

EXTENDING HOMEOMORPHISMS ON THE PSEUDO-ARC

BY

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1. Introduction. Appearing simultaneously with the definition of the term homogeneous by Sierpinski [7] was the following question proposed by Knaster and Kuratowski [4]: Is the simple closed curve the only nondegenerate, bounded, homogeneous plane continuum? Partial results were obtained by Mazurkiewicz [5] and Cohen [3] and incorrect solutions were presented by Waraskiewicz [8] and Choquet [2]. In 1948 Bing [1] settled the question by proving that the pseudo-arc was homogeneous.

This paper is the result of an effort to extend this result in a direction suggested by the following question. If H is a subset of the pseudo-arc M and T is a homeomorphism of H onto a subset $T(H)$ of M , what conditions will imply the existence of an extension of T to a homeomorphism of M onto M ? (Note that in case H is a single point, a condition is that M be homogeneous.) The main result of this paper answers this question in case H is closed and contains at most a finite number of components.

In particular, we obtain the following:

THEOREM 6. *Suppose $H_{1,1}, H_{1,2}, \dots, H_{1,n}$ are disjoint subcontinua of the pseudo-arc M and that M is irreducible between each pair of them. Suppose T is a homeomorphism of $H_{1,1} + H_{1,2} + \dots + H_{1,n}$ onto $H_{2,1} + H_{2,2} + \dots + H_{2,n}$ where $H_{2,1}, H_{2,2}, \dots, H_{2,n}$ are subcontinua of M such that M is irreducible between each pair of them. Then T can be extended to a homeomorphism of M onto M .*

THEOREM 7. *Suppose $\mathcal{K}_1, \mathcal{K}_2$ are closed subsets of the pseudo-arc M and each have the same finite number of components $H_{1,1}, H_{1,2}, \dots, H_{1,n}$ and $H_{2,1}, H_{2,2}, \dots, H_{2,n}$ respectively. Suppose also that T is a homeomorphism of \mathcal{K}_1 onto \mathcal{K}_2 . Let m be the maximum of 2 and $n-1$. Then a necessary and sufficient condition that T can be extended to a homeomorphism of M onto M is that for any m points P_1, P_2, \dots, P_m of \mathcal{K}_1 , there exists a homeomorphism of M onto M taking P_i onto $T(P_i)$ ($i=1, 2, \dots, m$).*

2. Some definitions and notation. A set M is homogeneous if, for each pair of its points P and Q , there exists a homeomorphism of M onto itself that carries P onto Q .

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A chain $D = [d_1, d_2, \dots, d_m]$ is a finite collection of open sets d_1, d_2, \dots, d_m such that d_i intersects d_j if $|i-j| \leq 1$ and the closure of d_i intersects the closure of d_j only if $|i-j| \leq 1$. d_i is called the i th link of D ; d_1 and d_m are the end links of D while links that are not end links are interior links. Two links are said to be adjacent if they intersect. If P and Q are points that belong to d_1 and d_m respectively but to no other links, D is a chain from P to Q .

A chain D contains a chain E if each link of E is a subset of a link of D . A chain D is a consolidation of a chain E if each link of D is the sum of links of E and D contains E .

A chain E each of whose links is a link of D is termed a subchain of D . $D(i, j)$ denotes the subchain of D whose end links are d_i and d_j . If P and Q are points each of which lies in exactly one link of D , then d_P and d_Q denote these links and $D(P, Q)$ is the subchain of D with end links d_P and d_Q . $d_P < d_Q$ means d_P precedes d_Q in the chain D .

D^* denotes the sum of the elements of D . The mesh of D (denoted by $\mu(D)$) is the maximum of the diameters of the links of D . If the mesh of D is less than ϵ , then D is an ϵ -chain. If D covers a space M , the Lebesgue number of D is the least upper bound of all numbers δ such that any subset of M of diameter less than δ is contained in a link of D .

A chain $E = [e_1, e_2, \dots, e_n]$ is crooked in a chain $D = [d_1, d_2, \dots, d_m]$ if D contains E and whenever e_h, e_k intersect d_r, d_s respectively with $|r-s| > 2$, then $E(h, k)$ is the sum of three subchains $E(h, i)$, $E(i, j)$ and $E(j, k)$ such that $(i-j)(h-k)$ is positive and e_i, e_j are subsets of the links of $D(r, s)$ adjacent to d_s, d_r respectively.

If $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ is a collection of ordered pairs of positive integers, the chain E follows the pattern $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ in the chain D provided that the x_i th link of E is a subset of the y_i th link of D ($i=1, 2, \dots, n$). Note that E may follow more than one pattern in D .

3. Some properties of crooked chains. The results of this section were originally proved by Bing [1]. However, the proof of Theorem 1 (below) as given there has presented some readers with difficulty and it is at the request of Professor Bing that a more detailed version is presented here. For the proofs of Lemmas 1 and 2 the reader is referred to [1].

LEMMA 1. *If D, E , and F are chains such that D is a consolidation of E and F is crooked in E , F is crooked in D .*

LEMMA 2. *If D, E , and F are chains such that F is contained in E and E is crooked in D , F is crooked in D .*

THEOREM 1. *Suppose x_1, x_2, \dots, x_n is a collection of positive integers such that $h = x_1 \leq x_i \leq x_n = k$ and $|x_i - x_{i+1}| \leq 1$ ($i=1, 2, \dots, n-1$). Suppose also that D_1, D_2, \dots is a sequence of chains from P to Q such that for each positive integer i , D_{i+1} is crooked in D_i , the closure of each link of D_{i+1} is a compact sub-*

set of a link of D_i , and the mesh of D_i is less than $1/i$. Let $d(i)_r$ denote the r th link of D_i . Suppose further that the subchain $D_2(u, v)$ of D_2 is contained in the subchain $D_1(h, k)$ of D_1 and the closures of $d(2)_u$ and $d(2)_v$ are mutually exclusive subsets of $d(1)_h$ and $d(1)_k$ respectively. Then for each integer w there is an integer j greater than w and a chain $E = [e_1, e_2, \dots, e_n]$ following the pattern $(1, x_1), (2, x_2), \dots, (n, x_n)$ in D_1 such that E is a consolidation of the links of D_j contained in $D_2(u, v)$ and no interior link of E intersects $d(2)_u + d(2)_v$.

Proof. The proof will be given by an induction on n and considering four cases. Let w be any integer. Since $|x_i - x_{i+1}| \leq 1$, n is at least as great as $k - h + 1$.

CASE 1. Suppose $n = k - h + 1$. Since the closures of $d(2)_u$ and $d(2)_v$ are mutually exclusive compact subsets of $d(1)_h$ and $d(1)_k$ respectively, there exists an integer $j > w$ such that any link of D_j that intersects $d(2)_u$ or $d(2)_v$ is contained in $d(1)_h$ or $d(1)_k$ respectively and no link of D_j intersects both $d(2)_u$ and $d(2)_v$. Now let e_1 be the sum of the links of D_j that are contained in the intersection of $D_2^*(u, v)$ and $d(1)_h$. Let e_n (the last link of E) be the sum of the links of D_j that are contained in the intersection of $D_2^*(u, v)$ and $d(1)_k$. Let e_i ($i = 2, 3, \dots, n-1$) be the sum of the remaining links of D_j that are contained in the intersection of $D_2^*(u, v)$ and $d(1)_{x_i}$. This proves the theorem for the case $n = k - h + 1$.

Now suppose k, h, n are fixed positive integers such that $n > k - h + 1$ and that the theorem holds for all positive integers less than n . We shall prove that it holds for this value of n .

CASE 2. Consider first the situation where $x_1 = x_2$. By the induction hypothesis, there exists a positive integer $k > w$ and a chain $F = [f_1, f_2, \dots, f_{n-1}]$ such that F is a consolidation of the links of D_k in $D(u, v)$, only the first link of F intersects $d(2)_u$, only the last link of F intersects $d(2)_v$, and F follows the pattern $(1, x_2), (2, x_3), \dots, (n-1, x_n)$ in D_1 .

Let j be an integer greater than k and such that there exists a link of D_j that is contained in f_1 but does not intersect $d(2)_u$.

Define $E = [e_1, e_2, \dots, e_n]$ as follows. e_1 is the sum of all those links of D_j that are contained in f_1 and intersect $d(2)_u$ plus those links of D_j that are contained in $d(2)_u$ but are not contained in any link of F . e_2 is the sum of all those links of D_j that are contained in f_1 but that are not contained in e_1 . In general, e_i is the sum of all those links of D_j that are contained in f_{i-1} but that are not contained in $e_1 + e_2 + \dots + e_{i-1}$ ($i = 2, 3, \dots, n-1$). Finally, e_n is the sum of all those links of D_j that are contained in $f_{n-1} + d(2)_v$ but that are not contained in $e_1 + e_2 + \dots + e_{n-1}$.

CASE 3. Now we consider the case where $x_1 \neq x_2$ but where there exists an integer $r \neq 2$ such that $x_1 = x_r$. Then there also exists an integer t with $1 < t < r < n$ such that $x_r < x_t$ and $x_i < x_t$ ($i = 1, 2, \dots, r$).

By hypothesis, the closure of each link of D_2 is a compact subset of a

link of D_1 and in particular the closures of $d(2)_u, d(2)_v$ are mutually exclusive subsets of $d(1)_h, d(1)_k$ respectively. Since $\mu(D_i) < 1/i$, there exists an integer $m > w$ such that each five-linked subchain of D_m that is contained in $D_2(u, v)$ is such that the closure of the sum of the links of this subchain is contained in a link of $D_1(h, k)$, any five-linked subchain of D_m one of whose links intersects $d(2)_u$ or $d(2)_v$ is contained in $d(1)_h$ or $d(1)_k$ respectively, and the sum of no four consecutive links of D_m intersects $d(2)_u$ and $d(2)_v$.

Now suppose $d(2)_v$ is not contained in $D_1(h, x_i)$. Then there exists a subchain of D_{m+1} that is contained in $D_2(u, v)$ and whose end links intersect $d(2)_u$ and $d(1)_{x_{i+1}}$ respectively. Let $D_{m+1}(p, q)$ be such a subchain of D_{m+1} that satisfies the additional condition that no interior link of $D_{m+1}(p, q)$ intersects $d(2)_u$ or $d(1)_{x_{i+1}}$. Because of the way m was chosen and since D_{m+1} is crooked in D_m , it is true that $D_{m+1}(p, q)$ is the sum of three chains $D_{m+1}(p, y), D_{m+1}(y, z), D_{m+1}(z, q)$ such that $(q-p)(z-y)$ is positive, the closure of $d(m+1)_y$ is contained in $d(1)_{x_i}$, and the closure of $d(m+1)_z$ is contained in $d(1)_h$. Furthermore since no interior link of $D_{m+1}(p, q)$ intersects $d(1)_{x_{i+1}}$, $D_{m+1}(p, z)$ is contained in $D_1(h, x_i)$ and no link of $D_{m+1}(p, z)$ intersects $d(2)_v$.

On the other hand, suppose $d(2)_v$ is contained in $d(1)_{x_i}$. Then there exists a subchain $D_{m+1}(p, q)$ of D_{m+1} that is contained in $D_2(u, v)$ whose end links intersect $d(2)_u, d(2)_v$ respectively, and none of whose interior links intersect $d(2)_u$ or $d(2)_v$. Again because of the way m was chosen and since D_{m+1} is crooked in D_m , we find that $D_{m+1}(p, q)$ is the sum of three chains $D_{m+1}(p, y), D_{m+1}(y, z), D_{m+1}(z, q)$ satisfying the same conditions as in the preceding paragraph. These properties of $D_{m+1}(p, q)$ will be used in what follows in order to be sure that certain sets are non-null.

We now divide the links of D_{m+1} that are contained in $D_2(u, v)$ into a number of classes as follows.

A_1 is the set of all links that intersect $d(2)_u$. A_2 is the set of all links whose closures lie in $d(1)_{x_i}$ but which do not intersect $d(2)_v$. ($d(m+1)_y$ and $d(m+1)_{q-1}$ belong to A_2 .) A_3 is the set of all links of subchains of D_{m+1} that are irreducible with respect to having one end link in A_1 and another in $A_1 + A_2$. (Note that the links of $D_{m+1}(p, y)$ are in A_3 .)

B_1 is the set of all links that are not in $A_1 + A_2 + A_3$ but whose closures are in $d(1)_h$. ($d(m+1)_z$ is in B_1 .) Note that $A_1 + A_2 + A_3$ and B_1 are disjoint. B_2 is the set of all links of subchains of D_{m+1} that are irreducible with respect to having one end link in B_1 and another in A_2 . (Note that the links of $D_{m+1}(z, q-1)$ are in B_2 .)

