MODULES OVER A COMPLETE DISCRETE VALUATION RING

BY

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1. **Introduction.** In this paper we prove a structure theorem for reduced countably generated R-modules of finite rank, where R is a complete discrete valuation ring (e.g., the p-adic integers). In addition, we obtain an existence theorem for such modules of rank 1. These two theorems are applied to cancellation, direct summand, and unique factorization problems. Our results are generalizations of theorems of Kaplansky-Mackey [2] and Rotman [3].

We observe that classification of modules up to isomorphism is not appropriate for mixed modules. Instead, we use a slightly weaker relation, almost isomorphism (which we define below). For example, although it is not true that a direct summand of a completely decomposable module is again completely decomposable, it is often true that the summand is almost isomorphic to a completely decomposable module.

2. **Prerequisites.** We give a summary of basic definitions in this section. A detailed account may be found in [1].

R is a *discrete valuation ring* (DVR) if it is a local principal ideal domain. R becomes a topological ring by defining the neighborhoods of 0 to be the powers of the prime ideal (p); R is *complete* if it is complete as a metric space. An arbitrary sequence in R either contains a convergent subsequence or contains a subsequence whose terms are of the form $u_n p^k$, where k is a fixed nonnegative integer and the u_n's are *incongruent units* (i.e., the difference of any two u_n's is also a unit). Henceforth R shall denote a complete DVR, and *module* shall mean unitary R-module.

A module M is *divisible* if $pM = M$; M is *reduced* if M contains no divisible submodules (except $\{0\}$). Define a decreasing transfinite chain of submodules of M as follows: $p^0M = M$. Let α be an ordinal: if $\alpha = \beta + 1$, $p^\alpha M = p(p^\beta M)$; if α is a limit ordinal, $p^\alpha M = \bigcap_{\beta<\alpha} p^\beta M$. If M is reduced, there is a least ordinal λ such that $p^\lambda M = 0$; λ is the *length* of M. If x is a nonzero element of M, there is an ordinal α such that $x \in p^\alpha M$ and $x \notin p^{\alpha+1}M$. α is the *height of* x, denoted $h(x)$. If $x = 0$, $h(x) = \infty$, where ∞ is larger than any ordinal. The *Ulm sequence of* x, denoted Ux, is the sequence of ordinals and ∞'s, $h(p^n x)$. Let S be a submodule of M, and let $x \in S$. Since S has its own decreasing chain of submodules, we may measure the height of x in S, $h_S(x)$, as well as its height in M; clearly $h_S(x) \leqq h(x)$. S is a *pure* submodule in case $h_S(x) = h(x)$ for all $x \in S$.

Let M_α be the submodule of M consisting of all elements in M of order

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(p) and height $\geqq\alpha$. The quotient module $M_\alpha/M_{\alpha+1}$ is a vector space over $R/(p)$; its dimension is the αth *Ulm invariant of* M, denoted $f(M;\alpha)$ or simply $f(\alpha)$.

A *basis* of M is a maximal independent subset. The *rank* of M is the cardinality of a basis. Note that if M is torsion, rank $M=0$. (This notion of rank is called the torsion-free rank by some authors.)

If A is a subset of a module M, $\{A\}$ is the submodule of M generated by A.

3. Structure and existence theorems.

DEFINITION. A *KM module* is a reduced countably generated module of finite rank(1).

Our proof of the structure theorem for KM modules is an adaptation of the proof of Kaplansky and Mackey. They discovered that two KM modules of rank 1 are isomorphic if and only if they have the same Ulm invariants and elements of infinite order have equivalent Ulm sequences; (we shall define an equivalence relation below which will reduce to Kaplansky and Mackey's definition in the rank 1 case). In order to generalize their result, it is first necessary to generalize their second invariant.

Let $x_1, \cdots, x_s$ be an ordered basis of M, and let R^s be the cartesian product of s copies of R. Now $x_1, \cdots, x_s$ determines an ordinal valued function on R^s by $g(r_1, \cdots, r_s)=h(\sum r_i x_i)$. Let $x_1', \cdots, x_s'$ be another ordered basis of M, and let g' be the function it determines. Since rank $M=s$, there is an s by s nonsingular matrix (a_{ij}) over R such that $p^m x_i' = \sum_{j=1}^s a_{ij}x_j$. Hence, given $r_1, \cdots, r_s \in R$, $\sum_{i=1}^s p^m r_i x_i' = \sum_{j=1}^s \sum_{i=1}^s r_i a_{ij} x_j$, and so

$$g'(p^m(r_1, \cdots, r_s)) = g((r_1, \cdots, r_s)(a_{ij})), \tag{1}$$

where the argument of the right side is obtained by matrix multiplication.

DEFINITION. Two ordinal valued functions g and g' on R^s are *equivalent* in case they satisfy (1).

This relation is an equivalence relation, and the equivalence class of any g determined by an ordered basis of M clearly depends only on M. We denote this equivalence class by $S(M)$. If rank $M=1$, $S(M)$ is the invariant of Kaplansky and Mackey.

We can now state our structure theorem.

THEOREM 1. *Two KM modules M and M' are isomorphic if and only if they have the same Ulm invariants and $S(M)=S(M')$.*

In order to prove this theorem, we use the following definitions and lemmas.

