

ON CANONICAL CONFORMAL MAPS OF MULTIPLY CONNECTED DOMAINS

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In a recent paper [11], J. L. Walsh has established the existence of two new canonical domains for the conformal mapping of multiply connected regions. The canonical maps may be characterized by the fact that they transform certain harmonic domain functions into functions which are extendible harmonically to the entire plane, with the exception of a finite number of points, one in each of the components of the complement of the canonical domain. Two other existence proofs have been given, by H. Grunsky [6] and J. A. Jenkins [7]. All of these proofs rely on uniformization. In part I of the present paper we give a geometric and constructive proof of existence, based primarily on potential-theoretic considerations. In part II we consider the relation between the new canonical mapping functions and extremal problems, and show thereby that the maps are natural generalizations of those which take, respectively, a doubly connected domain onto an annulus and simply connected domain onto the unit circle. Finally, in part III, by specializing the existence proof to the case of a doubly connected domain, we obtain a new construction procedure for the conformal map of such a domain onto an annulus, and show it to converge more rapidly than one commonly used at present.

The author is deeply grateful to Professors J. L. Walsh, L. V. Ahlfors, and R. Osserman for their inspiration and help.

I. Existence of the canonical map. Throughout this discussion let the term "harmonic measure of an annular domain" be taken to mean the harmonic measure of the outer boundary curve with respect to the domain; that is, the function which is harmonic in the domain and takes the values 0, 1 on the inner and outer boundary curves respectively. Also, let the term "contour" denote an analytic Jordan curve.

THEOREM 1. *Let C_1 be a Jordan curve in the z -plane, and D its exterior. Let C be an arbitrary contour in D containing C_1 in its interior, and let $D' \subset D$ denote the doubly connected region bounded by C_1 and C . There exists a map $w=f(z)$, one to one and conformal in D and mapping D onto the exterior of a contour γ_1 , such that the harmonic measure of the image of D' is extendible harmonically to the interior of γ_1 except for a single point, where it has a logarithmic singularity. This map is unique up to a linear transformation.*

Presented to the Society, January 30, 1958; received by the editors December 2, 1959.

Proof. We will construct a sequence of maps which extend the harmonic measure farther and farther into the interior of the image of C_1 . We will then show that the limit of this sequence satisfies the requirements of Theorem 1.

A. CONSTRUCTION OF THE SEQUENCE. We may assume that C_1 is the unit circumference, for C_1 may always be mapped onto $|z|=1$ by a preliminary transformation one to one and conformal exterior to C_1 , and taking ∞ onto ∞ . Let $R = \min_{z \in C} |z|$; we have $R > 1$. The harmonic measure $u(z)$ of D' is extendible by reflection up to C_r , the reflection of C in $|z|=1$, which lies interior to $|z|=1/R$.

Let first $w_1^* = g_1^*(z)$ be the conformal map of the exterior of C_r onto $|w_1^*| > 1/R$, with $g_1^*(\infty) = \infty$. Let C^* be the image of C under $g_1^*(z)$, and let $g_1^*(z) = a_0 + a_1z + b_1/z + b_2/z^2 + \dots$ in a neighborhood of $z = \infty$. By Schwarz's Lemma we have $|a_1| \geq 1$ and $|w_1^*| \geq |z|$ for all z exterior to C_r ; thus, setting $R_1 = \min_{w_1^* \in C^*} |w_1^*|$, we obtain $R_1 \geq R$.

It is important, however, to normalize the map at ∞ . We set $w_1 = g_1(z) \equiv g_1^*(z)/a_1$, let $C^{(1)}$ be the image of C under $w_1 = g_1(z)$, and $K_1 = 1/|a_1| \leq 1$. The net result is

$$w_1 = \sigma_0^{(1)} + z + \frac{\sigma_1^{(1)}}{z} + \frac{\sigma_2^{(1)}}{z^2} + \dots, \quad \text{in a neighborhood of } z = \infty.$$

The map $w_1 = g_1(z)$ is one to one and conformal exterior to C_r , and takes the exterior of C_r onto $|w_1| > K_1/R$ and C onto a contour $C^{(1)}$ which contains in its interior the disk $|w_1| < K_1R_1$. The function $w_1(z) - z$ is analytic without singularities exterior to C_r , including at $z = \infty$, and thus by the maximum principle

$$\begin{aligned} |w_1 - z| &\leq |w_1(z) - z|, \text{ for } z \text{ in } C_r \\ &\leq |w_1| + |z|, \text{ for } z \text{ in } C_r \\ &\leq K_1/R + 1/R = (K_1 + 1)/R. \end{aligned}$$

Now again, by reflection, the harmonic measure is extendible up to $C_r^{(1)}$, the reflection of $C^{(1)}$ in $|w_1| = K_1/R$, which lies interior to $|w_1| = K_1/R^2R_1$.

Iterating this procedure, we may define ($n = 2, 3, 4, \dots$)

$$w_n = g_n(w_{n-1}) = \sigma_0^{(n)} + w_{n-1} + \frac{\sigma_1^{(n)}}{w_{n-1}} + \frac{\sigma_2^{(n)}}{w_{n-1}^2} + \dots$$

in a neighborhood of $w_{n-1} = \infty$.

The map $w_n = g_n(w_{n-1})$ is one to one and conformal exterior to $C_r^{(n-1)}$, and it takes the exterior of $C_r^{(n-1)}$ onto $|w_n| > K_1K_2 \dots K_n/R_{n-1}R_{n-2}^2 \dots R_1^{2(n-2)}R^{2(n-1)}$, with $R_{n-1} > R_{n-2} > \dots > R_1 > R > 1$ and $K_i \leq 1$; the image under this map of $C^{(n-1)}$ is a contour $C^{(n)}$ which contains in its interior the disk

$|w_n| < K_1 K_2 \cdots K_n R_n$ where $R_n \geq R_{n-1}$. The function $w_n(w_{n-1}) - w_{n-1}$ is analytic without singularities exterior to $C_r^{(n-1)}$, including at $w_{n-1} = \infty$, and thus by the maximum principle

$$(1) \quad \begin{aligned} |w_n - w_{n-1}| &\leq [|w_n(w_{n-1}) - w_{n-1}|, \text{ for } w_{n-1} \text{ in } C_r^{(n-1)}] \\ &\leq |w_n| + |w_{n-1}|, \text{ for } w_{n-1} \text{ in } C_r^{(n-1)} \\ &\leq K_1 K_2 \cdots K_{n-1} (K_n + 1) / R_{n-1} R_{n-2}^2 \cdots R^{2^{(n-1)}}. \end{aligned}$$

By reflection, the harmonic measure will be extendible up to $C_r^{(n)}$, the reflection of $C^{(n)}$ in $|w_n| = K_1 \cdots K_n / R_{n-1} R_{n-2}^2 \cdots R^{2^{(n-1)}}$, which lies interior to $|w_n| = K_1 \cdots K_n / R_n R_{n-1}^2 \cdots R_1^{2^{(n-1)}} R^{2^n}$.

B. LIMIT OF THE SEQUENCE. Consider now the mappings $w_n = g_n(w_{n-1}) = f_n(z)$ all as functions of z defined, one to one, and conformal in the domain E , consisting of the exterior of the contour C_r . By the normalization, $w_n = f_n(z) = A_0^{(n)}/z + A_1^{(n)}/z + A_2^{(n)}/z^2 + \cdots$ in a neighborhood of $z = \infty$. Thus all the functions $f_n(z) - z$ are analytic without singularities in the entire domain E and we have by (1)

$$\begin{aligned} |f_n(z) - z| &= |w_n - w_{n-1} + w_{n-1} - w_{n-2} + \cdots + w_1 - z| \\ &\leq |w_n - w_{n-1}| + |w_{n-1} - w_{n-2}| + \cdots + |w_1 - z| \\ &\leq \frac{K_1 K_2 \cdots K_{n-1} (K_n + 1)}{R_{n-1} R_{n-2}^2 \cdots R_1^{2^{(n-1)}} R^{2^{(n-1)}}} + \cdots + \frac{K_1 (K_2 + 1)}{R_1 R^2} + \frac{K_1 + 1}{R} \\ &\leq B \quad \text{for all } n. \end{aligned}$$

Hence the functions $f_n(z) - z$ form a normal family in E .

