A FLUX INTEGRAL THEOREM FOR FUNCTIONS WHICH HAVE HARMONIC SUPPORT(1)

BY

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1. **Introduction.** This paper deals with logarithmic potential functions defined in plane domains and with the problem of the computation of mass distributions by means of the Gauss flux integral formula. For potentials which have harmonic support a formulation is possible wherein the flux integral is taken over boundaries consisting of a finite set of rectifiable Jordan curves. As defined in [10] a real valued function u is said to have harmonic support in a domain D if and only if u is subuniformly bounded(2) in D and has at each point of D a support function which is harmonic and single valued in D. The latter statement is understood to mean that for each point ζ in D there is a harmonic function h such that $h(\zeta)=u(\zeta)$ and $h\leqq u$ in D. Denote by (hs) the class of such functions for a domain D.

The result may be considered as an extension, for the class of potentials under consideration, of the formulations by Evans [7] and F. Riesz [18] in the following sense. Evans described a generalized flux integral valid on curves which support no mass with a subsequent determination of the mass on a general region as a limit of integrals on the boundaries of a sequence of expanding regions. Riesz' method for the class of subharmonic functions is to obtain the mass as a limit of flux integrals of best harmonic majorants. The present result allows a direct evaluation of a flux integral on an arbitrary rectifiable boundary without requiring the use of approximating regions.

To obtain such a theorem one is first confronted with the possible lack of a normal derivative in the flux integral. There are (hs)-functions for which the normal derivative fails to exist on quite simple curves, for example $u=\sup(1, |z|)$ on $|z|=1$. On the other hand, as is shown in §2, forward and backward directional derivatives do exist (Theorem 1) and are subuniformly bounded (Theorem 2). It is these derivatives which will appear in the flux integrals. Consequently, much of the preliminary work ofthis paper concerns their properties.

To visualize these properties it is helpful to make comparisons with the one dimensional analogue of the class (hs), namely the functions of one real

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(2) With G. MacLane we use the phrase "subuniformly . . . in D" instead of "uniformly . . . on each compact subset of D."

variable which have linear support. These are, to be sure, just convex functions which also serve as the one dimensional analogue of the class of subharmonic functions which in turn contains (hs) as a proper subclass. The differentiability properties appear to be associated with the presence of supports rather than with the property of dominance by chords.

It is convenient and perhaps revealing to enlarge the class of functions under consideration temporarily in §3 to include all those having forward derivatives which are subuniformly bounded. The concept of cone differentiability is introduced as a generalization of Stolz-Fréchet differentiability. The functions of the enlarged class have this property (Theorem 3) and a chain rule is valid (Theorem 4).

The chain rule facilitates the derivation in §4 of a sharp inequality for the forward derivatives of an (hs)-function (Theorem 5). For subsequent application two results are obtained in §5 concerning the limits of the derivatives on a sequence of points (Lemmas 2, 3). An immediate consequence is the Lebesgue integrability of the forward normal derivative on a smooth curve (Lemma 4). An example illustrates the inapplicability of Riemann integration.

Another point of comparison with convexity in one dimension arises in §6 where it is shown that the forward tangential derivative on a smooth curve is continuous from the front and its limit from the rear is the backward tangential derivative (Theorem 6). The right derivative of a convex function is continuous from the right and its limit from the left is the left derivative. Theorem 6 is instrumental in proving that the forward normal derivative is integrable on a rectifiable Jordan curve (Theorem 7) and the case can then be made to justify calling its integral a flux integral.

Let Δ be domain having a rectifiable boundary Γ such that $\Delta \cup \Gamma$ is contained in a domain where $u \in (hs)$. Let $\partial/\partial n$ denote differentiation normal to Γ and to the exterior of Δ and $\partial/\partial n_+$, $\partial/\partial n_-$ the corresponding forward, backward derivatives. Denoting by $-\mu$ the mass distribution for u, it is shown in § 7 that

$$\int_\Gamma \frac{\partial u}{\partial n_-} ds = 2\pi\mu(\Delta) \quad \text{and} \quad \int_\Gamma \frac{\partial u}{\partial n_+} ds = 2\pi\mu(\Delta \cup \Gamma)$$

(Theorem 8). This result permits a general statement of the limit properties of the flux integrals on families of curves (Theorem 9).

The integral theorem is used in §8 to give a concise proof of and to sharpen an inequality for the mass distribution (Theorem 10). Theorem 8 also leads easily to a formula for the distribution of mass on a rectifiable curve which is related to a formula of Poisson for the density of a linear distribution (Theorem 11).

2. **Forward and backward directional derivatives.** We shall refer to the direction of a ray whose argument is α as the direction α. In the definition to

follow u is a real or complex valued function defined in a domain which contains ζ.

DEFINITION 1. The forward directional derivative of u in the direction α at ζ is

$$\frac{\partial u(\zeta)}{\partial s_+(\alpha)} = \lim_{s\searrow 0} \frac{u(\zeta + se^{i\alpha}) - u(\zeta)}{s}$$

and the backward directional derivative is

$$\frac{\partial u(\zeta)}{\partial s_-(\alpha)} = \lim_{s\nearrow 0} \frac{u(\zeta + se^{i\alpha}) - u(\zeta)}{s}.$$

Notice that the backward derivative in the direction α is the negative of the forward derivative in the direction $\alpha+\pi$ so that the dual terminology could be dispensed with. However subsequent formulation of results is more convenient if it is retained. If $\partial u/\partial s_- = \partial u/\partial s_+$ then u has a derivative in the direction α and the customary notation $\partial u/\partial s$ will be used.

An almost indispensible tool in the derivations to follow is the notion of continuous convergence introduced by Carathéodory (see [5, pp. 173–179]). A sequence of functions $\{u_n\}$ defined on a set A is continuously convergent with respect to A at a limit point ζ of A if and only if $\{u_n(z_n)\}$ is convergent for each sequence $\{z_n\} \subset A \ni z_n \to \zeta$. Continuous convergence at each point of a compact set is equivalent to uniform convergence to a continuous limit function on that set.

THEOREM 1. *If $u \in (hs)$ in a domain D then u has a forward and a backward directional derivative at each point ζ of D and in each direction α. For each ζ and α there exist harmonic support functions h_1 and h_2 at ζ such that*

$$\frac{\partial u(\zeta)}{\partial s_-(\alpha)} = \frac{\partial h_1(\zeta)}{\partial s(\alpha)} \quad \text{and} \quad \frac{\partial u(\zeta)}{\partial s_+(\alpha)} = \frac{\partial h_2(\zeta)}{\partial s(\alpha)}.$$

If h is any support at ζ then

$$\frac{\partial u(\zeta)}{\partial s_-(\alpha)} \leqq \frac{\partial h(\zeta)}{\partial s(\alpha)} \leqq \frac{\partial u(\zeta)}{\partial s_+(\alpha)}.$$

Proof. Let $\mathcal{H}$ denote the family of support functions h at ζ. Let $\mathcal{H}_\alpha$ denote the family of derivatives $\{\partial h/\partial s\}$ in the fixed direction α in D. The family $\mathcal{H}$ is subuniformly bounded (above and below), normal and compact in D. The last property means that the limit of a sequence of supports at ζ is necessarily a support at ζ. The functions of the family $\mathcal{H}_\alpha$ are harmonic and it follows that the family is subuniformly bounded, normal and compact in D. Hence $\exists h_1, h_2 \in \mathcal{H} \ni$

$$\frac{\partial h_1(\zeta)}{\partial s} = \inf_{h\in\mathfrak{H}} \frac{\partial h(\zeta)}{\partial s} \quad\text{and}\quad \frac{\partial h_2(\zeta)}{\partial s} = \sup_{h\in\mathfrak{H}} \frac{\partial h(\zeta)}{\partial s}\cdot$$

We proceed to show that $\partial u(\zeta)/\partial s_+$ exists and is equal to $\partial h_2(\zeta)/\partial s$. Choose any sequence $z_n=\zeta+s_ne^{i\alpha}$ such that $s_n\searrow 0$. Let h_n denote a support at z_n, $n=1, 2, \cdots$. The sequence $\{h_n\}$ is normal and subuniformly bounded and consequently contains a subuniformly convergent subsequence $\{h_{n_i}\}$ with limit h. Continuous convergence of $\{h_{n_i}\}$ at ζ and continuity of u allow one to see readily that h is necessarily a support at ζ.

