GENERALIZED ARCHIMEDEAN GROUPS

BY

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Introduction. In a previous paper [3], the concept of archimedean group was generalized so as to obtain a larger class of groups that were referred to as *regularly ordered.* As it was shown in [3], these groups, unlike the archimedean ones, admit a formalization in the lower predicate calculus (LPC) of formal logic; yet they cannot be distinguished from archimedean groups by any properties formalizable in the LPC. Thus they can serve as an LPC-substitute for archimedean groups. This result was, however, based on several theorems stated without proof. It is our aim now to supply the proofs of these theorems. Another objective of this paper consists in giving a unified algebraic approach to regularly ordered groups, as a counterpart to the metamathematical study presented in [3].

Throughout this paper, the term "ordered group" stands for "totally ordered (and, hence, torsion-free) additive abelian group other than $\{0\}$." Such a group, A, is said to be *discretely* or *densely* ordered according as it does, or does not, contain a smallest positive element (called its *unit*, $\dot{1}$). A subset $S\subseteq A$ is called an *interval* if the relations $a<x<b$, $a, b\in S$, $x\in A$ always imply $x\in S$. An interval is referred to as infinite if the number of its elements is infinite. Whereas in [3] the concept of a regularly ordered group was defined separately for discretely and densely ordered groups, we now give a unified definition; its equivalence with that given in [3] will be proved later.

0.1. Definition. An ordered group A is said to be *regularly ordered, with respect to an integer* $n>0$ (briefly, "*n-regular*"), if every infinite interval of A contains at least one (and, hence, infinitely many) elements divisible by n.[1] If this holds for *every* n, we simply say that A is *regularly ordered.* A is referred to as *regularly discrete* (*regularly dense*) if its ordering is both regular and discrete (regular and dense, respectively).

From this definition we immediately derive the following corollary, to be used later.

0.2. Corollary. *Let a be an element of a regularly ordered group A, and let k, r be positive integers. Then every infinite interval S of A contains an element x of the form $x=ky$, with $y\equiv a \pmod r$, $y\in A$.*

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[1] As usual, we say that an element x is *divisible by n* (briefly, "*n-divisible*") in a group A, if A contains an element z with $x=nz$. Two elements x, $y\in A$ are said to be *congruent modulo n* (briefly, "*n-congruent*") in A, if $x-y$ is divisible by n. We then write $x\equiv y \pmod n$ in A. The negation of this formula is $x\not\equiv y \pmod n$ in A ("x and y are *n-incongruent in A*").

Proof. By k-regularity, S contains infinitely many elements x which are divisible by k. If x is treated as a variable ranging over all such elements of S, then, as is easily checked, the set of all elements of the form $x/k-a$ is, likewise, an infinite interval of A. Hence, by r-regularity, it contains an r-divisible element, so that we have $x/k \equiv a (\text{mod } r)$ for some $x \in S$. It now suffices to set $x/k=y$, to obtain the required result.

Some further preliminary definitions and corollaries are given in §1.

1. Preliminaries.

I. As in [3], we define the *nth congruence invariant* of an abelian group A, denoted by $[n]A$ or, briefly, by $[n]$, to be the maximum (possibly infinite) number of n-incongruent elements in A.(2) If n is a prime, $[n]$ is called a *prime invariant.* In the infinite case, we set $[n]=\infty$, without distinguishing between infinities of different cardinalities(3).

1.1. Corollary. *If a group A is isomorphic with the additive group of all integers, then $[n]A=n$, $n=1, 2, 3, \cdots$.*

In fact, for every n, the group A splits into exactly n residue classes modulo n; i.e., the quotient group A/nA has order n (cf. footnote 2).

II. Let A be an additive abelian group, S its subgroup, and $n>0$ an integer. We shall say that *S is n-serving in A* if, for all x, $y \in S$, the relation $x \equiv y (\text{mod } n)$ holds in S whenever it holds in A. If this is true for *every n*, we simply say that S is *serving in A* (Prüfer, [1]). S is said to be *n-basic in A*, if every element of A is n-congruent to some element of S. Again, if this holds for *every n*, we say that S is *basic in A*.

1.2. Corollary. *Let A_0 and A ($A_0 \subseteq A$) be regularly discrete groups with one and the same unit $\dot{1}$. Then A_0 is basic and serving in A.*

Proof. Let A_1 be the subgroup generated by the unit $\dot{1}$, and let $a \in A$. Clearly, all elements of the form $a-k\cdot\dot{1}$ ($k=1, 2, 3, \cdots$) constitute an infinite interval of A which, by n-regularity (n being any positive integer), must contain an n-divisible element, say $a-k_0\cdot\dot{1}$, so that $a \equiv k_0\cdot\dot{1} (\text{mod } n)$, and $k_0\cdot\dot{1} \in A_1$. Thus every element of A is n-congruent to some element of A_1. Since n is arbitrary here, A_1 is basic in A and, similarly, A_0 is basic in A. Moreover, if $a \in A_0$, the congruence $a \equiv k_0\cdot\dot{1} (\text{mod } n)$ easily implies that n divides a (in both A_0 and A) if and only if n divides the integer k_0. It follows that n divides a either in both A_0 and A, or in none of these groups. This shows that A_0 is serving in A, as asserted.

1.3. Corollary. *A subgroup S of a torsion-free abelian group A is serving*

(2) This maximum number always exists and is unique. In fact, as is readily seen, $[n]A$ coincides with the (finite or infinite) order of the quotient group A/nA, where nA is the subgroup of all elements divisible by n.

(3) The symbol ∞ will be treated as a positive number with the usual conventions as to operations and inequalities. In particular, we set $\log_p \infty = \infty$ for every $p>1$.

(basic) in A if and only if, for every prime p, S is p-serving (p-basic) in A. (For basic subgroups, this holds also if A is not torsion-free.)

Proof. Finite induction easily shows that, whenever S is p-serving, it is also p^k-serving ($k=1, 2, 3, \cdots$). It is also readily seen that, whenever S is both m-serving and n-serving (m and n being relatively prime), then S is also (mn)-serving. Similarly for basic subgroups. The required result then is obtained by considering each integer n as factored into prime powers.

1.4. Corollary. *Let S be a subgroup of an abelian group A, and $n>0$ an integer. Then, if S is n-serving in A, we have $[n]S \leqq [n]A$. If, however, S is n-basic in A, we have $[n]S \geqq [n]A$.*

For, let $[n]S = m \leqq \infty$; i.e., S contains exactly m elements incongruent modulo n. If S is n-serving, these elements are n-incongruent in A as well, whence $[n] \geqq m = [n]S$. If, however, S is n-basic, then, clearly, A cannot contain more n-incongruent elements than S; hence $[n]A \leqq [n]S$.

1.5. Corollary. *If A is discretely ordered, then $[n]A \geqq n$, $n=1, 2, 3, \cdots$.*

In fact, let A_1 be the subgroup generated by the unit $\dot{1}$ of A. Then, as is readily seen, A_1 is serving in A and isomorphic with the ordered group of all integers. By 1.1 and 1.4, we then have $n = [n]A_1 \leqq [n]A$, as asserted.