Consider the infinite sequence of functions:

$$f_1(z) = [f_1(z) - z] + z; \quad f_2(z) = [f_2(z) - z] + z; \quad \cdots$$

all one to one and conformal in E . Select first a subsequence $f_{m_1}(z), f_{m_2}(z), \cdots$ such that the sequence of functions: $[f_{m_1}(z) - z], [f_{m_2}(z) - z], \cdots$ converges uniformly on every compact (on the Riemann sphere) subset of E . Then $f(z) = \lim_{i \rightarrow \infty} [f_{m_i}(z) - z] + z$ is a nonconstant, hence univalent, analytic function on E .

We will have occasion, in the subsequent discussion, to use the following elementary lemma, whose statement and proof we reproduce from [11].

LEMMA 1. *Let $g_n(z)$ be a sequence of functions analytic and univalent in a domain R , converging uniformly on every compact subset of R to a function $g(z)$ which is nonconstant and therefore univalent. Let S be the image of R under the map $w = g(z)$, and S^* be a compact subset of S . Then, for all n sufficiently large, the image of R under the map $w = g_n(z)$ covers S^* .*

Proof of Lemma 1. Let Γ be a Jordan curve, or the union of a finite num-

ber of Jordan curves in R , whose image under the map $w = g(z)$ lies in $S - S^*$ and separates S^* from the boundary of S . Since Γ forms a compact subset of R , we have $g_n(z) \rightarrow g(z)$ uniformly on Γ , thus for z in Γ and all n sufficiently large we have, uniformly for w in S^* ,

$$\left| \frac{g_n(z) - g(z)}{g(z) - w} \right| < 1,$$

whence by Rouché's theorem, $g_n(z) - w$ has the same number of zeros in the region bounded by Γ as does $g(z) - w$, namely one. Lemma 1 is established.

Denote now by $f_n(z)$ the subsequence selected above which converges to $f(z)$ on E , uniformly on every compact (on the Riemann sphere) subset of E . Let the map $w = f(z)$ take the domains D and D' onto domains Δ and Δ' respectively, and the contours $C_1: |z| = 1$ and C onto contours γ_1 and γ respectively. Similarly, let the map $w = f_n(z)$ take the domains D and D' onto domains Δ_n and Δ'_n respectively. By Lemma 1, the curve γ is contained in the domain Δ_n for some n sufficiently large and so, by construction of the Δ_n , does not pass through the point $w = 0$. Furthermore, since on the compact subset C of $|z| > 1$ the functions $f_n(z)$ converge uniformly to $f(z)$ we have

$$[\arg f(z)]_C = \int_C \frac{g'(z)}{g(z)} dz = \lim_{n \rightarrow \infty} \int_C \frac{g'_n(z)}{g_n(z)} dz = \lim_{n \rightarrow \infty} [\arg f_n(z)]_C,$$

and $[\arg f_n(z)]_C = 2\pi$ for all n , by construction of the maps $w = f_n(z)$. It follows that the contour γ separates $w = 0$ from $w = \infty$.

Select next a region D_0 whose closure \overline{D}_0 is a compact subset of D' , and let Δ_0 be the image of D_0 under the map $w = f(z)$. By Lemma 1, any compact subset of Δ_0 is contained in all the domains $f_n(D_0)$ for n sufficiently large. Thus the maps $z = f_n^{-1}(w)$ are defined and univalent on every compact subset of Δ_0 , for n sufficiently large, and map it into D_0 ; they therefore form a normal family in Δ_0 , and we may restrict ourselves to a subsequence which approaches a limit $F(w)$ on Δ_0 , uniformly on every compact subset of Δ_0 . The function $F(w)$ is either univalent or identically constant on Δ_0 . On any given compact subset of Δ_0 , the function $F(w)$ is the uniform limit of the functions $f_n^{-1}(w)$ so that the image of that compact subset under the map $z = F(w)$ is the limit of its images under the maps $z = f_n^{-1}(w)$. But the maps $z = f_n^{-1}(w)$, as soon as n is sufficiently large, map the given subset into D_0 , hence their limit must be contained in \overline{D}_0 . We conclude that the image of Δ_0 under the map $z = F(w)$ is contained in \overline{D}_0 . Thus, for $w_0 \in \Delta_0$, we have $\lim_{n \rightarrow \infty} f_n^{-1}(w_0) = F(w_0) \in \overline{D}_0$ so that for n sufficiently large all the points $f_n^{-1}(w_0)$ lie in a compact subset of D' , where the functions $f_n(z)$ converge uniformly to $f(z)$. Then by Carathéodory's criterion for continuous convergence [4], we have $f[F(w_0)] = \lim_{n \rightarrow \infty} f_n[f_n^{-1}(w_0)] = w_0$. Similarly, for $z_0 \in D_0$, we have $\lim_{n \rightarrow \infty} f_n(z_0) = f(z_0) \in \Delta_0$, so that for n sufficiently large all the points $f_n(z_0)$ lie in a compact subset of Δ_0 , where the functions $f_n^{-1}(w)$ converge uniformly to $F(w)$. Therefore,

as above, $F[f(z_0)] = \lim_{n \rightarrow \infty} f_n^{-1}[f_n(z_0)] = z_0$. Consequently $F(w)$ is not constant and coincides with the inverse function $f^{-1}(w)$ of $f(z)$, for $w \in \Delta_0$ and $z \in D_0$.

Let $u(z)$ be, as before, the harmonic measure of D' (i.e., the harmonic measure of the contour C with respect to D'); similarly, let $v(w)$ and $v_n(w)$ be the harmonic measures respectively of the domains Δ' and Δ'_n . On Δ'_n we have $v_n(w) \equiv u[f_n^{-1}(w)]$, since both functions are harmonic in Δ'_n and assume the same values on the boundary of Δ'_n ; similarly, $v(w) \equiv u[f^{-1}(w)]$.

Denote by R the domain bounded by the contour γ and the point $w=0$. By Lemma 1, every compact subset of Δ' lies in all the Δ'_n , for n sufficiently large; furthermore, by construction, the function $v_n(w)$ is extendible harmonically across the inner boundary of Δ'_n , up to the curve $C_r^{(n)}$, which lies interior to $|w| = K_1 \cdots K_n / R_n \cdots R^{2^n}$. Hence the functions $v_n(w)$ are defined and harmonic on any given compact subset of R for n sufficiently large, and $v_n(w) < 1$ everywhere in R . The functions $v_n(w)$ thus form a normal family in R , and we may find a subsequence converging on R , uniformly on every compact subset of R , to a limiting function $V(w)$ which is either harmonic in R or identically equal to $-\infty$. The latter cannot be the case, since from the preceding discussion, on the subdomain Δ_0 of R , $\lim_{n \rightarrow \infty} f_n^{-1}(w) = f^{-1}(w)$, so that, since $u(z)$ is continuous, we have for $w \in \Delta_0$

$$V(w) = \lim_{n \rightarrow \infty} v_n(w) \equiv \lim_{n \rightarrow \infty} u[f_n^{-1}(w)] = u[f^{-1}(w)] \equiv v(w) \neq -\infty.$$

Thus $V(w)$ is harmonic in R . But since, as we have just shown, the harmonic functions $V(w)$ and $v(w)$ coincide on the open subdomain Δ_0 of Δ' , they coincide everywhere on their common domain of definition Δ' , and the function $V(w)$ is seen to be the required harmonic extension of the harmonic measure $v(w)$ of Δ' (the image of D' under the map $w=f(z)$) to the interior of γ_1 , with the exception of the single point $w=0$. Since the harmonic function $V(w)$ approaches $-\infty$ in the neighborhood of $w=0$, it is of the form $A \log |w| + V_1(w)$, with $V_1(w)$ harmonic in the neighborhood of $w=0$ [10, p. 675]. We have $A = \int_{\gamma_1} \partial V / \partial n \, ds = \int_{\gamma} \partial v / \partial n \, ds$; A is therefore a conformal invariant, equal to the period of the conjugate function of the harmonic measure of D' around any curve in D' homologous to C .

C. UNIQUENESS OF THE MAP. Suppose that $w=f(z)$ and $W=F(z)$ are two conformal maps of D onto the exteriors of contours γ_1 and Γ_1 in the w and W -planes respectively, such that the harmonic measures $v(w)$ and $V(W)$ of the images δ' and Δ' of D' are extendible harmonically everywhere to the interiors of γ_1 and Γ_1 , with the exception of the points w_0 and W_0 , at which they have logarithmic singularities $A \log |w-w_0|$ and $A \log |W-W_0|$; the coefficient A is necessarily the same for v and V , since we have shown it to be a conformal invariant. Then $W=F[f^{-1}(w)] \equiv h(w)$ is a conformal map of the exterior of γ_1 onto the exterior of Γ_1 , which takes δ' onto Δ' . We will prove that this map is linear by showing that it can be extended to be one to one and conformal everywhere in the (extended) w -plane.