Now on the one hand

$$\frac{u(z_{n_i}) - u(\zeta)}{s_{n_i}} \leqq \frac{h_{n_i}(z_{n_i}) - h_{n_i}(\zeta)}{s_n} \tag{2.1}$$

and on the other hand

$$\frac{h_{n_i}(z_{n_i}) - h_{n_i}(\zeta)}{s_{n_i}} = \frac{\partial h_{n_i}(\zeta_i)}{\partial s}$$

where $\zeta_i=t\zeta+(1-t)z_{n_i}$, $0<t<1$. The sequence $\{\partial h_{n_i}/\partial s\}$ converges subuniformly to $\partial h/\partial s$ in D and hence is continuously convergent at ζ. Thus $\partial h_{n_i}(\zeta_i)/\partial s\rightarrow\partial h(\zeta)/\partial s$. The fact that h is a support at ζ implies $\partial h(\zeta)/\partial s \leqq\partial h_2(\zeta)/\partial s$ and taking the limit in (2.1) we find

$$\limsup_{i\to\infty} \frac{u(z_{n_i}) - u(\zeta)}{s_{n_i}} \leqq \frac{\partial h_2(\zeta)}{\partial s}\cdot$$

Since any sequence $\{z_n\}$ contains a subsequence for which the last inequality is valid we conclude that

$$\limsup_{s\searrow 0} \frac{u(\zeta + se^{i\alpha}) - u(\zeta)}{s} \leqq \frac{\partial h_2(\zeta)}{\partial s}\cdot \tag{2.2}$$

To obtain an inequality for the limit infimum we observe that $[h_2(\zeta+se^{i\alpha})-h_2(\zeta)]/s\leqq[u(\zeta+se^{i\alpha})-u(\zeta)]/s$ in view of the fact that h_2 is a support at ζ. Thus

$$\frac{\partial h_2(\zeta)}{\partial s} \leqq \liminf_{s\searrow 0} \frac{u(\zeta + se^{i\alpha}) - u(\zeta)}{s}\cdot \tag{2.3}$$

Combining (2.2) and (2.3) we find

$$\frac{\partial u(\zeta)}{\partial s_+} = \lim_{s\searrow 0} \frac{u(\zeta + se^{i\alpha}) - u(\zeta)}{s} = \frac{\partial h_2(\zeta)}{\partial s}\cdot$$

The equality $\partial u(\zeta)/\partial s_-=\partial h_1(\zeta)/\partial s$ follows directly from the above result.

$$\frac{\partial u(\zeta)}{\partial s_-(\alpha)} = -\frac{\partial u(\zeta)}{\partial s_+(\alpha+\pi)} = -\sup_{h\in\mathcal{H}} \frac{\partial h(\zeta)}{\partial s(\alpha+\pi)} = \inf_{h\in\mathcal{H}}\left(-\frac{\partial h(\zeta)}{\partial s(\alpha+\pi)}\right)$$
$$= \inf_{h\in\mathcal{H}} \frac{\partial h(\zeta)}{\partial s(\alpha)} = \frac{\partial h_1(\zeta)}{\partial s(\alpha)}\cdot$$

Proof complete.

The boundedness of the derivatives to be proved next is essential in much of the subsequent development.

THEOREM 2. *If $u\in(hs)$ in a domain D then its forward (backward) derivatives are subuniformly bounded in D.*

Proof. Let A be a compact subset of D and denote by $\mathcal{H}$ the family of all support functions of u which have a point of support on the set A. $\mathcal{H}$ is subuniformly bounded (above and below) in D. Let $\mathcal{H}'$ denote the family of derivatives of all functions of $\mathcal{H}$ in each direction α. Then $\mathcal{H}'$ is a subuniformly bounded family of harmonic functions and is thus bounded on A. But taking account of Theorem 1 any derivative $\partial u/\partial s_+$ at a point of A must be equal to a function of the set $\mathcal{H}'$ at that point. Hence the derivatives of u are uniformly bounded on A. Proof complete.

For an application later we state this theorem in a more general form.

THEOREM 2'. *If $\mathcal{U}$ is a subuniformly bounded family of functions of the class (hs) in a domain D then the forward (backward) derivatives of the functions of $\mathcal{U}$ are subuniformly bounded in D.*

3. **Cone differentiability.** For a function having a Stolz-Fréchet total derivative (see [11, p. 6]) the directional derivative may be computed by allowing the variable point in the difference quotient to move on a curve having a tangent in the desired direction. The corresponding result is valid for the forward derivative of an (hs)-function and although the result could be obtained directly it happens to be quite natural to consider a larger setting.

Let (fd) denote the class of complex valued functions which have a forward derivative at each point of D and in each direction α. For this class of functions it is meaningful to introduce an extension of the idea of differentiability.

DEFINITION 2. A function u is cone differentiable at ζ if and only if there is a neighborhood of ζ such that

$$\Delta u = u(z) - u(\zeta) = \rho q(\alpha) + \rho\epsilon(\rho, \alpha)$$

where $z=\zeta+\rho e^{i\alpha}$, $0\leqq\rho$, $q(\alpha)$ is a continuous function with period 2π, and $\epsilon(\rho, \alpha)$ is continuous and zero at $\rho=0$.

If u is Stolz-Fréchet differentiable then $q(\alpha)=a\cos\alpha+b\sin\alpha$ and if analytic then $q(\alpha)=ce^{i\alpha}$. If u is cone differentiable at ζ then it is continuous at ζ, has a forward derivative in each direction α at ζ and this derivative may be computed from $\Delta u/\rho$ by allowing z to vary on any curve which has the ray in the direction α as half tangent at ζ, i.e., $\arg(z-\zeta)\to\alpha$ as $z\to\zeta$.

A function $u\in(fd)$ may fail to be cone differentiable at some points of D, however this cannot occur if the derivatives are subuniformly bounded in D. To facilitate the proof of this result we state a "mean value" inequality.

LEMMA 1. *If $y(x)$ is real valued and continuous in the interval $a\leqq x\leqq b$ and has right and left derivatives at each point of the interval $a<x<b$, then there exists c such that $a<c<b$ and*

$$y'_-(c)\leqq\frac{y(b)-y(a)}{b-a}\leqq y'_+(c).$$

or

$$y'_+(c)\leqq\frac{y(b)-y(a)}{b-a}\leqq y'_-(c).$$

A proof may be patterned on the proof of the mean value theorem.

THEOREM 3. *If $u\in(fd)$ in a domain D and its derivatives are subuniformly bounded in D then u is cone differentiable in D.*

Proof. Let $\zeta\in D$ and define $q(\alpha)=\partial u/\partial s_+$ at ζ for each direction α. Define $\epsilon(\rho,\alpha)=\Delta u/\rho-q(\alpha)$ where $z=\zeta+\rho e^{i\alpha}$, $\Delta u=u(z)-u(\zeta)$ and $z\in N(\zeta)$, a neighborhood of ζ whose closure is in D. Choose any sequence of points $(\rho_n,\alpha_n)\ni\epsilon(\rho_n,\alpha_n)\to L$ and $\rho_n\to 0$. If we can show that L is necessarily zero then $\epsilon(\rho,\alpha)$ is continuous at $\rho=0$. We may suppose without loss of generality that $\alpha_n\to\alpha'$.

Denoting $z_n=\zeta+\rho_n e^{i\alpha_n}$ and $z'_n=\zeta+\rho_n e^{i\alpha'}$, form the increments $\Delta u_n=u(z_n)-u(\zeta)$, $\Delta u'_n=u(z'_n)-u(\zeta)$ and $\delta u_n=u(z_n)-u(z'_n)$. Then $\delta u_n=\Delta u_n-\Delta u'_n$ and the difference quotient

$$\begin{aligned}\frac{\delta u_n}{|z_n-z'_n|}&=\frac{\delta u_n}{2\rho_n\sin(|\alpha_n-\alpha'|/2)}\\&=\frac{1}{2\sin(|\alpha_n-\alpha'|/2)}\left(\frac{\Delta u_n}{\rho_n}-\frac{\Delta u'_n}{\rho_n}\right).\end{aligned}\tag{3.1}$$

Now in view of the hypothesis we have $|\partial u/\partial s_+|<M$ on $N(\zeta)$ so that applying Lemma 1 we conclude that any difference quotient of u is bounded by $2^{1/2}M$. Noting that the existence of $\partial u/\partial s_+$ implies $\lim_n\Delta u'_n/\rho_n=q(\alpha')$ we may multiply (3.1) by $\sin(|\alpha_n-\alpha'|/2)$ and take the limit to find that

$$\lim_{n\to\infty}\frac{\Delta u_n}{\rho_n}=q(\alpha').$$

Recalling now that $q(\alpha_n)=\Delta u_n/\rho_n-\epsilon(\rho_n,\alpha_n)$ we see that $\lim_n q(\alpha_n)=q(\alpha')-L$.

