tor}_1^R(A, C) = 0$ for any finitely generated left R-module C. Hence A is R-flat, completing the proof of the theorem.

Clearly every left Noetherian ring satisfies condition (d) of Theorem 2.1. Hence the theorem may be viewed as a generalization of Exercise 4 of [3, p. 122], which states that the direct product of a family of flat right modules over a left Noetherian ring is again flat. Indeed, the final part of the proof given above follows to some extent the proof suggested in that exercise.

We shall now present a purely ideal-theoretic characterization of the class of rings described in Theorem 2.1. This characterization is based upon a result concerning residual division in commutative rings, which was communicated to me by J. Eagon. We need first a couple of definitions and lemmas.

Definition 2.1. Let R be a ring, I be a left ideal in R, and C be a subset of R. Let $(I: C) = \{\lambda \in R / \lambda C \subseteq I\}$. Clearly $(I:C)$ is a left ideal in R. If $C = \{a_1, \cdots, a_r\}$, we shall sometimes write $(I: a_1, \cdots, a_r)$ for $(I:C)$.

LEMMA 2.1. *If* $C = \bigcup_\alpha C_\alpha$, *then* $(I:C) = \bigcap_\alpha (I:C_\alpha)$.

Proof. Trivial.

LEMMA 2.2. *Let* $I = Ra_1 + \cdots + Ra_n$ *be a left ideal in a ring* R, *and let* $a \in R$. *Set* $J = I + Ra$, *and let* F *be a free left* R-*module with basis* $x_1, \cdots, x_{n+1}$. *Define a homomorphism* $f: F \to J$ *by* $f(x_i) = a_i$ *for* $i \leqq n$ *and* $f(x_{n+1}) = a$. *Let* $K = \ker(f)$, *and set* $F' = Rx_1 + \cdots + Rx_n \subseteq F$ *and* $K' = K \cap F'$. *Then there exists a homomorphism* $g: K \to (I:a)$ *such that* $\ker(g) = K'$.

Proof. If $u \in K$, write $u = \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_u x_{n+1}$; then $\lambda_1 a_1 + \cdots + \lambda_n a_n + \lambda_u a = f(u) = 0$, and so $\lambda_u \in (I:a)$. Define g by $g(u) = \lambda_u$. Straightforward computations then complete the proof.

THEOREM 2.2. *The following statements are equivalent for any ring* R:

(a) *Every finitely generated left ideal in* R *is finitely related.*

(b) *If* I *is a finitely generated left ideal in* R, *then* $(I:a)$ *is finitely generated for any* $a \in R$.

(c) $(0:a)$ *is a finitely generated left ideal for any* $a \in R$, *and the intersection of any two finitely generated left ideals in* R *is again finitely generated.*

If any (and hence all) of these conditions hold, then $(I:C)$ *is a finitely generated left ideal in* R *for any finitely generated left ideal* I *and finite subset* C *of* R.

Proof. (a)⇒(b): This follows immediately from Lemma 2.2.

(b)⇒(a): Let $I = Ra_1 + \cdots + Ra_n$ be a left ideal in R. If $n = 1$, then the exact sequence $0 \to (0:a_1) \to R \to I \to 0$ shows that I is finitely related. Proceed by induction on n; assume $n > 1$ and the statement is true for $k < n$. Let F be a free left R-module with basis $x_1, \cdots, x_n$ and define an epimorphism $f: F \to I$ by $f(x_i) = a_i$. Let $K = \ker(f)$, and set $I' = Ra_1 + \cdots + Ra_{n-1} \subseteq I$, $F' = Rx_1 + \cdots + Rx_{n-1} \subseteq F$, and $K' = K \cap F'$. Then we obtain from Lemma 2.2 the following exact sequences:

$$0 \to K' \to F' \to I' \to 0, \qquad 0 \to K' \to K \to (I':a_n) \to 0.$$

$(I':a_n)$ is finitely generated by hypothesis, and we see from the first exact sequence above that K' is finitely generated by the induction assumption. It then follows from the second exact sequence above that K is finitely generated; i.e., I is finitely related.

(a)⇔(c): If $a \in R$ then the exact sequence $0 \to (0:a) \to R \to Ra \to 0$ shows that $(0:a)$ is finitely generated if and only if Ra is finitely related. Let I_1, I_2 be finitely generated left ideals in R, and consider the exact sequences $0 \to K_i \to F_i \to I_i \to 0$, $i = 1, 2$, where F_i is free of finite rank. Let K be the kernel of the obvious epimorphism of $F = F_1 \oplus F_2$ onto $I = I_1 + I_2$. We then get the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & I_1 \cap I_2 & & \\
 & & & & & & \downarrow & & \\
0 & \rightarrow & K_1 \oplus K_2 & \rightarrow & \mathrm{F} & \rightarrow & I_1 \oplus I_2 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & I & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

where the rows and column are exact and the map $F \rightarrow F$ is the identity. It then follows by standard diagram-chasing that there exists a homomorphism of K onto $I_1 \cap I_2$ such that the sequence $0 \rightarrow K_1 \oplus K_2 \rightarrow K \rightarrow I_1 \cap I_2 \rightarrow 0$ is exact. Thus, if K_1 and K_2 are finitely generated, then K is finitely generated if and only if $I_1 \cap I_2$ is likewise. That is, if I_1 and I_2 are finitely related, then $I = I_1 + I_2$ is finitely related if and only if $I_1 \cap I_2$ is finitely generated.

