VITALI'S THEOREM FOR INVARIANT MEASURES

BY

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1. **Introduction.** If $\mathfrak{U}$ is a family of closed sets each with positive k-dimensional Lebesgue measure, and if the subset A of the Euclidean space $X=R^k$ has the property that for each neighborhood V of each point $x\in A$ there exists $U\in\mathfrak{U}$ with $x\in U\subset V$, then $\mathfrak{U}$ is said to be a Vitali cover for A. The classical Vitali theorem for the case $k=1$ asserts that if $\mathfrak{U}$ is a Vitali cover for the set $A\subset X=R$, then there is in $\mathfrak{U}$ a sequence of pairwise disjoint elements whose union exhausts all of A but a Lebesgue null set. The conclusion follows also if $k>1$, provided one assumes that the Vitali cover $\mathfrak{U}$ of A is regular in the following sense: For each $x\in A$ there is a number $\alpha>0$ and a sequence U_n of elements of $\mathfrak{U}$ and a sequence S_n of spheres (i.e., "balls") for which $x\in U_n \subset S_n$, $\lim_{n\to\infty}\lambda_k S_n=0$, and $\lambda_k U_n/\lambda_k S_n\geqq\alpha$. ($\lambda_k$ denotes Lebesgue measure in the space $X=R^k$. The number α is called a parameter of regularity at x.)

The invariance under translation of the set-function λ_k suggests the point of view adopted in the present generalization of Vitali's theorem. We consider a group G acting transitively on a Hausdorff space X, the latter endowed with a measure μ for which $\mu(gB)=\mu B$ whenever $g\in G$ and B is a μ-measurable subset of X. A Vitali covering for a subset A of X is defined in the obvious way. Noticing that if S_α denotes the sphere of radius α centered at the origin ϕ in R^k, then $S_\alpha+S_\alpha\subset S_{2\alpha}$ and $\lambda_k S_\alpha/\lambda_k S_{2\alpha}=1/2^k$, we define regularity of a Vitali cover in terms of a set-theoretic "multiplication" defined between subsets of X. We replace the spheres S_α by what we call quasispheres. As a conceptual aid, the reader may regard the quasisphere S_n, of Definition 2.7, as corresponding to a sphere centered at ϕ of radius $1/2^n$. (See §4 for a precise treatment of this matter.)

The upshot of these considerations is our main result, Theorem 3.5, which may be informally stated as follows: If μ is G-invariant on X, and if $\mathfrak{U}$ is a regular Vitali cover for $A\subset X$, then $\mathfrak{U}$ admits a sequence of pairwise disjoint elements whose union exhausts all of A but a μ-null set. Our theorem includes the classical Vitali theorem in R^k, but the first example of §4 shows that even in R^2 the conventional regularity restrictions imposed upon the cover $\mathfrak{U}$ are unnecessarily stringent. This example shows that Vitali's conclusion may hold, in nontrivial circumstances, even when $\mathfrak{U}$ admits no positive parameter of regularity at any point $x\in A$.

Presented to the Society, February 25, 1961; received by the editors October 24, 1960.

[1] The research on which this paper is based was conducted while each of the two authors was supported in part by the Office of Ordnance Research, U. S. Army, under Contract DA-19-020-ORD-3778.

The question as to whether regular Vitali covers can exist other than in the case $X=R^k$ is settled affirmatively in §4. There we assert their existence whenever G satisfies at some point a "local equi-Lipschitz condition" (see §4) and X is locally Euclidean. A simple instance in which these hypotheses are satisfied is the case in which $G=O_3$ is the orthogonal group of rotations acting on the 2-sphere $X=S^2\subset R^3$, μ being taken as 2-dimensional Lebesgue measure in X; many other examples will occur to the reader.

Our theorem is readily applicable to the instance in which a locally compact Hausdorff group acts by left-translation on itself, μ being taken as Haar measure. We show that any Lie group G acting upon itself satisfies the "local equi-Lipschitz condition", and thus that every subset of G admits numerous Vitali covers which are regular relative to a naturally defined collection of quasispheres.

There exist many generalizations of the Vitali theorem; the interested reader is referred to [2, pp. 267–268] for a bibliography on this topic. None of these generalizations seems to be relevant to our present undertaking.