Let γ and Γ be the images of C in the w and W -planes respectively. We see that $[v(w)-1]/A$ is the Green's function of the interior of γ with pole at w_0 , from which we may construct in the usual way a conformal map $t=g(w)$ of the interior of γ onto the interior of the unit circle $|t| < 1$ which sends w_0 onto $t=0$. Since γ_1 is a level locus of the Green's function, the image of δ' under this map is the annulus $e^{-1/A} < |t| < 1$. By the identical argument, there exists a conformal map $T=G(W)$ of the interior of Γ onto the interior of the unit circle $|T| < 1$, by which Δ' is mapped onto the annulus $e^{-1/A} < |T| < 1$. The composed map $T=G\{h[g^{-1}(t)]\}$ now takes the above two annuli onto each other, with the unit circumferences corresponding; it is thus precisely a rotation, so that $G\{h[g^{-1}(t)]\} \equiv e^{i\theta_0}t$, or, equivalently, $h(w) \equiv G^{-1}[e^{i\theta_0}g(w)]$, for w in δ' . But the map $W=G^{-1}[e^{i\theta_0}g(w)]$ is one to one and conformal not only on δ' but everywhere interior to γ as well; it therefore provides the required extension of $h(w)$ to the entire w -plane. Thus $W=h(w)$ is a one to one conformal map of the w -plane onto the W -plane; it is therefore a linear transformation as was to be proved. Theorem 1 is established.

COROLLARY 1. *Let D be a region of the z -plane, bounded by the contour C ; to fix the ideas, let D be the exterior of C . Let $u(z)$ be a harmonic function defined in a neighborhood of C in D , and suppose $u(z)$ to be continuous and constant on C with no critical point on C . Then there exists a map $w=f(z)$ with $f(\infty)=\infty$, one to one and conformal in D , mapping it onto the exterior of a contour γ , and such that $v(w)=u[f^{-1}(w)]$, the transform of $u(z)$, is extendible harmonically to the interior of γ with the exception of one point, where it has a logarithmic singularity.*

Proof. Let $N \subset D$ be the domain of definition of $u(z)$. Suppose $u(z)=k$ on C ; $u(z)$ may be extended by reflection to be harmonic in a neighborhood of C , so that $\partial u/\partial n$ exists everywhere on C and is continuous there. Since $u(z)$ has no critical points on C , we have $\partial u/\partial n \neq 0$ on C , so that $\partial u/\partial n$ is either positive or negative everywhere on C . Thus, inside a sufficiently small neighborhood of C , the values of $u(z)$ in N are all either larger or smaller than k ; suppose that they are larger. Let C' be a Jordan curve in N , such that C' and C together bound an annular domain R contained in the above neighborhood of C , and let $\alpha = \min_{z \in C'} u(z)$. We have $\alpha > k$. Now select β so that $\alpha > \beta > k$, and consider in R the locus $\lambda: u(z)=\beta$. The locus λ is a closed curve in R . It cannot be made up of more than one closed Jordan curve, since two closed curves in an annular domain must either be homotopic to each other, or have one of them homotopic to zero: in each case there would exist a subdomain of R on whose boundary $u(z)=\beta$, which would imply that $u(z) \equiv \beta$ in R , contradicting the nature of $u(z)$. Consequently the locus λ has no critical points of $u(z)$, else it would be the union of more than one closed Jordan curve, which was seen to be impossible; it is therefore an analytic Jordan curve which, together with C , bounds an annular domain R' . The harmonic meas-

ure of R' is seen to be $v(z) \equiv [u(z) - k]/(\beta - k)$, and by Theorem 1 there exists a one to one conformal map $w=f(z)$ of the exterior of C onto the exterior of a contour γ , with $f(\infty) = \infty$, such that the transform $v[f^{-1}(w)]$ of $v(z)$ is extendible harmonically everywhere to the interior of γ with the exception of a single point where it has a logarithmic singularity. Since

$$v[f^{-1}(w)] \equiv \frac{u[f^{-1}(w)] - k}{\beta - k},$$

we see that the transform $u[f^{-1}(w)]$ of $u(z)$ under the map $w=f(z)$ is also extendible harmonically everywhere to the interior of γ with the exception of one point at which it has a logarithmic singularity.

If the values of $u(z)$ in R are smaller than k , let $\alpha = \max_{z \in C'} u(z)$. We have $k > \alpha$, and selecting β so that $k > \beta > \alpha$ we argue precisely as before. Corollary 1 is established.

THEOREM 2 (WALSH). *Let D be a region of the extended z -plane whose boundary consists of mutually disjoint Jordan curves $B_1, B_2, \dots, B_\mu; C_1, C_2, \dots, C_\nu, \mu\nu \neq 0$. There exists a conformal map of D onto a region Δ of the extended Z -plane, one to one and continuous in the closures of the two regions, where Δ is defined by*

$$(2) \quad 1 < |T(Z)| < e^\alpha, \quad T(Z) \equiv \frac{A(Z - a_1)^{M_1} \dots (Z - a_\mu)^{M_\mu}}{(Z - b_1)^{N_1} \dots (Z - b_\nu)^{N_\nu}},$$

with $\alpha, M_i, N_j > 0, \sum M_i = \sum N_j = 1$. The locus $|T(Z)| = 1$ consists of μ mutually disjoint Jordan curves B_i^* , respective images of the B_i , which separate Δ from the a_i ; the locus $|T(Z)| = e^\alpha$ consists of ν mutually disjoint Jordan curves C_j^* , respective images of the C_j , which separate Δ from the b_j .

Proof. Assume the Jordan curves $B_1, \dots, B_\mu; C_1, \dots, C_\nu$ to be all analytic and D to contain $z = \infty$, since that property may always be obtained by a preliminary conformal transformation of D . The harmonic measure $u(z)$ of the point set $\bigcup_{j=1}^\nu C_j$ with respect to the domain D may be extended, by reflection across the analytic boundary curves, to be harmonic in a neighborhood of the boundary of D . The function $u(z)$ cannot have a critical point on the boundary of D , since a level locus of a harmonic function which passes through a critical point consists in a neighborhood of that point of $2n$ (with $n > 1$) analytic arcs, radiating from the critical point and spaced at equal angles π/n one from the next; thus if a critical point lay on one of the boundary curves B_i or C_j , there would be points interior to D at which $u(z)$ had the values 1 or 0, contradicting the maximum principle. By Corollary 1, there exists a map $w_1 = f_1(z)$, one to one and conformal exterior to B_1 , with $f_1(\infty) = \infty$, taking D onto a domain Δ_1 and B_1 onto a curve β_1 , such that the function $u_1(w_1) \equiv u[f_1^{-1}(w_1)]$, which corresponds in the w_1 -plane to $u(z)$, is extend-

ible harmonically to the interior of β_1 with the exception of one point, where it has a logarithmic singularity.

The above procedure may next be applied to the image in the w_1 -plane of B_2 , and so on successively for all the B_i and C_j . Denote $w_{\mu+\nu}$ by Z ; the combined effect of the $\mu+\nu$ appropriate successive maps is to take D one to one and conformally onto a region Δ of the Z -plane, where the function which corresponds to $u(z)$ may be extended harmonically to the entire plane, with the exception of one point in each of the components of the complement of Δ , where it has a logarithmic singularity. Consequently, $u(z) \equiv \log |T(z)|/\alpha$, with $T(Z)$ of the form (2), and Δ may be defined by $1 < |T(Z)| < e^\alpha$. Theorem 2 is established.

We show next that the canonical conformal map of Theorem 2 is unique up to a linear transformation of the Z -plane.