The existence of $\partial u/\partial s_+$ in the direction α_n implies $\epsilon(\rho, \alpha_n) \to 0$ as $\rho \to 0$. For each α_n select $\rho_n' < \rho_n$ such that $|\epsilon(\rho_n', \alpha_n)| < 1/n$. We thus obtain a sequence of points $(\rho_n', \alpha_n) \ni \lim_n \epsilon(\rho_n'. \alpha_n) = 0$, $\rho_n' \to 0$ and $\alpha_n \to \alpha'$. As in the previous paragraph we must conclude that $\lim_n q(\alpha_n) = q(\alpha')$. Hence $L = 0$.

Since any sequence of α's contains a subsequence as above, the continuity of q has also been proven in the above argument. Proof complete.

Thus, in particular, (hs)-functions are cone differentiable. The support functions h are Soltz-Fréchet differentiable so that the corresponding functions $q_n(\alpha)$ at a point ζ have the form $a(h) \cos \alpha + b(h) \sin \alpha$. In view of the results of Theorem 1, the functions $q_u(\alpha)$ at ζ has the form $\sup_h(a(h) \cos \alpha + b(h) \sin \alpha)$. Thus $q_u(\alpha)$ has the same form as the Minkowski function of support of a closed convex set and the Phragmén-Lindelöf indicator function for an entire analytic function of exponential type (see [13, p. 585]). Restricted to an interval of length less than π these are examples of a generalized convexity introduced by Beckenbach [2] and developed by Peixoto [12], Bonsall [3] and others. Recently Armijo has obtained criteria for convexity with respect to a pair of functions.

Something can be said with regard to the occurrence of "conical points" as contrasted to those at which $q(\alpha) = a \cos \alpha + b \sin \alpha$. A theorem of Rademacher [16] asserts that if $u(z)$ is measurable in a domain D and $\limsup_{\rho \to 0} |\Delta u/\rho|$ is finite at each point then $u(z)$ has a total differential of Stolz-Fréchet almost everywhere in D. Thus a function cone differentiable in D can have "conical points" at most on a set of measure zero. For the case of convex functions Valiron [19] made a further subclassification which corresponds to the edge points and vertices of a polyhedron. He showed that the "convex points" of a convex function are countable. It seems likely, that some such result may extend to cone differentiable functions or at least (hs)-functions. As will be seen subsequently (Theorem 11) such a result would yield information regarding the mass distribution for the latter class of functions. A useful property of cone differentiability is the validity of a chain rule.

THEOREM 4. *Suppose $z(w)$ is cone differentiable at $w = \tau$ and $u(z)$ is cone differentiable at $\zeta = z(\tau)$. Then $v(w) = u[z(w)]$ is cone differentiable at τ. If $\partial z(\tau)/\partial s_+(\beta) = 0$ then $\partial v(\tau)/\partial s_+(\beta) = 0$. If $\partial z(\tau)/\partial s_+(\beta) \neq 0$ then*

$$\frac{\partial v(\tau)}{\partial s_+(\beta)} = \frac{\partial u(\zeta)}{\partial s_+(\alpha)} \left| \frac{\partial z(\tau)}{\partial s_+(\beta)} \right|$$

where $\alpha = \arg \partial z(\tau)/\partial s_+(\beta)$.

Proof. Write $w = \tau + re^{i\beta}$, $0 \leq r$, then $\Delta z = z(w) - \zeta = rq_z(\beta) + r\epsilon_z(r, \beta)$ where $q_z(\beta) = \partial z(\tau)/\partial s_+(\beta)$ is continuous and ϵ_z is continuous and zero at $r = 0$. Denoting $z(w) = \zeta + \rho e^{i\phi}$, $0 \leq \rho$, we have

$$\lim_{r \to 0} \frac{\rho}{r} = \lim_{r \to 0} \frac{|\Delta z|}{r} = \left| \frac{\partial z(\tau)}{\partial s_+(\beta)} \right| \tag{3.2}$$

where the convergence is uniform with respect to β. If $\partial z(\tau)/\partial s_+(\beta)\neq 0$ then

$$\lim_{r\to 0}\phi = \lim_{r\to 0}\arg\frac{\Delta z}{r} = \arg\frac{\partial z(\tau)}{\partial s_+(\beta)} = \alpha$$

but the convergence will not always be uniform with respect to β. Let A denote the closed set $\{\beta \mid \partial z(\tau)/\partial s_+(\beta)=0\}$. Then the convergence is uniform on any closed subset of the complement of A but $\lim \phi$ may fail to exist at points of A. Since u is cone differentiable we may write

$$(3.3)\qquad \Delta v = v(w) - v(\tau) = u[z(w)] - u(\zeta) = \rho q_u(\phi) + \rho\epsilon_u(\rho, \phi)$$

so that

$$(3.4)\qquad \frac{\partial v(\tau)}{\partial s_+(\beta)} = \lim_{r\to 0}\frac{\Delta v}{r} = \left(\lim_{r\to 0}\frac{\rho}{r}\right)q_u(\alpha) = \left|\frac{\partial z(\tau)}{\partial s_+(\beta)}\right|\frac{\partial u(\zeta)}{\partial s_+(\alpha)}$$

for any β in the complement of A and uniformly on closed subsets. At points of A it is clear because of the boundedness of $q_u(\phi)$ that $\partial v(\tau)/\partial s_+(\beta)=0$. Thus the values of the forward derivatives of v are those asserted in the statement of the theorem. To complete the proof of cone differentiability the uniform convergence of (3.4) for all β, taking the right side to be zero on A, must be demonstrated.

We may suppose $|q_u(\phi)+\epsilon_u(\rho, \phi)| < M$ for $0\leqq \rho<\rho_1$. Given $\epsilon>0$, there exists an open set B containing A and a number r_1 such that $\rho/r<\epsilon/2M$ for $\beta\in B$ and $0<r<r_1$. This is a consequence of the continuity of $\partial z(\tau)/\partial s_+(\beta)$ and the uniform convergence of (3.2). There is a number r_2 such that $0\leqq\rho<\rho_1$ for $0<r<r_2$. Using these estimates in (3.3) we find $|\Delta v/r| <\epsilon/2$ for $\beta\in B$ and $0<r<\min(r_1, r_2)$. It follows that $|\partial v(\tau)/\partial s_+(\beta)| \leqq \epsilon/2$ and $|\partial v(\tau)/\partial s_+(\beta) - \Delta v/r| <\epsilon$ for $\beta\in B$.

The same inequality is valid in the complement of B for a suitable interval $0<r<r_3$ because of the uniform convergence of (3.4). Thus defining $q_v(\beta) =\partial v(\tau)/\partial s_+(\beta)$ we have $|q_v(\beta)-\Delta v/r| <\epsilon$ for $0<r<\min(r_1, r_2, r_3)$ and all β. Hence $\Delta v = rq_v(\beta)+r\epsilon_v(r, \beta)$ where $\epsilon_v(r, \beta)$ is continuous and zero at $r=0$. It is clear that $q_v(\beta)$ is continuous on A and on the complement of A. Continuity at the boundary points of A results from the estimate of the previous paragraph. Proof complete.

In the particular case of an analytic function $z(w)$ we have the relation $\partial z(\tau)/\partial s(\beta) = e^{i\beta}z'(\tau)$ and the formula of Theorem 4 may be stated

$$(3.5)\qquad \frac{\partial v(\tau)}{\partial s_+(\beta)} = \frac{\partial u(\zeta)}{\partial s_+(\alpha)}\,|z'(\tau)|, \qquad \alpha = \beta + \arg z'(\tau)$$

where $\partial v(\tau)/\partial s_+(\beta)=0$ if $z'(\tau)=0$.

4. **A specific bound for the derivative.** As an application of (3.5) an inequality will be derived for the derivatives of an (hs)-function which will it-

self find application in §8. Use will be made of the Harnack inequality for nonpositive (hs)-functions in $|z|<1$ (see [10, Theorem 2])

$$u(0)\frac{1+|z|}{1-|z|} \leqq u(z) \leqq u(0)\frac{1-|z|}{1+|z|} \tag{4.1}$$

and of the fact that an (hs)-function of a holomorphic function is an (hs)-function.

Theorem 5. *If $u\in(hs)$ in $|z|<1$ and $u\leqq 0$, then*

$$\left|\frac{\partial u(z)}{\partial s_+}\right| \leqq -\frac{2u(z)}{1-|z|^2},$$

for $|z|<1$ and differentiation in each direction α.