DEFINITION. Let S be a submodule of M. An element $x\in M$ is *S-proper* if $h(x)\geqq h(x+s)$ for every $s\in S$. M has the *coset property* if every coset $x+S$ contains an S-proper element, whenever S is finitely generated.

(1) This is called a semi-KM module in [3].

LEMMA 1. *Let M and M' be reduced countably generated modules of finite rank. If each has the coset property, then $M \approx M'$ if and only if they have the same Ulm invariants and $S(M) = S(M')$.*

Proof. [3, Theorem 4.3].

LEMMA 2. *Every KM module has the coset property.*

Proof. Let S be a finitely generated submodule and let $x \in M$. If $x \in S$, then 0 has maximal height in the coset $x+S$. Hence we may assume $x \notin S$. Since S is finitely generated, it is the direct sum of cyclic modules. The proof will be an induction on the number c of cyclic summands.

Let $c=1$, so that $S=Ry$. We may assume we have a sequence $x+a_n y$ such that $h(x+a_n y)=\alpha_n$ is strictly increasing. Our task is to find an $a \in R$ such that $h(x+ay) \geqq \alpha_n$ for all n. Now $h(b_n y)=\alpha_n$, where $b_n = a_{n+1}-a_n$. Hence $h(b_{n+1}y) > h(b_n y)$. If $(p^{m(n)})$ is the smallest ideal containing b_n, then $m(n+1) > m(n)$, i.e., $m(n) \to \infty$, and so $b_n \to 0$. Hence $\{a_n\}$ is a Cauchy sequence. Therefore $a_n \to a$, since R is complete. Thus

$$x + ay = x + a_n y + (a - a_n)y. \tag{2}$$

If there is an n such that $h((a-a_n)y)=\beta<\alpha_n$, then $h(x+ay)=\beta$, and (2) implies $h((a-a_k)y)=\beta$ for large k. Therefore, for large k, all $a-a_k$ are associates, contradicting $a-a_k \to 0$. Hence $h((a-a_n)y) \geqq \alpha_n$ for all n, and $h(x+ay) \geqq \alpha_n$, for all n. Thus $x+ay$ is the desired element.

For the inductive step, suppose $h(x+a_{n1}y_1+ \cdots +a_{nc}y_c)=\alpha_n$ is strictly increasing.

CASE 1. $\{a_{n1}\}$ contains no convergent subsequence.

We may assume further that $\{a_{n1}\}$ consists of incongruent units. Now $h(a_{n+1\,1}(x+\sum a_{ni}y_i) - a_{n1}(x+\sum a_{n+1i}y_i)) = \alpha_n = h((a_{n+1\,1}-a_{n1})x + \sum b_{ni}y_i)$, where $b_{ni}=a_{n+1\,1}a_{ni}-a_{n1}a_{n+1i}$ and $i \geqq 2$. Since $a_{n+1\,1}-a_{n1}$ is a unit, and since multiplication by a unit does not alter heights, we may assume it is 1. But there are now only $c-1$ y's occurring, and so the inductive hypothesis applies. Hence there is an $s \in \sum_{i\geqq 2} Ry_i \subset S$ such that $h(x+s) \geqq \alpha_n$, for all n. The desired element is $x+s$.

CASE 2. Each sequence $\{a_{ni}\}$ contains a convergent subsequence.

By dropping to a subsequence we may assume $a_{ni} \to a_i$ for all i. Now

$$x + \sum a_i y_i = (x + \sum a_{ni}y_i) + \sum (a_i - a_{ni})y_i. \tag{3}$$

If there is an n such that $h(\sum(a_i-a_{ni})y_i)=\beta<\alpha_n$, then $h(x+\sum a_i y_i)=\beta$ and (3) implies $h(\sum(a_i-a_{ni})y_i)=\beta$ for large n. We show this last equation is impossible by proving the following statement.

(*) If $b_{ni} \to 0$, then it is impossible that $h(\sum_{i=1}^{m} b_{ni}y_i)$ is independent of n.

We prove (*) by induction on m. If $m=1$, (*) was proved in the initial step of the induction. Define $d_n = b_{n1}/b_{11}$. Since $b_{n1} \to 0$, we may assume $d_n \in (p^n)$. Hence we have

$$h\left(\sum_{i=1}^{m} b_{1i}y_i\right) = h\left(\sum_{i=1}^{m} b_{ni}y_i - \sum_{i=1}^{m} d_n b_{1i}y_i\right) = h\left(\sum_{i=2}^{m} (b_{ni} - d_n b_{1i})y_i\right).$$

Since $e_{ni} = b_{ni} - d_n b_{1i} \to 0$, the inductive hypothesis yields $h(\sum_{i=2}^{m} e_{ni}y_i)$ is not independent of n. This contradiction completes the proof.

Theorem 1 may be restated without mentioning $S(M)$.

DEFINITION. Two modules M and M' are *almost isomorphic* in case there exist torsion modules T and T' such that $M \oplus T \approx M' \oplus T'$.