THEOREM 3 (WALSH). *Let D be a region of the z -plane defined by the inequality*

$$1 < |R(z)| < e^\alpha, \quad R(z) \equiv \frac{A(z - a_1)^{m_1} \cdots (z - a_\mu)^{m_\mu}}{(z - b_1)^{n_1} \cdots (z - b_\nu)^{n_\nu}}, \quad \sum m_i = \sum n_j = 1,$$

and whose boundary consists of mutually disjoint Jordan curves $B_1, B_2, \dots, B_\mu; C_1, C_2, \dots, C_\nu$, where B_i separates a_i from D and C_j separates b_j from D . Let Δ be a region of the Z -plane defined by

$$1 < |R_1(Z)| < e^{\alpha'} \quad R_1(Z) \equiv \frac{A'(Z - a'_1)^{m'_1} \cdots (Z - a'_\mu)^{m'_\mu}}{(Z - b'_1)^{n'_1} \cdots (Z - b'_\nu)^{n'_\nu}},$$

$$\sum m'_i = \sum n'_j = 1,$$

and whose boundary consists of mutually disjoint Jordan curves $B'_1, B'_2, \dots, B'_\mu; C'_1, C'_2, \dots, C'_\nu$, where B'_i separates a'_i from Δ and C'_j separates b'_j from Δ . If there exists a one to one conformal transformation $Z=f(z)$ of D onto Δ so that each B_i corresponds to B'_i and each C_j corresponds to C'_j , then the transformation is a linear transformation of the complex variable z , defined throughout the extended planes z and Z . We have $\alpha=\alpha'$, $m_i=m'_i$, and $n_j=n'_j$.

Proof. Let $u(z)$ be the harmonic measure of the point set $\bigcup_{j=1}^{\nu} C_j$ with respect to D , and let $U(Z)$ be the harmonic measure of the point set $\bigcup_{j=1}^{\nu} C'_j$ with respect to Δ . By definition of the two domains, $u(z) \equiv \log |R(z)|/\alpha$ and $U(Z) \equiv \log |R_1(Z)|/\alpha'$, so that the two harmonic measures are extendible harmonically everywhere to the z and Z -planes, except for one point in each component of the complements of D and Δ , where they have logarithmic singularities. As in the proof of Theorem 2, the function $u(z)$ cannot have a critical point on the boundary of D . Therefore, applying the proof of Corollary 1 to the boundary contour B_1 , we may find a contour γ in D on which $u(z)$ is constant and which, together with B_1 , bounds an annular domain D' .

On Δ we have $U(Z) \equiv u[f^{-1}(z)]$, since both functions are harmonic in Δ and assume the same values on the boundary of Δ ; it follows that the image Γ of γ under $Z=f(z)$ is a contour in Δ on which $U(Z)$ has the same constant value and which, together with B'_1 , bounds an annular domain Δ' . Now proceeding precisely as in the uniqueness proof of Theorem 1, we may extend the map $Z=f(z)$ to be one to one and conformal in the interior of γ , with $f(a_1)=a'_1$.

Applying the identical argument to the remaining boundary contours of D , we see that the map $Z=f(z)$ is extendible to a one to one conformal map of the extended z -plane onto the extended Z -plane, and is therefore a linear transformation. By the construction, $Z=f(z)$ sends the points $z=a_i$ onto $Z=a'_i$ and $z=b_j$ onto $Z=b'_j$. The singularities of $u(z)$ at a_i and b_j are given by $m_i \log |z-a_i|/\alpha$ and $-n_j \log |z-b_j|/\alpha$ respectively; analogously for $U(Z)$. Since $u(z)$ and $U(Z)$ are transforms of each other under the linear map, they have the same singularities, so that $m_i/\alpha=m'_i/\alpha'$ and $n_j/\alpha=n'_j/\alpha'$. Adding these equalities we conclude that $\alpha=\alpha'$, $m_i=m'_i$, and $n_j=n'_j$. Theorem 3 is established.

THEOREM 4 (WALSH). *Let D be a region of the Z -plane whose boundary consists of mutually disjoint Jordan curves C_1, \dots, C_r , let $\alpha_1, \dots, \alpha_\mu$ be arbitrary distinct points of D , and let M_1, \dots, M_μ be arbitrary positive numbers with $\sum M_i=1$. Then there exists a conformal map of D onto a region Δ of the Z -plane, one to one and continuous in the closed regions, where Δ is defined by*

$$(3) \quad |T(Z)| < 1, \quad T(Z) \equiv \frac{A(Z-a_1)^{M_1} \cdots (Z-a_\mu)^{M_\mu}}{(Z-b_1)^{N_1} \cdots (Z-b_r)^{N_r}}, \quad N_j > 0, \quad \sum N_j = 1.$$

The a_i are the respective images of the α_i ; the locus $|T(Z)|=1$ consists of r analytic Jordan curves, respective images of the C_j , which separate Δ from the b_j .

Proof. Theorem 3 represents the limiting case of Theorem 2 as the curves B_i shrink to points. It may be established by applying without change the argument above to the function $u(z)$ which is harmonic in D except for singularities $M_i \log |z-\alpha_i|$ at the points $z=\alpha_i$, and which takes on the value zero on $\bigcup_{j=1}^r C_j$.

The map of Theorem 4 is likewise uniquely determined, except for a possible linear transformation.

THEOREM 5 (WALSH). *Let D be a region of the z -plane defined by the inequality*

$$|R(z)| < 1, \quad R(z) \equiv \frac{A(z-a_1)^{m_1} \cdots (z-a_\mu)^{m_\mu}}{(z-b_1)^{n_1} \cdots (z-b_r)^{n_r}}, \quad m_i > 0, \quad \sum m_i = \sum n_j = 1,$$

and whose boundary consists of mutually disjoint Jordan curves C_1, C_2, \dots, C_r , which separate the b_j respectively from D . Let Δ be a region of the Z -plane defined by the inequality

$$|R_1(Z)| < 1, \quad R_1(Z) \equiv \frac{A'(Z - a'_1)^{m_1} \cdots (Z - a'_\mu)^{m_\mu}}{(Z - b'_1)^{n'_1} \cdots (Z - b'_\nu)^{n'_\nu}}, \quad \sum n'_j = 1,$$

and whose boundary consists of mutually disjoint Jordan curves $C'_1, C'_2, \dots, C'_\nu$, which separate the b'_j respectively from Δ . If there exists a one to one conformal transformation of D onto Δ so that the a_i correspond respectively to the a'_i , then this correspondence can be continued beyond D and Δ as a linear transformation of the extended z -plane onto the extended Z -plane, and $n_j = n'_j$.

Proof. We establish Theorem 5 by applying without change the argument of Theorem 3, to the functions $u(z)$ and $U(Z)$ which are harmonic in D and Δ except for singularities $m_i \log |z - a_i|$ and $m_i \log |Z - a'_i|$ at the points $z = a_i$ and $Z = a'_i$ respectively and which take on the value zero on $\bigcup_{j=1}^r C_j$ and $\bigcup_{j=1}^r C'_j$.

II. The canonical map and extremal problems. We will now consider the relation between the canonical maps of Theorems 2 and 4 and extremal problems.

The conformal map $Z = f(z)$ of a doubly connected domain D onto an annulus $r_1 < |Z| < r_2$ may, by a simple application of Schwarz's Lemma, be seen to possess the following extremal characterization: let $Z = g(z)$ be any univalent map of D onto a bounded doubly connected domain D_g in the Z -plane whose inner and outer boundaries are Jordan curves B_g and C_g respectively, with $Z = 0$ interior to B_g . Set $M_g = \min_{Z \in B_g} |Z| / \max_{Z \in C_g} |Z|$; then the map $Z = f(z)$ has the property that

$$(4) \quad M_f \geq M_g.$$

This extremal property may be viewed somewhat differently. Consider the family H of functions $h(z)$ harmonic in D , continuous in the closure \bar{D} of D , and normalized so that $\int_\gamma \partial h / \partial n \, ds = 2\pi$, where γ is an analytic Jordan curve in D which separates the boundary curves B and C of D , and n is the normal on γ in the direction of C . For $h(z) \in H$, let $R_h = \max_{z \in \bar{D}} h(z) - \min_{z \in \bar{D}} h(z)$, and consider the problem of minimizing R_h over the family H . We will show (Corollary 2) that R_h is minimized in H by any function of the form $\alpha u(z) + k$, where $u(z)$ is the harmonic measure of the curve C with respect to D , the normalizing constant $\alpha = 2\pi / \int_\gamma \partial u / \partial n \, ds$, and k is an arbitrary constant. This extremal problem immediately induces one among the univalent functions of D . For every one of the univalent maps $Z = g(z)$ of D defined above, the function $\log |Z| \equiv \log |g(z)|$ is contained in H , by virtue of the normalization of the maps, while the function $\log |f(z)|$ coincides with the particular extremal function $\alpha u(z) + \log r_1$. Thus $R_{\log |f|} \leq R_{\log |g|}$, or

$$(5) \quad \frac{\max_{z \in \bar{D}} |f(z)|}{\min_{z \in \bar{D}} |f(z)|} \leq \frac{\max_{z \in D} |g(z)|}{\min_{z \in \bar{D}} |g(z)|}.$$

But by virtue of the assumption that B_θ and C_θ are the inner and outer boundary curves of D_θ , the quantities $\max_{z \in \bar{D}} |g|$ and $\min_{z \in \bar{D}} |g|$ coincide respectively with $\max_{z \in C_\theta} |g|$ and $\min_{z \in B_\theta} |g|$, so that (5) is equivalent to the inequality (4) above.