Proof. Suppose $|\zeta|<1$ and consider $v(w)=u[(w+\zeta)/(1+\bar{\zeta}w)]$. $v\in(hs)$ in $|w|<1$ and $v\leqq 0$ so that v satisfies (4.1). Thus

$$\frac{2v(0)}{1-|w|} \leqq \frac{v(w)-v(0)}{|w|} \leqq -\frac{2v(0)}{1+|w|}$$

and letting $w\to 0$ along the ray $\arg w=\alpha$ we find

$$2v(0) \leqq \frac{\partial v(0)}{\partial s_+(\alpha)} \leqq -2v(0). \tag{4.2}$$

Noting that $z'(0)=1-|\zeta|^2$ and $\arg z'(0)=0$ application of (3.5) yields

$$\frac{\partial v(0)}{\partial s_+(\alpha)} = \frac{\partial u(\zeta)}{\partial s_+(\alpha)}(1-|\zeta|^2).$$

Substituting this in (4.2) making the replacement $v(0)=u(\zeta)$ we obtain

$$\frac{2u(\zeta)}{1-|\zeta|^2} \leqq \frac{\partial u(\zeta)}{\partial s_+(\alpha)} \leqq -\frac{2u(\zeta)}{1-|\zeta|^2}.$$

Proof complete.

In view of the occurrence of equality in the Harnack inequality for certain harmonic functions, the same is true of Theorem 5. The result is thus sharp. For a function u with upper bound 1 the inequality takes the form

$$\left|\frac{\partial u(z)}{\partial s_+}\right| \leqq \frac{2[1-u(z)]}{1-|z|^2}. \tag{4.3}$$

For a comparison one may consider the function $|f|$ where f is analytic in $|z|<1$ and is bounded by 1. $|f|\in(hs)$ in $|z|<1$ but satisfies a somewhat more restrictive inequality than (4.3).

$$\left|\frac{\partial|f(z)|}{\partial s_+}\right| \leqq |f'(z)| \leqq \frac{1-|f(z)|^2}{1-|z|^2} = \frac{[1+|f(z)|][1-|f(z)|]}{1-|z|^2}.$$

One then compares the factors 2 and $1+|f(z)|$.

5. **Integrability of the forward normal derivative on a smooth curve.** Two lemmas prepare the way for the result of this section. They will also find later application.

LEMMA 2. *Suppose $\{h_n\}$ is a subuniformly convergent sequence of single valued functions harmonic in D having limit h. Suppose $\{z_n\} \subset D$ and $z_n \to \zeta$, $\zeta \in D$. Let α_n denote a sequence of angles having limit α. Then*

$$\lim_{n\to\infty} \frac{\partial h_n(z_n)}{\partial s(\alpha_n)} = \frac{\partial h(\zeta)}{\partial s(\alpha)}.$$

Proof. $\partial h_n/\partial x \to \partial h/\partial x$ and $\partial h_n/\partial y \to \partial h/\partial y$ subuniformly in D and are thus continuously convergent at ζ. Therefore

$$\frac{\partial h_n(z_n)}{\partial s(\alpha_n)} = \frac{\partial h_n(z_n)}{\partial x}\cos\alpha_n + \frac{\partial h_n(z_n)}{\partial y}\sin\alpha_n$$

$$\to \frac{\partial h(\zeta)}{\partial x}\cos\alpha + \frac{\partial h(\zeta)}{\partial y}\sin\alpha = \frac{\partial h(\zeta)}{\partial s(\alpha)}.$$

Proof complete.

LEMMA 3. *Suppose $\{z_n\} \subset D$ and $z_n \to \zeta$, $\zeta \in D$. Let $\{\alpha_n\}$ denote a sequence of angles having limit α. If $u \in (hs)$ in D then*

$$\frac{\partial u(\zeta)}{\partial s_-(\alpha)} \leq \liminf_{n\to\infty} \frac{\partial u(z_n)}{\partial s_-(\alpha_n)} \leq \limsup_{n\to\infty} \frac{\partial u(z_n)}{\partial s_+(\alpha_n)} \leqq \frac{\partial u(\zeta)}{\partial s_+(\alpha)}.$$

Proof. Let h_n denote a support function at $z_n \ni \partial u(z_n)/\partial s_+(\alpha_n) = \partial h_n(z_n)/\partial s(\alpha_n)$. The sequence $\{h_n\}$ is normal and subuniformly bounded in D. Any convergent subsequence is a support at ζ. Select an arbitrary subsequence of $\{z_n\}$ and from the corresponding subsequence of $\{h_n\}$ choose a convergent subsequence $\{h_{n_i}\}$ having limit h. Utilizing first Lemma 2 and then Theorem 1

$$\frac{\partial u(z_{n_i})}{\partial s_+(\alpha_{n_i})} = \frac{\partial h_{n_i}(z_{n_i})}{\partial s(\alpha_{n_i})} \to \frac{\partial h(\zeta)}{\partial s(\alpha)} \leqq \frac{\partial u(\zeta)}{\partial s_+(\alpha)}.$$

Since an arbitrary subsequence of $\{\partial u(z_n)/\partial s_+(\alpha_n)\}$ contains a convergent subsequence satisfying the above inequality, it follows that the inequality on the right in the lemma is valid.

The inequality on the left is obtained by writing the backward derivative in the direction α_n as the negative of the forward derivative in the direction

$\alpha_n+\pi$. The middle inequality is a simple consequence of Theorem 1. Proof complete.

A curve γ will be called smooth if it has a parametric representation $z(t)$, $a\leqq t\leqq b$, where $z'(t)$ is continuous and nonzero on the closed interval. Taking the orientation of γ according to increasing t we will assume for definiteness that the normal direction is that of the direction of the curve less $\pi/2$, i.e., $\arg z'(t)-\pi/2$. Forward and backward differentiation normal to γ will then be denoted by $\partial/\partial n_+$ and $\partial/\partial n_-$.

LEMMA 4. *Suppose $u\in(hs)$ in a domain D and γ is a smooth curve in D. Then $\partial u/\partial n_+$ and $\partial u/\partial n_-$ are Lebesgue integrable on γ.*

Proof. Since the normal direction is a continuous function on γ it follows from Lemma 3 that $\partial u/\partial s_+$ is upper semicontinuous and $\partial u/\partial n_-$ is lower semicontinuous on γ. According to Theorem 2 the functions are bounded on γ so that $\int_\gamma \partial u/\partial n_+ ds$ and $\int_\gamma \partial u/\partial n_- ds$ both exist in the Lebesgue sense. Proof complete.

To illustrate the insufficiency of the Riemann integral we construct a function $u\in(hs)$ such that $\partial u/\partial n_-$ has a set of discontinuities of positive measure on $|z|=1$. The function $\partial u/\partial n_-$ will be similar to the characteristic function of the complement of a Cantor type set of positive measure. Determine a sequence of arcs on $|z|=1$ as follows. Locate on the circle first an arc of length $\pi/2$ and then one of length $\pi/2^2$ in the center of the complement of the first arc. Now locate two arcs of length $\pi/2^4$ in the centers of the arcs complementary to the first two arcs. At the nth stage locate 2^{n-2} arcs of length $\pi/2^{2n-2}$ in the centers of the arcs complementary to all preceding arcs. Let A denote the union of the interiors of these arcs for all n. Then the measure of A on $|z|=1$ is π. Denoting the complement of A relative to the unit circle as B then the measure of B is π. Let u be the supremum of the function $h\equiv 0$ and all linear functions each of which takes the value -1 at $z=0$ and the value zero at two ends of an arc of A. A geometric visualization of this function as the envelope of a family of planes enables one to see easily that $\partial u/\partial n_- > 1$ at points of A and $\partial u/\partial n_-=0$ at points of B. The set A is dense on the circle so that each point of B is a point of discontinuity.

6. **Integrability of the forward normal derivative on a rectifiable curve.** The integrability theorem will be obtained with the help of integrals on a family of smooth curves. To relate these integrals we need yet another theorem regarding limits of the derivatives of an (hs)-function. This time the derivatives will be taken in the direction of a smooth curve. Again a lemma on harmonic functions will prove convenient.