2. Preliminary discussion and definitions.

2.1. Standing conventions. X will denote a Hausdorff space, G a group which acts transitively on X. By this we mean that G is a subgroup of the group of all one-to-one mappings of X onto itself such that, whenever $x\in X$ and $y\in X$, there is a $g\in G$ for which $gx=y$. We consider on X a countably additive, non-negative measure μ which is G-invariant, in the sense that gU is μ-measurable and $\mu gU=\mu U$ whenever U is a μ-measurable subset of X and $g\in G$. We suppose also that for each $x\in X$ and each $\epsilon>0$ there exists a μ-measurable neighborhood V of x for which $\mu V<\epsilon$. The outer measure determined by μ will be denoted μ^*.

2.2. Remark. The requirements of 2.1 are satisfied, for example, if μ is Haar measure on a locally compact nondiscrete Hausdorff group $G=X$ which acts on itself by left-translation. We do not specifically suppose μ to be a regular Borel measure which assigns measure zero to each point of X, since such an assumption does not shorten our discussion.

2.3. Definition. ϕ will denote a point of X, arbitrary but henceforth fixed.

2.4. Definition. For $B\subset X$ and $C\subset X$, let $B^{-1}C=\bigcup\{gC\mid g\in G,\ \phi\in gB\}$.

2.5. Remark. If ϕ' had replaced ϕ, and if h is any element of G for which $h\phi=\phi'$, then $B^{-1}C$ is replaced by the set $D=\bigcup\{gC\mid g\in G,\ \phi'\in gB\}$ $=\bigcup\{hgC\mid g\in G,\ \phi'\in hgB\}=\bigcup\{hgC\mid g\in G,\ \phi\in gB\}=hB^{-1}C$. Thus a change in the choice of ϕ results simply in a translation of the set $B^{-1}C$.

In case $X=G$ is a group acting by translation upon itself, $B^{-1}C$ has its usual meaning provided ϕ is chosen to be the identity element; that is, $B^{-1}C$ represents the set of all elements of the form $g^{-1}h$ with $g\in B$ and $h\in C$. In the discussion which follows we shall be concerned chiefly with the situation where $B=C$; $B^{-1}B$ may be described as the union of all translates of B which contain ϕ.

Purely as a notational convenience, we introduce the following definition.

2.6. DEFINITION. For $B \subset X$ and $C \subset X$, let

$$BC = \bigcup \{gC \mid g \in G, g\phi \in B\}.$$

The following two definitions describe the covers to which our theorem applies.

2.7. DEFINITION. A sequence S_n of subsets of X will be called a sequence of quasispheres if, for some $\epsilon > 0$ and each positive integer n:

$$(1) \qquad \overset{-1}{S}_{n+1} S_{n+1} \subset S_n;$$

$$(2) \qquad \mu^* S_{n+1} > \epsilon \mu^* S_n.$$

2.8. DEFINITION. Let $A \subset X$, and let $\mathfrak{U}$ be a family of closed, μ-measurable subsets of X each with positive μ-measure. Then $\mathfrak{U}$ is said to be a regular Vitali cover for A if there exists a sequence S_n of quasispheres and a real-valued function M on A such that, for each $x \in A$ and each neighborhood V of x, there is a $U \in \mathfrak{U}$, a $g \in G$ and an integer n for which $x \in U \subset V$, $gU \subset S_n$ and $\mu^* S_n \leqq M(x) \mu U$.

2.9. REMARK. The concept of a regular Vitali cover has been defined in terms of the set-theoretic multiplication of Definition 2.4, which depends in turn upon the choice of $\phi \in X$. But from Remark 2.5 and the fact that $(hB)^{-1}(hC) = B^{-1}C$ for each $h \in G$, it is easy to see that if a cover $\mathfrak{U}$ is a regular Vitali cover for A for a particular choice of ϕ, then it is a regular Vitali cover for every choice of ϕ.

The function M of Definition 2.8 corresponds to the parameter of regularity of the classical theory. This observation motivates the following definition.

2.10. DEFINITION. A family $\mathfrak{U}$ of closed, μ-measurable subsets of X is said to be a regular Vitali cover with constant parameter of regularity if it satisfies the conditions of the preceding definition with some constant function M.