Let C_1 be the sum of the remaining links of D_{m+1} .

We shall now construct three chains A, B, C using these sets.

The chain A is defined as follows. The first link of A is the sum of the links in A_1 together with those links that belong to A_3 but not to A_2 and whose closures are contained in $d(1)_{x_i}$. The i th link of A is the sum of those links that belong to A_3 but to neither A_1 nor A_2 and whose closures are contained

in $d(1)_{x_1+(i-1)}$ ($i=2, 3, \dots, x_t-x_1$). The last link of A is the sum of those links that belong to A_2 or A_3 but not A_1 and whose closures are contained in $d(1)_{x_t}$. Then A is a chain that is contained in $D_1(x_1, x_t)$ and the closures of its first and last links are disjoint subsets of $d(1)_{x_1}$ and $d(1)_{x_t}$ respectively.

The chain B is defined in a similar way using the elements of B_1, B_2 , and A_2 so as to obtain a chain B that is contained in $D(x_1, x_t)$, whose first and last links contain B_1^* and A_2^* respectively, and the closures of these first and last links are mutually exclusive subsets of $d(1)_{x_1}$ and $d(1)_{x_t}$ respectively. In addition, only the last links of A and B intersect.

Finally, the links in B_1 and C_1 are added together to obtain a chain C as follows. The first link of C is the sum of the links of B_1 plus the links in C_1 whose closures are contained in $d(1)_{x_1}$. The i th link of C is the sum of the links in C_1 whose closures are contained in $d(1)_{x_1+(i-1)}$ ($i=2, 3, \dots, x_n-x_1+1$). Then the closures of the first and last links of C are disjoint subsets of $d(1)_{x_1}$ and $d(1)_{x_n}$ respectively, only the last link of C intersects $d(2)_*$, only the first links of B and C intersect, and A^* and C^* are disjoint.

By making use of the induction hypothesis we can find an integer j_1 greater than m and a chain F such that F is a consolidation of the links of D_{j_1} that are contained in A_3 , only the first and last links of F contain points of A_1^* and A_2^* respectively, and F follows the pattern $(1, x_1), (2, x_2), \dots, (t, x_t)$ in D_1 . For the same reason, there exists an integer j_2 greater than m and a chain G such that G is a consolidation of the links of D_{j_2} that are contained in B , only the first and last links of G contain points of A_2^* and B_1^* respectively, and G follows the pattern $(1, x_t), (2, x_{t+1}), \dots, (r+1-t, x_r)$ in D_1 . Similarly, there exists an integer j_3 greater than m and a chain H such that H is a consolidation of the links of D_{j_3} that are contained in C , only the first and last links of H contain points of B_1^* and $d(2)_*$ respectively, and H follows the pattern $(1, x_r), (2, x_{r+1}), \dots, (n-1-r, x_n)$ in D_1 .

Let j be the maximum of j_1, j_2, j_3 . Then we can define the desired chain E as follows. E will be a consolidation of the links of D_j that are contained in $D_2(u, v)$ where e_1 is the sum of those links of D_j in $D_2(u, v)$ that are contained in either $d(2)_u$ or the first link of F ; e_i is the sum of the links of D_j in $D_2(u, v)$ that are contained in the i th link of F ($i=2, 3, \dots, t-1$); e_t is the sum of the links of D_j in $D_2(u, v)$ that are contained in the last link of F or the first link of G ; the next $(r+1-t)-2$ links of E are the sum of the links of D_j in $D_2(u, v)$ that are contained in some particular interior link of G ; the next link of E is the sum of the links of D_j in $D_2(u, v)$ that are contained in the last link of G or the first link of H ; the next $(n-1-r)-2$ links of E are the sum of the links of D_j in $D_2(u, v)$ that are contained in some particular interior link of H ; and the last link of E is the sum of the links of D_j in $D_2(u, v)$ that are subsets of $d(2)_v$ or the last link of H . This gives us the desired chain E if for some integer i greater than 1, $x_i=x_1$.

CASE 4. Now assume that $x_i \neq x_1$ ($i=2, 3, \dots, n$). By the symmetry of

the preceding arguments, we can also assume that $x_i \neq x_n$ ($i = 1, 2, \dots, n-1$). Hence if $x_n - x_1 = 1$, n can only equal 2 and then the theorem follows by Case 1.

Two possibilities remain: $x_n - x_1 = 2$ and $x_n - x_1 > 2$. We shall consider this latter case first.

Let U be the set of all links of $D_2(u, v)$ that are not contained in $d(1)_{x_1}$. Then we define a chain $F = [f_1, f_2, \dots, f_{x_n - x_1}]$ as follows. f_i is the sum of those elements of U whose closures are contained in

$$d(1)_{x_1+i} \quad (i = 1, 2, \dots, x_n - x_1).$$

Since $x_n - x_1 > 2$, the closures of f_1 and $f_{x_n - x_1}$ are disjoint subsets of $d(1)_{x_1+1}$ and $d(1)_{x_n}$ respectively and $d(2)_v$ is contained in $f_{x_n - x_1}$.

By the induction hypothesis, there is an integer j greater than m (as defined in Case 3) and a chain $G = [g_1, g_2, \dots, g_{n-1}]$ such that G is a consolidation of the links of D_j that are contained in F^* , only g_1 intersects f_1 , only g_{n-1} intersects $f_{x_n - x_1}$, and G follows the pattern $(1, x_2), (2, x_3), \dots, (n-1, x_n)$ in D_1 . (Note that since only g_{n-1} intersects $f_{x_n - x_1}$, only g_{n-1} intersects $d(2)_v$.) Now we define the desired chain E as follows: e_1 is the sum of all the links of D_j in the intersection of $D_2^*(u, v)$ and $d(1)_{x_1}$ and $e_{i+1} = g_i$ ($i = 1, 2, \dots, n-1$).

Now suppose $x_n - x_1 = 2$. Then $d(1)_{x_1+1} = d(1)_{x_2} = \dots = d(1)_{x_{n-1}} = d(1)_{x_n-1}$. Let $F = [f_1, f_2, f_3]$ be a chain defined as follows: f_i is the sum of the links of $D_2(u, v)$ whose closures are contained in $d(1)_{x_1+(i-1)}$ ($i = 1, 2, 3$). Note that the closures of f_1, f_3 are disjoint and contain $d(2)_u, d(2)_v$ respectively. Since $\mu(D_i) < 1/i$, there exists an integer j greater than m such that any subchain of D_j contained in $D_2(u, v)$ and with end links in f_1, f_3 respectively has at least n links.

We classify the links of D_j that are contained in $D_2(u, v)$ as follows. Let A_1 be the set of those links of D_j in $D_2(u, v)$ that are contained in f_1 or intersect $d(2)_u$. Let A_2 be the set of those links of D_j in $D_2(u, v)$ that are contained in f_1 or intersect $d(2)_v$. Let B_i ($i = 1, 2$) be the set of links of D_j in $D_2(u, v)$ that are interior links of subchains of D_j that have their end links in A_i but have no interior link in $A_1 + A_2$. Let C be the set of links of D_j in $D_2(u, v)$ that remain. Notice that each link in C is an interior link of a subchain of D_j that has one end link in A_1 and another in A_2 while no interior link of this subchain is an element of A_1 or A_2 .

We define the chain E as follows. The first link of E is A_1^* ; the last link is A_2^* ; the second link contains B_1^* ; the next to last link contains B_2^* ; the links in C are consolidated so that E has the correct number of links.

This completes the proof of the theorem.

4. Description and some properties of the pseudo-arc M . Let X and Y be two points in a compact metric space. Let W_1, W_2, \dots be a sequence of chains from X to Y such that for each positive integer 1, (1) W_{i+1} is crooked in W_i , (2) the mesh of W_i is less than $1/i$, and (3) any link of W_i that contains

a link of W_{i+1} also contains the closure of that link. Then the intersection of W_1^*, W_2^*, \dots is the pseudo-arc M .

Throughout the rest of this paper, space is M and therefore open sets are sets that are open in M . We shall also suppose that we have given a definite sequence W_1, W_2, \dots of chains satisfying conditions 1-3 whose intersection is M .

Some properties of the pseudo-arc M that we shall make use of are contained in Theorems 2 and 3. Theorem 2 is due to Moise [6] and Theorem 3 to Bing [1].

THEOREM 2. *The pseudo-arc is homeomorphic to each of its nondegenerate subcontinua.*

THEOREM 3. *If the pseudo-arc M is irreducible between the points P and Q there exists a sequence of chains D_1, D_2, \dots from P to Q such that for each positive integer i , D_i is a consolidation of some W_m , D_{i+1} is crooked in D_i , the mesh of D_i is less than $1/i$, and any link of D_i that contains a link of D_{i+1} also contains the closure of that link. In addition, if $D_i, D_{i'}$ are consolidations of $W_m, W_{m'}$, respectively, then m is less than m' if i is less than i' .*

COROLLARY. *Suppose that the pseudo-arc M is irreducible between the points P and Q and that D is a chain from P to Q that covers M and is a consolidation of some W_m . Then there exists a sequence of chains $D = E_1, E_2, \dots$ from P to Q such that for each positive integer i , E_i is a consolidation of some W_m , E_{i+1} is crooked in E_i , the mesh of E_{i+1} is less than $1/i+1$, and any link of E_i that contains a link of E_{i+1} also contains the closure of that link. Also if $E_i, E_{i'}$ are consolidations of $W_m, W_{m'}$, respectively, then m is less than m' if i is less than i' .*

Proof. Suppose D is a consolidation of W_k . Let ϵ be the Lebesgue number of W_{k+1} . Then there exists a sequence of chains D_1, D_2, \dots from P to Q satisfying the conclusions of Theorem 3. Let i be an integer such that $1/i$ is less than minimum $(1, \epsilon)$. Then D_i is contained in W_{k+1} and by Lemma 2, D_i is crooked in D . Let $D_i = E_2$ and $D_{i+j} = E_{2+j}$ ($j = 1, 2, \dots$). Then E_1, E_2, \dots is the desired sequence. (The fact that any link of E_1 that contains a link of E_{i+1} also contains the closure of that link follows from the fact that each E_n is a consolidation of some W_m .)

5. Some special chains. In this section we shall show how to construct some chains that cover the pseudo-arc M in certain prescribed ways.

LEMMA 3. *Suppose H_1, H_2, \dots, H_n are nondegenerate disjoint subcontinua of the pseudo-arc M such that M is irreducible between each pair of them. Suppose further that H_i is irreducible between the points P_i and Q_i ($i = 1, 2, \dots, n$). Then for each positive number ϵ , there exists an ϵ -chain D from P_1 to Q_n that covers M and such that D is a consolidation of some W_m , $D(P_i, Q_i)$ contains H_i ($i = 1, 2, \dots, n$), $D^*(P_i, Q_i)$ does not intersect $D^*(P_j, Q_j)$ if $i \neq j$ and $d_{Q_1} < d_{P_2} < d_{Q_2} < d_{P_3} < \dots < d_{P_n}$.*

Proof. By Theorem 3, there exists a sequence of chains D_1, D_2, \dots from P_1 to Q_n each of which covers M and such that for each positive integer i , D_i is a consolidation of some W_m , D_{i+1} is crooked in D_i , the mesh of D_i is less than $1/i$ and any link of D_i that contains a link of D_{i+1} also contains the closure of that link.

Now the proof will proceed by an induction on n . Consider the case $n=2$. Since H_1, H_2 are disjoint pseudo-arcs (Theorem 2), there exist (by Theorem 3) ϵ -chains $E_i = [e(i)_1, e(i)_2, \dots, e(i)_{m_i}]$ ($i=1, 2$) from P_i to Q_i that cover H_i and such that E_1^* is disjoint from E_2^* .

Let $F = [f_1, f_2, \dots, f_u]$ be D_{j-1} where j is chosen so large that the mesh of F is less than the minimum of the Lebesgue numbers of E_1 and E_2 and any 4-linked subchain of F one of whose links intersects H_i is contained in a link of E_i .

Let f_{r_1} be the last link of F to contain Q_1 , f_{s_1} the last link of F to intersect H_1 , and f_{t_1} the last link of F such that $F(1, t_1)$ is contained in E_1 . Then $5 \leq r_1 \leq s_1 \leq t_1 - 3$. Similarly, let f_{t_2} be the first link of F such that $F(t_2, u)$ is contained in E_2 , f_{s_2} the first link of F to intersect H_2 , and f_{r_2} the first link of F to contain P_2 . Then $t_1 + 1 < t_2 \leq s_2 - 3 < s_2 \leq r_2 \leq u - 4$.

Now let $G = [g_1, g_2, \dots, g_v]$ be D_j . Note that $G(=D_j)$ is crooked in $F(=D_{j-1})$.