III. A subset S of an abelian group A is said to be *n-independent* (in A) if, for any finite number m of distinct elements $x_1, x_2, \cdots, x_m \in S$, and any integers $q_1, q_2, \cdots, q_m$, the relation $\sum_{i=1}^{m} q_i x_i \equiv 0 \pmod{n}$ in A always implies $q_i \equiv 0 \pmod{n}$, $i=1, 2, \cdots, m$. S is referred to as *maximal* if it is not contained in any other n-independent subset of A. The existence of (possibly empty) maximal n-independent subsets, for every n, is easily proved by Zorn's lemma. Also easily derived is the following

1.6. Corollary. *In a torsion-free abelian group, every n-independent subset is also n^k-independent ($k=1, 2, 3, \cdots$).*

1.7. Corollary. *If an abelian group A is not divisible*(4) *by a prime p, then A contains at least one nonempty maximal p-independent subset.*

Proof. Take any element $x \in A$ not divisible by p. Then $\{x\}$ is clearly a nonempty p-independent subset of A. The rest follows by Zorn's lemma.

1.8. Corollary. *Let A be an abelian group, and p a prime. Then all maximal p-independent subsets of A contain exactly the same (finite or infinite) number of elements. This number (called the pth prime rank of A) equals $\log_p [p]A$.*

Proof. Let S be a maximal p-independent subset of A, and let m be the

(4) We say that a group A is *divisible by an integer $n>0$*, or that *n divides A*, if all elements of A are n-divisible (in A). If this holds for *every n*, A is referred to as *(completely) divisible*.

number of its elements. Then it easily follows that A contains exactly p^m p-incongruent elements(5). In our notation, this means that $[p]A = p^m$, or $m = \log_p [p]A$. As the value of $[p]A$ is uniquely determined, all is proved.

IV. Given a subgroup A_0 of an abelian group A, and $c_0 \in A$, we define the *c_0-extension of A_0 in A* to be the subgroup of all elements $x \in A$ satisfying equations of the form $tx = sc_0 + b$, with $b \in A_0$ and t, s integers ($t > 0$).

1.9. COROLLARY. *Let A_1 be the c_0-extension of a subgroup A_0 in A. Then:*

(i) *A_1 is a serving subgroup of A which contains A_0 and c_0.*

(ii) *A_0 is serving in A_1 if and only if A_0 is serving in A.*

(iii) *If A_0 and A are regularly discrete, and have the same unit $\dot{1}$, then A_1 has the same unit and, moreover, $[n]A_0 = [n]A_1 = [n]A$, $n = 1, 2, 3, \cdots$.*

Proof. Assertions (i) and (ii) easily follow from the definition (IV). Since $A_0 \subseteq A_1$, we see, in case (iii), that $\dot{1} \in A_1$, so that A_1 has the same unit. By 1.2, A_0 is basic and serving in A. From this, and (i), (ii), it follows, by 1.4, that $[n]A_0 = [n]A_1 = [n]A$, for all n, as required.

V. As is well known, every torsion-free (ordered) group A can be embedded in a smallest *divisible extension*, i.e., a smallest divisible group which contains it and is, likewise, torsion-free (ordered, respectively). The elements of that extension (which is unique, up to isomorphism) are all of the form x/n, with $x \in A$ and n a positive integer(6).

VI. For further references, we list some familiar properties of direct sums.

1.10. THEOREM. *Let M be the direct sum of two abelian groups, A and B, with $A \cap B = \{0\}$. Then:* (a) *If A and B are torsion-free (ordered), so is M.*

(b) *Every element $u \in M$ admits a unique representation $u = x + y$ ($x \in A$, $y \in B$).*

(c) *For any $x \in A$ and $y \in B$, we have $x + y \equiv 0 \pmod{n}$ if and only if $x \equiv 0 \pmod{n}$ in A, and $y \equiv 0 \pmod{n}$ in B.*

(d) *A and B are serving subgroups of M.*

NOTE. If A and B are ordered, M is assumed to be ordered *lexicographically.*

2. **The class of regularly ordered groups.** We shall now study more closely the class of regularly ordered groups.

2.1. THEOREM. (a) *A densely ordered group A is regularly dense if and only if, for every integer $n > 0$, and any a, $b \in A$ ($a < b$), there is an $x \in A$ with $a < nx < b$.*

(b) *A discretely ordered group A is regularly discrete if and only if $[n]A = n$, for every integer $n > 0$.*

(5) For the method of the proof of this assertion, see e.g., [4, p. 214].

(6) All these assertions can be proved by a procedure analogous to the familiar process of forming the quotient field of an integral domain.

Proof. Assertion (a) follows immediately from Definition 0.1 if one considers that, in a densely ordered group, every interval with endpoints a, b ($a<b$) is infinite. Thus we proceed to prove (b).

I. *Necessity.* Let A be regularly discrete; further, let A_1 be the subgroup generated by the unit $\dot{1}$ of A. Since A_1 is isomorphic to the group of all integers, we have $[n]A_1=n$, $n=1, 2, 3, \cdots$ (see 1.1). Moreover, by 1.2, A_1 is basic in A. Hence, by 1.4, and 1.5, $n=[n]A_1\geqq[n]A\geqq n$, for all n, as required.

II. *Sufficiency.* Suppose that $[n]A=n$, $n=1, 2, \cdots$, further, let S be any infinite interval of A, and $a\in S$. As is readily seen, S contains either all elements of the form $a+k\cdot\dot{1}$, or all elements of the form $a-k\cdot\dot{1}$ ($k=1,2,3,\cdots$), say the latter. Now consider the n elements $k\cdot\dot{1}$ ($k=1, 2, 3, \cdots, n$). They are clearly n-incongruent with each other. On the other hand, the assumption $[n]A=n$ implies that A cannot contain *more* than n elements incongruent modulo n. Hence the element a (selected above) must be n-congruent with one of the n elements $k\cdot\dot{1}$ ($k=1, 2, \cdots, n$). Thus we have $a\equiv k_0\cdot\dot{1}$ (mod n) for some $k_0>0$. But then the element $a-k_0\cdot\dot{1}\in S$ is divisible by n. Thus every infinite interval S of A contains an n-divisible element, and all is proved.

2.1.1. Corollary. *Let A_0 and A ($A_0\subseteq A$) be regularly discrete groups having the same unit $\dot{1}$, and let $c_0\in A$. Then the c_0-extension of A_0 in A is, likewise, regularly discrete and has the same unit $\dot{1}$.*

For the proof, combine 2.1(b) with 1.9(iii).

2.2. Theorem. *For an ordered group A to be regularly ordered, it suffices that A be p-regular for every prime p.*

For the proof, we first establish the following fact.

2.2.1. *If A is both m-regular and n-regular, then A is (mn)-regular.*

Let S be an infinite interval of A. Due to m-regularity, S contains infinitely many elements divisible by m. If x is a variable ranging over all m-divisible elements of S, then, as is easily seen, all quotients x/m again constitute an infinite interval of A. One of these quotients, say x_0/m ($x_0\in S$), must be divisible by n, due to the n-regularity of A. Hence the element $x_0\in S$ is divisible by mn. This establishes 2.2.1. To complete the proof of 2.2, one only has to consider each integer as factored into prime powers and apply 2.2.1 to the arising products.

Theorems 2.1 and 2.2, when combined, show that our Definition 0.1 is equivalent to the definition of regularly ordered groups given in [3, (3.3)].

We now proceed to prove some theorems announced in [3].

2.3. Lemma. *Let $b>0$ be an element of a densely ordered group A. Then, for every integer $n>0$, there is an element $x\in A$ such that $0<nx<b$.*

For the proof, select any $n+1$ elements $x_i\in A$ such that $0<x_1<x_2<\cdots<x_{n+1}<b$, and set $x=\min(x_{i+1}-x_i)$. Then x is the required element.