Suppose now that (a) holds. Let I_1 and I_2 be finitely generated left ideals in R, and set $I = I_1 + I_2$. Then, by hypothesis, I_1, I_2, and I are all finitely related, and so it follows from the above discussion that $I_1 \cap I_2$ is finitely generated. Hence (c) is satisfied.

Assume on the other hand that (c) holds, and let $I = Ra_1 + \cdots + Ra_n$ be a finitely generated left ideal in R. If $n = 1$, it follows from a previous remark that I is finitely related. Proceed by induction on n; assume $n > 1$ and (a) holds for $k < n$. Set $I_1 = Ra_1 + \cdots + Ra_{n-1}$ and $I_2 = Ra_n$. We have from the induction assumption that I_1 and I_2 are finitely related. Then, since $I_1 \cap I_2$ is finitely generated, it follows from our preceding arguments that I is finitely related, and thus (a) is satisfied.

The final assertion of the theorem follows easily from Lemma 2.1 in conjunction with conditions (b) and (c). The proof of the theorem is hence complete.

We have already observed that a left Noetherian ring satisfies the conditions of Theorems 2.1 and 2.2. Less trivial examples of such rings are the following:

(a) A left semi-hereditary ring; i.e., a ring in which every finitely-generated left ideal is projective [3, p. 14].

(b) The ring of polynomials in any finite or infinite set of variables with coefficients in a field.

(c) The ring of formal power series in any finite or infinite set of variables with coefficients in a field, with the restriction that only finitely many of the variables appear in the expansion of any given element of the ring.

Finally, we remark that, if R is a ring satisfying the conditions of Theorems 2.1 and 2.2 and I is a two-sided ideal in R which is finitely generated as a left ideal, then R/I also satisfies these conditions; this follows immediately from Theorem 2.2.

3. **Direct products of projective modules.** The problems concerning direct products of projective modules, to which we now turn, are more difficult. We shall consider a more general situation, which leads us to a rather ambitious generalization of the theorem of Baer mentioned in the introduction. We shall show, roughly speaking, that if the direct product of a "large" number of copies of a ring R can be embedded in a certain way in a direct sum of left R-modules, each of which is generated by a "small" number of elements, then R must satisfy the descending chain condition on principal right ideals. First we introduce several concepts which will be needed in the proof of the main result.

DEFINITION 3.1. Let R be a ring, A be a left R-module, and A' be a submodule of A. A' will be called a *pure* submodule of A if $A' \cap aA = aA'$ for all $a \in R$.

REMARKS. 1. It is trivial to verify that every direct summand of A is a pure submodule of A.

2. The concept of purity is well known in the theory of abelian groups (see [7, p. 14]).

DEFINITION 3.2. Let R be a ring, and A be a left R-module. Let $\{C_\beta\}$ be a family of left R-modules (where β traces some index set) and let $f_\beta \in \mathrm{Hom}_R(A, C_\beta)$. The family $\{f_\beta\}$ will be called a Φ-family of homomorphisms if the following conditions are satisfied for any $x \neq 0$ in A:

(a) $f_\beta(x) = 0$ for almost all β.

(b) $f_\beta(x) \neq 0$ for some β.

The following theorem is the principal result of this section.

THEOREM 3.1. *Let R be a ring, and J be an infinite set of cardinality ζ, where $\zeta \geq \mathrm{card}(R)$. Set $A = \prod_{\alpha \in J} R^{(\alpha)}$, where $R^{(\alpha)} \approx R$ is a left R-module. Suppose that A is a pure submodule of a left R-module of the form $C = \sum_\beta \oplus C_\beta$, where each C_β is generated by a subset of cardinality less than or equal to ζ. Then R must satisfy the descending chain condition on principal right ideals.*

Proof. Since J is an infinite set, it follows easily that there exists a descending sequence $J = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$ of subsets of J such that $\bigcap_{n=0}^\infty J_n = \emptyset$ and $\mathrm{card}(J_n) = \zeta$ for all n. Let $A_n = \prod_{\alpha \in J_n} R^{(\alpha)}$, which may be viewed, in the obvious way, as a direct summand of A. Let $f_\beta \in \mathrm{Hom}_R(A, C_\beta)$ be the restriction to A of the projection of C onto C_β; it is then easily verified that $\{f_\beta\}$ is a Φ-family of homomorphisms on A.