The analogous formulation may be given for the canonical map $Z=f(z)$ in the domain D of Theorem 2. We first consider the family H of functions $h(z)$ harmonic in D , continuous in the closure \bar{D} of D , and so normalized that $\int_\gamma \partial h / \partial n \, ds = 2\pi$, where γ is an analytic Jordan curve in D separating the curves $\bigcup_{i=1}^\mu B_i$ from the curves $\bigcup_{j=1}^v C_j$. Defining R_h as above, we will show (Corollary 2) that R_h is again minimized in H by any function of the form $\alpha u(z) + k$, where $u(z)$ is the harmonic measure of the curves $\bigcup_{j=1}^v C_j$ with respect to D , the normalizing constant $\alpha = 2\pi / \int_\gamma \partial u / \partial n \, ds$, and k is an arbitrary constant. Since the function $\log |T[f(z)]|$, where $Z=f(z)$ and $T(Z)$ are as in Theorem 2, coincides with the specific extremal function $\alpha u(z)$, we have in particular $R_{\log|T(f)|} \leq R_h$. From this we will below obtain an inequality which refers explicitly to univalent maps of D and whose form is modeled after (4).

THEOREM 6. *Let D be a region of the extended z -plane whose boundary consists of mutually disjoint Jordan curves $B_1, \dots, B_\mu; C_1, \dots, C_v, \mu v \neq 0$. Let $u(z)$ be the harmonic measure of the set $\bigcup_{j=1}^v C_j$ with respect to the domain D , and let $v(z)$ be any function harmonic in D , continuous in the closure \bar{D} of D , and such that $v(z) \geq 0$ on $\bigcup_{i=1}^\mu B_i$ and $v(z) \leq 1$ on $\bigcup_{j=1}^v C_j$. Let γ be an analytic Jordan curve in D which separates the curves $\bigcup_{i=1}^\mu B_i$ from the curves $\bigcup_{j=1}^v C_j$. Then*

$$\int_\gamma \partial u / \partial n \, ds \geq \int_\gamma \partial v / \partial n \, ds,$$

where n is the normal on γ in the direction of the curves $\bigcup_{j=1}^v C_j$. Equality occurs if and only if $v(z) \equiv u(z)$ in \bar{D} .

Proof. We may assume the curves B_i and $C_j, i=1, \dots, \mu; j=1, \dots, v$, to be all analytic Jordan curves, for this property may always be obtained by a preliminary conformal transformation of D . Let $p(z) \equiv u(z) - v(z)$; then $p(z)$ is harmonic in D , with $p(z) \leq 0$ for z in $\bigcup_{i=1}^\mu B_i$ and $p(z) \geq 0$ for z in $\bigcup_{j=1}^v C_j$. Let $q(z)$ be the function harmonic in D which has value zero on $\bigcup_{j=1}^v C_j$ and which coincides with $p(z)$ on $\bigcup_{i=1}^\mu B_i$, and let $r(z) \equiv p(z) - q(z)$. Now $r(z) = 0$ for z in $\bigcup_{i=1}^\mu B_i$ and $r(z) = p(z) \geq 0$ for z in $\bigcup_{j=1}^v C_j$. The function $r(z)$ may be extended by reflection across the part of the boundary made up of the curves B_i to be harmonic in a neighborhood of $\bigcup_{i=1}^\mu B_i$, so that the normal derivative of $r(z)$ exists almost everywhere on $\bigcup_{i=1}^\mu B_i$. By the maximum principle, $r(z) \geq 0$ in D , so that $\partial r(z) / \partial n_i \geq 0$ for $Z \in \bigcup_{i=1}^\mu B_i$, with n_i the interior normal. Hence

$$0 \leq \int_{\cup_i B_i} \partial r / \partial n_i ds = \int_{\gamma} \partial r / \partial n ds,$$

with n the normal on γ in the direction of the curves $\cup_{j=1}^p C_j$, or, from the definition of $r(z)$,

$$(6) \quad \int_{\gamma} \partial p / \partial n ds \geq \int_{\gamma} \partial q / \partial n ds.$$

Now, arguing as with $r(z)$, the function $q(z)$ may be extended to be harmonic in a neighborhood of the curves $\cup_{j=1}^p C_j$, so that its normal derivative exists almost everywhere on $\cup_{j=1}^p C_j$. By the maximum principle, $q(z) \leq 0$ in D , so that $\partial q(z) / \partial n_e \geq 0$ for $z \in \cup_{j=1}^p C_j$, with n_e the exterior normal. Thus

$$(7) \quad 0 \leq \int_{\cup_j C_j} \partial q / \partial n_e ds = \int_{\gamma} \partial q / \partial n ds.$$

Combining (6) and (7) we obtain

$$\int_{\gamma} \partial p / \partial n ds \geq 0,$$

or

$$(8) \quad \int_{\gamma} \partial u / \partial n ds \geq \int_{\gamma} \partial v / \partial n ds.$$

If equality occurs in (8), by retracing the steps of the proof we see that we must have $\partial q / \partial n = 0$ almost everywhere for $z \in \cup_{j=1}^p C_j$, and $\partial r / \partial n = 0$ almost everywhere for $z \in \cup_{i=1}^m B_i$. The function $q(z)$ is harmonic in a neighborhood of the curves C_j and has value 0 for $z \in \cup_{j=1}^p C_j$. Choose a point z_0 in $\cup_{j=1}^p C_j$ and a simply connected neighborhood N of z_0 such that $q(z)$ is harmonic in N , and let $q^*(z)$ be the conjugate harmonic function of $q(z)$ in N . On any arc A of $\cup_{j=1}^p C_j$ contained in N we have $\partial q^*(z) / \partial s = \partial q(z) / \partial n = 0$, so that the function $q(z) + iq^*(z)$, analytic in N , is constant on the arc A in N , hence identically constant in N . Thus $q(z) \equiv 0$ in N , whence $q(z) \equiv 0$ in \bar{D} . Arguing analogously, $r(z) \equiv 0$ in \bar{D} . But this implies that $p(z) \equiv 0$ for $z \in \cup_{i=1}^m B_i$ and $z \in \cup_{j=1}^p C_j$, thus also $p(z) \equiv 0$ in \bar{D} , whence $v(z) \equiv u(z)$ in \bar{D} . Theorem 6 is established.

COROLLARY 2. *Let H be the family of functions $h(z)$ harmonic in D , continuous in the closure \bar{D} of D , and normalized so that $\int_{\gamma} \partial h / \partial n ds = 2\pi$, where γ is an analytic Jordan curve in D separating the curves $\cup_{i=1}^m B_i$ from the curves $\cup_{j=1}^p C_j$, and n is the normal on γ in the direction of $\cup_{j=1}^p C_j$. Let $R_h = \max_{z \in \bar{D}} h(z) - \min_{z \in \bar{D}} h(z)$, and let $w(z) \equiv 2\pi u(z) / \int_{\gamma} \partial u / \partial n ds$ with $u(z)$, as before, being the*

harmonic measure of the curves $\bigcup_{j=1}^r C_j$ with respect to D . Then $R_w \leq R_h$, for all $h(z) \in H$, with equality if and only if $h(z) \equiv w(z) + k$, k a constant.

Proof. If $h(z) \in H$, the quantity $R_h \neq 0$, else $h(z) \equiv c$ in D , which contradicts the normalization requirement. Consider then, for every $h(z) \in H$, the function $h^*(z) = [h(z) - \min_{z \in \bar{D}} h(z)]/R_h$; we have $0 \leq h^* \leq 1$ in \bar{D} , so that, by Theorem 6,

$$(9) \quad \int_{\gamma} \partial u / \partial n \, ds \geq \int_{\gamma} \partial h^* / \partial n \, ds = \int_{\gamma} \partial h / \partial n \, ds / R_h = 2\pi / R_h.$$

Now $w(z)$ has constant boundary values, 0 and $2\pi / \int_{\gamma} \partial u / \partial n \, ds$ respectively, on the boundary of D , and, as in Theorem 6, the latter quantity is positive. Thus $R_w = 2\pi / \int_{\gamma} \partial u / \partial n \, ds$ or, from (9), $2\pi / R_w \geq 2\pi / R_h$, whence $R_w \leq R_h$. Equality occurs if and only if it occurs in (9), that is, if and only if $h^*(z) \equiv u(z)$ (Theorem 6). By definition this is equivalent to $h(z) - \min_{z \in \bar{D}} h(z) \equiv R_h u(z) \equiv w(z)$, whence $h(z) \equiv w(z) + \min_{z \in \bar{D}} h(z)$. Corollary 2 is established.