THEOREM 1'. *Two KM modules M and M' are isomorphic if and only if they are almost isomorphic and they have isomorphic torsion submodules.*

Proof. The necessity is trivial. For the sufficiency, suppose $M \oplus T \approx M' \oplus T'$. Let $x_1, \cdots, x_s$ be an ordered maximal independent subset of M, and let $y_1, \cdots, y_s$ be an ordered maximal independent subset of $M \oplus T$. There exists an integer $k \geqq 0$ such that $p^k y_i \in M$ for all i. Hence $x_1, \cdots, x_s$ and $p^k y_1, \cdots, p^k y_s$ determine equivalent ordinal functions. But $p^k y_1, \cdots, p^k y_s$ and $y_1, \cdots, y_s$ also determine equivalent ordinal functions. Hence $S(M) = S(M \oplus T)$. Similarly $S(M') = S(M' \oplus T')$. Therefore $S(M) = S(M')$, since $M \oplus T$ and $M' \oplus T'$ are isomorphic. Since M and M' have the same Ulm invariants, the result now follows from Theorem 1.

Incidentally, we have also shown that two KM modules M and M' are almost isomorphic if and only if $S(M) = S(M')$.

We now have two ways of classifying modules—up to isomorphism and up to almost isomorphism. Thus there are two possible existence theorems: we may prescribe $S(M)$ and the torsion submodule (the *fine* existence theorem), or we may only prescribe $S(M)$ (the *crude* existence theorem). Clearly the fine existence theorem is stronger than the crude one, but we present both because of the simplicity of the proof of the latter. Note that if rank $M = 1$, $S(M)$ is just the equivalence class of the Ulm sequence of any element in M of infinite order.

Let M be a KM module, $x \in M$; let $g(n)$ by the Ulm sequence of x and let $f(\alpha)$ be the Ulm invariants of M.

DEFINITION. g has a *gap at n* if $g(n+1) > g(n) + 1$.

The following lemma of Kaplansky provides a link between $f(\alpha)$ and $[g]$, where $[g]$ is the equivalence class of g.

LEMMA 3. *If g has a gap at n, $f(g(n)) \neq 0$.*

Proof. [1, Lemma 22].

Motivated by Lemma 3, we make the following definition.

DEFINITION. Let f be a function from the ordinals to the cardinals, and let g be a monotone increasing sequence of ordinals (which may be ∞ from some point on). The functions f and g are *consistent* if $f(g(n)) \neq 0$ whenever g has a gap at n.

LEMMA 4. *Let f be the Ulm invariants of M, and let g be consistent with f. If $g(n+1)=\infty$, then there exists an $x\in M$ with $Ux=g$.*

Proof. [1, Lemma 24].

THEOREM 2[2] (CRUDE EXISTENCE THEOREM). *Let $g(n)$ be a monotone increasing sequence of countable ordinals. Then there exists a reduced countably generated module M of rank 1 such that $S(M)=[g]$. Any two such modules are almost isomorphic.*

Proof. Let T be a reduced torsion module whose Ulm invariants f are consistent with g. (T exists by the Ulm-Zippin Theorem). Let Π denote the product of countably many copies of T. By Lemma 4, for each n there is an $x_n\in T$ whose Ulm sequence is $g(0), g(1), \cdots, g(n), \infty, \infty, \cdots$. Set $x=(x_n)$; (x is the element in Π whose nth coordinate is x_n). Now x has infinite order and $Ux=g$. Let $N=\{y\in\Pi: ry\in Rx$ for some $r\in R$, r depending on $y\}$. N is a pure submodule of Π of rank 1 which contains x. But N is not countably generated. However for each n, there are countably many elements $y_{n\alpha}$ which exhibit the fact that $h(p^n x)=g(n)$. Let M be the submodule of N generated by the $y_{n\alpha}$ for all n and α; M is the desired module.

Suppose M' is another module satisfying the conditions of the theorem. Let $V(V')$ be the torsion submodule of $M(M')$. Then $S(M\oplus V')=S(M)=S(M')=S(M'\oplus V)$ and $V\oplus V'\approx V'\oplus V$. Hence $M\oplus V'\approx M'\oplus V$, and M and M' are almost isomorphic.

Notice that in the above proof we have no control over the torsion submodule of the constructed module. Prescription of the torsion submodule is the cause of the difficulty in proving the fine existence theorem.

Let M be a reduced module with torsion submodule T. If the length of T is λ, then the length of $M=\lambda$ or $\lambda+\omega$ since $p^\lambda M$ is a reduced torsion-free module. If f is a function from the ordinals to the countable cardinals, the *length of* f shall be the least ordinal λ such that $f(\alpha)=0$ for $\alpha\geqq\lambda$. Such a function is a *Zippin function* if, between any two limits ordinals $<\lambda$, there are infinitely many α such that $f(\alpha)\neq 0$. (f is the set of Ulm invariants of a torsion module if and only if f is a Zippin function.)

THEOREM 3 (FINE EXISTENCE THEOREM). *Let f be a Zippin function of length λ, where λ is a countable ordinal. Let g be a monotone increasing sequence, $g(n)<\lambda+\omega$, such that f and g are consistent. Then there exists a KM module M of rank 1 such that $S(M)=[g]$ and $f(M;\alpha)=f(\alpha)$.*

Proof. There are three cases to consider: (1) $g(n)=\lambda+n$ for all n (the *prolonged* case); (2) $g(n)<\lambda$ and g has finitely many gaps; (3) $g(n)<\lambda$ and g has infinitely many gaps.

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(2) Note that the completeness of R is not used in either of our existence theorems; hence our theorems are more general than stated. However, we do not know if the prescribed invariants form a complete set of invariants when R is incomplete.