Suppose that the theorem is false. Then there exists a strictly descending chain $R = a_0R \supsetneq a_1R \supsetneq a_2R \supsetneq \cdots$ of principal right ideals in R. For each $k \geq 0$, define $A_{nk} = a_kA_n/a_{k+1}A_n$, $C_{\beta k} = a_kC_\beta/a_{k+1}C_\beta$, $R_k = a_kR/a_{k+1}R$, and $R_k^{(\alpha)} = a_kR^{(\alpha)}/a_{k+1}R^{(\alpha)}$, all of which we shall view as left modules over the ring Z of rational integers. Observe that $A_{0,k} = a_kA/a_{k+1}A$, A_{nk} is a direct summand of $A_{0,k}$, $R^{(\alpha)}$ is Z-isomorphic to R_k, and, finally, A_{nk} may be identified, in the obvious way, with $\prod_{\alpha\in J_n} R_k^{(\alpha)}$. Each f_β induces a Z-homomorphism $f_{\beta k}: A_{0,k} \to C_{\beta k}$. Suppose that $\bar{x}=x+a_{k+1}A$ is in $A_{0,k}$, and $f_{\beta k}(\bar{x})=0$ for all β. Then $f_\beta(x)\in a_{k+1}C_\beta$ for all β, and so $x\in a_{k+1}C$. But since A is a pure submodule of C, we have that $x\in a_{k+1}A$, and so $\bar{x}=0$. It then follows easily that, for fixed k, the family $\{f_{\beta k}\}$ is a Φ-family of Z-homomorphisms.

By hypothesis $R_k\neq 0$ for all $k\geqq 0$, and so R_k contains at least two elements. Since $\text{card}(J_n)=\zeta$ for all n, and $A_{nk}\approx\prod_{\alpha\in J_n} R_k^{(\alpha)}$, we get that $\text{card}(A_{nk}) \geqq 2^\zeta>\zeta$ for all n, k, by Cantor's theorem. Furthermore, since each C_β can be generated by a subset of cardinality less than or equal to ζ, and since $\text{card}(R)\leqq\zeta$, it follows that $\text{card}(C_\beta)\leqq\zeta^2=\zeta$, and so a fortiori $\text{card}(C_{\beta k})\leqq\zeta$.

Given $\beta_1, \cdots, \beta_r$ and $k\geqq 0$, let $C'=C_{\beta_1 k}\oplus\cdots\oplus C_{\beta_r k}$. Consider the Z-homomorphism $g: A_{nk}\to C'$ defined by

$$g(\bar{x}) = (f_{\beta_1 k}(\bar{x}), \cdots, f_{\beta_r k}(\bar{x}))$$

where $\bar{x}\in A_{nk}$. Since $\text{card}(C')\leqq r\zeta=\zeta<\text{card}(A_{nk})$, there exists $\bar{x}\neq 0$ in A_{nk} such that $g(\bar{x})=0$. Then $f_{\beta k}(\bar{x})\neq 0$ for some β, since $\{f_{\beta k}\}$ is a Φ-family of homomorphisms. Obviously $\beta\neq\beta_1, \cdots, \beta_r$. We have thus shown that the following condition holds:

(*) For any n, $k\geqq 0$ and any $\beta_1, \cdots, \beta_r$, there exists $\bar{x}\in A_{nk}$ and $\beta\neq\beta_1, \cdots, \beta_r$ such that $f_{\beta k}(\bar{x})\neq 0$.

We shall now construct an element $x\in A$ with the property that $f_\beta(x)\neq 0$ for infinitely many β. This will contradict the fact that $\{f_\beta\}$ is a Φ-family of homomorphisms on A, and will hence complete the proof of the theorem. We shall first construct inductively a sequence $x_0, x_1, x_2, \cdots$ of elements of A and a sequence $f_{\beta_0}, f_{\beta_1}, f_{\beta_2}, \cdots$ selected from the given family of homomorphisms such that the following conditions are satisfied:

(i) $x_n\in a_nA_n$.

(ii) $f_{\beta_n}(x_n)\not\equiv 0 \pmod{a_{n+1}C_{\beta_n}}$.

(iii) $f_{\beta_n}(x_k)=0$ for $k<n$.