2.4. THEOREM. *All archimedean groups are regularly ordered.*

Proof. Let A be an archimedean group. If A is discretely ordered, then, as is readily seen, the archimedean property implies that all elements of A are integral multiples of its unit $\dot{1}$. Hence A is isomorphic with the additive group of all integers, and our assertion follows from 1.1 and 2.1(b). Now suppose that A is densely ordered, and let S be an infinite interval of A, and n a positive integer. We shall show that S contains an n-divisible element and thus complete the proof. Obviously, it suffices to consider the case $0 \notin S$. In this case, all elements of S have the same sign, say positive. Let then $a, b \in S$, $0 < a < b$. Taking in 2.3 for b the minimum of a and $b-a$, we see that there is an $x \in A$ with 2.4.1. $0 < nx < a$, $0 < nx < b-a$.

The archimedean property entails that there are positive integers m with $mnx > a$. Let m_0 be the least of them. Then it is easily deduced from 2.4.1 that $a < m_0nx < b$, so that, by the definition of an interval, the element $y = m_0nx$ belongs to S (for so do a and b). As y is also divisible by n, all is proved.

From this theorem it follows, in particular, that every group consisting of real numbers in their natural order is regularly ordered (for it is archimedean). On the other hand, it is easy to give examples of regularly ordered groups which are not archimedean. Such is, for instance, the additive group of all complex numbers, ordered lexicographically, and, more generally, every nonarchimedean *divisible* group (ordered divisible groups are, obviously, regularly dense). The direct sum of any ordered divisible groups is regularly dense, though not archimedean, under lexicographic ordering. There are also regularly *discrete* nonarchimedean groups. Such is, for instance, the lexicographically ordered additive group of all complex numbers $a+bi$ with b an integer. Thus the class of all regularly ordered groups is more general than that of all archimedean groups. The following existence theorem is important.

2.5. THEOREM. *Let $p_1, p_2, \cdots, p_n, \cdots$ be the ascending sequence of all primes; further, let $m_1, m_2, \cdots, m_n, \cdots$ be an arbitrary sequence in which each m_n is either a non-negative integer or ∞. Then there always exists a densely ordered archimedean (and, hence, regularly dense) group M such that*

$$[p_n]M = p_n^{m_n}, \qquad n = 1, 2, 3, \cdots$$

Proof. As a preparatory step, we fix some *transcendental* real number e and set up an infinite two-entry matrix (a_{nj}), $n, j = 1, 2, 3, \cdots$, whose elements are either 0 or *distinct* powers of e, defined as follows:

(1) $a_{nj} = 0$ whenever $j > m_n$, where m_n is the nth term of the sequence $\{m_n\}$ specified in the theorem. If $m_n = 0$, all a_{nj} are 0 (for that n).

(2) If $j \leq m_n$, we set $a_{nj} = e^{k_{nj}}$ where k_{nj} is some non-negative integer. The

exponents k_{nj} are supposed to be distinct, but, otherwise, may be chosen arbitrarily. We shall consider the k_{nj} and a_{nj} as fixed.

Now let R be the ordered additive group of all reals, and let M be its subgroup generated by the set of all fractions of the form a_{nj}/r where a_{nj} is as defined above and r is an integer *not divisible by the prime* p_n. As is easily seen, M is a densely ordered archimedean group containing all a_{nj}. Moreover, using the fact that e is transcendental, it is easy to show that the set S_{n_0} of all *nonzero* elements a_{n_0j} ($j \leqq m_{n_0}$) (i.e., the set of all nonzero elements in the n_0th row of the matrix (a_{nj})) is a maximal p_{n_0}-independent subset of M. It follows, by 1.8, that S_{n_0} contains exactly $\log_{p_{n_0}}[p_{n_0}]M$ elements. On the other hand, by the definition of the matrix (a_{nj}), its n_0th row contains exactly m_{n_0} nonzero elements, and this is the number of elements in S_{n_0}. We conclude that $\log_{p_n}[p_n]M = m_n$, for every n; that is $[p_n]M = p_n^{m_n}$. Thus the group M has all the required properties, and the theorem is proved. More concisely, it could be formulated thus:

2.5′. *For any choice of the prime ranks* (*see* 1.8), *there exists a densely ordered archimedean group M with prime ranks so chosen.*

Finally, Corollary 1.8 yields the following immediate inference.

2.6. THEOREM. *In any abelian group, every prime invariant* $[p_n]$ *is a* (*possibly infinite*) *power of the corresponding prime* p_n.

Theorems 2.4, 2.5 and 2.6 establish three of the results announced in [3] (where they appear as Theorems 3.4, 3.5 and 3.6, respectively). To prove the next of the required theorems (3.7 of [3]), we need some additional preliminary remarks concerning what we shall define as *linear systems* and *closed subgroups*. These remarks constitute a natural generalization of some concepts and propositions of linear algebra.

3. **Linear systems. Closed subgroups.**

I. As in [3], we define a *linear system* to be any finite system of linear equations, inequalities, congruences and (or) incongruences of the form

3.1. $$\sum_{j=1}^{n} q_{ij}x_j = a_i, \qquad i = 1, 2, \cdots, m_1,$$

3.2. $$\sum_{j=1}^{n} q_{ij}x_j > a_i, \qquad i = m_1 + 1, m_1 + 2, \cdots, m_2,$$

3.3. $$\sum_{j=1}^{n} q_{ij}x_j \equiv a_i \pmod{r_i}, \qquad i = m_2 + 1, m_2 + 2, \cdots, m_3,$$

3.4. $$\sum_{j=1}^{n} q_{ij}x_j \not\equiv a_i \pmod{r_i}, \qquad i = m_3 + 1, m_3 + 2, \cdots, m_4,$$

where the q_{ij} and r_i are integers ($r_i > 0$), the x_j are unknowns or variables, and the a_i are given elements of an ordered group A; the a_i will be called the *constants* of the system. We shall denote by (q_{ij}) the matrix of the coefficients q_{ij} occurring in the *equations* 3.1 (if any), and by (q_{ij}, a_i) the "extended" matrix which arises from (q_{ij}) by adjoining the column of the constants a_i, $i=1, 2, \cdots, m_1$. (Note that only the coefficients and constants of the *equations* 3.1, not those occurring in 3.2, 3.3 and 3.4, are spoken of.) A linear system with a nonempty set of equations 3.1 is said to be of type I, II or III, according as (I) the matrices (q_{ij}) and (q_{ij}, a_i) are of the same rank n (= the number of the unknowns x_j), (II) the ranks of these matrices are equal, but less than n, or (III) their ranks are different. Two linear systems L_1 and L_2 will be referred to as *conjugate* if, whenever an ordered group A contains their constants, either both L_1 and L_2, or none of them, can be solved in A. Equivalent systems are a special case.

II. Since every completely divisible torsion-free group is a vector space over the field of all rational numbers, it is clear that the ordinary theory of linear equations (in particular, Cramer's rule) is valid in every such group. Some familiar rules still remain valid for nondivisible groups and for more general linear systems, of the form 3.1–3.4. We list them here.

3.5. THEOREM. (i) *A linear system of type* I *has at most one solution in every ordered group A containing its constants a_i. Moreover, the set of equations* 3.1 *contained in such a system is equivalent to some system of the form*

3.5.1. $$kx_j = c_j, \qquad j = 1, 2, \cdots, n,$$

where k is a positive integer, and the c_j are elements of the subgroup of A generated by the constants a_i occurring in 3.1.