We first dispose of case (2), assuming the existence of prolonged modules. Since g has only finitely many gaps, we may assume $g(n)=\mu+n$, all n, where $\mu<\lambda$ is a limit ordinal([3]). We may write $f=f_1+f_2$, where the f_i are Zippin functions and the length of $f_1=\mu$. Let T_i be the torsion module with Ulm invariants f_i and let M' be the KM module of rank 1 with torsion submodule T_1 and $S(M')=[g]$. Then $M=M'\oplus T_2$ is the desired module.

We now construct prolonged modules. For $k=1, 2, \cdots$, let T_k be a torsion module with the following properties([4]):

(i) $p^\lambda T_k$ is cyclic of order (p^k) (and so T_k has length $\lambda+k$);

(ii) $\sum_{k=1}^{\infty} f(T_k; \alpha)=f(T; \alpha)$ for all $\alpha<\lambda$.

Let $\Pi=\prod_{k=1}^{\infty} T_k$. The elements of Π are sequences $u=(u_k)$ of elements $u_k\in T_k$. Now $h(u)=\min_k h(u_k)$. Let Σ denote $\sum_{k=1}^{\infty} T_k$, N the torsion submodule of Π and y_k a generator of $p^\lambda T_k$. Set $y=(y_1, y_2, \cdots, y_k, \cdots)$. Given an element $u\in\Pi$, let $u^1=u$ and $u^n=(0, \cdots, 0, u_n, u_{n+1}, \cdots)$. An element u is *regular* in case:

(R1) $h(u_{k+1})>h(u_k)$ for sufficiently large k;

(R2) $h(u_k)$ is not a limit ordinal for infinitely many k;

(R3) $h(pu_k)=h(u_k)+1$ for infinitely many k.

An element u is *quasi-regular* in case it satisfies (R1) and (R2).

LEMMA 5. *If pu is quasi-regular, then there exists a regular element v such that $pu=pv$.*

Proof. Let $z=pu=(z_1, z_2, \cdots)$. We can assume $h(z_1)<h(z_2)<\cdots$. If $h(z_k)=\alpha+1$, choose v_k in T_k such that $pv_k=z_k$ and $h(v_k)=\alpha$. If $h(z_k)$ is a limit ordinal, choose v_k in T_k such that $pv_k=z_k$, $h(v_k)$ is not a limit ordinal, and $h(v_k)>h(z_{k-1})$. Since z is quasi-regular, $v=(v_1, v_2, \cdots)$ has properties (R1) and (R3). Now v has property (R2) unless $h(z_k)=\alpha_k+1$ for all $k\geqq m$ where α_k is a limit ordinal. In such a case, there are infinitely many ordinals between $h(z_k)$ and $h(z_{k+1})$ for any $k\geqq m$. Hence we may rechoose $v_{2k}(2k>m)$ so that v satisfies condition (R2).

LEMMA 6. *There exists a quasi-regular element u such that $pu=y$ and* $\lim h(u_k)=\lambda$.([5])

Proof. Let $\alpha_1, \alpha_2, \cdots$ be a monotone increasing sequence of ordinals such that $\sup \alpha_k=\lambda$. Given any ordinal $\beta<\lambda$, there is a $u_k\in T_k$ such that $pu_k=y_k$, $h(u_k)$ is not a limit ordinal, and $h(u_k)\geqq\max(\alpha_k, \beta)$. This allows us to find inductively elements $u_k\in T_k$ such that $u=(u_1, u_2, \cdots)$ satisfies the conditions of the lemma.

([3]) If $\mu=0$, the module is simply a direct sum of a torsion module and an infinite cyclic module.

([4]) For a technical reason (see Lemma 6 below) we shall assume that λ is a limit ordinal. There is no loss of generality from this restriction, for if λ is not a limit ordinal, we may use the scheme of the last paragraph to reduce it to a limit ordinal.

([5]) λ is assumed to be a limit ordinal. See footnote ([4]).

DEFINITION. A submodule B of Π is *allowable* if:

(A1) $B \cap N = \Sigma$;

(A2) rank $B = 1$;

(A3) any element $b \in B$ of infinite order with $h(b_k) \geqq \lambda$ for infinitely many k is congruent to a multiple of y mod Σ;

(A4) any element $b \in B$ with $h(b_k) < \lambda$ for large k is either regular or is quasi-regular, $\lim h(b_k) = \lambda$ and $pb - y \in \Sigma$.

The submodule generated by Σ and y is allowable. Since the ascending union of allowable submodules is allowable, there exists a maximal allowable submodule M_1.

The next lemmas are concerned with the purity of M_1.

LEMMA 7. *If $pu \in M_1$, there is an element $v \in M_1$ with $pu = pv$.*

Proof. We may assume $u \notin M_1$. By the maximality of M_1, the submodule $M' = \{M_1, u\}$ is not allowable. Suppose M' violates (A1). Then $ru - w \in N$, where $r \in R$ and $w \in M_1$. r must be a unit, so that we may assume $u - w \in N$. But $p(u-w) \in N \cap M_1 = \Sigma$. Hence $p(u-w) = \sigma$, where $\sigma \in \Sigma$. Since Σ is pure in Π, there is a $\sigma' \in \Sigma$ with $p\sigma' = \sigma$. Hence $pu = p(w + \sigma')$ and $w + \sigma' \in M_1$.