We proceed as follows: Let x_0 be any element of A which is not in a_1A. Since $\{f_{\beta,0}\}$ is a Φ-family of homomorphisms on A_{00}, there exists β_0 such that $f_{\beta_0}(x_0)\not\equiv 0 \pmod{a_1C_{\beta_0}}$. Proceeding by induction, let $n>0$ and assume that there exist $x_0, x_1, \cdots, x_{n-1}$ and $\beta_0, \beta_1, \cdots, \beta_{n-1}$ such that conditions (i)–(iii) are satisfied. Since $\{f_\beta\}$ is a Φ-family of homomorphisms on A, there are only a finite number $f_{\gamma_1}, \cdots, f_{\gamma_r}$ from this family with the property that $f_{\gamma_i}(x_k)\neq 0$ for some $k<n$. Applying condition (*), we get that there exist $\bar{x}_n\in A_{nn}$ and $\beta_n\neq\gamma_1, \cdots, \gamma_r$ such that $f_{\beta_n n}(\bar{x}_n)\neq 0$. Select $x_n\in a_nA_n$ which maps onto $\bar{x}_n$. Since $f_{\beta_n n}(\bar{x}_n)\neq 0$, we have that $f_{\beta_n}(x_n)\not\equiv 0 \pmod{a_{n+1}C_{\beta_n}}$. Finally, $f_{\beta_n}(x_k)=0$ for $k<n$, since $\beta_n\neq\gamma_1, \cdots, \gamma_r$. Thus x_n and f_{β_n} satisfy conditions (i)–(iii). It then follows that the sequences $x_0, x_1, x_2, \cdots$ and $f_{\beta_0}, f_{\beta_1}, f_{\beta_2}, \cdots$ can be constructed so that conditions (i)–(iii) hold. Note that, because of (ii) and (iii), $\beta_n\neq\beta_k$ for $k<n$.

Now write $x_n = \{x_n(\alpha)\}$, where α traces J. If $\alpha \in J$, there exists $n \geqq 0$ (depending on α) such that $\alpha \notin J_n$, since $\bigcap_{n=0}^{\infty} J_n = \emptyset$. Since $x_k \in A_n$ for $k \geqq n$, it follows that $x_k(\alpha) = 0$ for $k \geqq n$. Thus the element $x(\alpha) = \sum_{k=0}^{\infty} x_k(\alpha) = x_0(\alpha) + x_1(\alpha) + \cdots + x_{n-1}(\alpha)$ is a well-defined element of $R^{(\alpha)}$. Set $x = \{x(\alpha)\} \in A$. This element will give us our contradiction.

Since $x_k \in a_k A$, we have that $x_k(\alpha) \in a_k R^{(\alpha)}$ for all $\alpha \in J$; hence, for any $n \geqq 0$, $x(\alpha) = x_0(\alpha) + x_1(\alpha) + \cdots + x_n(\alpha) + a_{n+1} y_n(\alpha)$ for some $y_n(\alpha) \in R^{(\alpha)}$, since $a_{n+1} R^{(\alpha)} \supseteq a_{n+2} R^{(\alpha)} \supseteq \cdots$. Setting $y_n = \{y_n(\alpha)\} \in A$, we then get that $x = x_0 + x_1 + \cdots + x_n + a_{n+1} y_n$ for all $n \geqq 0$. Therefore, by conditions (ii) and (iii), we obtain that $f_{\beta_n}(x) \equiv f_{\beta_n}(x_n) \not\equiv 0 \pmod{a_{n+1} C_{\beta_n}}$, and thus $f_{\beta_n}(x) \neq 0$ for all n. Since $\beta_n \neq \beta_k$ for all $k < n$, we have reached a contradiction to the fact that $\{f_\beta\}$ is a Φ-family of homomorphisms on A. It then follows that R satisfies the descending chain condition on principal right ideals, completing the proof of the theorem.

We mention, as an immediate corollary to the preceding theorem, the following slight improvement of the theorem of Baer mentioned in the introduction.

COROLLARY. *Let A be the direct product of a countably infinite family of copies of the ring Z of rational integers. Then A cannot be a pure subgroup of a direct sum of countably generated abelian groups. In particular, A is not free.*

Proof. Assume the statement is false. Then, letting ζ be the cardinality of Z, we see immediately that the hypotheses of Theorem 3.1 are satisfied, and so Z possesses the minimum condition on ideals, which is absurd. The conclusion follows.

Theorem 3.1 has led us to focus our attention on rings satisfying the descending chain condition on principal right ideals, which will play a central role in the remainder of our discussion. These rings have been rather thoroughly analyzed by Bass [1]. We shall have use for a part of his main theorem concerning these rings; this we shall state, without proof, after a preliminary definition.

DEFINITION 3.3. Let R be a ring, and I be a left or right ideal in R. I will be called *left T-nilpotent* if, for any sequence $a_1, a_2, \cdots$ of elements of I, there exists $n > 0$ such that $a_1 a_2 \cdots a_n = 0$ (right T-nilpotence requires that $a_n a_{n-1} \cdots a_1 = 0$ for some n).

THEOREM 3.2. *For any ring R, the following statements are equivalent*:

(a) *R satisfies the descending chain condition on principal right ideals.*

(b) *The Jacobson radical N of R is left T-nilpotent, and R/N is semisimple with minimum condition on ideals.*

(c) *Every flat left R-module is projective.*

(d) *Every right R-module contains a simple submodule.*

Proof. See [1].