LEMMA 5. *Suppose $\{h_n\}$ is a subuniformly convergent sequence of single valued functions harmonic in a domain D having limit h. Let γ be a smooth oriented curve contained in D. If $\partial/\partial s$ denotes differentiation in the direction of γ*

at each of its points then $\partial h_n/\partial s \to \partial h/\partial s$ *uniformly on* γ. *Moreover, if* $\{z_n\} \subset \gamma$ *and* $z_n \to \zeta$ *then*

$$\lim_{n\to\infty} \frac{h_n(z_n) - h_n(\zeta)}{|z_n - \zeta|} = \frac{\partial h(\zeta)}{\partial s}.$$

Proof. Let $\zeta \in \gamma$ and $\{z_n\} \subset \gamma \ni z_n \to \zeta$. Letting α_n denote the direction of γ at z_n and α its direction at ζ we have $\alpha_n \to \alpha$ because of the smoothness of γ. According to Lemma 2

$$\frac{\partial h_n(z_n)}{\partial s(\alpha_n)} \to \frac{\partial h(\zeta)}{\partial s(\alpha)}.$$

But then $\partial h_n/\partial s$ converges continuously to $\partial h/\partial s$ at each point of γ and hence uniformly on γ.

To prove the second part of the lemma consider the sequence of functions

$$\phi_n(z) = \frac{h_n(z) - h_n(\zeta)}{|z - \zeta|}.$$

We will first show that $\phi_n(z)$ converges uniformly to

$$\phi(z) = \frac{h(z) - h(\zeta)}{|z - \zeta|}$$

on γ with the point ζ deleted. For convenience it will be shown for the portion of γ corresponding to $t \in (\tau, 1]$ where $\zeta = z(\tau)$. Write

$$h_n[z(t)] - h_n[z(\tau)] = \int_\tau^t \frac{dh_n[z(\lambda)]}{dt} d\lambda,$$

$$h[z(t)] - h[z(\tau)] = \int_\tau^t \frac{dh[z(\lambda)]}{dt} d\lambda,$$

and $z(t) - z(\tau) = w(t)(t - \tau)$. Then

$$\phi_n(z) - \phi(z) = \frac{\int_\tau^t (dh_n/dt - dh/dt) d\lambda}{|w(t)|(t - \tau)}.$$

As a special case of Theorem 4 the integrand may be written

$$\frac{dh_n}{dt} - \frac{dh}{dt} = \left(\frac{\partial h_n}{\partial s} - \frac{\partial h}{\partial s}\right) |z'(t)|.$$

According to the first part of this lemma the parentheses may be made uniformly small for n sufficiently large. The function $w(t)$ is continuous and nonzero for $t\in(\tau, 1]$ and $w(t)\to z'(\tau)\neq 0$ as $t\searrow\tau$. Hence $|w(t)|\geqq\delta>0$ for $t\in(\tau, 1]$. On taking M as the upper bound for $|z'(t)|$ we find

$$|\phi_n(z)-\phi(z)| \leqq \frac{\epsilon M(t-\tau)}{\delta(t-\tau)} = \eta$$

for $t\in(\tau, 1]$ and $n>n(\eta)$. Uniform convergence on the deleted t-interval implies continuous convergence at τ. Thus (see [5, §178]) $\lim \phi_n(z_n)=\lim \phi(z_n) =\partial h(\zeta)/\partial s$. Proof complete.

A function v will be called continuous from the front at $\zeta=z(\tau)$ on an oriented curve γ with parametrization $z(t)$ if $v[z(t)]\to v(\zeta)$ as $t\searrow\tau$. Continuity from the rear is defined in terms of increasing t.

THEOREM 6. *Let γ be a smooth oriented curve in a domain D and $\partial/\partial s$ denote differentiation in the direction of γ at each of its points. If $u\in(hs)$ in D then $\partial u/\partial s_+$ is continuous from the front and $\partial u/\partial s_-$ is continuous from the rear on γ. Moreover, if $\zeta=z(\tau)$ then*

$$\lim_{t\nearrow\tau} \frac{\partial u(z)}{\partial s_+} = \frac{\partial u(\zeta)}{\partial s_-} \quad \text{and} \quad \lim_{t\searrow\tau} \frac{\partial u(z)}{\partial s_-} = \frac{\partial u(\zeta)}{\partial s_+}\cdot$$

Proof. Let $\{t_n\}\subset[0, 1]$ be a decreasing sequence with limit τ. Denoting $z_n=z(t_n)$, $\zeta=z(\tau)$ and the direction of γ at z_n by α_n, we wish to show that

$$\lim_{n\to\infty} \frac{\partial u(z_n)}{\partial s_+(\alpha_n)} = \lim_{n\to\infty} \frac{\partial u(z_n)}{\partial s_-(\alpha_n)} = \frac{\partial u(\zeta)}{\partial s_+}\cdot$$

If α denotes the direction of γ at ζ then $\alpha_n\to\alpha$ and as an immediate consequence of Lemma 3 and Theorem 1 we have at least the following information concerning the backward derivative.

$$(6.1)\qquad \frac{\partial u(\zeta)}{\partial s_-(\alpha)} \leqq \liminf_{n\to\infty} \frac{\partial u(z_n)}{\partial s_-(\alpha_n)} \leqq \limsup_{n\to\infty} \frac{\partial u(z_n)}{\partial s_-(\alpha_n)} \leqq \frac{\partial u(\zeta)}{\partial s_+(\alpha)}\cdot$$

Hence an arbitrary subsequence of $\{\partial u(z_n)/\partial s_-\}$ will contain a convergent subsequence. Select a support function at each point of the corresponding subsequence of $\{z_n\}$ whose derivative is equal to the backward derivative of u at that point (see Theorem 1). Again take a subsequence such that the support functions converge subuniformly in D. We then have subsequences $\{z_{n_i}\}$ and $\{h_i\}$ such that $h_i\to h$ subuniformly in D, $\partial u(z_{n_i})/\partial s_-=\partial h_i(z_{n_i})/\partial s$ and because of (6.1)

$$\lim_{i\to\infty} \frac{\partial h_i(z_{n_i})}{\partial s} \leqq \frac{\partial u(\zeta)}{\partial s_+}\cdot$$

Referring to Lemma 5, the sequence $\{\partial h_i/\partial s\}$ is continuously convergent at ζ which means that the limit on the left is equal to $\partial h(\zeta)/\partial s$. Letting h^*

denote the support at ζ whose derivative at ζ is equal to the forward derivative of u we see that $\partial h(\zeta)/\partial s \leqq \partial h^*(\zeta)/\partial s$. If it can be shown that equality must occur then the proof of the theorem follows readily.

Denote $k=h-h^*$ and $k_i=h_i-h^*$. Then k and the functions of the sequence $\{k_i\}$ are harmonic in D, $k_i \to k$ subuniformly in D and $\partial k(\zeta)/\partial s \leqq 0$. Also $k_i(z_{n_i})=h_i(z_{n_i})-h^*(z_{n_i})=u(z_{n_i})-h^*(z_{n_i}) \geqq 0$ because h_i is a support at z_{n_i} and h^* is a support. Similarly $k_i(\zeta)=h_i(\zeta)-h^*(\zeta)=h_i(\zeta)-u(\zeta) \leqq 0$. Hence

$$\frac{k_i(z_{n_i}) - k_i(\zeta)}{|z_{n_i} - \zeta|} \geqq 0.$$

According to Lemma 5 this sequence of difference quotients has the limit $\partial k(\zeta)/\partial s$. The derivative of k is thus non-negative. Together with the previous result we conclude $\partial k(\zeta)/\partial s=0$ or $\partial h(\zeta)/\partial s=\partial h^*(\zeta)/\partial s$.

Collecting the above results we find

$$\lim_{i\to\infty} \frac{\partial u(z_{n_i})}{\partial s_-} = \lim_{i\to\infty} \frac{\partial h_i(z_{n_i})}{\partial s} = \frac{\partial h(\zeta)}{\partial s} = \frac{\partial h^*(\zeta)}{\partial s} = \frac{\partial u(\zeta)}{\partial s_+}.$$

Since $\{\partial u(z_{n_i})/\partial s_-\}$ is a subsequence of an arbitrary subsequence of $\{\partial u(z_n)/\partial s_-\}$ it follows that the original sequence is convergent. Thus $\lim_n \partial u(z_n)/\partial s_-=\partial u(\zeta)/\partial s_+$. Again drawing on the inequalities of Lemma 3

$$\liminf_{n\to\infty} \frac{\partial u(z_n)}{\partial s_-} \leqq \liminf_{n\to\infty} \frac{\partial u(z_n)}{\partial s_+} \leqq \limsup_{n\to\infty} \frac{\partial u(z_n)}{\partial s_+} \leqq \frac{\partial u(\zeta)}{\partial s_+}$$

so that $\lim_n \partial u(z_n)/\partial s_+=\partial u(\zeta)/\partial s_+$. Because the original sequence $\{t_n\}$ was chosen arbitrarily

$$\lim_{t\searrow\tau} \frac{\partial u(z)}{\partial s_-} = \lim_{t\searrow\tau} \frac{\partial u(z)}{\partial s_+} = \frac{\partial u(\zeta)}{\partial s_+}.$$

The corresponding expression for increasing t is

$$\lim_{t\nearrow\tau} \frac{\partial u(z)}{\partial s_-} = \lim_{t\nearrow\tau} \frac{\partial u(z)}{\partial s_+} = \frac{\partial u(\zeta)}{\partial s_-}.$$

Proof complete.