3. The main theorem.

3.1. LEMMA. *Let $A \subset X$, and let $\mathfrak{U}$ be a collection of closed μ-measurable subsets of X each of which has positive μ-measure. Let K be an open subset of X for which $A \subset K$ and $\mu^* K < \infty$, and suppose that for each $x \in A$ and each neighborhood V of x there exists $U \in \mathfrak{U}$ for which $x \in U \subset V$. Suppose there is a real number P and a sequence C_n of subsets of X for which*

(1) $n\mu^*(C_n C_n) < P$ for each n; and

(2) if $U \in \mathfrak{U}$ and $g \in G$ with $\phi \in gU$ and $\mu U < 1/n$, then $gU \subset C_n$.

Then there is a (possibly finite) sequence U_k of pairwise disjoint elements of $\mathfrak{U}$ for which $\mu^*(A \setminus \bigcup_{k=1}^{\infty} U_k) = 0$. The sequence may be chosen so that $\bigcup_{k=1}^{\infty} U_k \subset K$.

Proof. Insofar as the proof of Banach (see [1]) is applicable in our generalized context, we present it here as reported in [3, §39] by Munroe.

We may as well suppose, discarding certain elements of $\mathfrak{U}$ if necessary, that $U \subset K$ whenever $U \in \mathfrak{U}$.

The sequence U_k is defined recursively. Choose any $U_1 \in \mathfrak{U}$, and suppose that U_k has been chosen for $1 \leqq k \leqq n$. If $A \subset \bigcup_{k=1}^n U_k$, the construction terminates. Otherwise, since the elements of $\mathfrak{U}$ are closed, there are a point $x \in A \setminus \bigcup_{k=1}^n U_k$ and a $U \in \mathfrak{U}$ for which $x \in U$ and $U \cap \bigcup_{k=1}^n U_k = \emptyset$. Then, with $\delta_n = \sup\{\mu U \mid U \in \mathfrak{U},\ U \cap \bigcup_{k=1}^n U_k = \emptyset\}$, we have $\delta_n > 0$. We select $U_{n+1} \in \mathfrak{U}$ with $U_{n+1} \cap \bigcup_{k=1}^n U_k = \emptyset$ and $\mu U_{n+1} > \delta_n/2$.

To show the sequence U_k is as required, we set $T = \limsup T_n = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty T_n$ where, for each positive integer n, we define

$$T_n = \bigcup \{U \in \mathfrak{U} \mid \mu U < 2\mu U_n,\ U \cap U_n \neq \emptyset\}.$$

We will show (1) $A \setminus \bigcup_{k=1}^\infty U_k \subset T$ and (2) $\mu^* T = 0$.

Let $x \in A \setminus \bigcup_{k=1}^\infty U_k$, and choose a positive integer m. Then $x \notin \bigcup_{k=1}^m U_k$, so there is a $U \in \mathfrak{U}$ with $x \in U$ and $U \cap \bigcup_{k=1}^m U_k = \emptyset$. Since the measurable sets U_k are pairwise disjoint subsets of K, it follows from our hypothesis $\mu^* K < \infty$ that $\mu U_k \to 0$. If $U \cap U_k = \emptyset$ for each $k > m$, then $\mu U \leqq \delta_k < 2\mu U_{k+1}$ for each $k > m$, so that $\mu U = 0$. From this contradiction it follows that there is a smallest integer N for which $U \cap U_N \neq \emptyset$. Then $\mu U \leqq \delta_{N-1} < 2\mu U_N$, so that $x \in U \subset T_N$. Since $N > m$, we have $x \in \bigcup_{n=m}^\infty T_n$. Thus (1) is established.