We may now assume there is a $w \in M_1$ such that $u - w$ violates (A3) or (A4). Since rank $M' = 1$, either $p^{s+s'}u = p^s w$ or $p^s u = p^{s+s'}w$, where s and s' are non-negative integers. In the latter case, set $w' = p^{s'}w$. Then $p^s(u - w') = 0$ so that $u - w' \in N$. Therefore $p(u - w') \in \Sigma$. The purity of Σ yields an element σ of Σ with $p\sigma = p(u - w')$. Then $pu = p(w' + \sigma)$ and $w' + \sigma \in M_1$. In order to complete the argument, it will suffice to prove the relation $p^{s+s'}u = p^s w$ is impossible if $s' > 0$. Adjusting w by an element of Σ, we may assume $p^{s'}u = w$. But now $u - w = (1 - p^{s'})u$ is a multiple of u by a unit in the ring. Hence $u - w$ violates (A3) or (A4) if and only if u violates (A3) or (A4). But Lemmas 5 and 6 permit us to assume u is regular or quasi-regular—a contradiction.

LEMMA 8. *The submodule M_1 is pure.*

Proof. Since Σ is a pure submodule, we need only consider the elements of infinite order. If the height of a quasi-regular element z is preserved, then the height of the element y is also preserved, for $\lim h(z_k) = \lambda$. Therefore, it is only necessary to consider the regular and quasi-regular elements of M_1. Let h_1 denote the height taken in M_1. Assume that, for any regular or quasi-regular element z in M_1, $h_1(z) = h(z)$ if $h_1(z) < \alpha$. By Lemma 7, the induction hypothesis is true for $\alpha = 1$. Let z be an element in M_1 with $h_1(z) = \alpha$.

CASE 1. $\alpha = \beta + 1$. There is a regular element w in M_1 such that $pw = z$ and $h_1(w) = \beta = h(w)$. If $h(z) > \alpha$, then there is an element u in Π such that $pu = z$ and $h(u) \geqq \alpha$. Let $w(n) = u - u^n + w^n$. Then $w(n)$ belongs to M_1 and $pw(n) = z$. Hence $h_1(w(n)) \leqq \beta$ and $h_1(w(n)) = h(w(n))$, for all positive integers n. For all sufficiently large n, we have $h(u) = h(u - u^n)$ and $h(w^n) = h(w_n) > h(w)$ by the

regularity of w. Hence $\beta \geqq h(w(n)) = \min(h(u-u^n),\ h(w^n)) = h(w^n) = h(w_n) > h(w) = \beta$ for all large n; this is impossible.

CASE 2. α is a limit ordinal. Let w be a regular element in M_1 such that $pw = z$. Then $h_1(w) < \alpha$ and $h_1(w) = h(w)$. If $h(z) > \alpha$, then there is an element u in Π such that $pu = z$ and $h(u) \geqq \alpha$. Form the elements $w(n)$ as in Case 1. Then we have $\alpha > h(w(n)) = h(w_n)$ for all large n. Since α is a limit ordinal, $\alpha > h(w_n) + 1$ as well; this contradicts the property (R3) of the element w. The proof of Lemma 8 is complete.

It follows from Lemma 8 that $S(M_1)$ is the equivalence class of the sequence $\{\lambda + n\}$. However the torsion submodule of M_1 is too large. Let $\Sigma_0 = \sum_{k=1}^{\infty} p^\lambda T_k$, and set $M = M_1/\Sigma_0$. The elements $p^n y$, $n = 0, 1, 2, \cdots$ are Σ-proper, hence Σ_0-proper. Furthermore, all regular and quasi-regular elements are Σ_0-proper, for the heights of these elements are, by (A4), less than λ. Hence, if ϕ: $M_1 \to M_1/\Sigma_0$ is the natural homomorphism, $h(\phi(p^n y)) = \lambda + n$ for all n; [3, Lemma 3.1]. Thus $S(M)$ is the equivalence class of $\{\lambda + n\}$. Furthermore, the torsion submodule of M is $\Sigma/\Sigma_0 = \sum_{k=1}^{\infty} T_k/p^\lambda T_k$. By property (ii) of the definition of T_k and by [1, Exercise 35], $\Sigma/\Sigma_0 \approx T$. This completes the construction of prolonged modules.

We now construct the modules of Case 3. Let T be the given torsion module and let $g(n)$ be the given sequence of ordinals consistent with T; let $n_1 < n_2 < \cdots$ be the sequence of gaps of g. For $k = 1, 2, \cdots$, define torsion modules T_k such that

(i) $\sum_{k=1}^{\infty} T_k = T$;

(ii) $f(T_k; g(n_k)) \neq 0$.

Let Π denote $\prod_{k=1}^{\infty} T_k$, and let N denote the torsion submodule of Π.

LEMMA 9. *There exists an element $x = (x_k)$ in Π such that all $p^n x$ are regular and $h(p^n x) = g(n)$, for $n = 0, 1, 2, \cdots$.*

Proof. Since the sequence g has no gaps between 0 and n_1, $g(n) = g(0) + n$ for $0 \leqq n \leqq n_1$. Let x_1 be an element in T_1 such

$$Ux = (g(0), g(1), \cdots, g(n_1), \infty, \cdots).$$

In general, we find inductively elements x_k in T_k with the following properties:

(i) the order of x_k is (p^{n_k+1});

(ii) $h(p^{n_k} x_k) = g(n_k)$;

(iii) except the natural gap at n_k, the Ulm sequence of x_k has at most one gap; if it has a gap at $m_k < n_k$ then $h(p^{m_k+1} x_k)$ is a limit ordinal;

(iv) $h(p^n x_k) > h(p^n x_{k-1})$ for $0 \leqq n \leqq n_{k-1}$.

