A NOTE ON ABSTRACT INTEGRATION

BY

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I. **Introduction.** The principal purpose of this note is to distinguish between the basic nature of S-type (Stieltjes) and L-type (Lebesgue) integration with respect to a finitely additive set function (measure). Interest in these types of integration was given impetus by the fundamental representation theorem of Hildebrandt, Fichtenholz, and Kantorovitch (cf. [2]), and, since that time, a considerable amount of work has been done with these integrals along the lines of developing a formal theory and as a powerful analytic tool in the study of linear spaces.

In the case of the formal theory, the usual practice has been to start with a set algebra (ring) instead of a sigma algebra (ring) and define both of the integrals in this basic setting. (This is natural since the "integrator" need not be completely additive and, at any rate, on a sigma algebra the two integrals are, for essentially bounded functions, identical.) This procedure has worked well for the S-integral; however, for the L-integral a great deal of difficulty arises. For example, in this primitive setting, the L-integral (in general) is neither an absolutely continuous, nor a linear, nor a homogeneous operation. These difficulties, and the fact that, for a formal theory, no one of the three usual ways that the class of measurable functions can be defined is any more desirable than the others [in general, each yields a class of measurable functions distinct from the others], reflect the artificiality inherent in the usual ways of defining the class of measurable functions.

In this paper, we use a fourth class of function (called continuous) and, by considering the relationships between these four classes, establish a theorem (Theorem 2.2) that, in effect, says *the natural setting of the L-type integral is a sigma algebra*. Also, we show that the class of continuous functions (which includes the various types of bounded measurable functions) can be characterized entirely in terms of the S-integral (Theorem 2.1), and this, together with the definition of continuity, implies that *the natural setting of the S-type integral is a set algebra*. We conclude the paper by deriving a necessary and sufficient condition, in the case of a set algebra, in order that each continuous function belong to at least one of the classes of measurable functions and show, by example, that a set algebra need not be a sigma algebra in order to satisfy this condition.

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II. **Notation and terminology.** Throughout this paper we shall use the notation and terminology adopted in [1]. Hence, (X, S) will denote a set algebra (of a set X) and $H(X, S)$ will denote the set of bounded finitely additive set functions on S. There are many ways in which a maximal proper ideal, in S, can be characterized and, for the purpose of this paper, we will use two characterizations: (1) a proper ideal J in S is maximal if and only if $E\in S$ implies one, and only one, of E and $E'=X-E\in J$, and (2) a subset J of S is a maximal proper ideal if and only if there exists (uniquely) $g\in H(X, S)$ such that $g(E)=0$ if $E\in J$ and $g(E)=1$ if $E\notin J$. If g has the properties of (2) we say that g is a two-valued jump function. For the definitions of the S- and L-type integrals see [2] or [3]. All functions considered in this paper are assumed to be real valued.

Definition 2.1. If f is a function on X, then f is said to be an (X, S)-continuous function if, for each $e>0$, there exists a partition $\{E_i\}_{i=1}^{n}$ of X such that $E_i\in S$ and $O(f, E_i)<e$ for $i\leqq n$ where $O(f, E_i)=\text{lub}_{x,y\in E_i}|f(x)-f(y)|$ (i.e., $O(f, E)$ denotes the oscillation of f on E). The collection of (X, S)-continuous functions is denoted by $C(X, S)$.

Theorem 2.1. *Let (X, S) be a set algebra and let f be a function on X. Then f is an (X, S)-continuous function if and only if $s\int_X f\,dg$ exists for each $g\in H(X, S)$.*