To prove (2), we first find an integer n_0 for which $2\mu U_n < 1$ whenever $n \geqq n_0$. Let $n \geqq n_0$ be fixed, and let z be a fixed point in U_n. We will show that if N is the largest integer for which $2\mu U_n < 1/N$, and if h is any element of G for which $hz = \phi$, then $T_n \subset h^{-1} C_N C_N$. For each $x \in T_n$ there exist $U \in \mathfrak{U}$ and $y \in U \cap U_n$ for which $x \in U$ and $\mu U < 2\mu U_n$. Choosing a point $g \in G$ for which $gy = \phi$, we see from hypothesis (2) of the present lemma that $gU \subset C_N$ and $hU_n \subset C_N$, so that $gx \in C_N$ and $hy \in C_N$. Then $\phi = (gh^{-1})hy \in gh^{-1} C_N$, so that $hg^{-1} C_N \subset C_N C_N$ by 2.6. Hence $hx \in C_N C_N$ for each $x \in T_n$. Now since $2\mu U_n \geqq 1/(2N)$, we have $\mu^* T_n \leqq \mu^* C_N C_N < P/N \leqq 4P\mu U_n$ for each $n \geqq n_0$, so that $\sum_{n=n_0}^\infty \mu^* T_n \leqq 4P \sum_{n=n_0}^\infty \mu U_n \leqq 4P\mu^* K < \infty$. That $\mu^* T = 0$ now follows from the relations $\mu^* T \leqq \mu^* \bigcup_{n=k}^\infty T_n \leqq \sum_{n=k}^\infty \mu^* T_n$, valid for each k. This concludes the proof of the lemma.

The next two lemmas concern the properties of the product $B^{-1}C$.

3.2. LEMMA. *If $g \in G$ and $U \subset X$, then $g\phi \in U^{-1}U$ if and only if $g^{-1}\phi \in U^{-1}U$.*

Proof. If $g\phi \in U^{-1}U$, then there is an $h \in G$ for which $g\phi \in hU$ and $\phi \in hU$. But then $\phi \in g^{-1}hU$ and $g^{-1}\phi \in g^{-1}hU$, so that $g^{-1}\phi \in U^{-1}U$ by 2.4. The replacement of g by g^{-1} in this argument proves the converse.

3.3. LEMMA. *Let $U_\iota \subset X$ for each $\iota \in I$, and let $C = \bigcup_{\iota \in I} U_\iota^{-1} U_\iota$. Then $CD = C^{-1}D$ for each $D \subset X$.*

Proof. We have, using 3.2,

$$C^{-1}D = \left(\bigcup_{\iota \in I} U_\iota^{-1}U_\iota\right)^{-1}D = \bigcup \left\{gD \mid g \in G, g^{-1}\phi \in \bigcup_{\iota\in I} U_\iota^{-1}U_\iota\right\}$$
$$= \bigcup \left\{gD \mid g \in G, g\phi \in \bigcup_{\iota\in I} U_\iota^{-1}U_\iota\right\} = \left(\bigcup_{\iota\in I} U_\iota^{-1}U_\iota\right)D = CD.$$

We now prove as a lemma a special case of our theorem.

3.4. Lemma. *Let $\mathfrak{U}$ be a regular Vitali cover for A with constant parameter of regularity M. Let K be an open subset of X for which $A\subset K$ and $\mu^*K<\infty$. Then there exists a (possibly finite) sequence U_k of pairwise disjoint elements of $\mathfrak{U}$ for which $\mu^*(A\setminus\bigcup_{k=1}^\infty U_k)=0$. The sequence may be chosen so that $\bigcup_{k=1}^\infty U_k\subset K$.*

Let ϵ and S_n be as in Definition 2.8. We may clearly suppose, discarding certain elements from $\mathfrak{U}$ if necessary, that for each $U\in\mathfrak{U}$ there are a $g\in G$ and an S_k with $gU\subset S_k$ and $\mu^*S_k\leqq M\mu U$.

Since $\phi\in B^{-1}B$ for each nonempty subset B of X, we have $\phi\in S_{k+1}^{-1}S_{k+1}\subset S_k$ for each k. Denoting by e the identity element of the group G, we have $S_{k+1}=eS_{k+1}\subset\bigcup\{gS_{k+1}\mid g\in G,\ \phi=g\phi\}=\{\phi\}^{-1}S_{k+1}\subset S_{k+1}^{-1}S_{k+1}\subset S_k$, so that the real sequence μ^*S_k is monotone decreasing. Since $\mathfrak{U}$ is a Vitali cover for the set A, each point of which admits neighborhoods of arbitrarily small μ-measure, we see from the conclusion of the preceding paragraph that $\mu^*S_k\to 0$.