Although this result appears as a generalization of the corresponding properties of the derivatives of a convex function of a real variable, there is one property which is not preserved. The functions $\partial u/\partial s_+$ and $\partial u/\partial s_-$ are not monotonic on γ. The monotonicity generalizes instead as a property of the flux integral which, as may be observed in the next paragraph is indistinguishable from the derivative itself in the real variable case.

A convex function of a real variable $y(x)$ may be represented as the potential of a positive mass distribution η by means of the integral

$$y(x) = \frac{1}{2}\int_a^b |x-t|\,d\eta + m(x),$$

where $m(x)$ is a linear function. Arsove states [1, p. 332] that these one dimensional potentials have been discussed by Brelot in his lectures. The analogue of the Gauss flux integral theorem in one dimension takes the form

$$y'_-(\beta) - y'_+(\alpha) = \eta((\alpha, \beta)),$$

or

$$y'_+(\beta) - y'_-(\alpha) = \eta([\alpha, \beta]),$$

where $a<\alpha<\beta<b$. This may be compared with Theorem 8.

Turning now to the integrability theorem we consider a rectifiable Jordan (simple closed continuous) curve γ. Let $z(w)$ map $|w|<1$ conformally onto the inside (bounded component of the complement) of γ. $z(w)$ is continuous in $|w|\leqq 1$ and $z'(\rho e^{i\phi})$ has a nonzero radial limit for almost all values of ϕ and at the corresponding boundary points the mapping is conformal. A development of these results may be found in Golusin [9, Kap. IX, X]. It follows that the images of almost all radii are smooth curves orthogonal to γ. Let δ_ϕ denote the image of each such radius and γ_ρ denote the image of $|w|=\rho$, $0<\rho\leqq 1$. We follow the previous convention of letting $\partial/\partial n$ denote differentiation normal to the curves γ_ρ. This, however, now represents tangential differentiation on the curves δ_ϕ.

THEOREM 7a. *If $u\in(hs)$ in a domain D and γ is a rectifiable Jordan curve in D then $\partial u/\partial n_-$ is Lebesgue integrable on γ and*

$$\lim_{\rho\nearrow 1}\int_{\gamma_\rho}\frac{\partial u}{\partial n_-}ds = \lim_{\rho\nearrow 1}\int_{\gamma_\rho}\frac{\partial u}{\partial n_+}ds = \int_\gamma \frac{\partial u}{\partial n_-}ds.$$

Proof. Consider the integrals

$$\int_{\gamma_\rho}\frac{\partial u}{\partial n_-}ds = \int_0^{2\pi}\frac{\partial u}{\partial n_-}\,|z'(\rho e^{i\phi})|\,\rho d\phi$$

for $\rho_0\leqq\rho<1$ where the choice of ρ_0 should be made to insure that each γ_ρ is contained in D. From Theorem 6 we have the result

$$\lim_{\rho\nearrow 1}\frac{\partial u[z(\rho e^{i\phi})]}{\partial n_-} = \frac{\partial u[z(e^{i\phi})]}{\partial n_-}$$

for almost all ϕ. The function $z'(w)$ is representable by a Poisson Integral on $|w|=1$ and according to a theorem of Evans [8, p. 46] the integrals

$$\int_0^\phi |z'(\rho e^{i\alpha})|\,d\alpha$$

are absolutely continuous uniformly with respect to ρ, $0<\rho<1$. Because of the boundedness (Theorem 2) of $\partial u/\partial n_-$ we may conclude the same for the integrals

$$\int_0^{\phi} \frac{\partial u}{\partial n_-} \left| z'(\rho e^{i\alpha}) \right| \rho d\alpha$$

for $\rho_0 \leqq \rho < 1$. This fact together with the convergence of the integrand almost everywhere is sufficient to assert (see [14, pp. 445–446)] that $\partial u[z(e^{i\phi})]/\partial n_- |z'(e^{i\phi})|$ is integrable and the limit may be taken under the sign of integration. This proves the integrability of $\partial u/\partial n_-$ on γ and one of the limits of the theorem.

The other limit is obtained by writing

$$\int_{\gamma_\rho} \frac{\partial u}{\partial n_+} ds = \int_0^{2\pi} \frac{\partial u}{\partial n_+} \left| z'(\rho e^{i\phi}) \right| \rho d\phi$$

and again making use of Theorem 6. Proof complete.

The curve network may be established on the other side of γ by starting with a conformal mapping of $|w| > 1$ onto the outside of γ. There are now curves γ_ρ which are the images of $|w| = \rho$, $1 < \rho$, by means of the transformation. The consequence is the other part of Theorem 7.

THEOREM 7b. *If $u \in (hs)$ in a domain D and γ is a rectifiable Jordan curve in D, then $\partial u/\partial n_+$ is Lebesgue integrable on γ and*

$$\lim_{\rho \searrow 1} \int_{\gamma_\rho} \frac{\partial u}{\partial n_+} ds = \lim_{\rho \searrow 1} \int_{\gamma_\rho} \frac{\partial u}{\partial n_-} ds = \int_{\gamma} \frac{\partial u}{\partial n_+} ds.$$

A reversal of the orientation of the curves $\{\gamma_\rho\}$ and the curves $\{\delta_\phi\}$ corresponds to a reversal of the direction of differentiation. The limit formulas of Theorem 7 are then altered by interchanging the plus and minus signs.

In §5 an example was constructed in which the function $\partial u/\partial n_-$ had a set of discontinuities of positive measure because of the behavior (or misbehavior) of the function u. One can also construct an example in which the discontinuities of the normal derivative are due to the behavior of the curve. Let u be a nonconstant linear function and γ be a rectifiable Jordan curve whose direction is discontinuous on a set of positive measure. The pathology of the combination of a "rough" function and a "rough" curve is difficult to imagine. Daniell [6] has considered even more general boundary integrals while restricting the function to be differentiable and Bray [4] has considered more general functions on an elementary curve.

7. **A flux integral theorem.** An (hs)-function may be approximated subuniformly by (hs)-functions which have any order of differentiability (see [10, Theorem 5]). Making use of flux integrals on an approximating sequence

having continuous derivatives of second order and certain results of F. Riesz it will be possible to obtain the desired theorem. Two lemmas on approximating sequences will precede the principal theorem of this paper.

LEMMA 6. *Let $\{u_n\}$ be a sequence of (hs)-functions in a domain D which converges subuniformly to u. Then*

$$\frac{\partial u}{\partial s_-} \leqq \liminf_{n\to\infty} \frac{\partial u_n}{\partial s_-} \leqq \limsup_{n\to\infty} \frac{\partial u_n}{\partial s_+} \leqq \frac{\partial u}{\partial s_+},$$

at each point of D and each direction.

Proof. We consider a fixed point $\zeta \in D$ and differentiation $\partial/\partial s$ in a fixed direction α. Let $\{h_n\}$ be a sequence of supports for $\{u_n\}$ at ζ such that $\partial u_n/\partial s_- = \partial h_n/\partial s$. A convergent subsequence $\{h_{n_i}\}$ of an arbitrary subsequence of $\{h_n\}$ will have a limit h which is a support for u at ζ. Hence

$$\lim_{i\to\infty} \frac{\partial u_{n_i}}{\partial s_-} = \lim_{i\to\infty} \frac{\partial h_{n_i}}{\partial s} = \frac{\partial h}{\partial s} \geqq \frac{\partial u}{\partial s_-},$$

where the inequality comes from Theorem 1. Since an arbitrary subsequence of $\{\partial u_n/\partial s_-\}$ contains such a subsequence we conclude the left inequality of the lemma. The right inequality is then obtained by a reversal of direction of differentiation. The middle inequality is a simple consequence of Theorem 1. Proof complete.

Riesz states this result on term by term differentiation of a convergent sequence for the case of convex functions of a single variable in a footnote [18, p. 353].