Let g_{w_1} be the first link of G to intersect f_{t_1} , g_{v_1} the last link to intersect f_{t_2} . Since G is crooked in F , $G(1, w_1)$ is the sum of three subchains $G(1, x_1)$, $G(x_1, y_1)$, $G(y_1, w_1)$ such that y_1 is greater than x_1 , g_{x_1} is contained in f_{t_1-1} and g_{y_1} is contained in f_2 . Similarly, $G(w_2, v)$ is the sum of subchains $G(w_2, y_2)$, $G(y_2, x_2)$, $G(x_2, v)$ where x_2 is greater than y_2 , g_{y_2} is contained in f_{u-1} , and g_{x_2} is contained in f_{t_2+1} . Since $t_1 \geq s+3$, f_{t_1-1} follows f_{s+1} . And since f_{s_1} was the last link of F to intersect H_1 , g_{x_1} (which lies in f_{t_1-1}) does not intersect H_1 . Hence, $G(1, x_1)$ contains H_1 . Similarly, $G(x_2, v)$ contains H_2 .

Now $t_1 - 1 > r_1 > 2$ so f_{r_1} lies between f_{t_1-1} and f_2 . And since g_{x_1}, g_{y_1} are contained in f_{t_1-1}, f_2 respectively, there exists a link g_{z_1} between g_{x_1} and g_{y_1} that is contained in f_{r_1} and hence in $e(1)_{m_1}$. Similarly, there exists a link g_{z_2} between g_{y_2} and g_{x_2} such that g_{z_2} is contained in $e(2)_{m_2}$.

We now obtain the desired chain $D = [d_1, d_2, \dots, d_m]$ as follows. For $i=1, 2, \dots, m_1$, d_i is the sum of the links of $G(1, z_1)$ that are contained in the i th link of E_1 . The last m_2 links of D are each the sum of the links of $G(z_2, v)$ that are contained in a link of E_2 . The remaining (interior) links of D are the links of $G(z_1+1, z_2-1)$. This proves the lemma for $n=2$.

Now suppose the lemma is true for all integers less than n . Then there exists an ϵ -chain $E = [e_1, e_2, \dots, e_m]$ from P_2 to Q_n that covers M and such that $E(P_i, Q_i)$ covers H_i ($i=2, 3, \dots, n$), $E^*(P_i, Q_i)$ does not intersect $E^*(P_j, Q_j)$ ($i \neq j$; $2 \leq i, j \leq n$) and $e_{Q_2} < e_{P_3} < e_{Q_3} < \dots < e_{P_n}$.

Since the composants of M are dense, there exists a point X of $M - H_1$ that lies in the same composant as H_1 and only in the first link of E . Using

the induction hypothesis (for the case $n=2$) we find a sequence of chains F_1, F_2, \dots from P_1 to Q_n each of which covers M and such that for each positive integer k , F_k is a consolidation of some W_m , $F_k(P_1, Q_1)$ contains H_1 , and the mesh of F_k is less than $1/k$. Then for some k , F_k is contained in E , any link of F_k containing X does not intersect H_1 , and $F_k(P_1, X)$ does not contain a point of $H_2 + H_3 + \dots + H_n$. (The truth of this latter statement follows from the fact that M is irreducible from H_1 to each of H_2, H_3, \dots, H_n .)

Then if $f(k)_x$ is the first link of F_k to contain X the required chain D is obtained as follows. The first $x-1$ links of D are the links of $F_k(1, x-1)$ while the $x+i$ th link of D ($i=0, 1, \dots, m-1$) is the sum of the links of $F_k(X, Q_n)$ that are contained in the $i+1$ st link of E . This completes the proof of the lemma.

Theorem 4 below shows that given a finite set of subcontinua of M such that M is irreducible between each pair, there exists an ϵ -chain D satisfying the conclusions of Lemma 3 and in addition contains another chain which follows a certain pattern in D relative to these continua. The description of this pattern is contained in the definitions which follow.

If $D = [d_1, d_2, \dots, d_n]$ is a chain from P to Q , R a point that lies only in d_i ($1 < i < n$), and $E = [e_1, e_2, \dots, e_m]$ a chain from P to R that is contained in D , we say that E follows a v -pattern to R in D if (1) for each link e_x contained in d_{i-j} there exists a link e_y ($y > x$) contained in d_{i+j} ($j=1, 2, \dots$, minimum $(i-1, n-i)$), and (2) for each link e_x contained in d_{i+j+1} there exists a link e_y ($y > x$) contained in d_{i-j} ($j=1, 2, \dots$, minimum $(i-1, n-i-1)$). (See Figure 1.)

Suppose H is a proper subcontinuum of M irreducible from R to S and $D = [d_1, d_2, \dots, d_n]$ a chain covering M such that $D(R, S)$ contains H and $d_1 < d_R < d_n$. Suppose $E = [e_1, e_2, \dots, e_m]$ is a chain that is contained in D whose end links are each subsets of only d_1 , and such that $E(R, S)$ contains H . Let e_i be the link of E containing R . Then E is said to follow a v -pattern in D relative to H, R, S provided E is the sum of three subchains $E(1, i), E(i, j), E(j, m)$ such that (1) $E(1, i)$ follows a v -pattern to R in D , (2) $E(i, j)$ covers H , is contained in $D(R, S)$ and e_i, e_j are each subsets of only d_R , and (3) $E(j, m)$ follows a v -pattern to X in D where X is any point that is contained only in e_j .

Suppose H_1, H_2, \dots, H_n are nondegenerate disjoint subcontinua of the pseudo-arc M and H_i is irreducible between the points P_i and Q_i ($i=1, 2, \dots, n$). Suppose D is a chain from P_1 to Q_n that covers M and such that (1) $D(P_i, Q_i)$ contains H_i ($i=1, 2, \dots, n$), (2) $D^*(P_i, Q_i)$ does not intersect $D^*(P_j, Q_j)$ if $i \neq j$, and (3) $d_{Q_1} < d_{P_2} < d_{Q_2} < \dots < d_{P_n}$. Suppose $E = [e_1, e_2, \dots, e_m]$ is a chain from P_1 to Q_n that covers M , is contained in D , and such that $E(P_i, Q_i)$ contains H_i ($i=1, 2, \dots, n$) with $e_{Q_1} < e_{P_2} < e_{Q_2} < \dots < e_{P_n}$. If E is the sum of $n-1$ subchains $E(i_1, i_2), E(i_2, i_3), \dots, E(i_{n-2}, i_{n-1}), E(i_{n-1}, m)$ where $e_1 = e_{i_1}, e_{i_2}, \dots, e_{i_{n-2}}$ are each contained in d_1

and $E(i_j, i_{j+1})$ follows a v -pattern in D relative to

$$H_{j+1}, P_{j+1}, Q_{j+1}, \quad (j = 1, 2, \dots, n-2)$$

then D is said to have property S relative to $H_1, P_1, Q_1, \dots, H_n, P_n, Q_n$ and E is said to be a V -chain of D relative to $H_1, P_1, Q_1, \dots, H_n, P_n, Q_n$.

The next theorem shows that we can construct such chains D and E of arbitrarily small mesh provided only that M is irreducible between each pair of H_1, H_2, \dots, H_n .

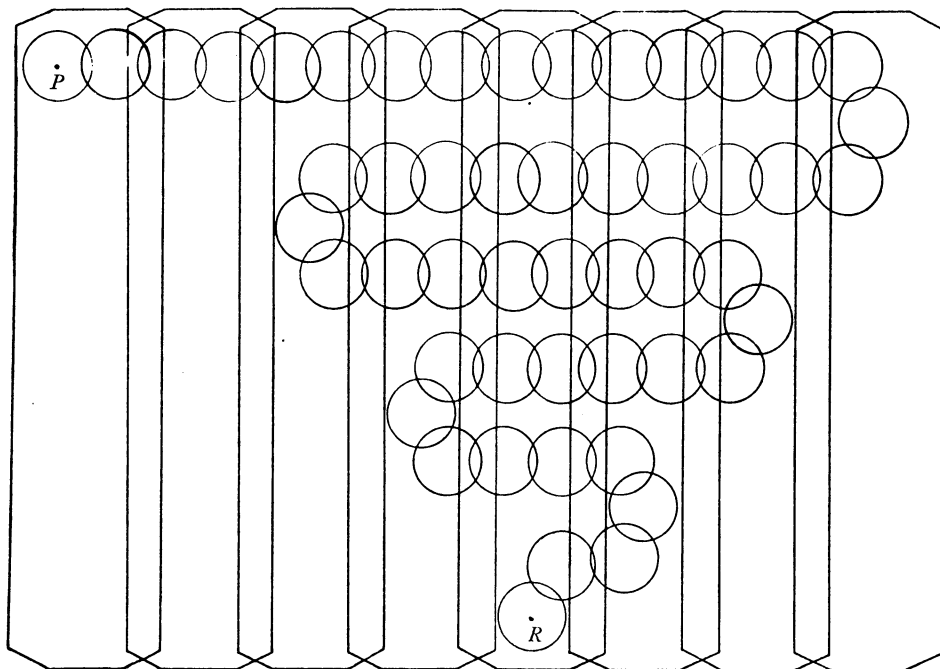


FIG. 1

THEOREM 4. Suppose H_1, H_2, \dots, H_n ($n \geq 3$) are nondegenerate proper subcontinua of the pseudo-arc M such that M is irreducible between each pair of them and such that H_α is irreducible between the points P_α, Q_α ($\alpha = 1, 2, \dots, n$). Then for each positive number ϵ there exist chains D, G each of which is a consolidation of some W_m and such that (1) D covers M and has mesh less than ϵ , (2) D has property S relative to $H_1, P_1, Q_1, \dots, H_n, P_n, Q_n$, (3) G covers M and follows a V -pattern in D relative to $H_1, P_1, Q_1, \dots, H_n, P_n, Q_n$ and (4) any link of D that contains a link of G contains the closure of that link.

Proof. The proof will be by an induction on n . In proving the theorem for $n=3$, we shall label the steps and then perform analogous operations for the case of arbitrary n .

Step 1. By Lemma 3, there exists a chain $C = [c_1, c_2, \dots, c_r]$ from P_2 to Q_3 that is a consolidation of W_{m_1} , covers M , and such that (1) $\mu(C) < \epsilon$, (2) $C(P_2, Q_2)$ and $C(P_3, Q_3)$ contain H_2 and H_3 respectively, and (3) $C^*(P_2, Q_2)$ and $C^*(P_3, Q_3)$ are disjoint. Let c_s be the link of C that contains Q_2 .

Step 2. By the corollary to Theorem 3 and Theorem 1, there exists a chain E' from P_2 to Q_3 that is a consolidation of W_{m_2} , covers M and follows the pattern $(1, 1), (2, 2), (3, 1), \dots, (m, x(m)), \dots, (y(r), r)$ in C where $y(r)$ is the number of links in E' , $y(2) = 4$, $y(r) = y(r-1) + 4r - 5$ and $x(m)$ is the minimum value of $\{|m - z(n)| + 1, n = 1, 2, \dots\}$ where $z(1) = 1$ and $z(n) = z(n-1) + 2[n/2]$ ($[n/2]$ denotes the greatest integer in $n/2$). If $r = 5$, E' follows the pattern indicated in Figure 2.

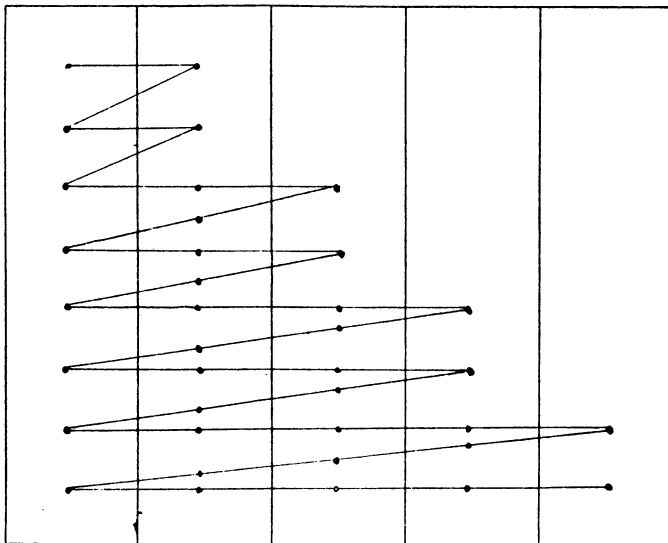


FIG. 2(2)

Note that any link of C that contains a link of E' also contains the closure of that link. Also note that E' is the sum of $4r - 5$ subchains $E'_1, E'_2, \dots, E'_{4r-5}$ where (1) $E'_{4k-3}, E'_{4k-2}, E'_{4k-1}, E'_{4k}$ ($k = 1, 2, \dots, r-2$) each have $k+1$ links and follow the pattern $(1, 1), (2, 2), \dots, (k+1, k+1)$ in C , (2) $E'_{4r-7}, E'_{4r-6}, E'_{4r-5}$ each have r links and follow the pattern $(1, 1), (2, 2), \dots, (r, r)$ in C , and (3) the last link of E'_j is also the last link of E'_{j+1} and the first link of E'_{j+1} is also the first link of E'_{j+2} ($j = 1, 3, 5, \dots, 4r-7$).