(ii) *In every linear system of type* II, *with $a_i \in A$, the set of the equations* 3.1 *is equivalent to some system of the form*

3.5.2. $$kx_{j+m} = c_j + \sum_{i=1}^{m} k_{ij}x_i, \qquad j = 1, 2, \cdots, n - m,$$

where again k is an integer >0, the c_j are as above, and $m<n$.(7)

(iii) *No linear system of type* III *has a solution in any ordered group.*

For the proof, it suffices to embed the group A in its divisible extension M and then apply Cramer's rule. The required results then easily follow if one takes into account that every element of M has the form a/q with $a \in A$ and q an integer.

3.6. THEOREM. *For every linear system L of type* II, *there exists a conjugate*

(7) The integer k in 3.5.1 and 3.5.2 is some multiple of the highest nonzero determinant of the matrix (q_{ij}). To simplify notation, we have assumed that this determinant is in the right lower corner of the matrix. The elements c_j are corresponding determinants (or their multiples) taken from (q_{ij}, a_j), in accordance with Cramer's rule. Clearly, these determinants, when expanded, are "linear combinations" of the a_i. Hence the c_j are in the subgroup generated by the a_i (we shall briefly say that they are "generated by the a_i").

system L' with no equations 3.1 *in it, and with all its constants in the subgroup generated by the constants a_i of L.*

Proof. The required system L' is obtained from L in three steps:

(1) Using Theorem 3.5(ii), we replace the equations 3.1 of L by 3.5.2, leaving the inequalities, congruences and incongruences (if any) unchanged. This, obviously, yields an equivalent and, hence, conjugate system (call it L_1).

(2) We now adjoin to L_1 the following system of congruences:

3.6.1. $$c_j + \sum_{i=1}^{m} k_{ij}x_i \equiv 0 \pmod{k}, \qquad j = 1, 2, \cdots, n - m,$$

where the c_j, k_{ij}, k and the unknowns x_i are the same as in 3.5.2. This again yields an equivalent system (call it L_2). For, every solution of L_2 clearly satisfies also L_1; conversely, every solution of L_1 satisfies, in particular, the equations 3.5.2 which *imply* 3.6.1, so that L_2 is solved as well. Moreover, as follows from Theorem 3.5, the constants of L_1 and L_2 are generated by those of L, as required.

(3) The last step is to remove the equations from L_2. We achieve this by using the equations 3.5.2, first of all, to eliminate the $n - m$ unknowns x_{j+m} ($j=1, 2, \cdots, n-m$) from the whole system(8). Having done so, we drop the equations 3.5.2, retaining, however, the congruences 3.6.1 instead. This leads to a new linear system L_3 with no equations in it and with its constants again generated by the constants of the original system L (obviously, the elimination process described in footnote 8 satisfies these requirements). Moreover, every solution of L_2 clearly yields also a solution of L_3 and vice versa (for the whole process is reversible, due to the fact that ordered groups are torsion-free)(9). Thus L_2 and L_3 are conjugate, and L_3 is the required system L'.

III. The remaining theorems of this and the next sections deal with what we shall now define as *closed subgroups*.

3.7. Definition. A subgroup A_0 of an ordered group A is said to be *closed* in A if every linear system, with its constants a_i in A_0, can already be solved in A_0 whenever it has a solution in A. If this is true of linear systems of some specific kind, we shall say that A_0 is closed *with respect to systems of that particular kind*(10).

(8) This elimination process can be carried out also in nondivisible ordered (and, hence, torsion-free) groups by, say, multiplying all inequalities, congruences and incongruences of L_2 by k and then replacing everywhere kx_{j+m} by $c_j + \sum_{i=1}^{m} k_{ij}x_i$ (note that in the congruences and incongruences, the moduli r_i, too, must be multiplied by k).

(9) Note that the adjoined congruences 3.6.1 ensure, in their turn, the solvability of the equations 3.5.2. Thus any solution of L_3 yields a solution of L_2.

(10) Clearly, this is a generalization of Prüfer's concept of a serving subgroup which can now be defined as a subgroup closed with respect to all (single) equations of the form $qx=a$, or all congruences of the form $qx\equiv a \pmod{n}$.

3.8. THEOREM. *Every serving subgroup A_0 of an ordered group A is closed in A, with respect to all linear systems of type* I.

Proof. Let L be a linear system of type I, with its constants a_i in A_0, and let $(b_1, b_2, \cdots, b_n)$ be its solution in A (here, as before, n is the number of the unknowns x_j). To prove the theorem, it suffices to show that all b_j necessarily are elements of A_0. By 3.5(i), the b_j must satisfy some equations of the form 3.5.1, where the c_j are generated by the a_i. As all a_i are in A_0, so are all c_j. Thus we have $kb_j = c_j$, $c_j \in A_0$, $j = 1, 2, \cdots, n$. It follows that the elements c_j are divisible, in A, by the integer k. But, as A_0 is serving in A, this divisibility holds in A_0 as well. This means that the quotients $b_j = c_j/k$ are all in A_0, as required.

3.9. THEOREM. *For a subgroup A_0 of an ordered group A to be closed in A, it suffices that A_0 be closed with respect to all linear systems containing no equations* 3.1.

Proof. Let this condition be satisfied. Then A_0 is, in particular, closed with respect to all (single) congruences of the form $0 \cdot x \equiv a (\text{mod } r)$; which implies that A_0 is serving in A. Now let L be a linear system, with all its constants a_i in A_0, which has a solution in A (and, hence, is not of type III). We have to show that L can already be solved in A_0. If L contains no equations 3.1, this follows from our assumption. If L is of type I, it follows from 3.8. Finally, if L is of type II, we can replace it by a conjugate system L' with no equations in it (see 3.6). Since conjugate systems are solvable in exactly the same ordered groups, both L and L' have a solution in A and, hence (by our assumption), in A_0.

4. **Linear systems in regularly ordered groups.** Our aim in this section is to prove Theorem 3.7 of [3]. First, however, we shall consider a special case of it.

4.1. THEOREM. *If A_0 and A are regularly discrete groups having the same unit $\dot{1}$, and $A_0 \subseteq A$, then A_0 is closed in A, with respect to all linear systems in one unknown x.*

Proof. Let L be such a system, with its constants a_i in A_0, having a solution $x = c$ in A. We have to show that L can already be solved in A_0. Without loss of generality, we may assume (by a similar argument as in 3.9) that L contains no equations. Thus L has the form

4.1.1. $\quad q_i x > a_i, \qquad i = m_1 + 1, \cdots, m_2,$

4.1.2. $\quad q_i x \equiv a_i \ (\text{mod } r_i), \qquad i = m_2 + 1, m_2 + 2, \cdots, m_3,$

4.1.3. $\quad q_i x \not\equiv a_i \ (\text{mod } r_i), \qquad i = m_3 + 1, m_3 + 2, \cdots, m_4.$

By multiplying these formulae by suitable integers, we can make all r_i equal, say, to r, and also make all q_i in the *inequalities* 4.1.1 equal, say, to

$q>0$, reversing, if necessary, the inequality signs. Thus 4.1.1 is replaced by *two* sets of inequalities, of the form $qx>b_i$ and $qx<d_j$ ($q>0$; b_i, $d_j \in A_0$). Assuming that both sets are nonempty (otherwise only insignificant changes are required), we set $b = \max b_i$, and $d = \min d_j$. Then 4.1.1 is equivalent to

4.1.4. $$b < qx < d \qquad (q > 0;\ b,\ d \in A_0).$$

Since $x=c$ is a solution of L, we have, in particular,

4.1.5. $$b < qc < d.$$

As A_0 is basic in A (by 1.2), there is an element $c' \in A_0$ with $c' \equiv c \pmod r$, where r is the common value of the r_i. Clearly, c' satisfies 4.1.2 and 4.1.3 (for so does c). Fixing this $c' \in A_0$, we now consider two possible cases:

(I) *The interval* (b, d) *is infinite*[11]. Then, by 0.2, it contains an element $a \in A_0$, of the form $a = qx_0$, with $x_0 \equiv c' \pmod r$, $x_0 \in A_0$, $b < qx_0 < d$. As is readily seen, x_0 satisfies both 4.1.4 and 4.1.2–4.1.3. Thus $x = x_0$ is the required solution of L in A_0.