A ring satisfying the conditions of Theorem 3.2 has been designated as left perfect [1]; we shall adhere to this terminology throughout this paper.

We have now accumulated enough information to characterize completely those rings for which projective modules are preserved by direct products.

THEOREM 3.3. *For any ring R, the following statements are equivalent:*

(a) *The direct product of any family of projective left R-modules is projective.*

(b) *The direct product of any family of copies of R is projective as a left R-module.*

(c) *R is left perfect, and any finitely generated right ideal in R is finitely related.*

Proof. (a)⇒(b): Trivial.

(b)⇒(c): Since a projective module is a direct summand, and hence a pure submodule, of a free module, it follows from our assumptions that the hypotheses of Theoren 3.1 are satisfied. Thus, by that theorem, R is left perfect. Furthermore, since the direct product of any family of copies of R is a projective—and hence flat—left R-module, we may apply Theorem 2.1 to conclude that every finitely generated right ideal in R is finitely related.

(c)⇒(a): Since any projective module is flat, and since every finitely generated right ideal in R is finitely related, we obtain from Theorem 2.1 that the direct product of any family of projective left R-modules is at least flat. But since R is left perfect, it follows from Theorem 3.2 that every flat left R-module is projective, and therefore the afore-mentioned direct product is projective. This completes the proof of the theorem.

REMARKS. 1. We do not know whether a ring R satisfying the conditions of Theorem 3.3 must also have the property that the direct product of any family of free left R-modules is free.

2. A ring with the properties described in Theorem 3.3 need not satisfy the minimum condition on either left or right ideals. For example, a right hereditary, left perfect ring fulfills the conditions of Theorem 3.3, but need not satisfy either minimum condition. But for commutative rings such behavior cannot occur, as is seen from the following result.

THEOREM 3.4. *Let R be a commutative ring. Then R possesses the properties described in Theorem 3.3 if and only if R satisfies the minimum condition on ideals.*

Proof. If R satisfies the minimum condition on ideals, then R is necessarily left perfect and every finitely generated right ideal in R is finitely related. Conversely, assume that R fulfills the conditions of Theorem 3.3. Let N be the Jacobson radical of R. Since R is commutative and left perfect, it follows from Theorem 3.2 that R/N is a direct sum of fields; say $R/N=K_1\oplus \cdots \oplus K_n$. Let $\bar{e}_i$ be the unit of K_i; then, since N is T-nilpotent, there exists an idempotent e_i in R which maps onto $\bar{e}_i$ (see [1]). Set $R_i=e_iRe_i$; then R_i is a local ring (i.e., the non-units in R_i form a maximal two-sided ideal) and $R = R_1 \oplus \cdots \oplus R_n$. Clearly each R_i satisfies the conditions of Theorem 3.3, and R will possess the minimum condition on ideals if and only if each R_i does. It then follows that we may assume R to be a local ring, in which case N is the unique maximal ideal in R.

Now, by Theorem 3.2, R contains a simple ideal, which is necessarily isomorphic to R/N. But then R/N, being finitely generated, must be finitely related, by Theorem 3.3; hence N is finitely generated. Then N^k/N^{k+1} is a finitely generated R-module for all $k \geqq 0$, and thus, being semi-simple, must have finite length. Let $N = (a_1, \cdots, a_r)$. Since N is T-nilpotent, there exists $s > 0$ such that $a_i^s = 0$ for all $i \leqq r$. Since R is commutative, we then get easily that $N^{rs} = 0$. It then follows immediately that R has finite length, and thus satisfies the minimum condition on ideals. This completes the proof of the theorem.

4. **Applications.** We shall now apply our results on direct products of flat modules to derive a characterization of semi-hereditary rings. First we introduce a concept due to Bass [2, p. 62 and p. 64].

DEFINITION 4.1. Let R be a ring, and A be a left R-module. A will be called *torsion-less* if it is a submodule of a direct product of copies of R.

REMARKS. 1. If A is a left R-module, it is easy to see that A is torsion-less if and only if, for any $x \in A$, there exists $f \in \mathrm{Hom}_R(A, R)$ such that $f(x) \neq 0$.

2. If R is an integral domain and A is torsion-less, then A is clearly torsion-free, and the converse is true if A is finitely generated. It should be emphasized, however, that an arbitrary torsion-free R-module need not be torsion-less. For example, the quotient field of R is torsion-free, but is never torsion-less unless R is a field.

In the following theorem we shall denote the global weak dimension of a ring R by GWD(R).

THEOREM 4.1. *For any ring R, the following statements are equivalent*:

(a) *R is left semi-hereditary.*

(b) *$GWD(R) \leqq 1$, and the direct product of an arbitrary family of copies of R is flat as a right R-module.*

(c) *Every torsion-less right R-module is flat.*

Proof. (a)⇒(b): Let R be left semi-hereditary; then every finitely generated left ideal in R, being projective, is both flat and finitely related. The first conclusion implies, via a direct limit argument, that every left ideal in R is flat and hence GWD(R) $\leqq 1$; the second conclusion implies, via Theorem 2.1, that the direct product of any family of copies of R is a flat right R-module.