Step 3. By Lemma 3, there exists a chain E'' from Q_2 to Q_3 that is a consolidation of W_{m_1} , covers M , is contained in E' and such that $E''(Q_2, P_2)$ is contained in $C(1, s)$ and contains H_2 . In addition, any link of E' that contains a link of E'' also contains the closure of that link.

(2) In this figure the five rectangles with disjoint interiors correspond to the links of C while the broken line corresponds to the 1-nerve of E' .

Step 4. We now consolidate E'' so as to obtain a chain E by the following scheme. If e'' is a link of $E''(Q_2, P_2)$ and e'' lies in the j th link of C ($j=1, 2, \dots, s$), let e'' be contained in the $s-(j-1)$ st link of E . If e'' is a link of $E''(P_2, Q_3)$ and e'' lies in the j th link of E' , let e'' be contained in the $s+(j-1)$ st link of E . Then $E = [e_1, e_2, \dots, e_t]$ is a chain from Q_2 to Q_3 that is a consolidation of W_{m_3} , covers M , and is contained in C . Furthermore, contains $4r-4$ subchains $E_0, E_1, \dots, E_{4r-5}$ where

(1) $E_0 = E(P_2, Q_2)$ contains H_2 and follows the pattern $(1, 1), (2, 2), \dots, (s, s)$ in C ,

(2) the last link of E_j is also the last link of E_{j+1} and the first link of E_{j+1} is also the first link of E_{j+2} ($j=1, 3, 5, \dots, 4r-7$),

(3) the first link of E_0 is also the first link of E_1 ,

(4) $E_{4k-3}, E_{4k-2}, E_{4k-1}, E_{4k}$ ($k=1, 2, \dots, r-2$) each follow the pattern $(1, 1), (2, 2), \dots, (k+1, k+1)$ in C , and

(5) $E_{4r-7}, E_{4r-6}, E_{4r-5}$, each follow the pattern $(1, 1), (2, 2), \dots, (r, r)$ in C . Finally, any link of C that contains a link of E also contains the closure of that link.

Step 5. Let X_1 and X_2 be two points of M that are contained only in the last link of E and X_3 a point of M that lies only in the first link of C such that M is irreducible between each pair of $X_1, X_2, X_3, H_1, H_2, H_3$. Then by Lemma 3, there exists a chain F from P_1 to Q_3 such that (1) F is contained in E and covers M , (2) F is a consolidation of W_{m_4} , (3) $F(P_\alpha, Q_\alpha)$ contains H_α ($\alpha=1, 2, 3$), (4) $f_{Q_1} < f_{X_1} < f_{P_2} < f_{Q_2} < f_{X_2} < f_{X_3} < f_{P_3}$, and (5) if any link of E contains a link of F , it contains the closure of that link.

Step 6. We consolidate F so as to obtain the chain D as follows. If f_{X_1} is the $(i+1)$ st link of F , let the j th link of D be the j th link of F ($j=1, 2, \dots, i$). If f is a link of $F(X_1, X_2)$ and f lies in the j th link of E_k ($k=0, 1, \dots, 4r-5$), f is a subset of the $i+r \pm (j-1)$ st link of D where the plus sign is used if $k=0$ or if $k \equiv 1$ or $2 \pmod{4}$ and the minus sign is used otherwise. If f is a link of $F(X_2, X_3)$ and f is a subset of the j th link of C , f is contained in $d_{i+r-(j-1)}$. If f is a link of $F(X_3, Q_3)$ and f lies in the j th link of C , f is contained in $d_{i+r+(j-1)}$.

Then $D = [d_1, d_2, \dots, d_{i+2r-1}]$ is an ϵ -chain from P_1 to Q_3 that is a consolidation of W_{m_4} , covers M , and such that $D(P_\alpha, Q_\alpha)$ contains H_α ($\alpha=1, 2, 3$), $D^*(P_\alpha, Q_\alpha)$ does not intersect $D^*(P_\beta, Q_\beta)$ if $\alpha \neq \beta$ and $d_{P_\alpha} < d_{Q_\alpha}$ ($\alpha=1, 2, 3$). In the steps that follow, we shall show that D has property S relative to $H_1, P_1, Q_1, \dots, H_3, P_3, Q_3$.

Step 7. Here we consolidate subchains of F and obtain chains G'_1, G'_2, G'_3, G'_4 . These are then added together (end-to-end) to obtain a new chain G' . We then show (Steps 8 and 9) that G' is close to being a V -chain of D relative to $H_1, P_1, Q_1, \dots, H_3, P_3, Q_3$. Then using G' we construct a chain G (Step 10) and show (Step 11) that it is a V -chain of D relative to $H_1, P_1, Q_1, \dots, H_3, P_3, Q_3$.

If f_j is the j th link of $F(1, i)$ ($j=1, 2, \dots, i$), let G'_1 be the chain whose j th link is f_j .

If f is a link of $F(X_1, Q_2)$ and f lies in the j th link of $E = [e_1, e_2, \dots, e_t]$, let f be contained in the $t-j+1$ st link of G'_2 .

If f is a link of $F(Q_2, X_2)$ and f lies in the j th link of E , let f be contained in the j th link of G'_3 .

If f is a link of $F(X_2, Q_3)$ and f lies in the j th link of D ($j=i+1, i+2, \dots, i+2r-1$), f is a subset of the $(j-i)$ th link of G'_4 .

Let G' be the chain whose first i links are the links of G'_1 , whose $i+1$ st link is the first link of G'_2 , whose last link is the last link of G'_4 and whose remaining links are interior links of G'_2, G'_3, G'_4 or the sum of the last link of G'_2 and the first link of $G'_{\alpha+1}$ ($\alpha=2, 3$).

Then $G' = [g'_1, g'_2, \dots, g'_u]$ is a chain from P_1 to Q_3 that is a consolidation of W_m , covers M , is contained in D and such that $G'(P_\alpha, Q_\alpha)$ contains H_α ($\alpha=1, 2, 3$), $G'*(P_\alpha, Q_\alpha)$ does not intersect $G'*(P_\beta, Q_\beta)$ if $\alpha \neq \beta$, and $g'_{P_\alpha} < g'_{Q_\alpha}$ ($\alpha=1, 2, 3$).

Step 8. Before seeing how G' comes close to following a V -pattern in D , we shall show that G' follows a unique pattern in D . This will follow immediately after we show that the patterns followed by G'_1, G'_2, G'_3 and G'_4 are each unique.

The pattern followed by G'_1 is unique since the links of G'_1 are also links of D .

Let g' be any link of G'_2 . g' consists of all the links of $F(X_1, Q_2)$ that are contained in some particular link, call it e , of E . Since X_1, Q_2 are contained only in the last and first links respectively of E , so are f_{X_1}, f_{Q_2} contained only in these links. Hence, each link of E (in particular say e) contains some link of $F(X_1, Q_2)$ that is contained in no other link of E . Let f be a link of $F(X_1, Q_2)$ that is contained only in e . (Then f is also a subset of g' .) But by the definition of D , f is then contained in only a single link of D . Hence since g' contains f , g' can be contained in only a single link of D .

A similar argument applies to G'_3 .

Now let g' be any link of G'_4 . It consists of all links of $F(X_2, Q_3)$ that are contained in some particular link d of $D(i+1, i+2r-1)$. But since f_{X_2}, f_{Q_3} lie only in the first and last links respectively of $D(i+1, i+2r-1)$, d must contain some link f of $F(X_2, Q_3)$ that is contained in no other link of D . Then g' contains f and hence can be contained in only one link of D .

Therefore G' follows a unique pattern in D .

Step 9. We now begin to show that G' comes close to being a V -chain of D relative to $H_1, P_1, Q_1, \dots, H_3, P_3, Q_3$. Exactly how close will become apparent as we describe the pattern that G' follows in D .

$G'(P_1, P_2)$ follows a v -pattern to P_2 in D . To see this first suppose that g'_m is a link of $G'(P_1, P_2)$ that lies in the $(i+r-j)$ th link of D ($j=1, 2, \dots, r-1$).

We must prove that there exists a link g'_n of $G'(P_1, P_2)$ with $n > m$ and such that g'_n is contained in the $(i+r+j)$ th link of D .

Since g'_m is a link of $G'(P_1, P_2)$ and $j \leq r-1$, g'_m is a link of G'_2 and hence is the sum of the links of $F(X_1, Q_2)$ that are contained in a particular link of E . By the definition of D , the links of $F(X_1, Q_2)$ that are subsets of the $(i+r-j)$ th link of D ($j=1, 2, \dots, r-1$) are the links that are contained in the $(j+1)$ st links of those E_k such that $k > 0$ and $k \equiv 3$ or $4 \pmod{4}$. Since $k > 0$ and $K \equiv 3$ or $4 \pmod{4}$, E_{k-2} exists and (by the definition of $E_1, E_2, \dots, E_{4-r-k}$) has the same number of links as E_k . Since the links of E_{k-2} precede the links of E_k in E , the $(j+1)$ st link of E_{k-2} precedes the $(j+1)$ st link of E_k in E . Then if g'_n is the link of G'_2 that consists of the sum of the links of $F(X_1, Q_2)$ in the $(j+1)$ st link of E_{k-2} , g'_n follows g'_m in G'_2 by the definition of G'_2 . And since $k-2 \equiv 1$ or $2 \pmod{4}$, the links of $F(X_1, Q_2)$ that lie in the $(j+1)$ st link of E_{k-2} are contained in the $(i+r+j)$ th link of D . That is, g'_n is contained in d_{i+r+j} .

Now suppose g'_m is a link of $G'(P_1, P_2)$ that lies in the $(i+r+j+1)$ st link of D ($j=1, 2, \dots, r-2$). We must show that there exists a link g'_n of $G'(P_1, P_2)$ with $n > m$, and such that g'_n is contained in d_{i+r-j} .

Since g'_m is a link of $G'(P_1, P_2)$ and $j \geq 1$, g'_m is a link of G'_2 and hence the sum of the links of $F(X_1, Q_2)$ that are contained in some particular link of E . By the definition of D , the links of $F(X_1, Q_2)$ that are contained in the $(i+r+j+1)$ st link of D are those that are subsets of the $(j+2)$ nd links of those E_k for which $k=0$ or $k \equiv 1$ or $2 \pmod{4}$. Hence, g'_m is the sum of the links of $F(X_1, Q_2)$ that are contained in the $(j+2)$ nd link of some particular E_k where $k=0$ or $k \equiv 1$ or $2 \pmod{4}$. But since $j \geq 1$, $j+2 \geq 3$ and thus k is not 1 or 2 since E_1 and E_2 each have only two links. Also since P_2 is contained in the first link of E_0 , g'_m (being a link of $G'(P_1, P_2)$) is not the sum of the links of $F(X_1, Q_2)$ that lie in the $(j+2)$ nd link of E_0 . Thus $k \geq 5$ and $k-2$ exists and $k-2 \equiv 3$ or $4 \pmod{4}$. Since E_{k-2} has only one less link than E_k , E_{k-2} has a $(j+1)$ st link and this link precedes the $(j+2)$ nd link of E_k in E . Hence the sum of the links of $F(X_1, Q_2)$ that are contained in the $(j+1)$ st link of E_{k-2} is a link, call it g'_n , of G'_2 that follows g'_m in G'_2 . And by the definition of D , the links of $F(X_1, Q_2)$ that are contained in g'_n are subsets of the $(i+r-j)$ th link of D . Hence, g'_n is contained in d_{i+r-j} . This completes the proof that $G'(P_1, P_2)$ follows a v -pattern to P_2 in D .

Next we note that since $G'(P_2, Q_2)$ is a consolidation of the links of $F(X_1, Q_2)$ in E_0 , $G'(P_2, Q_2)$ is contained in $D(P_2, Q_2)$.

Since f_{Q_2}, f_{X_2} lie in only the first and last links of E respectively, there exists a point X_4 of M such that f_{X_4} is a link of $F(Q_2, X_2)$ and is contained in the first link of E_0 . Then $G'(Q_2, X_4)$ is contained in $D(P_2, Q_2)$ since it is a consolidation of the links of $F(Q_2, X_2)$ that lie in E_0 . In addition, g'_{X_4} is contained in d_{P_2} .