(II) *The interval* (b, d) *is finite.* Then all its elements have the form $b + k \cdot \dot{1}$ and, hence, belong to A_0 (for so does b). From 4.1.5 it then follows that $qc \in A_0$. Moreover, qc is, clearly, divisible by q in A. As A_0 is serving in A (by 1.2), this implies that the quotient $(qc)/q = c$ is an element of A_0. Thus, in this case, $x = c \in A_0$ is already the required solution of L in A_0.[12] This completes the proof.

4.2. LEMMA. *Let A_0 and A ($A_0 \subseteq A$) be regularly discrete groups having the same unit $\dot{1}$, and let $c_0 \in A$. Then A_0 is a closed subgroup of its c_0-extension in A (call that extension A_1).*

Proof. Let L be a linear system in n unknowns x_j, with its constants a_i in A_0, and let $(c_1, \cdots, c_n)$ be its solution in A_1. We shall show, in three steps, that L can already be solved in A_0, which will complete the proof.

(I) By the definition of a c_0-extension (§1, Part IV), the elements c_j satisfy some equations of the form

4.2.1. $$t_j c_j = a_j c_0 + b_j \qquad (b_j \in A_0),\ j = 1, 2, \cdots, n,$$

with t_j and s_j integers ($t_j > 0$). As $(c_1, \cdots, c_n)$ is a solution, we may substitute the c_j for the x_j in L, thus obtaining a set of *correct* formulae. Using 4.2.1, we then eliminate $c_1, c_2, \cdots, c_n$ from these formulae[13]. After simplifications, this process yields a set of (correct) formulae, of the form

4.2.2. $$k_i c_0 = a_i', \qquad i = 1, 2, \cdots, m_1,$$

[11] Note that (b, d) is infinite in A_0 if and only if it is infinite in A.

[12] Recall that $x = c$ was supposed to be a solution of L.

[13] Footnote 8 applies to this elimination process as well, with k replaced by some common multiple of the t_j, and other selfevident modifications.

4.2.3. $k_i c_0 > a'_i$, $i = m_1 + 1, m_1 + 2, \cdots, m_2$,

4.2.4. $k_i c_0 \equiv a'_i \pmod{r'_i}$, $i = m_2 + 1, m_2 + 2, \cdots, m_3$,

4.2.5. $k_i c_0 \not\equiv a'_i \pmod{r'_i}$, $i = m_3 + 1, m_3 + 1, \cdots, m_4$,

with k_i, r'_i integers ($r'_i > 0$), and all a'_i in A_0.

(II) We now introduce an auxiliary linear system L' which arises from formulae 4.2.1 through 4.2.5 if the c_j (including c_0) are replaced by $n+1$ unknowns, $x_0, x_1, \cdots, x_n$. In particular, equations 4.2.1 transform into

4.2.1°. $$t_j x_j = s_j x_0 + b_j \quad (b_j \in A_0),\ j = 1, 2, \cdots, n.$$

Clearly, $(c_0, c_1, \cdots, c_n)$ is a solution of L' in A_1. Moreover, if $(d_0, \cdots, d_n)$ is any solution of L' in A, then $(d_1, \cdots, d_n)$ is a solution of L; for the whole process described in step (I) is reversible (all groups being torsion-free). To prove the theorem, it then suffices to show that L' has a solution in A_0.

(III) For this purpose, we use 4.2.1° to eliminate the unknowns $x_1, \cdots, x_n$ from L', leaving only x_0 in it, and then replace equations 4.2.1° by a conjugate set of congruences, $s_j x_0 + b_j \equiv 0 \pmod{t_j}$, $j = 1, 2, \cdots, n$, where the s_j, b_j and t_j are as in 4.2.1°. This yields a linear system L'' in *one unknown* x_0 *only*; L'' consists of the now adjoined congruences and 4.2.2–4.2.5 (with c_0 replaced by x_0). An argument analogous to that of 3.6 shows that L' and L'' are conjugate systems. Moreover, L'' has a solution in A_1 (e.g., $x_0 = c_0$). Therefore, by 4.1, L'' (and, hence, also L') can be solved in A_0 as well (in fact, by 2.1.1, A_1 is regularly discrete and has the same unit as A_0; so all assumptions of 4.1 are satisfied). This completes the proof. We are now able to prove Theorem 3.7 of [3], even in a somewhat stronger version:

4.3. THEOREM. *Let A_0 and A ($A_0 \subseteq A$) be regularly discrete groups. Then A_0 is closed in A if and only if the two groups have the same unit,* $\dot{1}$.

Proof. (i) *Necessity.* If the units of A_0 and A are different, then, clearly, A_0 has the larger unit. Hence, the system of inequalities $0 < x < \dot{1}$ ($\dot{1} \in A_0$) can be solved in A, but not in A_0. Thus A_0 is not closed in A.

(ii) *Sufficiency.* Let A_0 and A have the same unit, and let L be a linear system, with its constants in A_0, and with a solution, $(c_1, c_2, \cdots, c_n)$, in A (here n is the number of the unknowns x_j). We have to show that L can already be solved in A_0. For this purpose, we inductively define n additional subgroups A_j, setting: *A_j equals the c_j-extension of A_{j-1} in A* ($j = 1, 2, \cdots, n$). By 2.1.1 and 1.2, all A_j are regularly discrete and serving subgroups of A, and have the same unit as A and A_0. Moreover, A_n contains all c_j; hence L has in A_n the same solution, $(c_1, \cdots, c_n)$.(14) By applying Lemma 4.2 recursively n times, we infer that L has a solution in A_0 as well, q.e.d.

The same method of proof (i.e., first proving the required theorems for

(14) Recall that, by 1.2, A_n is a serving subgroup of A. Therefore the congruences and incongruences contained in L are satisfied, both in A and A_n, by the same elements $c_1, c_2, \cdots, c_n$, supposed to constitute a solution of L.

systems in one unknown, then for c_0-extensions, and, finally, generalizing them by introducing the subgroups A_j as in 4.3) can also be used to establish the following two propositions (the details of the proof will be omitted)([15]).

4.4. *Every divisible subgroup A_0 of a divisible ordered group A is closed in A, with respect to all systems of inequalities of the form* 3.2.

4.5. *If A_0 is a serving subgroup of an ordered group A and has the same prime invariants as A, then A_0 is closed in A, with respect to all systems of congruences and (or) incongruences of the form* 3.3 *and* 3.4.

Our next objective is to prove a theorem analogous to 4.3 for *regularly dense* groups (this theorem is, likewise, a prerequisite of [3]). We need two lemmas.