(b)⇒(c): This follows immediately from Definition 4.1 and the fact that, if GWD(R) $\leqq 1$, then every submodule of a flat R-module is flat.

(c)⇒(b): If (c) holds, then every right ideal in R, being torsion-less, is flat, and so GWD(R) $\leqq 1$. For the same reason, the direct product of any family of copies of R is a flat right R-module.

(b)⇒(a): If (b) holds, then it follows immediately from Theorem 2.1 that every finitely generated left ideal in R is finitely related. But, since GWD(R) $\leqq 1$, such an ideal is flat, and hence is projective, by the corollary to Proposition 2.2. Thus R is left semi-hereditary. This completes the proof of the theorem.

REMARK. The implication (a)⇒(c) of Theorem 4.1 was proved by Bass, who applied a different method (see [2, p. 81]).

Theorem 4.1 yields easily several characterizations of Prüfer rings due to Hattori [5]. For purposes of the following discussion of this material, we recall that a Prüfer ring is a semi-hereditary integral domain; i.e., an integral domain in which every finitely generated ideal is invertible.

THEOREM 4.2. *Let R be an integral domain. Then the following conditions are equivalent*:

(a) *R is a Prüfer ring.*

(b) *If A and B are torsion-free R-modules, then $A\otimes_R B$ is torsion-free.*

(c) *If I and J are ideals in R, then $I\otimes_R J$ is a torsion-free R-module.*

(d) *GWD(R) $\leqq 1$.*

(e) *Every torsion-free R-module is flat.*

Proof. (a)⇒(b): See [3, p. 133].

(b)⇒(c): Trivial, since every ideal in R is torsion-free.

(c)⇒(d): Let I and J be ideals in R. Let $j_*: I\otimes_R J\rightarrow J$ be the homomorphism induced by the inclusion map $j: I\rightarrow R$. Let $u=a_1\otimes b_1+\cdots+a_r\otimes b_r$ be in $I\otimes_R J$, where $a_i\in I$ and $b_i\in J$; then $j_*(u)=a_1b_1+\cdots+a_rb_r$. If $\lambda\in I$, then $\lambda u=\lambda\otimes(a_1b_1+\cdots+a_rb_r)=\lambda\otimes j_*(u)$; hence, if $j_*(u)=0$, then $\lambda u=0$, from which it follows that $u=0$, since $I\otimes_R J$ is torsion-free. Therefore j_* is a monomorphism, and so $\mathrm{Tor}_1^R(R/I, J)=0$. This is true for all ideals I, J in R, and so it then follows easily that GWD(R) $\leqq 1$.

(d)⇒(e): Every finitely generated torsion-free R-module is a submodule of a free module, and is hence flat, since GWD(R) $\leqq 1$. It then follows from the standard direct limit argument that every torsion-free R-module is flat.

(e)⇒(a): By a previous remark, every torsion-less R-module is torsion-free, and is therefore flat. It then follows from Theorem 4.1 that R is semi-hereditary, and is hence a Prüfer ring. This completes the proof of the theorem.

We turn next to the module-theoretic characterization of Dedekind rings mentioned in the introduction.

DEFINITION 4.2. Let R be an integral domain, and A be a torsion R-module. A will be called a UT-module if, whenever $A\subseteq B$ and B/A is torsion-free, then A is a direct summand of B.

Kaplansky [8, p. 334] has proved that, if R is a Dedekind ring, then every R-module of bounded order is a UT-module. In the next theorem we shall prove that the converse is also true. It is curious that our argument uses, in an essential way, the results previously derived concerning direct products of projective modules.

THEOREM 4.3. *Let R be an integral domain, and assume that every R-module of bounded order is a UT-module. Then R is a Dedekind ring.*

Proof. We first show that R is a Prüfer ring, an observation due to Kaplansky. Let Z be the ring of rational integers, and C be the abelian group of rational numbers modulo the integers. Then, for any R-module A and ideal I in R, we have that

$$\mathrm{Ext}_R^1 \{A, \mathrm{Hom}_Z (R/I, C)\} \approx \mathrm{Hom}_Z \{\mathrm{Tor}_1^R (R/I, A), C\}$$

by [3, p. 119]. Let A be torsion-free; then, since $\mathrm{Hom}_Z(R/I, C)$ is an R-module of bounded order, it follows from our hypotheses and the above identity that $\mathrm{Tor}_1^R(R/I, A)=0$. This is true for any ideal I in R, and so any torsion-free R-module A is flat. We may then apply Theorem 4.2 to conclude that R is a Prüfer ring.