We will finally consider the pattern followed by $G'(X_4, Q_3)$ in D . Since

$G'(X_4, Q_3)$ is contained in $D(i+1, i+2r-1)$, $G'(Q_3, X_4)$ does not follow a v -pattern to X_4 in D . However, by using arguments analogous to those used in proving that $G'(P_1, P_2)$ follows a v -pattern to P_2 in D we can prove the following: (1) if g'_m is a link of $G'(X_4, Q_3)$ that is contained in the $(i+r-j)$ th link of D ($j=1, 2, \dots, r-1$), then there exists a link g'_n of $G'(X_4, Q_3)$ with $n < m$ and such that g'_n is a subset of the $(i+r+j)$ th link of D ; and (2) if g'_m is a link of $G'(X_4, Q_3)$ that is contained in the $(i+r+j+1)$ st link of D ($j=1, 2, \dots, r-2$), then there exists a link g'_n of $G'(X_4, Q_3)$ with $n < m$ and such that g'_n is a subset of the $(i+r-j)$ th link of D .

Figure 3 indicates roughly the pattern that G' follows in D and the relative positions of $P_\alpha, Q_\alpha, X_\alpha$ ($\alpha=1, 2, 3$) and X_4 .

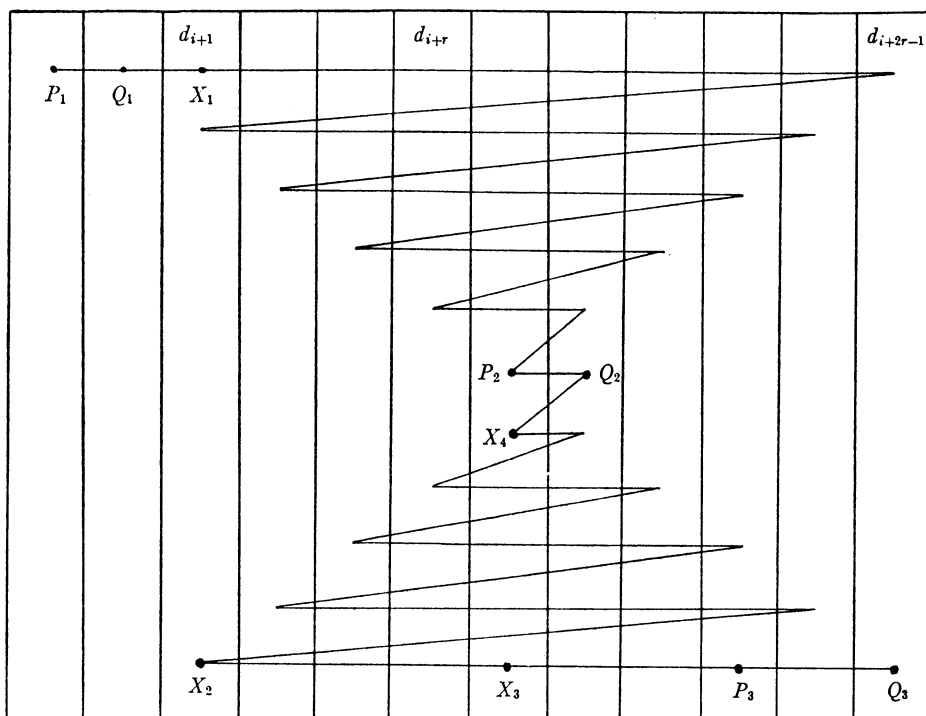


FIG. 3

Step 10. In this step we construct a chain G from P_1 to Q_3 that follows a V -pattern in D relative to $H_1, P_1, Q_1, \dots, H_3, P_3, Q_3$.

Since G' follows a unique pattern in D and the composants of M are dense, each link g'_j of G' contains a point Z_j that is contained only in the unique link of D containing g'_j and such that M is irreducible between each pair of the points $P_1, P_2, P_3, X_1, X_2, X_3, X_4, Z_1, Z_2, \dots, Z_u$. In addition, there exist points Y_1, Y_2 contained in only the last link of G' and the first link of D

respectively such that M is irreducible between each pair of the points $P_1, P_2, P_3, X_1, \dots, X_4, Z_1, Z_2, \dots, Z_u, Y_1, Y_2$.

Then by Lemma 3 there exists a chain G'' from P_1 to Q_3 such that (1) G'' is a consolidation of W_{m_6} , (2) G'' covers M and is contained in G' , (3) $G''(P_1, Y_1)$ contains $H_1 + H_2 + X_4$, (4) $G''(Y_2, Q_3)$ contains H_3 , and (5) g''_{z_j} is a link of $G''(P_1, Y_1)$ ($j = 1, 2, \dots, u$). Note that if any link of G' contains a link of G'' , then it also contains the closure of that link.

We now define chains G_1, G_2, G_3 and then add them together (end-to-end) to obtain the desired chain G .

Let G_1 be the chain whose j th link ($j = 1, 2, \dots, u$) is the sum of the links of $G''(P_1, Y_1)$ that are contained in the j th link of G' .

If g'' is a link of $G''(Y_1, Y_2)$ and g'' is in the j th link of D ($j = 1, 2, \dots, 1 + 2r - 1$), g'' is a subset of the $(i + 2r - j)$ th link of G_2 .

If g'' is a link of $G''(Y_2, Q_3)$ and g'' is in the j th link of F ($j = 1, 2, \dots, i + 2r - 1$), g'' is a subset of the j th link of G_3 .

Then G is the chain whose first and last links are the first and last links of G_1 and G_3 respectively and whose interior links are the interior links of G_α ($\alpha = 1, 2, 3$) or the sum of the last link of G_α and the first link of $G_{\alpha+1}$ ($\alpha = 1, 2$). Note that G is a consolidation of W_{m_6} and that if any link of D contains a link of G , it also contains the closure of that link.

Step 11. In the last step we show that G follows a V -pattern in D relative to $H_1, P_1, Q_1, \dots, H_3, P_3, Q_3$.

Since the j th link of G' ($j = 1, 2, \dots, u$) contains z''_{z_j} , the j th link of G_1 contains g_{z_j} and hence the j th link of G_1 is contained in only one link of D . Hence, G_1 follows a unique pattern in D and this pattern is identical to that followed by G' in D . And since G_1 contains the links of $G'(P_1, Y_1)$, G_1 contains $H_1 + H_2 + X_4$.

Therefore $G(P_1, P_2)$ follows a v -pattern to P_2 in D and $G(P_2, Q_2)$ and $G(Q_2, X_4)$ are contained in $D(P_2, Q_2)$. It remains to show that $G(Y_2, X_4)$ follows a v -pattern to X_4 in D .

Suppose g_m is a link of $G(Y_1, X_4)$ that lies in the $(i + r - j)$ th link of D ($j = 1, 2, \dots, r - 1$). (We shall consider the case where g_m is a link of $G(Y_2, Y_1)$ later.) Then g_m is a link of G_1 and hence is contained in a link g'_m of $G'(X_4, Q_3)$. Since G_1 and G' follow identical patterns in D , g'_m is also contained in the $(i + r - j)$ th link of D . In Step 9, we proved that in this case there exists a link g'_n of $G'(X_4, Q_3)$ with $n' < m'$ such that g'_n is a subset of the $(i + r + j)$ th link of D . Then the sum of the links of $G''(P_1, Y_2)$ that lie in g'_n will constitute a link g_n of $G(Y_1, X_4)$ such that g_n will be a subset of $(i + r + j)$ th link of D and follow g_m in the chain $G(Y_1, X_4)$.

In case g_m is a link of $G(Y_2, Y_1)$ and is contained in the $(i + r - j)$ th link of D ($j = 1, 2, \dots, r - 1$), there exists a link g_m of $G(Y_1, X_4)$ that follows g_m in $G(Y_1, X_4)$ and is contained in the same link of D . Then a repetition of the above argument yields a link g_n of $G(Y_1, X_4)$ (and therefore of $G(Y_2, X_4)$) that

follows g_m (and therefore g_m) in $G(Y_2, X_4)$ and is contained in the $(i+r+j)$ th link of D .

In the same way we can show that if g_m is a link of $G(Y_2, X_4)$ that is contained in the $(i+r+j+1)$ st link of D ($j=1, 2, \dots, r-2$), then there exists a link g_n of $G(Y_2, X_4)$ that follows g_m in $G(Y_2, X_4)$ and such that g_n is contained in the $(i+r-j)$ th link of D .

Hence, G is a V -chain of D relative to $H_1, P_1, Q_1, \dots, H_3, P_3, Q_3$ and the proof of Theorem 4 for the case $n=3$ is completed.

Now suppose the theorem is true for all integers less than n . We shall indicate the procedure that will yield an ϵ -chain D that has property S relative to $H_1, P_1, Q_1, \dots, H_n, P_n, Q_n$.

Step 1'. By the induction hypothesis, there exists an ϵ -chain C from P_2 to Q_n that covers M and has property S relative to $H_2, P_2, Q_2, \dots, H_n, P_n, Q_n$.

Steps 2', 3', 4'. Just as before we construct a chain E from Q_2 to Q_n that covers M , is contained in C , and in addition contains $4r-4$ subchains $E_0, E_1, \dots, E_{4r-5}$ satisfying conditions 1, 2, 3, 4, 5 of Step 4.

Step 5'. Using the same procedure as in Step 5 we construct a chain F from P_1 to Q_n such that (1) F is contained in E and covers M , (2) $F(P_\alpha, Q_\alpha)$ contains H_α ($\alpha=1, 2, \dots, n$), (3) $f_{Q_1} < f_{X_1} < f_{P_2} < f_{Q_2} < f_{X_2} < f_{X_3}$, and (4) $F(X_3, Q_n)$ contains $H_3 + H_4 + \dots + H_n$.

Step 6'. This step is almost identical to Step 6. The only change necessary is to note that the instructions on how to consolidate $F(X_3, Q_3)$ are now to be regarded as instructions on how to consolidate $F(X_3, Q_n)$.

This gives us the chain D and essentially the same methods used in Steps 7-11 show that D has property S relative to $H_1, P_1, Q_1, \dots, H_n, P_n, Q_n$.

6. Extending homeomorphisms. In this section we consider the problem of extending a homeomorphism that is defined between certain subsets of the pseudo-arc. In particular, Theorem 7 gives a necessary and sufficient condition that the homeomorphism can be extended to the entire pseudo-arc in case the subsets are closed and contain at most a finite number of components.

Theorem 5 below is stated without proof. It is a result of Bing's [1] we shall use to obtain the desired homeomorphisms.

THEOREM 5. *Suppose M is the pseudo-arc; $\epsilon_1, \epsilon_2, \dots$ is a sequence of positive numbers with a finite sum; and $D_{i,1}, D_{i,2}, \dots$ ($i=1, 2$) is a sequence of chains such that for each positive integer j , (1) $D_{i,j}$ covers M , (2) each link of $D_{i,j}$ intersects M , (3) $\mu(D_{i,j})$ is less than ϵ_j , (4) $D_{1,j}$ and $D_{2,j}$ each have the same number of links, and (5) if the m th link of $D_{i,1}$ intersects the n th link of $D_{i,j}$, then the distance between the m th link of $D_{i',1}$ and the n th link of $D_{i',j}$ is less than ϵ_j ($i, i'=1, 2$). Then these sequences of chains induce a homeomorphism T of M onto M such that if P is a point of M that is contained in the m th link of $D_{i,j}$, then $T(P)$ is contained in the m th link of $D_{i',j}$ ($i, i'=1, 2$).*

LEMMA 4. *Suppose $H_{1,1}, H_{1,2}, \dots, H_{1,n}$ ($n \geq 3$) are nondegenerate, disjoint subcontinua of the pseudo-arc M such that M is irreducible between each pair of them and $H_{1,j}$ is irreducible between the points $P_{1,j}, Q_{1,j}$ ($j = 1, 2, \dots, n$). Suppose also that T is a homeomorphism of $H_{1,1} + H_{1,2} + \dots + H_{1,n}$ onto $H_{2,1} + H_{2,2} + \dots + H_{2,n}$ such that $T(H_{1,j}) = H_{2,j}$, $T(P_{1,j}) = P_{2,j}$, $T(Q_{1,j}) = Q_{2,j}$ ($j = 1, 2, \dots, n$) where $H_{2,1}, H_{2,2}, \dots, H_{2,n}$ are disjoint subcontinua of M and M is irreducible between each pair of them. Then for each positive number ϵ , there exist chains D_1, D_2 such that (1) D_i is a consolidation of some W_{m_i} , (2) D_i is a chain from $P_{i,1}$ to $Q_{i,n}$ that covers M , (3) $\mu(D_2)$ is less than ϵ , (4) D_i has property S relative to $H_{i,1}, P_{i,1}, Q_{i,1}, \dots, H_{i,n}, P_{i,n}, Q_{i,n}$ ($i = 1, 2$), (5) D_1 has the same number of links as D_2 , (6) $P_{1,j}(Q_{1,j})$ and $P_{2,j}(Q_{2,j})$ lie in corresponding links of D_1, D_2 respectively, (7) $D_1(P_{1,j}, Q_{1,j})$ has the same number of links as $D_2(P_{2,j}, Q_{2,j})$ ($j = 1, 2, \dots, n$), (8) $D_1(Q_{1,j}, P_{1,j+1})$ has the same number of links as $D_2(Q_{2,j}, P_{2,j+1})$ ($j = 1, 2, \dots, n$), and (9) if the k th link of D_1 intersects $H_{1,1} + H_{1,2} + \dots + H_{1,n}$, then T maps their common part into the k th link of D_2 .*

Proof. The proof of this lemma is very similar to that of Theorem 4. In fact, the constructions described there will, with only minor modification, also yield the required chains D_1 and D_2 . Rather than repeat the proof of Theorem 4 in its entirety therefore, we shall simply indicate what added precautions must be taken at each step. The notation used here will correspond to that used in the proof of Theorem 4. Again we consider first the case $n = 3$.