4.6. LEMMA. *Let $(c_1, \cdots, c_n)$ be a solution of a system L' of inequalities of the form* 3.2, *in a densely ordered group A. Then A contains a positive element d such that the system L' is also satisfied by every other n-tuple of elements $(b_1, \cdots, b_n)$, with $c_j - d < b_j < c_j + d$, $b_j \in A$, $j = 1, 2, \cdots, n$.*

For the proof, let $D > 0$ be the least among the (positive) expressions $\sum_{j=1}^{n} q_{ij}c_j - a_i$, $i = m_1+1, m_1+2, \cdots, m_2$, with the q_{ij} and a_i as in 3.2; also let $q = \max |q_{ij}|$. By 2.3, there exists a positive element $d \in A$, with $0 < nqd < D$. Then it is easily verified that the element d has the required property.

4.7. LEMMA. *Every regularly dense subgroup A_0 of an ordered group A is closed in A, with respect to all systems of inequalities of the form* 3.2.

Proof. Let M_0 and M ($M_0 \subseteq M$) be the divisible extensions of A_0 and A, respectively (see §1, Part V). Also let L be a system of inequalities of the form 3.2, with all a_i in A_0, which has a solution in A and, a fortiori, in M. By 4.4, L has also a solution in M_0. This solution has the form $x_j = c_j/k$ $(j = 1, 2, \cdots, n)$, with k a positive integer and $c_j \in A_0$ (in fact, by the definition of a divisible extension, all elements of M_0 are such fractions; the number k may be assumed to be the common denominator of the fractions involved). Substitution of the c_j/k for the unknowns x_j in 3.2 yields

4.7.1. $$\sum_{j=1}^{n} q_{ij}c_j > ka_i, \qquad i = m_1 + 1, m_1 + 2, \cdots, m_2;\ c_j \in A_0.$$

Thus $(c_1, c_2, \cdots, c_n)$ is a solution, in A_0, of the auxiliary system

4.7.2. $$\sum_{j=1}^{n} q_{ij}x_j > ka_i, \qquad i = m_1 + 1, m_1 + 2, \cdots, m_2.$$

As A_0 is densely ordered, we can find a positive element $d \in A_0$ with properties specified in 4.6 (taking 4.7.2 for L'). Moreover, since the intervals $(c_j - d, c_j + d)$ are infinite, each of them contains (by the k-regularity of A_0)

([15]) Proposition 4.4 is actually a special case of A. Robinson's Theorem 3.1.5 proved in [2], and can also be proved in the same way as the latter.

a k-divisible element, call it kb_j ($b_j \in A_0$). Then, by 4.6, ($kb_1, \cdots, kb_n$) is another solution of 4.7.2 in A_0. Substituting the elements kb_j for the unknowns x_j in 4.7.2 and reducing by k, we see that ($b_1, \cdots, b_n$) is a solution in A_0 of the *original* system L. This completes the proof.

We are now able to establish the required additional result.

4.8. THEOREM. *For a regularly dense subgroup A_0 of an ordered group A to be closed in A, it suffices that A_0 be serving in A and have the same prime invariants as A, i.e., $[p]A_0 = [p]A$, for every prime p.*(16)

Proof. Let these conditions hold, and let L be a linear system of the form 3.2–3.4, with no equations in it(17), and with all its constants a_i in A_0. Given that L has a solution in A, we have to show that it can already be solved in A_0. By 4.7, A_0 contains, in any case, a solution, ($b_1, \cdots, b_n$), of the *partial system* 3.2, consisting of inequalities only. Similarly, by 4.5, there is a solution, ($c_1, \cdots, c_n$) ($c_j \in A_0$), of the *partial* system 3.3–3.4, consisting of the congruences and incongruences contained in L. Fixing these two solutions, we now use 4.6 to select a positive element $d \in A_0$ such that the inequalities 3.2 of L are also satisfied by every other n-tuple ($b_1', \cdots, b_n'$) with

4.8.1. $$b_j' \in (b_j - d, b_j + d), \qquad b_j' \in A_0, \qquad j = 1, 2, \cdots, n.$$

Now let $r>0$ be a common multiple of the moduli r_i in 3.3–3.4. As A_0 is regularly dense, each of the infinite intervals ($b_j - d$, $b_j + d$) contains an element b_j' with $b_j' \equiv c_j (\text{mod } r)$ in A_0, $b_j' \in A_0$ (see 0.2, with $k=1$). Then it easily follows that the b_j' ($j=1, 2, \cdots, n$) satisfy the congruences and incongruences of L (for so do the c_j). As the b_j' also satisfy the conditions 4.8.1, they constitute a solution of the inequalities 3.2 as well. Thus ($b_1', \cdots, b_2'$) is a solution, in A_0, of the whole system L. This completes the proof.

5. **Countable groups with the same prime invariants. (D/n)-extensions.** In order to prove the last of the required theorems (3.8 of [3]), we need some additional auxiliary concepts and propositions.

I. Let A and B be two countable disjoint torsion-free (possibly ordered) but not completely divisible groups such that $[p]A = [p]B$ for every prime p.(18) Let

5.1. $$p_1 < p_2 < p_3 < \cdots < p_k < \cdots$$

be the (finite or infinite) sequence of all primes which *do not* divide A and B. Using 1.7, select, for each k, a nonvoid maximal p_k-independent subset $S_k \subseteq A$, and also such a subset $T_k \subseteq B$. Due to countability, each S_k and T_k can be written as a (finite or infinite) sequence, say

(16) These conditions can also easily be proved to be necessary.

(17) We lose no generality by making this assumption (see 3.9).

(18) Note that, due to this assumption, A and B are divisible by exactly the same primes (for, p divides a group if and only if $[p]=1$ in that group).

5.2. $$S_k = \{a_{ki}\}, \qquad T_k = \{b_{ki}\}, \qquad a_{ki} \in A,\ b_{ki} \in B.$$

Since $[p_k]A = [p_k]B$, Corollary 1.8 implies that each S_k contains exactly as many elements as does the corresponding T_k, namely $\log_{p_k}[p_k]A = \log_{p_k}[p_k]B$. The one-to-one correspondence between S_k and T_k is, obviously, effected by the mapping $a_{ki} \rightarrow b_{ki}$ (see 5.2). Therefore, by embedding A and B in their direct sum M_0, we can define, for each k, a set D_k consisting of the differences of the form $a_{ki} - b_{ki}$, i.e.,

5.3. $$D_k = \{a_{k1} - b_{k1}, a_{k2} - b_{k2}, \cdots, a_{ki} - b_{ki}, \cdots \}.$$

The sets S_k, T_k, D_k and the sequences 5.2 and 5.3 will henceforth be considered as fixed. The subgroups generated by S_k, T_k, D_k will be denoted by $\overline{S}_k$, $\overline{T}_k$, $\overline{D}_k$, respectively. With these conventions, we have:

5.4. THEOREM. *The subgroups $\overline{S}_k$ and $\overline{T}_k$ are p_k-basic in A and B, respectively.*

We carry out the proof for $\overline{S}_k$. Suppose that $\overline{S}_k$ is not p_k-basic in A. Then there is an element $y \in A$ which is p_k-incongruent with any element of $\overline{S}_k$, so that, certainly, $y \notin S_k$. The maximality of S_k then implies that $(S_k \cup \{y\})$ is not p_k-independent in A, so that, for some integers q_i, and some distinct elements $x_i \in S_k$, we have

5.4.1. $$q_0 y + \sum_{i=1}^{m} q_i x_i \equiv 0 \pmod{p_k}, \qquad q_0 \not\equiv 0 \pmod{p_k}.$$

As q_0 is prime to p_k, there are integers s, t with $y = sq_0 y + p_k t y$, i.e., $y \equiv sq_0 y \pmod{p_k}$ whence, by 5.4.1, $y \equiv -s\sum_{i=1}^{m} q_i x_i \pmod{p_k}$, $x_i \in S_k$, which is impossible since y is p_k-incongruent with every element of $\overline{S}_k$.