Consider now the class $(hs)^2$ of (hs)-functions having continuous derivatives of second order in a domain D. If $u \in (hs)^2$ and Δ is a domain, $\Delta^- \subset D$, having boundary Γ consisting of a finite set of smooth Jordan curves then Gauss' flux integral formula is valid

$$\int_\Gamma \frac{\partial u}{\partial n} ds = 2\pi\mu(\Delta) \tag{7.1}$$

where $\partial/\partial n$ denotes normal differentiation to the exterior of Δ and $-\mu$ is the mass distribution for u as a logarithmic potential.

Consider now a domain Δ, $\Delta^- \subset D$, having boundary Γ consisting of a finite set of Jordan curves γ^i, $i=1, \cdots, n$ which are only required to be rectifiable. Orient Γ so that the normal direction is toward the exterior of Δ. Determine in the vicinity and on the Δ side of each γ^i a family of curves $\{\gamma^i_\rho\}$, $\rho<1$, as in Theorem 7 and assign the orientation corresponding to the curves γ^i. Choose ρ_0 such that the curves $\gamma^i_{\rho_0}$ are disjoint and $\gamma^i_\rho \subset D$ for $\rho_0 \leqq \rho < 1$, $i=1, \cdots, n$. Then the set Γ_ρ, $\rho_0 \leqq \rho < 1$, consisting of the curves γ^i_ρ, $i=1, \cdots, n$ is the boundary of a domain Δ_ρ, $\Delta^-_\rho \subset \Delta$.

LEMMA 7. *Suppose $u \in (hs)$ and $\{u_k\} \subset (hs)^2$ in a domain D and $u_k \to u$ subuniformly in D. In terms of the families $\{\Gamma_\rho\}$ and $\{\Delta_\rho\}$ described above, if $\rho_0 \leqq \rho < 1$ then*

$$\int_{\Gamma_\rho} \frac{\partial u}{\partial n_-} ds \leqq 2\pi \liminf_{k\to\infty} \mu_k(\Delta_\rho) \leqq 2\pi \limsup_{k\to\infty} \mu_k(\Delta_\rho) \leqq \int_{\Gamma_\rho} \frac{\partial u}{\partial n_+} ds$$

where $-\mu_k$ denotes the mass distribution for u_k.

Proof. The sequence $\{u_k\}$ is subuniformly bounded in D and as asserted in Theorem 2′ so are the derivatives of its members. Hence $|\partial u_k/\partial n| \leqq M$ on Γ_ρ so that we may apply Fatou's Lemma to write

$$\int_{\Gamma_\rho} \left(\liminf_{k\to\infty} \frac{\partial u_k}{\partial n}\right) ds \leqq \liminf_{k\to\infty} \int_{\Gamma_\rho} \frac{\partial u_k}{\partial n} ds$$

and

$$\int_{\Gamma_\rho} \left(\limsup_{k\to\infty} \frac{\partial u_k}{\partial n}\right) ds \geqq \limsup_{k\to\infty} \int_{\Gamma_\rho} \frac{\partial u_k}{\partial n} ds.$$

But from Lemma 6

$$\int_{\Gamma_\rho} \frac{\partial u}{\partial n_-} ds \leqq \int_{\Gamma_\rho} \left(\liminf_{k\to\infty} \frac{\partial u_k}{\partial n}\right) ds \leqq \int_{\Gamma_\rho} \left(\limsup_{k\to\infty} \frac{\partial u_k}{\partial n}\right) ds \leqq \int_{\Gamma_\rho} \frac{\partial u}{\partial n_+} ds.$$

Combining these three lines and taking (7.1) into account one obtains the inequality of the lemma. Proof complete.

The machinery now at hand together with two results from F. Riesz' theory [18] will yield the principal theorem. If u is subharmonic in D, then there exists a negative mass distribution $-\mu$ corresponding to u. μ is a nonnegative additive set function (see [17, p. 25]). Let h denote the best harmonic majorant of u on the region $A_{12} = (\Delta_{\rho_2} - \Delta_{\rho_1}) \cup \Gamma_{\rho_1}$, $\rho_0 \leqq \rho_1 < \rho_2 < 1$. A_{12} consists of n separate annular components. Then if $\rho_1 < \rho < \rho_2$

$$2\pi\mu(\Delta_{\rho_1}) \leqq \int_{\Gamma_\rho} \frac{\partial h}{\partial n} ds \leqq 2\pi\mu(\Delta_{\rho_2}) \tag{7.2}$$

and

$$\int_{\Gamma_\rho} \frac{\partial h}{\partial n} ds \to 2\pi\mu(\Delta), \tag{7.3}$$

as $\rho_1, \rho_2 \to 1$. Since an (hs)-function is subharmonic we rely on Riesz for the existence of the mass distribution.

THEOREM 8. *Suppose $u \in (hs)$ in the domain D and has corresponding mass distribution $-\mu$. Let Δ be a domain, $\Delta^- \subset D$, having boundary Γ consisting of a* *finite set of rectifiable Jordan curves*, γ^i, $i=1, \cdots, n$. *Then*

$$\int_\Gamma \frac{\partial u}{\partial n_-} ds = 2\pi\mu(\Delta) \quad \textit{and} \quad \int_\Gamma \frac{\partial u}{\partial n_+} ds = 2\pi\mu(\Delta \cup \Gamma).$$

Proof. Choose $\{u_k\} \subset (hs)^2 \ni u_k \to u$ subuniformly in D. Let $\{\Gamma_\rho\}$ and $\{\Delta_\rho\}$ be the families of curves and domains as formulated above. Let h_k be the best harmonic majorant for u_k on the region A_{12}, $k=1, 2, \cdots$ and let h be the best harmonic majorant for u on A_{12}. Then using (7.2)

$$2\pi\mu_k(\Delta_{\rho_1}) \leqq \int_{\Gamma_\rho} \frac{\partial h_k}{\partial n} ds \leqq 2\pi\mu_k(\Delta_{\rho_2})$$

where $-\mu_k$ is the mass distribution for u_k and $\rho_0 \leqq \rho_1 < \rho < \rho_2 < 1$. The uniform convergence of u_k to u on A_{12} implies $h_k \to h$ uniformly on A_{12}. Consequently $\partial h_k/\partial n \to \partial h/\partial n$ uniformly on Γ_ρ and

$$\int_{\Gamma_\rho} \frac{\partial h_k}{\partial n} ds \to \int_{\Gamma_\rho} \frac{\partial h}{\partial n} ds.$$

This implies

$$2\pi \limsup_{k\to\infty} \mu_k(\Delta_{\rho_1}) \leqq \int_{\Gamma_\rho} \frac{\partial h}{\partial n} ds \leqq 2\pi \liminf_{k\to\infty} \mu_k(\Delta_{\rho_2}),$$

which combined with Lemma 7 gives

$$\int_{\Gamma_{\rho_1}} \frac{\partial u}{\partial n_-} ds \leqq \int_{\Gamma_\rho} \frac{\partial h}{\partial n} ds \leqq \int_{\Gamma_{\rho_2}} \frac{\partial u}{\partial n_+} ds.$$

Letting $\rho_1, \rho_2 \nearrow 1$ and applying Theorem 7 and (7.3) these three integrals have the common limit

$$\int_\Gamma \frac{\partial u}{\partial n_-} ds = 2\pi\mu(\Delta).$$

Now establish a family of curves $\{\Gamma_\rho\}$, $1 < \rho \leqq \rho_3$ in the vicinity of Γ but this time in the exterior of Δ. Then $\Delta^- \subset \Delta_\rho$ and according to Theorem 7

$$\int_{\Gamma_\rho} \frac{\partial u}{\partial n_-} ds \to \int_\Gamma \frac{\partial u}{\partial n_+} ds,$$

as $\rho \searrow 1$. But from the previous paragraph the left side is just $2\pi\mu(\Delta_\rho)$ and since μ is an additive set function $\mu(\Delta_\rho) \to \mu(\Delta \cup \Gamma)$ as $\rho \searrow 1$. Thus

$$\int_\Gamma \frac{\partial u}{\partial n_+} ds = 2\pi\mu(\Delta \cup \Gamma).$$

Proof complete.

The validity of Theorem 8 and the properties of the set function μ permit a consideration of the flux integral on more general families of curves. Let $\{\Delta_\rho\}$ and $\{\Gamma_\rho\}$ be a family of domains and their boundaries respectively in a domain D. Suppose Γ_ρ consists of a set of n rectifiable Jordan curves γ^i, $i=1, \cdots, n$. Moreover suppose that Δ_ρ increases with ρ and is continuous, i.e., $\Delta_{\rho_1}\cup\Gamma_{\rho_1}\subset\Delta_{\rho_2}$ for $\rho_1<\rho_2$, $\Delta_\rho\nearrow\Delta_a$ as $\rho\nearrow a$ and $\Delta_\rho\searrow\Delta_a\cup\Gamma_a$ as $\rho\searrow a$. Then in support of a remark in §6 we may state the following theorem on flux integrals.