Proof. We will show that if f is not (X, S)-continuous then there exists a two-valued jump function g on S such that $s\int_X f\,dg$ does not exist. If, for $e>0$, we let $D(f, S, e)=[E\in S;$ if $\{E_i\}_{i=1}^{n}$ is a partition of E in S (i.e., $E_i\in S$ for $i\rightarrow n$), then $\text{lub}_{i\leq n} O(f, E_i)\geqq e]$, then, $f\in C(X, S)$ if and only if $D(f, S, e)=\theta$ for $e>0$. Suppose there exists $e>0$ such that $D(f, S, e)\neq\theta$. Let I denote the set of ideals in S such that $J\in I$ and $E\in J$ imply $E'=X-E\in D(f, S, e)$. There exists $J\in I$ which is maximal with respect to inclusion. Suppose there exists $E\in S$ such that each of E and $E'\notin J$. Then each of K and K_1, the ideals generated by J and E and J and E' respectively (i.e., $F\in K$ if and only if there exist $G\in J$ and $H\in S$ such that $F=G\cup(H\cap E)$), is a proper ideal in S which contains J as a proper subset. Therefore, each of K and $K_1\notin I$ and there exist $F\in K$ and $F_1\in K_1$ such that each of F' and $F_1'\notin D(f, S, e)$ which imply each of $E'\cap F'$ and $E\cap F_1'\notin D(f, S, e)$; however, each of $E'\cap F$ and $E\cap F_1\in J$. But, $X=E\cup E'=(E\cap F_1)\cup(E\cap F_1')\cup(E'\cap F)\cup(E'\cap F')$ $=(E\cap F_1)\cup(E'\cap F)\cup(E\cap F_1')\cup(E'\cap F')$, $(E\cap F_1)\cup(E'\cap F)\in J$ and $(E\cap F_1')\cup(E'\cap F')\notin D(f, S, e)$. This contradiction shows our supposition that J is not a maximal proper ideal in S is false. There exists $g\in H(X, S)$ such that $g(E)=0$ if $E\in J$ and $g(E)=1$ if $E\notin J$; $s\int_X f\,dg$ does not exist.

Remark. Perhaps it is of interest to note that, in a sense, Theorem 2.1 is an extension to the general case of the classical theorem which states that a function f on the interval $[a, b]$ is continuous if and only if the Stieltjes integral $\int_a^b f\,dg$ exists for every function g of bounded variation on $[a, b]$. Also, we note that, regarding $C(X, S)$ as a linear-normed-complete space, $C(X, S)$

is isomorphically isometric to the space of topologically continuous functions on $\beta(S)$, where $\beta(S)$ is the space of ultrafilters associated with S (i.e., the Stone-Čech type compactification of (X, S)). Finally, we see that a function f on X is in $C(X, S)$ if and only if there exists a sequence $\{f_i\}$ of (X, S)-simple functions such that $\lim_i \|f-f_i\| = \lim_i (\text{lub}_{x\in X} |f(x)-f_i(x)|) = 0$.

DEFINITION 2.2. If (X, S) is a set algebra, then

(1) $M(X, S) = [f \text{ on } X; -\infty \leqq a < b \leqq \infty \Rightarrow f^{-1}(a, b) \in S], f^{-1}(a, b) = [x\in X; a<f(x)<b]$,

(2) $LM(X, S) = [f \text{ on } X; -\infty < a < b \leqq \infty \Rightarrow f^{-1}[a, b) \in S], f^{-1}[a, b) = [x\in X; a\leqq f(x)<b]$,

(3) $RM(X, S) = [f \text{ on } X, -\infty \leqq a < b < \infty \Rightarrow f^{-1}(a, b]\in S], f^{-1}(a, b] = [x\in X; a<f(x)\leqq b]$,

(4) $G(X) = [f \text{ on } X; \|f\| = \text{lub}_{x\in X} |f(x)| < \infty]$, and

(5) $m(X, S) = G(X)\cap M(X, S)$, $Lm(X, S) = G(X)\cap LM(X, S)$, and $Rm(X, S) = G(X)\cap RM(X, S)$.

The following two lemmas follow readily from Definition 2.2.

LEMMA 2.1. *Let (X, S) be a set algebra. Then*

(1) *if $\{E_i\}_{i=1}^n$ is a finite collection of pairwise disjoint subsets of X and $\bigcup_{i\leq n} E_i\in S$ then either $E_i\in S$ for $i\leqq n$ or there exist at least two indices i and j such that each of E_i and $E_j \notin S$,*

(2) *each of $m(X, S)$, $Lm(X, S)$, and $Rm(X, S)$ is a subset of $C(X, S)$,*

(3) *if $f\in M(X, S)$ and P is a real number then $f^{-1}(P)\in S$,*

(4) *$M(X, S) = LM(X, S)\cap RM(X, S)$, and*

(5) *if f is a function on X, then $f\in M(X, S)$, $LM(X, S)$, or $RM(X, S)$ if and only if $f^n\in m(X, S)$, $Lm(X, S)$, or $Rm(X, S)$ respectively for each positive integer n where $f^n(x) = f(x)$ if $|f(x)| \leqq n$ and $f^n(x) = n\cdot f(x)\cdot |f(x)|^{-1}$ if $|f(x)| > n$.*