5.5. THEOREM. *Each D_k is p_k-independent in the direct sum M_0 of A and B.*

Proof. By 1.10 (c, d), the relation $\sum_{i=1}^{m} q_i(a_{ki} - b_{ki}) \equiv 0 \pmod{p_k}$ in M_0 implies $\sum_{i=1}^{m} q_i a_i \equiv 0 \pmod{p_k}$ in A whence (due to the p_k-independence of S_k in A) $q_i \equiv 0 \pmod{p_k}$, as required.

5.6. THEOREM. *For each p_k and any elements $d \in D_k$, $a \in A$ (or $a \in B$), the relation $a \equiv d \pmod{p_k}$ in M_0 implies $a \equiv 0 \pmod{p_k}$ in A (in B, respectively).*

Proof. Let $a \equiv d \pmod{p_k}$ in M_0 where $a \in A$ and $d = \sum_{i=1}^{m} q_i(a_{ki} - b_{ki})$. By 5.4, a is p_k-congruent with some element of $\overline{S}_k$, say with $\sum_{i=1}^{m} r_i a_{ki}$, where m and the a_{ki} may be assumed to be the same as in the expression for d (we need only use some zero-coefficients). We then have $a \equiv \sum_{i=1}^{m} r_i a_{ki} \equiv \sum_{i=1}^{m} q_i(a_{ki} - b_{ki}) \pmod{p_k}$ in M_0. These relations imply, by 1.10 (c, d) that

5.6.2. $$a \equiv \sum_{i=1}^{m} r_i a_{ki} \equiv \sum_{i=1}^{m} q_i a_{ki} \pmod{p_k} \text{ in } A$$

and

5.6.3. $$\sum_{i=1}^{m} q_i b_{ki} \equiv 0 \pmod{p_k} \text{ in } M_0 \text{ and } B.$$

As T_k is p_k-independent in B, 5.6.3 implies $q_i \equiv 0 (\mathrm{mod}\ p_k)$ for all q_i. This, combined with 5.6.2, yields $a \equiv 0 (\mathrm{mod}\ p_k)$, as required.

II. Now let M' be a divisible group, M its subgroup, D a nonempty subset of M, and $\overline{D}$ the subgroup generated by D. For every integer $n > 0$, we define the *(D/n)-extension of M in M'* to be the set M_n of all elements $u \in M'$ with

5.6*. $$u = x + (d/n^k) \qquad (x \in M, d \in \overline{D}, k = 0, 1, 2, \cdots),$$

where d/n^k is any one of the quotients of d by n^k in M'.[19]

5.7. THEOREM. *Let M_n be the (D/n)-extension of M in M', as above. Then*: (a) *M_n is a subgroup of M' which contains M*; (b) *In M_n, all elements of D are divisible by any powers of n*; (c) *M is q-serving in M_n whenever the integer q is prime to n*; (d) *M is basic in M_n.*

Proof. Assertions (a) and (b) immediately follow from 5.6*. To prove (c), suppose that $y \equiv z (\mathrm{mod}\ q)$ in M_n for some $y, z \in M$, with q prime to n. Then $y - z = qu$ $(u \in M_n)$, whence, by 5.6*, $n^k(y-z) = qn^k u = q(n^k x + d)$, $(x, d \in M)$, so that q divides $n^k(y-z)$ in M. As n^k is prime to q, q must divide $y-z$ in M. Thus the congruence $y \equiv z (\mathrm{mod}\ q)$ holds in M whenever it holds in M_n, q.e.d. For the proof of (d), it suffices to show that M is p-basic in M_n for every prime p (see 1.3). If p divides n, 5.6*, combined with 5.7(b), easily implies that every $u \in M_n$ is p-congruent (in M_n) to some $x \in M$, as required. If, however, p does not divide n, the same fact is obtained by combining 5.6* with the equation $ps + n^k t = 1$ (s, k are integers).

5.8. THEOREM. *With the same notation, if M' is torsion-free, and D is n-independent in M, then, for any $y \in M$ to be divisible by n in M_n, it is necessary and sufficient that y be n-congruent (in M) with some element of $\overline{D}$.*

Proof. Sufficiency follows from 5.7(b). To prove necessity, let n divide the element $y \in M$ in M_n; i.e., let $y = nu$ $(u \in M_n)$. Then 5.6* yields

5.8.1. $$y - nx = d/n^{k-1} \qquad (x, y \in M, d \in \overline{D});$$

that is, d is divisible by n^{k-1} in M. As $d \in \overline{D}$, we have

5.8.2. $$d = \sum_{i=1}^{m} q_i d_i \qquad (d_i \in D),$$

for some integers q_i, so that the divisibility of d by n^{k-1} implies

[19] Obviously, the expression d/n^k may be many-valued if M' is not torsion-free. This is why several quotients of d by n^k may be involved.

5.8.3. $$\sum_{i=1}^{m} q_i d_i \equiv 0 \pmod{n^{k-1}} \text{ in } M \qquad (d_i \in D).$$

But D is, by assumption, n-independent in M. As M is torsion-free, it follows, by 1.6, that D is also n^{k-1}-independent in M. Therefore relation 5.8.3 implies that all q_i are divisible by n^{k-1}. Hence (see 5.8.2) n^{k-1} divides d in $\overline{D}$, i.e., $d = n^{k-1}d'$, for some $d' \in \overline{D}$. This, combined with 5.8.1, yields $y \equiv d' \pmod{n}$ in M, as asserted.

After this preparatory work, we are now able to prove the last of the required results, namely, Theorem 3.8 of [3]. We shall, however, split it into two theorems (6.1 and 6.3) the first of which is more general (since it also applies to unordered groups), whereas the second is the one postulated in [3], with regard to regularly dense groups.

6. Two embedding theorems.

6.1. THEOREM. *Let A and B be two countable disjoint torsion-free (ordered) abelian groups, with $[p]A = [p]B$, for every prime p. Then there exists a torsion-free (ordered, respectively) group M such that* (a) *A and B are serving and basic subgroups in M;* (b) *M has the same prime invariants as A and B, i.e., $[p]M = [p]A = [p]B$, for every prime p.*[20]

Proof. Assertion (b) follows from (a), by 1.4; so it suffices to prove (a). Furthermore, it suffices to consider the case where A and B are not divisible (otherwise, their direct sum can serve as the required group M).

Let then again 5.1 be the sequence of the primes p_k which do not divide A and B, and let the sets S_k, T_k, D_k and the groups $\overline{S}_k$, $\overline{T}_k$, $\overline{D}_k$ be defined as in §5. Further, let M' be the divisible extension of the direct sum M_0 of A and B. We start with constructing an ascending sequence of groups

6.1.1. $$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k \subseteq \cdots,$$

where M_k is defined inductively by

6.1.2. M_k is the (D_k/p_k)-extension of M_{k-1} in M', $\qquad k = 1, 2, 3, \cdots.$

The inductive process 6.1.2 is infinite or finite according as the sequence 5.1 is infinite or finite. As is readily seen, each M_k contains A and B, and is a subgroup of M'. As M' is torsion-free (ordered), so are all M_k. The divisible extension of each M_k coincides with M'. Finite induction easily yields the following facts.