Suppose that there exists an element x_{k-1} in T_{k-1} satisfying (i)–(iii). To find x_k, we distinguish two cases.

CASE 1. $g(n_k) = \alpha + n_k$. Let x_k be an element in T_k of order (p^{n_k+1}) such that $h(p^n x_k) = \alpha + n$, $0 \leqq n \leqq n_k$. The element x_k has the properties (i)–(iv).

CASE 2. $g(n_k) = \alpha + m$, where $0 \leqq m < n_k$ and α is a limit ordinal. Since the

sequence g has no gaps between $n_{k-1}+1$ and n_k, $m \geqq n_k - n_{k-1} - 1$. Let the Ulm sequence of x_{k-1} be

$$(\beta, \beta+1, \cdots, \beta+m_{k-1}, \gamma, \gamma+1, \cdots, \gamma+n_{k-1}-m_{k-1}, \infty, \cdots),$$

where γ is either $\beta+m_{k-1}+1$ or a limit ordinal.

(2.1) $\gamma+n_k \leqq \alpha+m$. Then $\gamma<\alpha$. Since α is a limit ordinal, there is an ordinal α_0 between γ and α such that $\alpha_0 > g(n_{k-1})$ and $f(\alpha_0+m_k) \neq 0$, where $m_k = n_k - m - 1$. Let x_k be an element in T_k having the Ulm sequence $(\alpha_0, \alpha_0+1, \cdots, \alpha_0+m_k, \alpha, \alpha+1, \cdots, \alpha+m, \infty, \cdots)$. The element x_k satisfies (i)–(iv).

(2.2) $\gamma+n_k > \alpha+m$. Then $\alpha+m = \gamma+i$ for some positive integer $i<n_k$. In fact, since $\gamma+i = g(n_k) > g(n_{k-1}) + (n_k - n_{k-1}) = \gamma + (n_k - m_{k-1} - 1)$, $i > n_k - m_{k-1} - 1 \geqq n_k - n_{k-1} - 1$. The element x_k in T_k having Ulm sequence $(\beta+m_{k-1}-(n_k-i-1), \cdots, \beta+m_{k-1}, \gamma, \gamma+1, \cdots, \gamma+i, \infty \cdots)$, satisfies (i)–(iv).

In all cases the Ulm sequence of x_k does not have a gap between $n_{k-1}+1$ and n_k; this is clearly true for Case 1, while for Case 2, we have both m of (2.1) and i of (2.2) greater than $n_k - n_{k-1} - 1$. Therefore $h(p^n x) = g(n)$ for $n = 0, 1, 2, \cdots$.

The elements $p^n x$ are regular, unless the sequence of gaps $\{m_k\}$ is equivalent to a constant sequence, say $m_k = c$ for all $k \geqq l$. In this case $p^c x$ violates (R3) and $p^{c+1}x$ violates (R2). From the construction of x_k, we see that the equality $m_k = m_{k-1}$ can only occur in (2.1). In this case, we have $h(x_k) > g(n_{k-1})$. Thus, it is possible to rechoose x_{2k} $(2k > l)$ so that $p^n x$ are also regular.

DEFINITION. A submodule P of the module II is termed *permissible* if

(P1) rank $P = 1$;

(P2) $P \cap N = T$;

(P3) every element of infinite order in P is regular.

Analogous to the construction of prolonged modules we get the desired module through a maximal permissible submodule containing $\{T, x\}$. This completes the proof of Theorem 3.

4. **Applications.** $S(M)$ is a clumsy invariant. In spite of this clumsiness, the main theorems can be applied to solve isomorphism problems.

THEOREM 4 (CANCELLATION THEOREM). *Let T be a torsion KM module all of whose Ulm invariants are finite, and let M and M' be KM modules. If $T \oplus M \approx T \oplus M'$, then $M \approx M'$.*

Proof. $S(M) = S(T \oplus M) = S(T \oplus M') = S(M')$. By Ulm's Theorem, we may cancel T so that the torsion submodules of M and M' are isomorphic Hence $M \approx M'$, by Theorem 1.

DEFINITION. A module is *completely decomposable* if it is the direct sum of modules of rank 1.

DEFINITION. Let M be a KM module. A subset $x_1, \cdots, x_s$ of M is a *decomposition basis* of M if

(i) the x's are a basis;

(ii) $h(\sum r_i x_i) = \min h(r_i x_i)$ for all $r_i \in R$.

A *decomposition set* is an independent set (not necessarily a basis) which satisfies condition (ii).

DEFINITION. M is *almost completely decomposable* if it is almost isomorphic to a completely decomposable module.