To complete the proof, we need only show that R is Noetherian. To this end, let I be an ideal in R, and $x\neq 0$ be an element of I. Let $R^*=R/(x)$ and I^* be the image of I in R^*. Let A be the direct product of a family of copies of R; then A/xA is the direct product of the corresponding family of copies of R^*. Let $0\rightarrow K\rightarrow F\rightarrow A\rightarrow 0$ be an exact sequence, where F is free. Since A is torsion-free, we obtain easily that the induced sequence $0\rightarrow K/xK\rightarrow F/xF \rightarrow A/xA\rightarrow 0$ of R^*-modules is also exact. Also, F/xF is a free R^*-module.

Now consider the R-module F/xK. Since $(F/xK)/(K/xK)\approx F/K\approx A$, which is torsion-free, we get easily that the torsion submodule of F/xK is precisely K/xK, which has bounded order. Hence, by hypothesis, K/xK is a direct summand of F/xK, from which it follows without difficulty that K/xK is a direct summand of F/xF. Hence A/xA is R^*-projective. We have shown that the direct product of any family of copies of R^* is a projective R^*-module. Applying Theorem 3.4, we conclude that R^* satisfies the minimum condition on ideals, and is thus certainly Noetherian. Hence I^* is finitely generated, from which we get immediately that I is finitely generated. This is true for any ideal I in R, and so R is Noetherian. This completes the proof that R is a Dedekind ring.

As a final application of our results on direct products of modules, we shall present a brief (and incomplete) treatment of the problem of Köthe mentioned in the introduction. In our discussion we shall have use for the following well-known concept.

DEFINITION 4.3. Let R be a ring, and N be a Jacobson radical of R. R will be called *semi-primary* if N is nilpotent and R/N is semi-simple with minimum condition on ideals.

REMARKS. It follows immediately from Theorem 3.2 that a semi-primary ring is both left and right perfect. Also, it is well-known that a ring satisfying the minimum condition on either left or right ideals is semi-primary.

In the proof of our main theorem, we shall have use for the following preliminary results, which were communicated to me by H. Bass.

PROPOSITION 4.1. *For any ring R, the following statements are equivalent:*

(a) *R is left Noetherian.*

(b) *The direct limit of any directed system of injective left R-modules is injective.*

(c) *The direct sum of any family of injective left R-modules is injective.*

Proof. The implication (a)$\Rightarrow$(b) is well-known (see [3, p. 17, Exercise 8]), and the implication (b)$\Rightarrow$(c) is immediate. There remains to prove that (a) follows from (c).

Assuming (c) holds, let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of left ideals in R. Let Q_n be an injective module containing R/I_n. Set $I = \bigcup_{n=1}^{\infty} I_n$ and $Q = \sum_{n=1}^{\infty} Q_n$. Define a homomorphism $f: I \rightarrow Q$ by $f(a) = \sum_{n=1}^{\infty} f_n(a)$, where $f_n: I \rightarrow Q_n$ is the homomorphism induced by the canonical mapping of I onto I/I_n. Observe that f is well-defined; for, if $a \in I$, then $a \in I_n$ for some n, in which case $f_k(a) = 0$ for all $k \geqq n$. By hypothesis, Q is injective; hence there exists a homomorphism $g: R \rightarrow Q$ such that $g/I = f$. But then $f(I) \subseteq g(R) \subseteq Q_1 \oplus \cdots \oplus Q_n$ for some n, from which it follows easily that $I = I_{n+1}$. Hence every ascending chain of left ideals in R is finite, and so R is left Noetherian. This completes the proof of the proposition.

LEMMA 4.1. *Let R be a semi-primary ring. Suppose that there exists a cardinal number ζ such that, if Q is any injective left R-module, then $Q = \sum_\beta \oplus Q_\beta$, where each Q_β is generated by a subset of cardinality less than ζ. Then R satisfies the minimum condition on left ideals.*

Proof. If A is a left R-module, we shall denote the injective envelope of A by $E(A)$ (see [10, p. 373]).

Let S be a simple left R-module. Let I be a set, and for each $\alpha \in I$ let S_α be a copy of S. Set $Q = \prod_{\alpha \in I} E(S_\alpha)$. Then Q is an injective left R-module, and so, by hypothesis, $Q = \sum_\beta \oplus Q_\beta$ (where β traces some index set J) and furthermore each Q_β is generated by a subset of cardinality less than ζ. It is then clear that, by choosing I large enough, we may so arrange matters that J has arbitrarily large cardinality.

Let $p_\alpha: Q \rightarrow E(S_\alpha)$ be the projection of Q onto $E(S_\alpha)$. If S' is a simple submodule of Q, then, selecting $\alpha \in I$ such that $p_\alpha(S') \neq 0$, we have that $S' \approx p_\alpha(S') = S_\alpha \approx S$, since S' and S_α are simple and $E(S_\alpha)$ is an essential extension of S_α. Thus every simple submodule of Q is isomorphic to S.