Now for each positive integer n we set $C_n=\bigcup\{U^{-1}U\mid U\in\mathfrak{U},\ \mu U<1/n\}$. The sequence C_n clearly satisfies hypothesis (2) of 3.1, so by 3.1 the proof of the present lemma may be completed by showing that the real sequence $n\mu^*C_nC_n$ is bounded. To this end we restrict our attention to those positive integers n for which $n>M/\mu S_2$, and we associate with each such n the smallest integer k_n for which $\mu^*S_{k_n}\leqq M/n$. Then always $k_n\geqq 3$ and $S_k\subset S_{k_n}$ whenever $\mu^*S_k\leqq M/n$. Thus for each $U\in\mathfrak{U}$ with $\mu U<1/n$ there is a $g\in G$ for which $gU\subset S_{k_n}$. Hence $U^{-1}U=(gU)^{-1}(gU)\subset S_{k_n}^{-1}S_{k_n}\subset S_{k_n-1}$ whenever $U\in\mathfrak{U}$ and $\mu U<1/n$. Thus $C_n\subset S_{k_n-1}$, and from 3.3 we have $C_nC_n=C_n^{-1}C_n\subset S_{k_n-1}^{-1}S_{k_n-1}\subset S_{k_n-2}$. Hence $\mu^*C_nC_n\leqq\mu^*S_{k_n-2}<\mu^*S_{k_n-1}/\epsilon<\mu^*S_{k_n}/\epsilon^2\leqq M/(\epsilon^2 n)$ whenever $n>M/\mu S_2$, and the proof is complete.

The next result, our main theorem, is obtained by removing some of the hypotheses of the preceding lemma.

3.5. Theorem. *Let $\mathfrak{U}$ be a regular Vitali cover for A, and suppose that there is a sequence V_k of open subsets of X for which $A\subset\bigcup_{k=1}^\infty V_k$ and always $\mu^*V_k<\infty$. Then there is a (possibly finite) sequence U_k of pairwise disjoint elements of $\mathfrak{U}$ for which $\mu^*(A\setminus\bigcup_{k=1}^\infty U_k)=0$.*

Proof. Let S_n be the sequence of quasispheres, M the (not necessarily constant) function given by 2.8. We suffer no loss in generality in supposing that $V_k\subset V_{k+1}$ for each positive integer k. For each positive integer k, we define

$$A_k = A\cap M^{-1}(0,k]\cap V_k$$

and $\mathfrak{U}_k = \{ U \in \mathfrak{U} \mid gU \subset S_n$ for some $g \in G$ and some integer n with $\mu^* S_n < k\mu U\}$. Then $\mathfrak{U}_k$ is a regular Vitali cover for A_k with constant parameter of regularity k, so by 3.4 we can find pairwise disjoint elements $U_1, U_2, \cdots, U_{n_1}$, of $\mathfrak{U}_1$ for which $\mu^*(A_1 \setminus \bigcup_{k=1}^{n_1} U_k) < 1$. Proceeding inductively on the assumption that pairwise disjoint elements $U_1, \cdots, U_{k_n}$ of $\mathfrak{U}_n$ have been chosen so that $\mu^*(A_n \setminus \bigcup_{k=1}^{k_n} U_k) < 1/n$, we notice that $W_n = V_n \setminus \bigcup_{k=1}^{k_n} U_k$ is an open subset of X for which $A_n \setminus \bigcup_{k=1}^{k_n} U_k \subset W_n$ and $\mu^* W_n < \infty$. Thus there are by Lemma 3.4 pairwise disjoint elements $U_{k_n+1}, \cdots, U_{k_{n+1}}$ of $\mathfrak{U}_{n+1}$ for which

$$\mu^*\left[\left(A_n \setminus \sum_{k=1}^{k_n} U_k\right) \setminus \bigcup_{k=k_n+1}^{k_{n+1}} U_k\right] < \frac{1}{n+1}$$

and

$$\bigcup_{k=k_n+1}^{k_{n+1}} U_k \subset W_n = V_n \setminus \bigcup_{k=1}^{k_n} U_k.$$

From these computations it follows that

$$\mu^*\left(A_n \setminus \bigcup_{k=1}^{\infty} U_k\right) \leqq \mu^*\left(A_n \setminus \bigcup_{k=1}^{k_n} U_k\right) < 1/n \quad \text{for each positive integer } n,$$

which together with the relations $A = \bigcup_{n=1}^{\infty} A_n$ and $A_n \subset A_{n+1}$ $(n \geqq 1)$ yields the conclusion $\mu^*(A \setminus \bigcup_{k=1}^{\infty} U_k) = 0$.