THEOREM 5. *A KM module M is almost completely decomposable if and only if it contains a decomposition basis.*

Proof. Necessity is obvious. In order to prove sufficiency, let $x_1, \cdots, x_s$ be a decomposition basis for M. Let M_i, $i=1, 2, \cdots, s$, be a KM module of rank 1 such that $S(M_i) = [Ux_i]$, the equivalence class of Ux_i. Then $S(M) = S(\sum M_i)$. Hence M is almost isomorphic to $\sum M_i$.

Rotman [3] has shown that if one assumes M has no elements of infinite height, M is completely decomposable if and only if it contains a decomposition basis. This is false in the general case, as the following example shows.

Let T be a reduced torsion module of length 2ω with Ulm invariants $f: f(2n-1)=0$, $f(2n)=1$, $f(\omega+2n-1)=0$, $f(\omega+2n)=1$. Let M_1 have torsion submodule T and contain an x with $Ux = 2, 4, 6, \cdots$; let M_2 have torsion submodule T and contain an element y with $Uy = \omega+2, \omega+4, \omega+6, \cdots$. Let S = all pairs $(t, -t)$, where $t \in T$. Finally, set $N = (M_1 \oplus M_2)/S$. Now (ax, by) is S-proper in $M_1 \oplus M_2$, where $a, b \in R$. Hence the images of $(x, 0)$ and $(0, y)$ form a decomposition basis of N. N is not completely decomposable, for any decomposition set contains elements x' and y', with Ux' having gaps at almost all the even integers, and Uy' having transfinite ordinals. Now the torsion submodule of N is isomorphic to T. Suppose $N = N_1 \oplus N_2$, with torsion T_1 and T_2 respectively. $T_1 \oplus T_2 \approx T$. If $S(N_1) = [Ux']$, then $f(T_1, 2n) = 1$ for almost all n. Hence $f(T_2, n) \neq 0$ only finitely often. Hence $f(T_2, \alpha)$ is not a Zippin function, contradicting its being the Ulm invariants of T_2. Therefore N is not completely decomposable. On the other hand, the theorem tells us that N is almost completely decomposable.

DEFINITION. A module M is *homogeneous* if it is almost isomorphic to $\sum_{i=1}^{n} M_i$, where all the M_i have rank 1 and $S(M_i) = S(M_j)$, for all i and j.

LEMMA 10. *Let $M = \sum M_i$, all the M_i of rank 1 and $S(M_i) = S(M_j)$ for all i and j. If $x \in M$ has infinite order, then $[Ux] = S(M_1)$.*

Proof. Choose $x_i \in M_i$ of infinite order such that $Ux_i = Ux_j$ for all i and j. Then $p^k x = \sum r_i x_i$, $r_i \in R$, $k \geqq 0$. (We assume the sum is taken over all i such

that $r_i \neq 0$). Now $r_i = u_i p^{k_i}$ where u_i is a unit. Let $k_0 = \min k_i$. Then $h(p^{n+k}x) = \min h(p^{n+k_i}x_i) = h(p^{n+k_0}x_0)$. Hence

$$[Ux] = [Ux_0] = S(M_0) = S(M_1).$$

This lemma shows that the definition of homogeneity is independent of the choice of decomposition into summands of rank 1. Further, we may say M is of *type* S if there is a decomposition $M = \sum M_i$ with $S(M_i) = S$ for all i.

LEMMA 11. *Let M be a reduced module, and let $x_1, \cdots, x_s$ be a decomposition basis such that each x_i has the same Ulm sequence. Suppose also that $x_i = w_{i1}a_1 + \cdots + w_{is}a_s$, and, for all i, $|w_{i1}| \leqq |w_{11}|$. ($|\cdot|$ is the p-adic norm). Under these conditions, $y_i = w_{i1}x_1 - w_{11}x_i$, $i \geqq 2$, is a decomposition set, and each y_i is in the submodule generated by $a_2, \cdots, a_s$.*

Proof. [3, Lemma 6.6].

THEOREM 6. *Let M be a homogeneous KM module of type S. Any direct summand of M is homogeneous of type S.*

Proof. Let $M = A \oplus B$.

Choose $a_1, \cdots, a_{s-k}$ independent in A, $a_{s-k+1}, \cdots, a_s$ independent in B so that these elements form a basis for M. By Lemma 10, we may assume the a's have identical Ulm sequences. We are now in the situation of Lemma 11. Applying this lemma k times (after each application we must normalize the y's obtained so that they have identical Ulm sequences), we obtain $s-k$ independent elements in $\{a_{s-k+1}, \cdots, a_s\} \subset B$, which is a decomposition set in B. By the purity of B and the fact that rank $B = s-k$, these elements constitute a decomposition basis of B. By Theorem 5, B is almost completely decomposable. By Lemma 10, B is homogeneous and of type S.

We now consider uniqueness of the decomposition of a module into summands of rank 1.

DEFINITION. Let $M = \sum M_i = \sum N_i$, where rank M_i = rank N_i = 1 for all i. These two decompositions are *almost isomorphic* in case the indices may be so ordered that $S(M_i) = S(N_i)$ for all i.

THEOREM 7. *Let M be a completely decomposable KM module. Any two decompositions of M into summands of rank* 1 *are almost isomorphic.*

Proof. We call two decomposition sets $x_1, \cdots, x_t$ and $y_1, \cdots, y_t$ *dependent* in case each y_i is linearly dependent on the x's and each x_j is linearly dependent on the y's.