Now let T_β be a simple submodule of Q_β; that such exists follows from Theorem 3.2, or may be easily seen directly. We have just observed that

$T_\beta \approx S$. Since Q_β is a direct summand of Q and is hence injective, we may write $Q_\beta = E(T_\beta) \oplus C_\beta$. Then $\sum_{\beta\in J} E(T_\beta)$ is a direct summand of Q, and is therefore injective. Since $E(T_\beta) \approx E(S)$ for each $\beta \in J$, and since J may be made arbitrarily large by suitable choice of I, it then follows that the direct sum of any family of copies of $E(S)$ is injective.

Let N be the Jacobson radical of R. Since R/N is semi-simple with minimum condition on ideals, we may write $R/N = S_1 \oplus \cdots \oplus S_n$, where S_i is a simple left R-module. Every simple left R-module is isomorphic to S_i for some $i \leq n$. Hence, if A is a semi-simple left R-module, we have that $A = \sum_{i=1}^{n} (\sum_{\gamma_i} S_{\gamma_i})$, where, for fixed i, γ_i traces some index set J_i and $S_{\gamma_i} \approx S_i$. Then

$$E(A) = \sum_{i=1}^{n} E\left(\sum_{\gamma i} S_{\gamma_i} \right) = \sum_{i=1}^{n} \left(\sum_{\gamma i} E(S_{\gamma_i}) \right).$$

From this it follows easily that, if $A = \sum_\alpha A_\alpha$, then $E(A) = \sum_\alpha E(A_\alpha)$.

Now, if B is an arbitrary left R-module, let $s(B)$ denote the join of all simple submodules of B; $s(B)$ is usually referred to as the *socle* of B [10, p. 373]. Since R is semi-primary, it is easily seen that B is an essential extension of $s(B)$, and so $E(B) = E(s(B))$ [10, p. 373].

Finally, let $\{Q_\alpha\}$ be a family of injective left R-modules, and set $Q = \sum_\alpha Q_\alpha$. Then $E(s(Q_\alpha)) = Q_\alpha$, since Q_α is injective. It then follows from our previous discussion that $Q = \sum_\alpha E(s(Q_\alpha)) = E(\sum_\alpha s(Q_\alpha))$, and is hence injective. That is, the direct sum of an arbitrary family of injective left R-modules is again injective. We may then apply Proposition 4.1 to conclude that R is left Noetherian. Since R is semi-primary, it then follows easily that R satisfies the minimum condition on left ideals. This completes the proof of the lemma.

In the final theorem of this paper we summarize the rather meager information we are able to contribute concerning the aforementioned problem of Köthe.

THEOREM 4.4. *Let R be a ring with the property that every left R-module is a direct sum of finitely generated modules. Then R satisfies the minimum condition on left ideals, and every indecomposable injective left R-module has finite length.*

Proof. It is clear that R satisfies the hypotheses of Theorem 3.1. Hence, by that theorem, R is at least left perfect. Let N be the Jacobson radical of R; we recall that, by Theorem 3.2, N is left T-nilpotent and $S = R/N$ is semi-simple with minimum condition on ideals.

Let Q be the injective envelope of S, where the latter is viewed as a left R-module. For each $k \geq 0$, define $Q_k = \{x \in Q / N^k x = 0\}$; Q_k is a submodule of Q, and $S = Q_1 \subseteq Q_2 \subseteq \cdots$. Also, $Q_{k+1}/Q_k \approx \mathrm{Hom}_R(N^k/N^{k+1}, Q) \approx \mathrm{Hom}_R(N^k/N^{k+1}, S)$ [10, p. 373].

Let Q' be any submodule of Q. By hypothesis, Q' is a direct sum of finitely generated modules. But Q' is an essential extension of $Q' \cap S$, which is a semi-simple module of finite length. It then follows easily that Q' is the direct sum of a finite number of finitely generated modules, and hence is itself finitely generated. Thus every submodule of Q is finitely generated, and so Q satisfies the maximum condition on submodules. In particular, there exists $n > 0$ such that $Q_n = Q_{n+1} = \cdots$. Then $\mathrm{Hom}_S(N^n/N^{n+1}, S) \approx Q_{n+1}/Q_n = 0$, and so $N^n = N^{n+1}$. Since N is left T-nilpotent, it then follows that $N^n = 0$ (see [1]). Hence R is semi-primary. It is then clear that R satisfies the hypotheses of Lemma 4.1, which we apply to conclude that R satisfies the minimum condition on left ideals.

Since Q is finitely generated and R satisfies the minimum condition on left ideals, it follows that Q has finite length. It is easy to see that every indecomposable injective left R-module is isomorphic to a direct summand of Q, and hence has finite length. This completes the proof of the theorem.

It is not true that every ring with minimum condition on left ideals satisfies the conditions of Theorem 4.4. For further information concerning this situation, we refer the reader to [10].

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