We can now specialize the above extremal problem to the univalent maps of D , by restricting consideration to the subfamily of H consisting of only those functions which are each extendible harmonically everywhere to the complement, except for points, one in each of the components of the complement, of the image of D under some univalent map. Let $Z = g(z)$ be any univalent map of D onto a domain D_θ of the Z -plane, bounded by Jordan curves $\bigcup_{i=1}^s B_{i,\theta}$ and $\bigcup_{j=1}^r C_{j,\theta}$, with $B_{i,\theta}$ and $C_{j,\theta}$ the images of B_i and C_j respectively under the map $Z = g(z)$. Let $S_\theta(Z)$ be any function of the form

$$(10) \quad S_\theta(Z) \equiv \frac{C(Z - \alpha_1)^{s_1} \cdots (Z - \alpha_\mu)^{s_\mu}}{(Z - \beta_1)^{t_1} \cdots (Z - \beta_\nu)^{t_\nu}},$$

where $s_i, t_j > 0$, $\sum s_i = \sum t_j = 1$, and α_i and β_j are separated from D_θ by the curves $B_{i,\theta}$ and $C_{j,\theta}$ respectively. Since $S_\theta(Z)$ has neither zeros nor poles in the closure \bar{D}_θ of D_θ , the functions $\log |S_\theta(Z)| \equiv \log |S_\theta[g(z)]|$ are all harmonic in D and continuous in the closure of D . If γ is as in Corollary 2, the image Γ of γ under the map $Z = g(z)$ separates the curves $\bigcup_{i=1}^s B_{i,\theta}$ from the curves $\bigcup_{j=1}^r C_{j,\theta}$ in D_θ , so that, from the form of $S_\theta(Z)$, we have that $[\arg S_\theta(Z)]_\Gamma = 2\pi$. But

$$[\arg S_\theta(Z)]_\Gamma = \int_\Gamma \partial \arg [S_\theta(Z)] / \partial s \, ds = \int_\Gamma \partial \log |S_\theta(Z)| / \partial n \, ds,$$

so that functions $\log |S_\theta(Z)|$ satisfy the normalization condition of Corollary 2 as well, and hence are all in H . The function $\log |T_f(Z)| = \log |T[f(z)]|$, where $Z = f(z)$ and $T(Z)$ are as in Theorem 2, is in H and has boundary values 0 on the curves $\bigcup_{i=1}^s B_i$ and α on the curves $\bigcup_{j=1}^r C_j$, so that it coincides with the extremal function $w(z)$ of Corollary 2. Thus, by Corollary 2, $R_{\log |T_f(Z)|} \leq R_{\log |S_\theta(Z)|}$, or

$$(11) \quad \frac{\max_{z \in \bar{D}} \log |T[f(z)]|}{\min_{z \in \bar{D}} \log |T[f(z)]|} \leq \frac{\max_{z \in \bar{D}} \log |S_\theta[g(z)]|}{\min_{z \in \bar{D}} \log |S_\theta[g(z)]|}.$$

The functions $\log |S_\theta(Z)|$ are all harmonic on the respective domains D_θ and hence assume their maxima and minima on the boundary of D_θ . Any level locus $\log |S_\theta(Z)| = c$ is composed of a finite number of closed Jordan curves and separates points in the Z -plane where $\log |S_\theta(Z)| > c$ from points where $\log |S_\theta(Z)| < c$. Furthermore, every region bounded by curves of this locus must contain in its interior points of singularity of the function $\log |S_\theta(Z)|$, that is, one or more of the points α_i or β_j ; else the function would have constant boundary values in a region of harmonicity which implies that it is identically constant, contrary to its form (10). Let $\lambda = \max_{z \in \bar{D}_\theta} \log |S_\theta(Z)| = \max_{z \in \bar{D}} \log |S_\theta[g(z)]|$. The level locus $\log |S_\theta(Z)| = \lambda$ lies entirely in the complement of D_θ . No Jordan curve J of this locus can lie inside a component of the complement of D_θ bounded by one of the curves $B_{i,\theta}$ since J would then separate D_θ from $Z = \alpha_i$, hence also from points where $\log |S_\theta(Z)| < \lambda$; this would imply that $\log |S_\theta(Z)| > \lambda$ on D_θ , contradicting the definition of λ . Consequently the function $\log |S_\theta(Z)|$ assumes its maximum in \bar{D}_θ on the boundary curves $\bigcup_{j=1}^r C_{j,\theta}$; by the identical argument, $\min_{z \in \bar{D}} \log |S_\theta[g(z)]| = \min_{z \in \bigcup_i B_i} \log |S_\theta[g(z)]|$. We may therefore replace (11) by

$$(12) \quad \frac{\max_{z \in \bigcup_j C_j} \log |T[f(z)]|}{\min_{z \in \bigcup_i B_i} \log |T[f(z)]|} \leq \frac{\max_{z \in \bigcup_j C_j} \log |S_\theta[g(z)]|}{\min_{z \in \bigcup_j C_j} \log |S_\theta[g(z)]|}.$$

Equality occurs in (12) if and only if it occurs in (11), that is, if and only if (Corollary 2) $\log |S_\theta(g)| \equiv \log |T(f)| + k$ for z in D . If Δ is the canonical domain of Theorem 2: $1 < |T(Z^*)| < e^\alpha$ which corresponds to D in the Z^* -plane, with $T(Z^*)$ and $Z^* = f(z)$ as in Theorem 2, the map $Z = g[f^{-1}(Z^*)]$ provides a one to one conformal transformation of Δ onto D_θ . Furthermore, since $|e^{-k} S_\theta(g)| \equiv |T(f)|$, the above transformation maps the boundary curves $B_{i,\theta}$ and $C_{j,\theta}$ of D_θ onto the boundary curves B_i^* and C_j^* of Δ respectively, $i = 1, \dots, \mu$; $j = 1, \dots, \nu$, and the domain D_θ is defined by $1 < |e^{-k} S_\theta(Z)| < e^\alpha$. Theorem 3 may now be applied; it establishes that the map $Z = g[f^{-1}(Z^*)]$ is a linear transformation $Z = A(Z^*)$ of the complex variable Z^* , defined throughout the extended planes Z and Z^* , or, equivalently, that $g(z) \equiv A[f(z)]$. We have thus obtained in (12) an extremal statement modeled in form after (4) which singles out the canonical conformal map of Theorem 2 from among the univalent maps of D .

We turn next to a brief consideration of the number α which figures in Theorem 2; we will show it to be interpretable as an extremal length. The canonical domain associated with D is the domain $1 < |T(Z)| < e^\alpha$, where

$Z=f(z)$ and $T(Z)$ are as defined in Theorem 2. As we have already shown above, the function $\log |T[f(z)]|$ is contained in the family H of Corollary 2, and coincides there with the function $\alpha u(z)$, where $u(z)$ is the harmonic measure of the curves $\bigcup_{j=1}^n C_j$ with respect to D . We thus have $\alpha = 2\pi / \int_{\gamma} \partial u / \partial n \, ds$, whence α is a conformal invariant. Furthermore

$$\int_{\gamma} \partial u / \partial n \, ds = \int_{\bigcup_j C_j} \partial u / \partial n \, ds = \int_{\bigcup_i B_i + \bigcup_j C_j} u \partial u / \partial n \, ds,$$

with n the exterior normal, and by Green's theorem

$$\int_{\bigcup_i B_i + \bigcup_j C_j} u \partial u / \partial n \, ds = \iint_D (\partial u / \partial x)^2 + (\partial u / \partial y)^2 \, dxdy.$$

Hence $\alpha/2\pi = 1 / \iint_D (u_x)^2 + (u_y)^2 \, dxdy$, which is equal to the extremal length of the family of curves in D joining points of $\bigcup_{i=1}^m B_i$ to points of $\bigcup_{j=1}^n C_j$ [2]; as such, α represents a natural generalization of the modulus of a doubly connected domain. The relation of the number α to the geometry of the domain D may now be studied by using tools such as the Comparison and Composition Laws which apply to extremal length. In particular, α is a monotonic domain functional in the following sense.