We show by induction on t that if $x_1, \cdots, x_t$ and $y_1, \cdots, y_t$ are dependent decomposition sets, then (after rearrangement), $[Ux_i] = [Uy_i]$ for $1 \leqq i \leqq t$. If $t = 1$, the statement is obvious. Suppose $M = M_1 \oplus \cdots \oplus M_n = N_1 \oplus \cdots \oplus N_n$. Take elements of infinite order $x_j \in M_j$ and $y_i \in N_i$. We may assume that the following equations are satisfied:

$$y_i = x_1 + \sum_{j=2}^{t} a_{ij}x_j, \qquad 1 \leqq i \leqq m \leqq t;$$

(4)

$$y_i = \sum_{j=2}^{t} a_{ij}x_j, \qquad m+1 \leqq i \leqq t.$$

$$p^r x_1 = b_1y_1 + \cdots + b_my_m + \cdots + b_ny_t. \tag{5}$$

If we replace the y's in (5) by their expressions in (4), We obtain $p^rx_1 = (b_1+\cdots+b_m)x_1+z$, where z is a linear combination of x_i's, $i\geqq 2$. By independence, $z=0$ and $p^r=b_1+\cdots+b_m$. At least one $b_i\notin(p^{r+1})$ for $i\leqq m$, say $b_1\notin(p^{r+1})$. Since the x's and y's form decomposition sets, (4) and (5) yield $h(p^{r+k}y_1)\leqq h(p^{r+k}x_1)\leqq h(p^kb_1y_1)$ for all $k\geqq 0$. In particular, if $k=0$, $b_1=up^r$ where u is a unit in R. Hence $h(p^{r+k}x_1)=h(p^{r+k}y_1)$ and Ux_1 and Uy_1 are equivalent. There is no loss in generality if we even assume $Ux_1=Uy_1$.

Now if $i\leqq m$, $h(p^ky_i) = h(p^kx_1 + \sum p^ka_{ij}y_j) = \min h(p^kx_1),\ h(p^ka_{ij}x_j) = \min h(p^ky_1),\ h(p^ka_{ij}x_j)$. Hence $h(p^ky_i)\leqq h(p^ky_1)$ for all $i\leqq m$.

We claim $y_2-y_1, \cdots, y_m-y_1, y_{m+1}, \cdots, y_t$ form a decomposition set. Clearly this set is independent. Consider $h(\sum_{i=2}^{m} a_i(y_i-y_1)+\sum_{i=m+1}^{t} a_iy_i)$. We must show this is equal to $\min h(a_i(y_i-y_1))_{i\leqq m},\ h(a_iy_i)_{i\geqq m+1}$. Now we do know the height in question is equal to $\min h(\sum_{i=2}^{m} a_iy_1),\ h(a_iy_i)_{i\geqq 2}$.

CASE 1. This expression is $h(a_{i_0}y_{i_0})$ for some $i_0\geqq 2$. But $h(a_{i_0}y_{i_0})\leqq \min_{i\geqq 2} h(a_iy_i)\leqq \min h(a_i(y_i-y_1))_{i\leqq m},\ h(a_iy_i)_{i\geqq m+1}$ since $Uy_i\leqq Uy_1$ for $i\leqq m$.

CASE 2. This expression is $h(\sum_{i=2}^{m} a_iy_1)$.

But $h(\sum_{i=2}^{m} a_iy_1)\geqq \min_{i\leqq m} h(a_iy_1)\geqq \min_{i\geqq 2} h(a_iy_i)$ since $Uy_i\leqq Uy_1$, which returns us to Case 1.

Since $y_2-y_1, \cdots, y_m-y_1, y_{m+1}, \cdots, y_t$ and $x_2, \cdots, x_t$ are dependent decomposition sets, the inductive hypothesis allows us to match the Ulm sequences. Hence, after reordering, $[U(y_i-y_1)]=[Ux_i]$ for $i\leqq m$ and $[Uy_i]=[Ux_i]$ for $i\geqq m+1$. Since $[U(y_i-y_1)]=[Uy_i]$, we have completed the proof of the theorem.

An even stronger uniqueness assertion can be stated if the module M has no elements of infinite height.

THEOREM 8. *Let M be a completely decomposable KM module with no elements of infinite height. Then any two decompositions of M into summands of rank 1 have isomorphic refinements.*

Proof. Let $M=M_1\oplus\cdots\oplus M_n=N_1\oplus\cdots\oplus N_n$ be two decompositions, each M_i and N_j of rank 1. By Theorem 7, we may assume $S(M_i)=S(N_i)$ for all i. By the existence theorem, there are modules A_i of rank 1 such that:

(i) A_i is isomorphic to a summand of M_i and of N_i;

(ii) if W_i is the torsion submodule of A_i, all the Ulm invariants of W_i are 0's and 1's.

Hence there exist torsion modules T_i and T_i' such that $M_i\approx A_i\oplus T_i$ and

$N_i \approx A_i \oplus T_i'$, and so $M \approx \sum A_i \oplus \sum T_i \approx \sum A_i \oplus \sum T_i'$. Looking only at the torsion submodules, we see that $\sum W_i \oplus \sum T_i \approx \sum W_i \oplus \sum T_i'$. By our choice of W_i, Ulm's theorem allows us to cancel and obtain $\sum T_i \approx \sum T_i'$. Since there are no elements of infinite height, these have isomorphic refinements.

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