We construct the chain D_2 by the procedure described in Theorem 4 such that $\mu(D_2)$ is less than ϵ .

Let ϵ' be the Lebesgue number of D_2 and let δ be chosen so that if X, Y are points of $H_{1,1} + H_{1,2} + H_{1,3}$ whose distance apart is less than δ , then the distance from $T(X)$ to $T(Y)$ is less than ϵ' .

By using Lemma 3, we find a chain C' from $P_{1,2}$ to $Q_{1,3}$ that is a consolidation of some W_m , and covers M , and such that (1) $\mu(C') < \delta$, (2) $C'(P_{1,j}, Q_{1,j})$ covers $H_{1,j}$ ($j = 2, 3$), and (3) $C'(Q_{1,2}, P_{1,3})$ has at least as many links as $D_2(Q_{2,2}, P_{2,3})$.

We now consolidate C' to obtain a chain C which will play the role of the chain C obtained in Step 1 of the proof of Theorem 4. Suppose $D_2(P_{2,2}, Q_{2,2})$ has x_2 links. Then the k th link of C ($1 \leq k \leq x_2$) is the sum of all links of $C'(P_{1,2}, Q_{1,2})$ whose intersection with $H_{1,2}$ is mapped by T into the k th link of $D_2(P_{2,2}, Q_{2,2})$. If $D_2(P_{2,3}, Q_{2,3})$ has x_3 links, the last x_3 links of C are defined in a similar fashion. $C'(Q_{1,2}, P_{1,3})$ is consolidated in such a way that $C(Q_{1,2}, P_{1,3})$ has the same number of links as $D_2(Q_{2,2}, P_{2,3})$.

We now proceed exactly as in the proof of Theorem 4 until we reach Step 5. Then we require that the chain F constructed there have mesh less than δ and such that the number of links in $F(Q_{1,1}, X)$ plus the number of links in C be at least as great as the number of links in $D_2(Q_{2,1}, P_{2,2})$.

Step 6 is also modified as follows. If $D_2(P_{2,1}, Q_{2,1})$ has x_1 links, the k th link of D_1 is the sum of all links of $F(P_{1,1}, Q_{1,1})$ whose intersection with $H_{1,1}$ is mapped by T into the k th link of $D_2(P_{2,1}, Q_{2,1})$ ($k = 1, 2, \dots, x_1$). $F(Q_{1,2}, X)$ is consolidated so that $D_1(Q_{1,1}, P_{1,2})$ has the same number of links as $D_2(Q_{2,1}, P_{2,2})$ and $F(X, Q_{1,3})$ is consolidated according to the same scheme used in the proof of Theorem 4. The chain D_1 arrived at in this way is the one required.

Now suppose that this lemma is true for all integers less than n . The proof that the lemma is true for the integer n will be very similar to the proof of Theorem 4 for the corresponding case. And as before we shall only indicate the necessary modifications in the constructions that are needed to obtain chains satisfying the additional requirements.

Corresponding to Step 1' of the proof of Theorem 4, we find chains C_1, C_2 satisfying the conclusions of the lemma relative to $H_{1,2}, H_{1,3}, \dots, H_{1,n}$ and $H_{2,2}, H_{2,3}, \dots, H_{2,n}$ respectively.

Then we perform Steps 2'–6' exactly as in the proof of Theorem 4 to obtain a chain D_2 satisfying the conclusions of the lemma.

To obtain D_1 we modify these steps in a fashion similar to the way Steps 5 and 6 for the case $n = 3$ were modified. In particular, Steps 2' and 4' remain unaltered while Step 5' requires the following adjustment. The chain F used there is now required to satisfy the added condition that the number of links of $F(Q_{1,1}, X)$ plus the number of links of C_1 be at least as great as the number of links in $D_2(Q_{2,1}, P_{2,2})$ and also $\mu(F)$ must be less than δ .

Step 6' is modified in the same way that Step 6 for the case $n = 3$ was altered. More precisely, if $D_2(P_{2,1}, Q_{2,1})$ has x_1 links, the k th link of D_1 is the sum of all the links of $F(P_{1,1}, Q_{1,1})$ whose intersection with $H_{1,1}$ is mapped by T into the k th link of $D_2(P_{2,1}, Q_{2,1})$ ($k = 1, 2, \dots, x_1$) and $F(Q_{1,2}, X)$ is consolidated so that $D_1(Q_{1,1}, P_{1,2})$ has the same number of links as $D_2(Q_{2,1}, P_{2,2})$. $F(X, Q_{1,3})$ is consolidated in the same way as in Theorem 4. The chains D_1, D_2 arrived at in this way satisfy the conclusions of the lemma.

Proof of Theorem 6. Without loss of generality we assume that $T(H_{1,j}) = H_{2,j}$ ($j = 1, 2, \dots, n$). Suppose that for some j , $H_{1,j}$ (and $H_{2,j}$) is degenerate. Let $H'_{1,j}, H'_{2,j}$ be nondegenerate, proper subcontinua of M that contain $H_{1,j}, H_{2,j}$ respectively. By Theorem 2, $H'_{1,j}$ and $H'_{2,j}$ are each pseudo-arcs and homeomorphic to one another. Then since the pseudo-arc is homogeneous, there exists an extension of T taking $H_{1,1} + \dots + H'_{1,j} + \dots + H_{1,n}$ onto $H_{2,1} + \dots + H'_{2,j} + \dots + H_{2,n}$. Hence without loss of generality, we now assume that no $H_{1,j}$ is degenerate.

Let $P_{1,j}, Q_{1,j}$ be points that lie in different components of $H_{1,j}$ ($j = 1, 2, \dots, n$). Let $T(P_{1,j}), T(Q_{1,j})$ be denoted by $P_{2,j}, Q_{2,j}$ respectively and note that $P_{2,j}, Q_{2,j}$ lie in different components of $H_{2,j}$. Let $\mathfrak{H}_i = H_{i,1} + H_{i,2} + \dots + H_{i,n}$ ($i = 1, 2$).

We shall also assume that all the chains that appear in the proof of this

theorem will be consolidations of chains of the given sequence W_1, W_2, \dots .

Now the proof will proceed by an induction on n . We consider first the case $n=3$. (If $n=1$ or 2 , we can enlarge \mathcal{K}_i , so that the case $n=3$ applies.) We divide the proof into 10 steps so as to make more transparent the proof for the case of arbitrary n . The basic idea will be to construct sequences of chains $D_{i,1}, D_{i,2}, \dots$ which by Theorem 5, will induce the desired extension of T . To do this, a rather large number of auxiliary chains will have to be constructed.

Step 1. By Lemma 4, there exist chains $E_{1,1}$ and $D_{2,1}$ each of which covers M and such that (1) $E_{1,1}$ is a chain from $P_{1,1}$ to $Q_{1,3}$, (2) $\mu(E_{1,1})$ is less than $1/4$, (3) $E_{1,1}$ has property S relative to $H_{1,1}, P_{1,1}, Q_{1,1}, \dots, H_{1,3}, P_{1,3}, Q_{1,3}$, (1') $D_{2,1}$ is a chain from $P_{2,1}$ to $Q_{2,3}$, (2') $D_{2,1}$ has property S relative to $H_{2,1}, P_{2,1}, Q_{2,1}, \dots, H_{2,3}, P_{2,3}, Q_{2,3}$, (3') $D_{2,1}(P_{2,j}, Q_{2,j})$ has the same number of links as $E_{1,1}(P_{1,j}, Q_{1,j})$ ($j=1, 2, 3$), (4') $D_{2,1}(Q_{2,j}, P_{2,j+1})$ has the same number of links as $E_{1,1}(Q_{1,j}, P_{1,j+1})$ ($j=1, 2$), and (5') if $d(2, 1)_k$ is the k th link of $D_{2,1}$ and intersects \mathcal{K}_2 , then $T^{-1}[d(2, 1)_k \cap \mathcal{K}_2]$ is contained in the k th link of $E_{1,1}$. (Note that 3' and 4' together imply that $D_{2,1}$ has the same number of links as $E_{1,1}$.)

Step 2. Let $D'_{2,1}$ be a V -chain of $D_{2,1}$ relative to $H_{2,1}, P_{2,1}, Q_{2,1}, \dots, H_{2,3}, P_{2,3}, Q_{2,3}$. Let $E_{2,2}$ be a chain from $P_{2,1}$ to $Q_{2,3}$ that covers M and such that (1) $E_{2,2}$ is contained in $D'_{2,1}$, (2) any link of $D'_{2,1}$ that contains a link of $E_{2,2}$ also contains the closure of that link, (3) $\mu(E_{2,2})$ is less than $1/4$, (4) no link of $E_{2,2}$ intersects three links of $D_{2,1}$, and (5) $E_{2,2}$ has property S relative to $H_{2,1}, P_{2,1}, Q_{2,1}, \dots, H_{2,3}, P_{2,3}, Q_{2,3}$. (The existence of such a chain follows from Theorem 4.) Let $D'_{2,2}$ be a chain from $P_{2,1}$ to $Q_{2,3}$ that covers M and is a V -chain of $E_{2,2}$ relative to $H_{2,1}, P_{2,1}, Q_{2,1}, \dots, H_{2,3}, P_{2,3}, Q_{2,3}$.

Step 3. Since $E_{1,1}$ has property S relative to $H_{1,1}, P_{1,1}, Q_{1,1}, \dots, H_{1,3}, P_{1,3}, Q_{1,3}$, there exists a chain $F_{1,1}$ from $P_{1,1}$ to $Q_{1,3}$ that covers M and such that (1) $F_{1,1}$ is a V -chain of $E_{1,1}$ relative to $H_{1,1}, P_{1,1}, Q_{1,1}, \dots, H_{1,3}, P_{1,3}, Q_{1,3}$, (2) if f is any link of $F_{1,1}$ that intersects \mathcal{K}_1 , then $T[f \cap \mathcal{K}_1]$ is contained in a link of $D'_{2,2}$, (3) the sum of no three consecutive links of $F_{1,1}$ intersects three links of $E_{1,1}$, (4) $\mu(F_{1,1})$ is less than $1/8$, (5) any link of $E_{1,1}$ that contains a link of $F_{1,1}$ also contains the closure of that link.

We now consolidate $F_{1,1}$ and obtain a chain $E'_{1,2}$ as follows. If $d'(2, 2)$ is a link of $D'_{2,2}(P_{2,j}, Q_{2,j})$ ($j=1, 2, 3$), then the sum of the links f of $F_{1,1}(P_{1,j}, Q_{1,j})$ such that $T[f \cap \mathcal{K}_1]$ is contained in $d'(2, 2)$ is a link of $E'_{1,2}(P_{1,j}, Q_{1,j})$. The remaining links of $E'_{1,2}$ are simply the remaining links of F . Since the intersection of a link f of $F_{1,1}$ with \mathcal{K}_1 may be contained in a link of $E_{1,1}$ that does not contain all of f itself, $E'_{1,2}$ may not be contained in $E_{1,1}$. Therefore in the next step we define a chain $D_{1,1}$ that does contain $E'_{1,2}$ and which differs from $E_{1,1}$ only slightly.