4. EXAMPLES. The examples presented here are designed to indicate the applicability of our theorem to numerous instances in which a group acts transitively on a measure space. For the sake of simplicity we consider cases where each U in the cover $\mathfrak{U}$ is a translate of a quasisphere. It will be clear how more complicated examples can be constructed. We begin by stating a corollary to our theorem.

4.1. COROLLARY. *Let S_n be a sequence of closed, measurable quasispheres, and let A be contained in a countable union of open sets each of which has finite outer measure. Suppose that for each neighborhood V of each point x in A there exists an element $g \in G$ and a positive integer n for which $x \in gS_n \subset V$. Then there are a sequence g_k of elements of G and a sequence n_k of positive integers for which the sets $g_k S_{n_k}$ are pairwise disjoint and $\mu^*(A \setminus \bigcup_{k=1}^{\infty} g_k S_{n_k}) = 0$.*

From this corollary it follows, speaking loosely, that Theorem 3.5 will be applicable to any situation in which quasispheres can be introduced. Euclidean k-space R^k is one setting in which this can be done in such a way that the cover which results from a translation of the quasispheres to the points of a nonempty subset A of R^k is nowhere regular in the classical sense. By using a sequence of solid ellipses with increasing eccentricity, we give in Example 4.2 a possible construction for the case $k = 2$.

4.2. EXAMPLE. Let the topological group $G = X = R^2$ act on itself by translation, and let μ denote 2-dimensional Lebesgue measure. Choosing ϕ to be the identity element in R^2 we define

$$S_n = \{(x_1, x_2) \mid (x_1, x_2) \in R^2 \quad \text{and} \quad (2^n x_1)^2 + (2^{2n} x_2)^2 \leqq 1\},$$

for each positive integer n. To see that the closed sets S_n form a sequence of quasispheres in the sense of Definition 2.7, we observe that $S_{n+1}^{-1}S_{n+1} = 2S_{n+1} = \{(x_1, x_2) \mid (x_1, x_2) \in R^2$ and $(2^n x_1)^2 + (2^{2n+1}x_2)^2 \leqq 1\} \subset S_n$, and that $\mu S_{n+1} = \pi/2^{3n+2} > \epsilon\pi/2^{3n} = \epsilon\mu S_n$ if $\epsilon < 1/8$. The sequence S_n being fundamental at ϕ, we see that the hypotheses of Corollary 4.1 are satisfied for an arbitrary subset A of R^2. Since the smallest disk D_n in R^2 containing S_n has area $\pi/2^{2n}$, however, the sequence $\mu S_n/\mu D_n$ has limit zero and the classical Vitali theorem is not applicable.

We next show how to find quasispheres in a more general situation.

4.3. EXAMPLE. Suppose that the space X admits a point, which we may as well choose for ϕ, some neighborhood V of which admits a homeomorphism onto a neighborhood of the origin in R^k. If η is such a homeomorphism, arbitrary but henceforth fixed, then $\eta^{-1}(\eta x + \eta y)$, whenever it is defined, will be denoted by $x+y$; $x-y$ is defined similarly. When $x \in V$, we write $|x|$ for the distance from ηx to the origin. Then quasispheres can be defined as the image under η^{-1} of spheres in R^k, provided that the elements of G satisfy the following condition E-L, a sort of local equi-Lipschitz condition.

CONDITION E-L. There is an open subset W of X and a real number P for which $\phi \in W \subset V$ and for which $gy \in V$ and $|gx - gy| < P|x-y|$ whenever $x \in W$ and $y \in W$ and $gx = \phi$.