COROLLARY 3. *Let D be the domain of Theorem 2. Let D' be a domain bounded by disjoint Jordan curves B'_1, \dots, B'_m and C'_1, \dots, C'_n , $mn \neq 0$, such that $D' \subset D$ and the sets of curves $\bigcup_{i=1}^m B'_i$ and $\bigcup_{j=1}^n C'_j$ each separate the curves $\bigcup_{i=1}^m B_i$ from the curves $\bigcup_{j=1}^n C_j$; that is, the sets of curves $\bigcup_{i=1}^m B'_i$ and $\bigcup_{i=1}^m B_i$ together form the total boundary of a union of subregions of D . Let α' be the constant corresponding to α in the definition of the canonical domain of D' . Then $\alpha' \leq \alpha$.*

Proof. The proof is immediate from the theory of extremal length [1; 2; 3]. The number α is equal to the extremal length of the family F of curves in D joining points of $\bigcup_{i=1}^m B_i$ to points of $\bigcup_{j=1}^n C_j$. Similarly, α' is the extremal length of the family F' of curves in D' joining points of $\bigcup_{i=1}^m B'_i$ to points of $\bigcup_{j=1}^n C'_j$. By the Comparison Law applied to the families F and F' , we have $\alpha' \leq \alpha$. Corollary 3 may also be established without recourse to the theory of extremal length, by applying the argument of Theorem 6.

We turn next to the canonical conformal map of Theorem 4; we will show that it is the solution of an extremal problem which is a generalization of the one solved by the map of a simply connected domain onto the interior of the unit circle.

THEOREM 7. *Let $Z=f(z)$ be the canonical map of Theorem 4, taking D onto a region of the Z -plane defined by $|T(Z)| < 1$, with $T(Z)$ as in (3). Let $Z=g(z)$ be any one to one conformal map of D onto a domain D_0 in the Z -plane bounded by Jordan curves, and let $S_0(Z)$ be any function of the form*

$$S_\theta(Z) \equiv \frac{C(Z - c_1)^{M_1} \cdots (Z - c_\mu)^{M_\mu}}{(Z - d_1)^{t_1} \cdots (Z - d_\nu)^{t_\nu}},$$

where $t_j > 0$, $\sum t_j = 1$, $c_i = g(\alpha_i)$, the point $Z = d_j$ is separated from D_θ by the image of C_j under the map $Z = g(z)$ and such that

$$|S_\theta(Z)| < 1, \quad Z \in D_\theta.$$

Let $F(z) \equiv T[f(z)]$, and $G(z) \equiv S_\theta[g(z)]$; let $p_i(z) = F(z)/[f(z) - a_i]^{M_i}$ and $q_i(z) \equiv G(z)/[g(z) - c_i]^{M_i}$. Then

$$|f'(\alpha_i)^{M_i} p_i(\alpha_i)| \geq |g'(\alpha_i)^{M_i} q_i(\alpha_i)|, \quad i = 1, \dots, \mu,$$

with equality attained for any i if and only if

$$g(z) \equiv A[f(z)],$$

with A a linear transformation.

Proof. Consider for z in D the function $u(z) \equiv \log |F(z)| - \log |G(z)|$. The function $u(z)$ is harmonic for all z in D with the possible exception of the points $z = \alpha_i$, where $F(z) = G(z) = 0$. In a neighborhood of $z = \alpha_i$, however, $p_i(z)$ and $q_i(z)$ are analytic $\neq 0$, and consequently

$$\begin{aligned} \log |F(z)| &= M_i \log |f(z) - a_i| + \log |p_i(z)| \\ &= M_i \log |z - \alpha_i| + M_i \log |f'(\alpha_i)| + \log |p_i(\alpha_i)| + \delta_i(z), \end{aligned}$$

where $\delta_i(z)$ is harmonic in the neighborhood of $z = \alpha_i$ and $\lim_{z \rightarrow \alpha_i} \delta_i(z) = 0$. Similarly, in a neighborhood of $z = \alpha_i$,

$$\begin{aligned} \log |G(z)| &= M_i \log |g(z) - c_i| + \log |q_i(z)| \\ &= M_i \log |z - \alpha_i| + M_i \log |g'(\alpha_i)| + \log |q_i(\alpha_i)| + \epsilon_i(z), \end{aligned}$$

where $\epsilon_i(z)$ is harmonic in the neighborhood of $z = \alpha_i$ and $\lim_{z \rightarrow \alpha_i} \epsilon_i(z) = 0$. Thus the function $u(z)$ has only a removable singularity at $z = \alpha_i$, and

$$u(\alpha_i) = \log |f'(\alpha_i)^{M_i} p_i(\alpha_i)| - \log |g'(\alpha_i)^{M_i} q_i(\alpha_i)|.$$

For $z \in \bigcup_{j=1}^{\nu} C_j$ we have $\log |F(z)| = 0$ and $\log |G(z)| \leq 0$, thus $u(z) \geq 0$. By the maximum principle, $u(z) \geq 0$ at every interior point of D , with equality at any interior point only if $u(z) \equiv 0$ for z in D . Hence, $u(\alpha_i) \geq 0$, or

$$|f'(\alpha_i)^{M_i} p_i(\alpha_i)| \geq |g'(\alpha_i)^{M_i} q_i(\alpha_i)|$$

for $i = 1, \dots, \mu$. If equality occurs for any i , we have $|F(z)| = |G(z)|$, for z in D . Arguing as in the sequel to Corollary 2, we let $Z^* = f(z)$ be the canonical map of Theorem 4, taking D onto the canonical domain Δ : $|T(Z^*)| < 1$ (with $T(Z^*)$ as in Theorem 4), and we consider the map $Z = g[f^{-1}(Z^*)]$ which provides a one to one conformal transformation of Δ onto D_θ . Since $|F(z)| = |G(z)|$, the above transformation maps the points a_i onto the points

c_i , and the domain D_θ is defined by $|S_\theta(Z)| < 1$. Theorem 5 may now be applied to establish that the map $Z = g[f^{-1}(Z^*)]$ is a linear transformation $Z = A(Z^*)$ of the complex variable Z^* , defined throughout the extended Z and Z^* -planes, or, equivalently, that $g(z) \equiv A[f(z)]$. Theorem 7 is established.

For the case that only a single point α_i is selected from D , we have $\mu = 1$, and $M_1 = 1$; the extremal condition of Theorem 7 becomes

$$|f'(\alpha_1)p_1(\alpha_1)| \geq |g'(\alpha_1)q_1(\alpha_1)|.$$

But $|f'(\alpha_1)p_1(\alpha_1)| = |F'(\alpha_1)|$ and $|g'(\alpha_1)q_1(\alpha_1)| = |G'(\alpha_1)|$. Thus the extremal condition which characterizes the canonical mapping function for the above special case of Theorem 4 is $|F'(\alpha_1)| = \max_G |G'(\alpha_1)|$. This extremal problem is a generalization of that which determines the mapping of a simply connected domain onto the interior of the unit circle.

III. Mapping a doubly connected domain onto an annulus. We show next how the construction of Theorem 1 may be applied to a doubly connected region D to yield a rapidly converging method for performing the map of D onto an annulus. A construction by successive approximations commonly used at present has been suggested by Y. Komatu [8] and consists of mapping D onto doubly connected regions which are obtained each from the preceding by mapping alternately, and with appropriate normalization, the interiors of their outer and exteriors of their inner boundaries onto interiors and exteriors of circles, respectively. The rapidity of convergence of these approximations has been studied by D. Gaier [5], who has shown that $|F(z) - f_n(z)| < M_1 \rho_1^n$, where $w = F(z)$ is a conformal map of D onto an annulus, $w = f_n(z)$ is the approximation of Komatu constructed by means of n conformal transformations of simply connected domains, and M_1 and $\rho_1 < 1$ are constants depending only on D . We will, below, define maps $w = F_n(z)$ of D , likewise obtained by means of n conformal transformations of simply connected domains, and will demonstrate that $|F(z) - F_n(z)| < M \rho^n$, where M and $\rho < 1$ are constants depending only on D .

Let us again adopt the convention of using the term "harmonic measure of a doubly connected domain" to mean the harmonic measure of the outer boundary curve with respect to the domain.

Consider a doubly connected domain D in the z -plane with outer contour C and inner contour C_1 . We may assume that C is the unit circumference and C_1 surrounds $z=0$, for D may always be mapped onto such a domain by a preliminary conformal transformation. Let C^* be the reflection of C_1 in C ; the harmonic measure of D is extendible by reflection to the domain D^* , bounded by C_1 and C^* , and has the value 2 on C^* .