Step 4. The k th link of $D_{1,1}$ is the k th link of $E_{1,1}$ plus any links $e'(1, 2)$ of $E'_{1,2}$ such that $e'(1, 2) \cap \mathcal{K}_1$ is contained in the k th link of $E_{1,1}$. Note that $D_{1,1}$ is a chain from $P_{1,1}$ to $Q_{1,3}$ that covers M and whose mesh is less than

$1/2$. Furthermore, since the links of $D_{1,1}$, $E'_{1,2}$ are identical to those of $E_{1,1}$, $F_{1,1}$ respectively, except (perhaps) for those that intersect \mathcal{K}_1 , $D_{1,1}$ has property S relative to $H_{1,1}$, $P_{1,1}$, $Q_{1,1}$, \dots , $H_{1,3}$, $P_{1,3}$, $Q_{1,3}$ and $E'_{1,2}$ is a V -chain of $D_{1,1}$ relative to the same collection. In addition, if $e'(1, 2)_k$ is the k th link of $E'_{1,2}(P_{1,j}, Q_{1,j})$ (k arbitrary), T maps the intersection of $e'(1, 2)_k$ with \mathcal{K}_1 into the k th link of $D'_{2,2}(P_{2,j}, Q_{2,j})$ ($j=1, 2, 3$). Hence, $E'_{1,2}(P_{1,j}, Q_{1,j})$ follows a pattern in $D_{1,1}$ that $D'_{2,2}(P_{2,j}, Q_{2,j})$ follows in $D_{2,1}$ ($j=1, 2, 3$). Finally, we note that $D_{1,1}$ has the same number of links as $D_{2,1}$ and $P_{1,j}$, $Q_{1,j}$ are contained in the links of $D_{1,1}$ that correspond to those of $D_{2,1}$ that contain $P_{2,j}$, $Q_{2,j}$ respectively ($j=1, 2, 3$).

In Step 5 we describe more explicitly a pattern that $D_{2,2}(P_{2,1}, P_{2,2})$ follows in $D_{2,1}$ and in Step 6 we construct a chain $E''_{1,2}$ from $P_{1,1}$ to $Q_{1,3}$ that we shall use (Step 7) to construct a chain $D'_{1,2}(P_{1,1}, P_{1,2})$ from $P_{1,1}$ to $P_{1,2}$ that follows a pattern in $D_{1,1}$ that $D'_{2,2}(P_{2,1}, P_{2,2})$ follows in $D_{2,1}$.

Step 5. Suppose that $D_{2,1}$ has s links and that $P_{2,2}$ lies in the r th link of $D_{2,1}$. Since $D'_{2,1}$ follows a V -pattern in $D_{2,1}$ relative to $H_{2,1}$, $P_{2,1}$, $Q_{2,1}$, \dots , $H_{2,3}$, $P_{2,3}$, $Q_{2,3}$, $D'_{2,1}$ is the sum of four subchains $D'_{2,1}(1, t)$, $D'_{2,1}(t, u)$, $D'_{2,1}(u, v)$, $D'_{2,1}(v, w)$ where $D'_{2,1}(1, t)$ follows a v -pattern to $P_{2,2}$ in $D_{2,1}$, $D'_{2,1}(t, u)$ covers $H_{2,2}$ and is contained in $D_{2,1}(P_{2,2}, Q_{2,2})$ while $d'(2, 1)_t$ and $d'(2, 1)_u$ are each contained only in $d(2, 1)_{P_{2,2}}$, $d'(2, 1)_v$ is contained only in $d(2, 1)_1$, and $D'_{2,1}(v, u)$ follows a v -pattern to X_2 in $D_{2,1}$ where X_2 is a point that is contained only in $d'(2, 1)_u$.

Since $D'_{2,1}(1, t)$ follows a v -pattern to $P_{2,2}$ in $D_{2,1}$, $D'_{2,1}(1, t)$ is the sum of $2(s-r)$ subchains $D'_{2,1}(x_{2,m}, x_{2,m+1})$ ($m=1, 2, \dots, 2(s-r)$) where (1) $x_{2,1}=1$, $x_{2,2(s-r)+1}=t$, (2) $d'(2, 1)_{x_{2,2m+1}}$ is contained only in the $(2r-s-m)$ th link of $D_{2,1}$ ($m=1, 2, \dots, s-r$), (3) $d'(2, 1)_{x_{2,2m}}$ is contained only in the $s-(m-1)$ st link of $D_{2,1}$ ($m=1, 2, \dots, s-r$), and (4) $D'_{2,1}(x_{2,2m}, x_{2,2m+1})$ is contained in $D_{2,1}(s-(m-1), 2r-s+m)$ ($m=1, 2, \dots, s-r$).

Similarly, $D'_{2,1}(v, u)$ is the sum of $2(s-r)$ subchains

$$D'_{2,1}(y_{2,m}, y_{2,m+1}) \quad (m=1, 2, \dots, 2(s-r))$$

where (1) $y_{2,1}=v$, $y_{2,2(s-r)+1}=u$, (2) $d'(2, 1)_{y_{2,2m+1}}$ is contained only in the $(2r-s+m)$ th link of $D_{2,1}$ ($m=1, 2, \dots, s-r$), (3) $d'(2, 1)_{y_{2,2m}}$ is contained only in the $s-(m-1)$ st link of $D_{2,1}$ ($m=1, 2, \dots, s-r$), and (4) $D'_{2,1}(y_{2,2m}, y_{2,2m+1})$ is contained in $D_{2,1}(s-(m-1), 2r-s+m)$ ($m=1, 2, \dots, s-r$).

Now let $z_{2,m}$ be the last link of $D'_{2,2}(P_{2,1}, P_{2,2})$ to be contained in $x_{2,m} + y_{2,m}$ ($m=2, 3, \dots, 2(s-r)$). Let $z_{2,1}$ be the first link of $D'_{2,2}(P_{2,1}, P_{2,2})$ and $z_{2,2(s-r)+1}$ the last. Then $D'_{2,2}(P_{2,1}, P_{2,2})$ is the sum of $2(s-r)$ subchains $D'_{2,2}(z_{2,m}, z_{2,m+1})$ ($m=1, 2, \dots, 2(s-r)$) where (1) $z_{2,1}$, $z_{2,2(s-r)+1}$ are the first and last links of $D'_{2,2}(P_{2,1}, P_{2,2})$ respectively, (2) $d'(2, 2)_{z_{2,2m+1}}$ is contained only in the $(2r-s+m)$ th link of $D_{2,1}$ ($m=1, 2, \dots, s-r$), (3) $d'(2, 2)_{z_{2,2m}}$ is contained only in the $s-(m-1)$ st link of $D_{2,1}$ ($m=1, 2, \dots, s-r$), and (4) $D'_{2,2}(z_{2,2m}, z_{2,2m+1})$ is contained in

$$D_{2,1}(s - (m - 1), 2r - s + m) \quad (m = 1, 2, \dots, s - r).$$

Step 6. Recall (Step 4) that $E'_{1,2}$ is a V -chain of $D_{1,1}$ relative to $H_{1,1}, P_{1,1}, Q_{1,1}, \dots, H_{1,3}, P_{1,3}, Q_{1,3}$ such that $E'_{1,2}(P_{1,j}, Q_{1,j})$ follows a pattern in $D_{1,1}$ that $D'_{2,2}(P_{2,j}, Q_{2,j})$ follows in $D_{2,1}$ ($j=1, 2, 3$) and in addition T maps the intersection of any link of $E'_{1,2}(P_{1,j}, Q_{1,j})$ with \mathcal{H}_1 into the corresponding link of $D'_{2,2}(P_{2,j}, Q_{2,j})$ ($j=1, 2, 3$).

By using Theorem 3, we find that there exists a chain $F_{1,2}$ and points Y, Z of M such that $F_{1,2}$ is a chain from $P_{1,1}$ to $Q_{1,3}$ that covers M , $F_{1,2}$ is contained in $E'_{1,2}$, any link of $E'_{1,2}$ that contains a link of $F_{1,2}$ also contains the closure of that link, $F_{1,2}(P_{1,j}, Q_{1,j})$ contains H_j ($j=1, 2, 3$), $f(1, 2)_Y, f(1, 2)_Z$ are contained in the first and last links respectively of $E'_{1,2}$, $f(1, 2)_{Q_{1,1}} < f(1, 2)_Y$, $f(1, 2)_Z < f(1, 2)_{P_{1,3}}$, $F_{1,2}(P_{1,1}, Y)$ is contained in $E'_{1,2}(P_{1,1}, Q_{1,1})$, $F_{1,2}(Z, Q_{1,3})$ is contained in $E'_{1,2}(P_{1,3}, Q_{1,3})$, and $F_{1,2}(P_{1,2}, Q_{1,2})$ is contained in $E'_{1,2}(P_{1,2}, Q_{1,2})$.

We obtain $E''_{1,2}$ by consolidating $F_{1,2}$ as follows. If $e'(1, 2)$ is a link of $E'_{1,2}(P_{1,1}, Q_{1,1})$ the sum of the links of $F_{1,2}(P_{1,1}, Q_{1,1})$ contained in $e'(1, 2)$ is a link of $E''_{1,2}$. Every interior link of $F_{1,2}(Q_{1,1}, Y)$ is a link of $E''_{1,2}$. If $e'(1, 2)$ is any link of $E'_{1,2}$, the sum of the links of $F_{1,2}(Y, Z)$ that are contained in $e'(1, 2)$ is a link of $E''_{1,2}$. Every interior link of $F_{1,2}(Z, P_{1,3})$ is a link of $E''_{1,2}$. Finally, if $e'(1, 2)$ is a link of $E'_{1,2}(P_{1,3}, Q_{1,3})$, then the sum of the links of $F_{1,2}(P_{1,3}, Q_{1,3})$ that are contained in $e'(1, 2)$ is a link of $E''_{1,2}$.

The chain $E''_{1,2}$ arrived at in this way has the following properties: (1) $E''_{1,2}$ is a chain from $P_{1,1}$ to $Q_{1,3}$ that covers M and is contained in $D_{1,1}$, (2) any link of $D_{1,1}$ that contains a link of $E''_{1,2}$ contains the closure of that link, (3) $E''_{1,2}(P_{1,j}, Q_{1,j})$ covers H_j and follows a pattern in $D_{1,1}$ that $D'_{2,2}(P_{2,j}, Q_{2,j})$ follows in $D_{2,1}$ ($j=1, 2, 3$), (4) T maps the intersection of any link of $E''_{1,2}(P_{1,j}, Q_{1,j})$ with \mathcal{H}_1 into the corresponding link of $D'_{2,2}(P_{2,j}, Q_{2,j})$ ($j=1, 2, 3$), (5) $E''_{1,2}(Y, Z)$ follows every pattern in $D_{1,1}$ that $E'_{1,2}$ does, and (6) $E''_{1,2}(P_{1,1}, Y)$, $E''_{1,2}(Z, Q_{1,3})$ are contained in $D_{1,1}(P_{1,1}, Q_{1,1})$, $D_{1,1}(P_{1,3}, Q_{1,3})$ respectively.

Recall (Step 4) that $D_{1,1}$ has the same number of links as $D_{2,1}$ (i.e., s , Step 5) and $P_{1,2}$ lies in the link of $D_{1,1}$ that corresponds to the link of $D_{2,1}$ containing $P_{2,2}$ (i.e., r , also Step 5). Then since $E'_{1,2}$ is a V -chain of $D_{1,1}$ relative to $H_{1,1}, P_{1,1}, Q_{1,1}, \dots, H_{1,3}, P_{1,3}, Q_{1,3}$ and $E''_{1,2}(Y, Z)$ follows every pattern in $D_{1,1}$ that $E'_{1,2}$ does, we find (compare with Step 5) that $E''_{1,2}(Y, Z)$ is the sum of $4(s-r) + 3$ subchains $E''_{1,2}(x_{1,m}, x_{1,m+1})$, $E''_{1,2}(P_{1,2}, Q_{1,2})$, $E''_{1,2}(Q_{1,2}, X_1)$, $E''_{1,2}(y_{1,m}, y_{1,m+1})$ ($m=1, 2, \dots, 2(s-r)$) and $E''_{1,2}(y_{1,1}, y_{1,2(s-r)+2})$, where (1) $x_{1,1} = e''(1, 2)_Y$, $e''(1, 2)_{x_{1,2(s-r)+1}}$ is $e''(1, 2)_{P_{1,2}}$, (2) $e''(1, 2)_{x_{1,2m+1}}$ is contained only in the $(2r - s + m)$ th link of $D_{1,1}$ ($m+1, 2, \dots, s-r$), (3) $e''(1, 2)_{x_{1,2m}}$ is contained only in the $s - (m-1)$ st link of $D_{1,1}$ ($m=1, 2, \dots, s-r$), (4) $E''_{1,2}(x_{1,2m}, x_{1,2m+1})$ is contained in $D_{1,1}(s - (m-1), 2r - s + m)$ ($m=1, 2, \dots, s-r$), (5) $E''_{1,2}(P_{1,2}, Q_{1,2})$ is contained in $D_{1,1}(P_{1,2}, Q_{1,2})$, (6) X_1 is a point of M such that $e''(1, 2)_{X_1}$ is contained in $d(1, 1)_{P_{1,2}}$ and $E''_{1,2}(Q_{1,2}, X_1)$ is contained in $D_{1,1}(P_{1,2}, Q_{1,2})$, (7) $e''(1, 2)_{y_{1,1}}$ is

contained in $d(1, 1)_1$, (8) $e''(1, 2)_{x_1}$ is $e''(1, 2)_{y_{1,2(s-r)+1}}$, (9) $e''(1, 2)_{y_{1,2m+1}}$ is contained only in the $(2r - s + m)$ th link of $D_{1,1}$ ($m = 1, 2, \dots, s - r$), (10) $e''(1, 2)_{y_{1,2m}}$ is contained only in the $s - (m - 1)$ st link of $D_{1,1}$ ($m = 1, 2, \dots, s - r$), (11) $E_{1,2}(y_{1,2m}, y_{1,2m+1})$ is contained in $D_{1,1}(s - (m - 1), 2r - s + m)$ ($m = 1, 2, \dots, s - r$), and (12) $e''(1, 2)_{y_{1,2(s-r)+2}}$ is $e''(1, 2)_z$.