Indeed, let r be a positive number for which $x \in W$ whenever $|x| < r$, and for which the sphere of radius r about the origin in R^k is contained in ηV. Now define

$$S_n = \{x \mid |x| \leqq r/(2P)^n\},$$

for each positive integer n. Our discussion will be complete if we can show that the sequence S_n satisfies conditions (1) and (2) of Definition 2.7.

To show $S_{n+1}^{-1}S_{n+1} \subset S_n$, which is (1), we notice that the constant P of condition E-L exceeds 1, so that always $S_n \subset W$. If $x \in S_{n+1}^{-1}S_{n+1}$, so that there exist points y and z in S_{n+1} and $g \in G$ for which $gy = x$ and $gz = \phi$, then we have

$$|x| = |gy - gz| < P|y-z| \leqq P(|y| + |z|) \leqq 2Pr/(2P)^{n+1} = r/(2P)^n;$$

thus $x \in S_n$ and (1) of Definition 2.7 is established.

To prove (2), we notice that for each positive integer n the sets ηS_n, ηS_{n+2} are spheres in R^k, the radius of the former being $(2P)^2$ times the radius of the latter. Thus there is an integer m, dependent only on k and P, for which $\eta S_n \subset \bigcup_{i=1}^m \eta x_i + \eta S_{n+2}$ for appropriately chosen points $x_i \in S_n$. Now if $y \in S_{n+2}$, then the distance in R^k from the origin to $\eta x_i + \eta y$ does not exceed $r/(2P)^n + r/(2P)^{n+2} < r$, so that $\eta^{-1}(\eta x_i + \eta y) = x_i + y$ exists and lies in V. Choosing $g_i \in G$ so that $g_i x_i = \phi$, we have

$$|g_i(x_i + y)| = |g_i(x_i + y) - g_ix_i| < P|(x_i + y) - x_i| = P|y|$$
$$\leqq Pr/(2P)^{n+2} < r/(2P)^{n+1}.$$

Thus $g_i(x_i+S_{n+2}) \subset S_{n+1}$, and from the relation $S_n \subset \bigcup_{i=1}^m x_i + S_{n+2}$ it follows that

$$\mu^* S_n \leqq \sum_{i=1}^m \mu^*(x_i + S_{n+2}) \leqq \sum_{i=1}^m \mu^* S_{n+1} = m\mu^* S_{n+1}.$$

4.4. Example. We consider finally the case in which a Lie group $G = X$ acts upon itself by translation. Retaining the notation introduced in 4.3, we recall from Smith [4] that the homeomorphism η may be chosen to introduce on X a coordinate system which is right regular, in the sense that there is a function F on $V \times V$, jointly continuous at (ϕ, ϕ), for which

$$uv = u + v + |v| F(u, v),$$

whenever each of the points, u, v, uv, $F(u, v)$ and $|v| F(u, v)$ lies in V. (By $|v| F(u, v)$, we mean $\eta^{-1}[|v| \eta F(u, v)]$.) The fact that our theorem is applicable in this situation will be proved by showing that if η is so chosen, then condition E-L is satisfied. The condition now reads as follows: There is an open subset W of X and a real number P for which $\phi \in W \subset V$ and for which $x^{-1}y \in V$ and $|x^{-1}y| < P|x - y|$ whenever $x \in W$ and $y \in W$.

Now if x and y are arbitrary points of V and if $z \in X$, we could write

$$y = xx^{-1}y = x + x^{-1}y + |x^{-1}y| z, \tag{1}$$

then

$$y - x - |x^{-1}y| z = x^{-1}y, \tag{2}$$

whence $|x^{-1}y| \leqq |y - x| + |x^{-1}y| |z|$ and $(1 - |z|)|x^{-1}y| < |y - x|$, provided that each of the points appearing in equations (1) and (2) is well-defined (i.e., lies in V). In the present case, a real number $P > 1$ being given, we choose W so small that (1) and (2) are meaingful for $z = F(x, x^{-1}y)$, and so small that $|F(x, x^{-1}y)| < (P-1)/P$ whenever $x \in W$ and $y \in W$. With this choice of W we have $(1/P)|x^{-1}y| < (1 - |z|)|x^{-1}y| < |y - x| = |x - y|$ whenever $x \in W$ and $y \in W$, so that condition E-L is satisfied.

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