LEMMA 2.2. *If (X, S) is a sigma algebra, then*

(1) *$M(X, S) = LM(X, S) = RM(X, S)$, and*

(2) *$m(X, S) = C(X, S)$.*

Proof of (2). If $f\in C(X, S)$, then there exists a sequence $\{f_n\}$ of (X, S)-simple functions such that $\|f - f_n\| < n^{-1}$; $a < b$ implies $f^{-1}(a, b) = \bigcup_{j\geq 1} \bigcap_{i\geq j} f_i^{-1}(a+j^{-1}, b-j^{-1})$.

THEOREM 2.2. *Let (X, S) be a set algebra, $m = m(X, S)$, $Lm = Lm(X, S)$, $Rm = Rm(X, S)$, and $C = C(X, S)$. Then $C = m = Lm = Rm$ if and only if (X, S) is a sigma algebra. Moreover, if (X, S) is not a sigma algebra $Lm \neq Rm$, $m \subset Lm$, $m \subset Rm$ ($m \neq Lm$, $m \neq Rm$), and each of Lm and Rm is a proper subset of C.*

Proof. If (X, S) is not a sigma algebra then there exists a sequence $\{E_i\}$ of pairwise disjoint elements of S such that $\bigcup E_i \notin S$. Let $f_L(x) = 2^{-i}$ if $x\in E_i$ and $f_L(x) = 0$ if $x\in X - \bigcup E_i$ and let $f_R = -f_L$; $f_L\in Lm - Rm$ and $f_R\in Rm - Lm$ ($f\in LM(X, S)$ if and only if $-f\in RM(X, S)$).

DEFINITION 2.3. Let (X, S) be a set algebra. Then by

(1) $\{E_i\} \uparrow$ in S we mean that $\{E_i\}$ is a nondecreasing (i.e., $E_i \subset E_{i+1}$) sequence of elements of S,

(2) $\{E_i\} \downarrow$ in S we mean that $\{E_i\}$ is a nonincreasing sequence of elements of S, and

(3) the statement that S has property Q we mean that if $\{E_i\} \uparrow$ in S, $\{F_i\} \uparrow$ in S and $(\bigcup E_i) \cap (\bigcup F_i) = \theta$ then at least one of $\bigcup E_i$ and $\bigcup F_i \in S$.

LEMMA 2.3. *Let (X, S) be a set algebra and let S have property Q. Then*

(1) *if $\{E_i\} \downarrow$ in S, $\{F_i\} \uparrow$ in S, and there exists a positive integer j such that $E_j \cap (\bigcup F_i) = \theta$ then at least one of $\bigcap E_i$ and $\bigcup F_i \in S$, and*

(2) *if $\{E_i\} \downarrow$ in S, $\{F_i\} \downarrow$ in S, and there exists a positive integer j such that $E_j \cap (\bigcap F_i) = \theta$ then at least one of $\bigcap E_i$ and $\bigcap F_i \in S$.*

Proof. (1) Let $G_i = E_j - E_i$; $\{G_i\} \uparrow$ in S, $(\bigcup G_i) \cap (\bigcup F_i) = \theta$, and $\bigcap E_i = E_j - \bigcup G_i$ ($\bigcap E_i \in S$ if and only if $\bigcup G_i \in S$).

(2) $\bigcap F_i = (\bigcap F_i) \cap E_j' = \bigcap (F_i \cap E_j')$. Let $G_i = F_i \cap E_j'$ and $H_i = E_j - E_i$; $\{G_i\} \downarrow$ in S, $\{H_i\} \uparrow$ in S, $\bigcap G_i = \bigcap F_i$, $\bigcup H_i = E_j - \bigcap E_i$, and $G_1 \cap (\bigcup H_i) = \theta$.