Step 7. In this step we make repeated use of Theorem 1 in order to obtain a chain $D'_{1,2}(P_{1,1}, P_{1,2})$ from $P_{1,1}$ to $P_{1,2}$ that follows a pattern in $D_{1,1}$ that $D'_{2,2}(P_{2,1}, P_{2,2})$ follows in $D_{2,1}$.

By the corollary to Theorem 3, there exists a sequence of chains $E''_{1,2} = C_1, C_2, \dots$ from $P_{1,1}$ to $Q_{1,3}$ such that for each positive integer m , C_{m+1} is crooked in C_m , any link of C_m that contains a link of C_{m+1} also contains the closure of that link, and the mesh of C_m is less than $1/m$.

Then by using Theorem 1, we find that there exists an integer j_0 and a chain F_0 such that F_0 is a consolidation of the links of C_{j_0} that are contained in $E''_{1,2}(Y, Q_{1,1})$. F_0 follows a pattern in $D_{1,1}(P_{1,1}, Q_{1,1})$ that $E''_{1,2}(P_{1,1}, Q_{1,1})$ does, and no interior link of F_0 intersects $e''(1, 2)_r + e''(1, 2)_{q_{1,1}}$.

Also there exists an integer j_1 and a chain F_1 such that F_1 is a consolidation of the links of C_{j_1} that are contained in $E''_{1,2}(x_{1,1}, x_{1,z})$, F_1 follows a pattern in $D_{1,1}$ that $D'_{2,2}(Z_{2,1}, Z_{2,2})$ follows in $D_{2,1}$, and no interior link of F_1 intersects $e''(1, 2)_{x_{1,1}} + e''(1, 2)_{x_{1,z}}$.

Similarly, there exists integers $j_2, j_3, \dots, j_{2(s-r)}$ and chains $F_2, F_3, \dots, F_{2(s-r)}$ such that F_m is a consolidation of the links of C_{j_m} that are contained in $E''_{1,2}(x_{1,m}, x_{1,m+1})$, F_m follows a pattern in $D_{1,1}$ that $D'_{2,2}(Z_{2,m}, Z_{2,m+1})$ follows in $D_{2,1}$, and no interior link of F_m intersects

$$e''(1, 2)_{x_{1,m}} + e''(1, 2)_{x_{1,m+1}} \quad (m = 2, 3, \dots, (s - r)).$$

Let j be an integer at least as great as the maximum of $j_0, j_1, \dots, j_{2(s-r)}$. Then by using the chains $F_0, F_1, \dots, F_{2(s-r)}$ found above, we obtain a chain $D'_{1,2}(P_{1,1}, P_{1,2})$ from $P_{1,1}$ to $P_{1,2}$ that is a consolidation of the links of C_j that are contained in $E''_{1,2}(P_{1,1}, P_{1,2})$ and such that (1) $D'_{1,2}(P_{1,1}, P_{1,2})$ follows a pattern in $D_{1,1}$ that $D'_{2,2}(P_{2,1}, P_{2,2})$ follows in $D_{2,1}$, and (2) if $d'(1, 2)_k$ is a link of $D'_{1,2}(P_{1,1}, Q_{1,1})$, T maps the intersection of this link with \mathcal{H}_1 into the k th link of $D'_{2,2}(P_{2,1}, Q_{2,2})$.

Step 8. By following a procedure similar to that gone through above, we finally obtain a chain $D'_{1,2}$ from $P_{1,1}$ to $Q_{1,3}$ that covers M and such that (1) $D'_{1,2}$ follows a pattern in $D_{1,1}$ that $D'_{2,2}$ follows in $D_{2,1}$, (2) $D'_{1,2}(P_{1,j}, Q_{1,j})$ covers H_j ($j = 1, 2, 3$), and (3) if $d'(1, 2)_k$ is a link of $D'_{1,2}(P_{1,j}, Q_{1,j})$, T maps the intersection of $d'(1, 2)_k$ with H_j into the corresponding link of $D'_{2,2}(P_{2,j}, Q_{2,j})$ ($j = 1, 2, 3$).

Step 9. In this step we consolidate $D'_{1,2}$ and obtain a chain $D_{1,2}$ that will correspond to the chain $E_{2,2}$.

If the i th link of $D'_{2,2}$ is contained in the j th link of $E_{2,2}$, let the j th link of $D_{1,2}$ contain the i th link of $D'_{1,2}$. Though $D_{1,2}$ may not be contained in $D_{1,1}$, this much is true. If the i th link of $D'_{2,2}$ is also contained in the k th link of $D_{2,1}$, then the j th link of $D_{1,2}$ is contained in the sum of the k th link of $D_{1,1}$ and one of the adjacent links.

Thus, if the j th link of $E_{2,2}$ intersects the k th link of $D_{2,1}$, the distance between the j th link of $D_{1,2}$ and the k th link of $D_{1,1}$ is less than $\mu(D_{1,1})$ and if the j th link of $D_{1,2}$ intersects the k th link of $D_{1,1}$ the distance between the j th link of $E_{2,2}$ and the k th link of $D_{2,1}$ is less than $\mu(D_{2,1})$.

Note also that if a link of $D_{1,2}$ intersects \mathcal{K}_1 , T maps the intersection of this link with \mathcal{K}_1 into the corresponding link of $E_{2,2}$.

Since $D'_{2,2}$ follows a V -pattern in $E_{2,2}$ relative to $H_{2,1}, P_{2,1}, Q_{2,1}, \dots, H_{2,3}, P_{2,3}, Q_{2,3}$, $D_{1,2}$ follows a V -pattern in $D_{1,2}$ relative to $H_{1,1}, P_{1,1}, Q_{1,1}, \dots, H_{1,3}, P_{1,3}, Q_{1,3}$ and $D_{1,2}$ has property S relative to the same collection.

Step 10. In this step we indicate how this process may be continued so as to obtain the chains $D_{i,1}, D_{i,2}, \dots$ ($i = 1, 2$) that we shall use in defining the desired extension of T .

In general, we proceed exactly as before (Steps 1–9) with one added precaution. Recall that each link $d(2, 2)$ of $D_{2,2}$ (Step 4) will be the sum of a link $e(2, 2)$ of $E_{2,2}$ plus (perhaps) some links $f(2, 2)$ of $F_{2,2}$ that intersect $e(2, 2)$. In obtaining $F_{2,2}$ then we require that the mesh of $F_{2,2}$ be less than $1/8$ and in addition also so small that if $e(2, 2)$ and $d(2, 1)$ are any two links of $E_{2,2}, D_{2,1}$ respectively that are a positive distance apart, then (1) $d(2, 2)$ (the link of $D_{2,2}$ containing $e(2, 2)$) is also at a positive distance from $d(2, 1)$, and (2) $d(2, 2)$ is contained in every link of $D_{2,1}$ that contained $e(2, 2)$. With this added precaution, the following statements are true: (1) $\mu(D_{2,2}) < 1/2$, (2) $D_{2,2}$ and $D_{1,2}$ have the same number of links, (3) if the j th link of $D_{2,2}$ intersects the k th link of $D_{2,1}$, the distance between the j th link of $D_{1,2}$ and the k th link of $D_{1,1}$ is less than $2\mu(D_{1,1})$, (4) if the j th link of $D_{1,2}$ intersects the k th link of $D_{1,1}$, the distance between the j th link of $D_{2,2}$ and the k th link of $D_{2,1}$ is less than $\mu(D_{2,1})$ and (5) if a link of $D_{1,2}$ intersects \mathcal{K}_1 , T maps the intersection of this link with \mathcal{K}_1 into the corresponding link of $E_{2,2}$.

In general then we obtain chains $D_{2,1}, D_{1,1}, D_{1,2}, D_{2,2}, D_{1,3}, \dots$ such that (1) $D_{i,j}$ is a chain from $P_{i,1}$ to $Q_{i,3}$ that covers M ($i = 1, 2, j = 1, 2, \dots$),

$$(2) \quad \mu(D_{1,2j-1}) < \frac{1}{2^j}, \quad \mu(D_{1,2j}) < \frac{1}{2^{j-1}}, \quad \mu(D_{2,2j}) < \frac{1}{2^j}, \quad \mu(D_{2,2j+1}) < \frac{1}{2^{j-1}},$$

$$(j = 1, 2, \dots),$$

(3) $D_{1,j}$ and $D_{2,j}$ each have the same number of links, (4) if the k th link of $D_{i,j+1}$ intersects the m th link of $D_{i,j}$, the distance between the k th link of $D_{i',j+1}$ and the m th link of $D_{i',j}$ is less than $2\mu(D_{i',j})$ ($i, i' = 1, 2, j = 1, 2, \dots$),

and (5) if the k th link of $D_{1,2j}$ intersects \mathcal{K}_1 , T maps their common part into the k th link of $D_{2,2j}$ ($j=1, 2, \dots$), (6) if the k th link of $D_{2,2j-1}$ intersects \mathcal{K}_2 , T^{-1} maps their common part into the k th link of $D_{1,2j-1}$ ($j=1, 2, \dots$).

Then Theorem 5 applies to this sequence of chains and the homeomorphism of M onto M induced by them is the desired extension of T .

The proof for the case of n an arbitrary positive integer is almost identical to the proof for the case $n=3$. In fact, the only changes necessary in Steps 1-7 are purely notational. Step 8 is complicated only to the extent that some extra applications of Theorem 1 are necessary to obtain the chain $D_{1,2}$. And the changes in Steps 9 and 10 are again only changes in notation. As an easy consequence we have

Proof of Theorem 7. The necessity of the condition is obvious.

In proving sufficiency, we first consider the case that M is irreducible between each pair of components of \mathcal{K}_1 . Then the condition stated implies that M is irreducible between each pair of components of \mathcal{K}_2 and the desired extension exists by Theorem 6.

In case M is not irreducible between each pair of components of \mathcal{K}_1 , various possibilities arise. We will consider one of these cases in detail. In particular, suppose that M contains a proper subcontinuum A containing $H_{1,1}$, $H_{1,2}$, and $H_{1,3}$ and that M is irreducible between each pair of the sets A , $H_{1,4}$, \dots , $H_{1,n}$. Suppose further that A contains a proper subcontinuum B that is irreducible about $H_{1,1}$ and $H_{1,2}$ and that A is irreducible from B to $H_{1,3}$. Then by choosing points P_1, P_2, P_3 in $H_{1,1}, H_{1,2}, H_{1,3}$ respectively, we see that the hypothesized homeomorphism of M onto M taking P_i onto $T(P_i)$ ($i=1, 2, 3$) implies the existence of a proper subcontinuum A' of M containing $H_{2,1}, H_{2,2}$, and $H_{2,3}$. In addition we note that A' contains a proper subcontinuum B' that is irreducible about $H_{2,1}$ and $H_{2,2}$ and that A' is irreducible from B' to $H_{2,3}$. In addition the existence of a homeomorphism of M onto M that agrees with T on a set of points P_1, P_4, \dots, P_n (where P_i is arbitrarily chosen in $H_{1,i}$) shows that M is irreducible between each pair of the sets $A', H_{2,4}, \dots, H_{2,n}$. Then by Theorem 6 there exists a homeomorphism T_1 of B onto B' that agrees with T on $H_{1,1}$ and $H_{1,2}$. Again by Theorem 6, there exists a homeomorphism T_2 of A onto A' that agrees with T_1 on B and with T on $H_{1,3}$. A final application of Theorem 6 yields a homeomorphism of M onto M that agrees with T_2 on A and with T on $H_{1,4}, H_{1,5}, \dots, H_{1,n}$. This homeomorphism is the desired extension of T .

The remaining possibilities are disposed of by an extension of the techniques used above.

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