Now let $Z = f(z)$ be the function obtained in Theorem 1 which is one to one and conformal in the entire exterior of C_1 , with $f(\infty) = \infty$, and which maps C_1 onto some contour γ_1 and D onto a domain Δ whose harmonic measure is extendible harmonically to the interior of γ_1 , except for the single point

$Z=0$, where it has a logarithmic singularity. Let γ and γ^* be the images under $Z=f(z)$ of C and C^* respectively.

PROPOSITION 1. *Let $w=h(Z)$ be the conformal map of the interior of γ^* onto $|w|<1$, with $h(0)=0$, and $h'(0)>0$. Then the map $w=h(Z)$ takes Δ onto an annulus in the w -plane.*

Proof. Let $U(Z)$ be the harmonic measure of Δ . By definition of Δ , and as in the uniqueness proof of Theorem 1, the Green's function $\log |h(Z)|$ of the interior of γ^* with pole at $Z=0$ coincides with a constant multiple of $[U(Z)-2]$. Since the contours γ and γ_1 are level loci of the latter function, the image of Δ under $w=h(Z)$ is an annulus. Proposition 1 is established.

Thus we see that in order to effect the map of D onto an annulus, the map $Z=f(z)$ of Theorem 1 need be followed by only a single additional conformal transformation. This would suggest that an approximation to the desired map of D onto an annulus, obtained by n iterated conformal mappings of simply connected regions, might be constructed by using $n-1$ such mappings to determine the approximation $f_{n-1}(z)$ to $f(z)$, and following this by the map of the interior of $C^{*(n-1)}$ [the image of C^* under $Z=f_{n-1}(z)$] onto the interior of the unit circle. We shall study the rapidity of convergence of this approximation process.

Let $Z=f_n(z)$ be, as in Theorem 1, the approximation to $Z=f(z)$ obtained by n successive conformal maps and reflections. Let C_1^n, C^n, C^{*n} be the images in the Z -plane of C_1, C, C^* respectively under the map $Z=f_n(z)$. Let $w=h_n(Z)$ map the interior of C^{*n} onto $|w|<1$, with $h_n(0)=0$ and $h'_n(0)>0$, and let $F_{n+1}(z)\equiv h_n[f_n(z)]$. Let $w=h(Z)$ be as defined above, and let $F(z)\equiv h[f(z)]$; by Proposition 1, the map $w=F(z)$ takes D onto an annulus.

THEOREM 8. *Let the maps $w=F_n(z)$ and $w=F(z)$ be as defined above. Then*

$$|F(z) - F_n(z)| < M\rho^{2^n}, \quad z \in D,$$

where M and $\rho < 1$ are constants depending only on D .

Proof. The functions $\log |h_n(Z)|$ and $\log |h(Z)|$ are Green's functions with pole at $Z=0$ of the domains interior to the contours C^{*n} and γ^* respectively; we will estimate their difference by means of Hadamard's variational formula. Since C^{*n} and γ^* have continuously turning tangents, the normal displacement $[9]\delta_n(s)$ of C^{*n} with respect to γ^* is a continuous function of s , and Hadamard's formula yields

$$(13) \log |h_n(Z)| - \log |h(Z)| = \frac{1}{2\pi} \int_{\gamma^*} \frac{\partial g(t, Z)}{\partial n_t} \frac{\partial g(t, 0)}{\partial n_t} \delta_n(s) ds_t + O(\delta_n(s)),$$

where $t \in \gamma^*$, the function $g(t, Z)$ is the Green's function of the interior of γ^* with pole at Z , the normal n_t points outward on γ^* , and Z is in the common domain of definition of h_n and h . By (1) we have, for z exterior to C_1 ,

$$\begin{aligned}
 |f_n(z) - f(z)| &= |(f_n - f_{n+1}) + (f_{n+1} - f_{n+2}) + \cdots| \\
 &\leq |f_n - f_{n+1}| + |f_{n+1} - f_{n+2}| + \cdots \\
 (14) \quad &\leq \frac{K_1 \cdots K_n (K_{n+1} + 1)}{R_n R_{n-1}^2 \cdots R^2} + \frac{K_1 \cdots K_{n+1} (K_{n+2} + 1)}{R_{n+1} R_n^2 \cdots R^{2^{n+1}}} + \cdots, \\
 &\quad \text{with } K_i < 1, \text{ and } R_{n+1} > R_n > \cdots > R_1 > R > 1, \\
 &\leq A_1 / R^{2^{n+1}}, \quad \text{with } A_1 \text{ independent of } n.
 \end{aligned}$$

Set $\rho = 1/R$, $\epsilon_n = \rho^{2^n}$, and $\phi_n(Z) \equiv \log |h_n(Z)| - \log |h(Z)|$. Let A_i represent henceforth constants independent of n . Let Γ_1 and Γ_2 be two contours in the Z -plane separating γ^* from γ and Γ_1 from γ respectively. By applying (14) to $z \in C^*$, we obtain

$$|\delta_n(s)| < A_2 \epsilon_{n+1},$$

so that for $n > N_0$, the contour Γ_1 lies interior to C^* . For $n > N_0$ and Z interior to Γ_1 , the left-hand side of (13) is defined and since Γ_1 and γ^* are disjoint, the partial derivatives $\partial g(t, Z)/\partial n_t$ and $\partial g(t, 0)/\partial n_t$, for $t \in \gamma^*$, are uniformly bounded in Z . Thus from (13)

$$(15) \quad |\phi_n(Z)| < A_3 \epsilon_{n+1}, \quad Z \text{ interior to } \Gamma_1, \text{ all } n > N_0.$$

We next obtain bounds on the conjugate harmonic function $\psi_n(Z)$ of $\phi_n(Z)$ from the expression

$$(16) \quad \psi_n(Z) - \psi_n(Z_0) = \int_C \left(-\frac{\partial \phi_n}{\partial Y} dX + \frac{\partial \phi_n}{\partial X} dY \right),$$

where $Z = X + iY$ and C is an analytic arc interior to Γ_1 joining Z to Z_0 . By definition of $\phi_n(Z)$ we have

$$\psi_n(Z) \equiv \arg [h_n(Z)/h(Z)]$$

whence, by the normalization of h_n and of h at $Z=0$, we obtain $\psi_n(0)=0$. The interior of Γ_2 is, with its closure, contained in the interior of Γ_1 , hence the partial derivatives of the harmonic function $\phi_n(Z)$ are uniformly bounded there and, from (15),

$$\left| \frac{\partial \phi_n}{\partial X} \right|, \left| \frac{\partial \phi_n}{\partial Y} \right| < A_4 \epsilon_{n+1}, \quad Z \text{ interior to } \Gamma_2, \text{ all } n > N_0.$$

From (16), therefore

$$(17) \quad |\psi_n(Z)| < A_5 \epsilon_{n+1}, \quad Z \text{ interior to } \Gamma_2, \text{ all } n > N_0.$$

We combine the above bounds as follows:

$$\log [h_n(Z)/h(Z)] \equiv \phi_n(Z) + i\psi_n(Z),$$

whence

$$(18) \quad h(Z) - h_n(Z) \equiv h(Z)[1 - e^{\phi_n(Z) + i\psi_n(Z)}].$$

By applying (14) to $z \in C$, we see that, for $n > N_1$, the contour C^n lies interior to Γ_2 . Thus the bounds (15) and (17) are valid for Z on and interior to C^n as soon as $n > N_2 = \max(N_0, N_1)$, whence, from (18),

$$(19) \quad |h(Z) - h_n(Z)| < A_6 \epsilon_{n+1}, \quad Z \text{ on and interior to } C^n, \text{ all } n > N_2.$$

Now by definition

$$(20) \quad \begin{aligned} |F(z) - F_{n+1}(z)| &= |h[f(z)] - h[f_n(z)] + h[f_n(z)] - h_n[f_n(z)]| \\ &\leq |h[f(z)] - h[f_n(z)]| + |h[f_n(z)] - h_n[f_n(z)]|. \end{aligned}$$

The function $h(Z)$ is defined and analytic for Z interior to γ^* , thus its derivative is uniformly bounded for Z interior to Γ_2 . This fact, combined with (14), yields

$$|h[f(z)] - h[f_n(z)]| < A_7 \epsilon_{n+1}, \quad z \in D,$$

whence, with (19) we obtain from (20)

$$|F(z) - F_{n+1}(z)| < M \epsilon_{n+1} = M \rho^{2^{n+1}},$$

for $z \in D$ and all $n > N_2$. Theorem 8 is established.

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