6.1.3. *M_0 is a basic subgroup in each M_k.* (For the proof, apply 5.7(d).)

6.1.4. *M_k is p-serving in M_{k+n} $(n = 1, 2, \cdots)$ for every prime p other than p_{k+i} $(i = 1, 2, \cdots, n)$.* (Apply 5.7(c).)

6.1.5. *If A and B are divisible by some prime p, so is each M_k.* (Apply 5.6*.)

(20) The zero elements of A and B must, of course, be identified when they are embedded in M.

6.1.6. *Each* D_k *is* p_k*-independent in* M_n, *for* $n=0, 1, \cdots, k-1$. (Use 5.5 and 5.7(c).)

6.1.7. *A and B are serving in each* M_k. (Apply 1.3, 1.10(d), 5.8 and 5.6.) Furthermore, we have:

6.1.8. *If* p *is one of the primes* p_k $(k=1, 2, \cdots, n)$, *then*: (a) *For every element* $a\in A$ $(a\in B)$, *there is an element* $b\in B$ $(b\in A)$ *with* $a\equiv b(\text{mod } p)$ *in* M_n. (b) *A and B are p-basic in* M_n.

In fact, if $p=p_n$, every element $a\in A$ is, by 5.4, p-congruent to some element of $\overline{S}_n$, say

6.1.8′. $$a \equiv \sum_{i=1}^{m} q_i a_{ni} \ (\text{mod } p) \text{ in } A \text{ and in } M_n.$$

Now take, for each a_{ni}, the corresponding b_{ni} (see 5.2), and let

$$b = \sum_{i=1}^{m} q_i b_{ni}, \qquad d = \sum_{i=1}^{m} q_i(a_{ni} - b_{ni}), \qquad b \in B, d \in \overline{D}_n.$$

This, combined with 6.1.8′ yields $d\equiv a-b(\text{mod } p)$ in M_n, whence, by 5.7(b), $a-b\equiv 0\equiv d(\text{mod } p)$ in M_n, as asserted. If, however, $p=p_k$ $(1\leqq k<n)$ then, by what has been proved, for every $a \in A$, there is a $b \in B$ with $a\equiv b(\text{mod } p)$ in M_k. As $M_k\subseteq M_n$, this congruence holds in M_n as well. This establishes (a). Now take any element $u\in M_n$. By 6.1.3 and 1.10(b), there are elements $x\in A$ and $y\in B$ such that

6.1.8″. $$u \equiv x + y \ (\text{mod } p) \text{ in } M_n, \qquad (x + y) \in M_0.$$

On the other hand, by (a), we have $y\equiv x'(\text{mod } p)$ in M_n, for some $x'\in A$. This, combined with 6.1.8″, yields $u\equiv x+x'(\text{mod } p)$, where $(x+x')\in A$. Thus every element $u\in M_n$ is p-congruent with some element of A, i.e., A is p-basic in M_n. Similarly for B.

Now let M be the union of all M_k. Then the group M is torsion-free (ordered, respectively), for $M\subseteq M'$. From 6.1.7 it easily follows that A and B are serving subgroups of M. By 1.3, it remains to show that A and B are p-basic in M, for all primes values of p. Let then p be any prime, and $u\in M$. We consider two cases.

(I) *p is none of the primes* p_k. Then A and B are divisible by p; so are also all M_k (see 6.1.5). Hence, M too is divisible by p, so that all elements of M are p-congruent. It follows (trivially) that A and B are p-basic in M.

(II) *p is one of the primes* p_k, *say* $p=p_{k_0}$. By assumption, $u\in M$, i.e., u belongs to some M_k, say $u\in M_n$. If $k_0\leqq n$, then 6.1.8 implies that A and B are p-basic in M_n, so that u is p-congruent with some $x\in A$ and some $y\in B$. If, however, $n<k_0$, then $M_n\subseteq M_{k_0}$, and the same conclusion follows from 6.1.8. Thus, in all possible cases, u is p-congruent with some elements of A and B. This shows that A and B are basic in M, so that the group M has all the required properties. This completes the proof.

6.2. NOTE. It follows from this proof that the group M has the following additional properties.

(i) *Every element $u \in M$ satisfies some equation of the form $nu = a+b$ where $a \in A$, $b \in B$, and n is a positive integer.*

(ii) *If A and B are ordered, so is M. Moreover, the elements of A and B follow each other in M in the same order as they do in the direct sum M_0 of A and B (ordered lexicographically).*

Both properties follow from the fact that M is a subgroup of M' in which they hold. In the sequel, we shall assume that M_0 is ordered in such a way that the positive elements of A are greater than those of B (this amounts to selecting one of the two possible lexicographic orderings in M_0).

6.3. THEOREM. *Let A and B be two disjoint countable regularly dense groups having the same prime invariants. Then they can always be embedded in a regularly dense group M in such a way that M has the same prime invariants as A and B, and contains A and B as serving and basic subgroups.*

Proof. Construct the group M exactly as in Theorem 6.1. Then M has the same prime invariants as A and B, and contains A and B as serving and basic subgroups. It remains to show that M is regularly dense. Let then q be an integer >0, and u_1, $u_2 (u_1 < u_2)$ two elements of M. By 6.2(i), we have $n_1 u_1 = a_1 + b_1$ and $n_2 u_2 = a_2 + b_2$ ($a_1, a_2 \in A$; $b_1, b_2 \in B$); obviously, we may set $n_1 = n_2 = n$. Thus

6.3.1. $$nu_1 = a_1 + b_1, \quad nu_2 = a_2 + b_2; \quad a_1, a_2 \in A; \quad b_1, b_2 \in B; \quad n > 0.$$

As $u_1 < u_2$, we have $a_1 - a_2 < b_2 - b_1$ so that $a_1 - a_2$ cannot be positive (this would contradict our convention as to the ordering of M). Consider two cases:

(I) $a_1 - a_2 < 0$. In this case, due to the regular density of A, there is an element $y \in A$ with $a_1 < nqy < a_2$ (see 2.1(a)). Again, our convention as to the ordering of M easily implies that $a_1 + b_1 < nqy < a_2 + b_2$, or, by 6.3.1, $u_1 < qy < u_2$ ($y \in M$), which is the required condition, by 2.1(a).

(II) $a_1 = a_2 = a$. In this case, using 6.3.1, we obtain

6.3.2. $$nu_1 = a + b_1, \qquad nu_2 = a + b_2, \qquad b_1 < b_2.$$

As B is basic in M, a is (qn)-congruent with some $b \in B$, so that $a = b + qnx$ ($x \in M$) whence, by 6.3.2,

6.3.3. $$nu_1 = (b + b_1) + qnx, \qquad nu_2 = (b + b_2) + qnx, \qquad b_1 < b_2.$$

Here $b + b_1 < b + b_2$ and b, b_1, $b_2 \in B$. Since B is regularly dense, there is an element $x' \in B$ with $b + b_1 < qnx' < b + b_2$. This, combined with 6.3.3, yields $u_1 < q(x + x') < u_2$, $(x + x') \in M$, and the regular density of M is again established, by 2.1(a). Thus the theorem is proved.

Theorem 6.3 is a stronger version of the last of the required propositions (3.8 of [3]). Thus our objective of supplying the proofs of the theorems announced in [3] has been achieved.

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