THEOREM 9. *If $u\in(hs)$ in a domain D and $\{\Gamma_\rho\}$ is a family of boundaries as described in the previous paragraph, then $\int_{\Gamma_\rho}\partial u/\partial n_- ds$ and $\int_{\Gamma_\rho}\partial u/\partial n_+ ds$ are nondecreasing functions of ρ which are respectively continuous from the left and continuous from the right. Moreover $\int_{\Gamma_\rho}\partial u/\partial n_+ ds\to\int_{\Gamma_a}\partial u/\partial n_- ds$ as $p\nearrow a$ and $\int_{\Gamma_\rho}\partial u/\partial n_- ds\to\int_{\Gamma_a}\partial u/\partial n_+ ds$ as $\rho\searrow a$.*

8. **Consequences of the integral theorem.** It was shown in [10, Theorem 6] that an (hs)-function bounded by 1 in $|z|<1$ and zero at the origin has a mass distribution $-\mu$ which satisfies $\mu(C_\rho)\leqq 2\rho/(1-\rho^2)$ where $C_\rho=\{z\mid |z|<\rho\}$, $0<\rho<1$. Theorems 5 and 8 permit a sharpening of this result and a concise proof of it. Let $\gamma_\rho=\{z\mid |z|=\rho\}$.

THEOREM 10. *Suppose $u\in(hs)$ in $|z|<1$, $u<1$ and $u(0)=0$. Then*

$$\mu(C_\rho) < \frac{2\rho}{1-\rho^2} \quad \textit{and} \quad \mu(C_\rho\cup\gamma_\rho) \leqq \frac{2\rho}{1-\rho^2}\,.$$

Moreover, if $\mu(C_a\cup\gamma_a)=2a/(1-a^2)$ then $\mu(C_a)=0$.

Proof. According to (4.3)

$$\frac{\partial u(z)}{\partial r_+} \leqq \frac{2[1-u(z)]}{1-|z|^2},$$

for $|z|<1$ where $z=re^{i\theta}$. Thus from Theorem 8

$$\mu(C_\rho\cup\gamma_\rho) = \frac{1}{2\pi}\int_{\gamma_\rho}\frac{\partial u}{\partial r_+}ds \leqq \frac{1}{\pi(1-\rho^2)}\int_{\gamma_\rho}(1-u)ds$$

$$= \frac{2\rho}{1-\rho^2}\left(1-\frac{1}{2\pi}\int_0^{2\pi}u(\rho e^{i\theta})d\theta\right) \leqq \frac{2\rho}{1-\rho^2}[1-u(0)] = \frac{2\rho}{1-\rho^2},$$

where the last inequality utilizes the fact that u is subharmonic so that its mean value on a circle exceeds its value at the center.

Suppose now that $\mu(C_a\cup\gamma_a)=2a/(1-a^2)$. Let h denote the harmonic support for u at $z=0$. Then using both properties $h\leqq u$ and $h(0)=u(0)$ we have

$$\frac{2a}{1-a^2} = \mu(C_a \cup \gamma_a) = \frac{1}{2\pi}\int_{\gamma_a} \frac{\partial u}{\partial r_+}\,ds \leqq \frac{1}{\pi(1-a^2)}\int_{\gamma_a}(1-u)ds$$

$$\leqq \frac{1}{\pi(1-a^2)}\int_{\gamma_a}(1-h)ds = \frac{2a}{1-a^2}\left[1-h(0)\right] = \frac{2a}{1-a^2}\,.$$

Hence $\int_{\gamma_a} u ds = \int_{\gamma_a} h ds$ or $\int_{\gamma_a}(u-h)ds = 0$. But $u-h$ is continuous and nonnegative so that $u-h\equiv 0$ on γ_a. Since $u(0)-h(0)=0$ the maximum principle implies $u-h\equiv 0$ or $u\equiv h$ in C_a. Hence $\mu(C_a)=0$.

It remains to show that $\mu(C_\rho) < 2\rho/(1-\rho^2)$. Suppose to the contrary that $\mu(C_a) = 2a/(1-a^2)$ for some a. This implies successively $\mu(C_a \cup \gamma_a) = 2a/(1-a^2)$ and $\mu(C_a)=0$. The contradiction then implies the assertion. Proof complete.

It may be observed that if equality occurs in Theorem 10 for one value of ρ then it can occur for no other value. For if $\mu(C_a\cup\gamma_a) = 2a/(1-a^2)$ then $\mu(C_\rho\cup\gamma_\rho)=0$ for $0<\rho<a$. That the inequalities are sharp is illustrated by an example previously discussed in [10, § 4]. Let $0<a<1$ and define

$$h_\alpha(z) = \mathcal{R}\left(\frac{2(ze^{i\alpha}-a)}{(1-a)(1+ze^{i\alpha})}\right),$$

$0 \leqq \alpha < 2\pi$, where $\mathcal{R}$ denotes the real part. Defining $u(z) = \sup_\alpha[0, h_\alpha(z)]$ one finds $\mu[C_a \cup \gamma_a] = 2a/(1-a^2)$.

Theorem 8 yields directly a formula for the determination of the portion of the distribution on a curve. The result is in a certain sense related to the formula of Poisson for the linear density of a distribution on a curve (see [15, p. 256]).

It will be convenient to make use of the fact that each point of a simple rectifiable continuous curve is accessible by means of a rectifiable curve. The author has been unable to locate a proof of this in the literature but one can be constructed on making use of the properties of the conformal transformation of its complement into the unit disc.

THEOREM 11. *If $u\in(hs)$ in a domain D and γ is a simple rectifiable continuous curve in D, then*

$$\int_\gamma \left(\frac{\partial u}{\partial n_+} - \frac{\partial u}{\partial n_-}\right) ds = 2\pi\mu(\gamma),$$

where $-\mu$ is the mass distribution for u in D.

Proof. Construct two simple rectifiable curves γ_2 and γ_3 with the same end points as γ but which are otherwise disjoint from γ and from each other. The choice may be made, moreover, such that the inside of each of the Jordan curves formed by the pairs (γ, γ_2), (γ, γ_3) and (γ_2, γ_3) is contained in D. Relable γ as γ_1 and denote the insides of the three Jordan curves respectively as Δ_3, Δ_2 and Δ_1. Theorem 8 then gives the relations

$$\int_{\gamma_i}\left(\frac{\partial u}{\partial n_+}-\frac{\partial u}{\partial n_-}\right)ds+\int_{\gamma_j}\left(\frac{\partial u}{\partial n_+}-\frac{\partial u}{\partial n_-}\right)ds=\int_{\gamma_i\cup\gamma_j}\left(\frac{\partial u}{\partial n_+}-\frac{\partial u}{\partial n_-}\right)ds$$

$$=\int_{\gamma_i\cup\gamma_j}\frac{\partial u}{\partial n_+}ds-\int_{\gamma_1\cup\gamma_j}\frac{\partial u}{\partial n_-}ds=2\pi\mu(\Delta_k\cup\gamma_i\cup\gamma_j)-2\pi\mu(\Delta_k)$$

$$=2\pi\mu(\gamma_i\cup\gamma_j)=2\pi\mu(\gamma_i)+2\pi\mu(\gamma_j)$$

where for each $k=1, 2, 3$, (i, j, k) is one of the permutations of $(1, 2, 3)$. Use has been made of the fact that the integrals $\int_{\gamma_i}(\partial u/\partial n_+-\partial u/\partial n_-)ds$ are independent of the orientation of γ_i to choose the orientations for each k so that the normal direction is toward the exterior of Δ_k. The last equality is valid because of the absence of point masses as guaranteed by Theorem 10. The solution of the system of three equations yields

$$\int_{\gamma_i}\left(\frac{\partial u}{\partial n_+}-\frac{\partial u}{\partial n_-}\right)ds=2\pi\mu(\gamma_i),\qquad i=1, 2, 3.$$

Proof complete.

Thus if one defines a linear density of μ with respect to γ as the derivative λ of μ with respect to arc length on γ, then $2\pi\lambda=\partial u/\partial n_+-\partial u/\partial n_-$ almost everywhere on γ. Simple examples demonstrate, however, that in contrast to the Poisson formula one cannot guarantee equality everywhere even though λ be continuous and γ be smooth and have curvature.

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