NOTE ON A THEOREM OF GROSSWALD

BY

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In E. Grosswald's paper *On some algebraic properties of the Bessel polynomials*, appearing in the Transactions [1], he has proven the following theorem:

THEOREM. *For even n, the Bessel Polynomial $Y_n(X)$ has no real zero.*

Grosswald proves this theorem by establishing four lemmas. However, his proof of Lemma 4 is incorrect in that (13) does not follow from (12). Further, Equation (11) is not true for $n=2$ and $u=3/4$ (the numbers (11), (12) and (13) refer to Grosswald's paper [1]). However, the theorem is true and we submit the following proof.

In the proof we utilize the following recurrence relations [2]:

$$Y_{n+1} = (2n+1)XY_n + Y_{n-1}, \tag{1}$$

$$X(Y_n' + Y_{n-1}') = n(Y_n - Y_{n-1}), \tag{2}$$

$$(nx+1)Y_n' + Y_{n-1}' = n^2 Y_n. \tag{3}$$

All coefficients of the Bessel Polynomials are positive so that:

$$Y_{2n}(X) > 0 \qquad \text{for } X \geqq 0.$$

Grosswald's Lemma 3 shows that for $X \leqq -1$, $Y_{2n}(X) > 0$. We prove by induction:

$$Y_{2n}(X) > 0 \qquad \text{for } -1 < X < 0.$$

For $n=1$, $Y_{2n}(X) = Y_2(X) = 1+3X+3X^2 > 0$. Assume $Y_{2i}(X) > 0$ and consider the three classes of X in $(-1, 0)$:

(I) $Y_{2i+1}(X) = 0$, (II) $Y_{2i+1}(X) < 0$, (III) $Y_{2i+1}(X) > 0$.

If $X \in$ Class I the recurrence relation (1) yields

$$\begin{aligned} Y_{2i+2}(X) &= (4i+3)XY_{2i+1}(X) + Y_{2i}(X) \\ &= Y_{2i}(X) > 0. \end{aligned}$$

If $X \in$ Class II,

$$XY_{2i+1}(X) > 0,$$

and (1) again implies $Y_{2i+2}(X) > 0$.

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In considering Class III, we note that all zeros of the Bessel Polynomials are simple. See [1, p. 199].

If $X \in$ Class III, assume $Y_{2i+2}(X) < 0$. Since $Y_{2i+2}(0) > 0$, $Y_{2i+2}(-1) > 0$ a minimum of Y_{2i+2} exists, i.e.,

$$Y_{2i+2}(\omega) < 0,$$
$$Y'_{2i+2}(\omega) = 0.$$

By the previous results $\omega \in$ Class III. We write (3) as

$$(2iX + 2X + 1)Y'_{2i+2} + Y'_{2i+1} = (2i + 2)^2 Y_{2i+2}.$$

From the recurrence relation above $Y'_{2i+1}(\omega) < 0$. Now consider the recurrence relation (2) written as

$$X(Y'_{2i+2} + Y'_{2i+1}) = (2i + 2)(Y_{2i+2} - Y_{2i+1}).$$

Evaluating the left side of the above relation at the point ω we see it is positive. Evaluating the right-hand side at ω we find it is negative. From this absurdity we arrive at a contradiction, so that there is no point of Class III such that $Y_{2i+2} < 0$. This completes the proof.

For a different proof of the above theorem we refer the reader to Burchnall [3].

References

1. E. Grosswald, *On some algebraic properties of the Bessel polynomials*, Trans. Amer. Math. Soc. vol. 71 (1951) pp. 197–210.

2. H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 100–115.

3. J. L. Burchnall, *The Bessel polynomials*, Canad. J. Math. vol. 3 (1951) pp. 62–68.

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