LEMMA 2.4. *Let (X, S) be a set algebra, S have property Q, and $f \in C(X, S)$. Then*

(1) *if each of (a, b) and (c, d) is a segment and $(a, b) \cap (c, d) = \theta$, then at least one of $f^{-1}(a, b)$ and $f^{-1}(c, d) \in S$,*

(2) *if $c \notin [a, b]$, then at least one of $f^{-1}(c)$ and $f^{-1}(a, b) \in S$,*

(3) *if $c \neq P$, then at least one of $f^{-1}(c)$ and $f^{-1}(P) \in S$.*

Proof. Since $f \in C(X, S)$, there exists a sequence $\{f_i\}$ of (X, S)-simple functions such that $\|f - f_i\| < (2(i+1))^{-2}$ which implies $f^{-1}(a + 2^{-1}((i-1)^{-1} + i^{-1}), b - 2^{-1}((i-1)^{-1} + i^{-1})) \subset f_i^{-1}(a + i^{-1}, b - i^{-1}) \subset f^{-1}(a + 2^{-1}(i^{-1} + (i+1)^{-1}), b - 2^{-1}(i^{-1} + (i+1)^{-1}))$; this implies $\{f_i^{-1}(a + i^{-1}, b - i^{-1})\} \uparrow$ in S and $\bigcup f_i^{-1}(a + i^{-1}, b - i^{-1}) = f^{-1}(a, b)$. The preceding, together with property Q, shows (1). To get (2) we note that $f^{-1}(c) = \bigcap f_i^{-1}(c - i^{-1}, c + i^{-1}) = \bigcap E_n$ where $E_n = \bigcap_{i \leq n} f_i^{-1}(c - i^{-1}, c + i^{-1})$ and then apply Lemma 2.3–1 to $\{E_n\} \downarrow$ in S and $\{f_i^{-1}(a + i^{-1}, b - i^{-1})\} \uparrow$ in S. We get (3) in a similar fashion from Lemma 2.3-2.

DEFINITION 2.4. If (X, S) is a set algebra, f is a real valued function on X, and P is a real number, then P is said to have property U with respect to f if at least one of the following conditions is satisfied: (1) $f^{-1}(P) \notin S$, (2) $e > 0$ implies there exists (a, b) such that $|P - a| < e$, $|P - b| < e$, and $f^{-1}(a, b) \notin S$.

LEMMA 2.5. *Let (X, S) be a set algebra, S have property Q, f be a real valued function on X, P be a real number such that P has property U with respect to f, and each of a, b, and c be a real number distinct from P such that $a < b$. Then each of $f^{-1}(c)$, $f^{-1}(a, b)$, $f^{-1}[a, b)$, and $f^{-1}(a, b] \in S$.*

Proof. Parts (2) and (3) of Lemma 2.4 imply $f^{-1}(c) \in S$. If $P \notin (a, b)$, then $P \notin [a, b]$ and parts (1) and (2) of Lemma 2.4 imply $f^{-1}(a, b) \in S$. If $P \in (a, b)$ then each of $f^{-1}(-\infty, a), f^{-1}(a), f^{-1}(b)$, and $f^{-1}(b, \infty) \in S$ and, hence, $f^{-1}(a, b) \in S$ (Lemma 2.1-1). Finally, $f^{-1}[a, b) = f^{-1}(a) \cup f^{-1}(a, b) \in S$ and $f^{-1}(a, b] = f^{-1}(a, b) \cup f^{-1}(b) \in S$.

LEMMA 2.6. *Let (X, S) be a set algebra, S have property Q, and $f \in C(X, S) - Lm(X, S)$. Then there exists uniquely a point P such that* (1) *$f^{-1}(P) \notin S$ and* (2) *$-\infty \leqq a < P$ implies $f^{-1}(a, P) \notin S$.*

Proof. There exists $[c, d)$ such that $f^{-1}[c, d) \notin S$. We want to find a point P in $[c, d]$ which has property U with respect to f. If $f^{-1}(c) \notin S$, let $P = c$; suppose $f^{-1}(c) \in S$; then $f^{-1}(c, d) \notin S$, if $f^{-1}(2^{-1}[c+d]) \notin S$, let $2^{-1}[c+d] = P$; otherwise, exactly one of $f^{-1}(c, 2^{-1}[c+d])$ and $f^{-1}(2^{-1}[c+d], d) \notin S$ (Lemma 2.4-1), denote that one by (c_1, d_1) and repeat the preceding inductively. If there exists a positive integer i such that $f^{-1}(2^{-1}[c_i+d_i]) \notin S$, fine; otherwise, let $P = \bigcap [c_i, d_i]$. It is impossible that $P \in (c, d)$ since $P \in (c, d)$ would imply $f^{-1}[c, d) \in S$ (Lemma 2.5). If $P = d$, then $f^{-1}(c) \in S$ (Lemma 2.5) which implies $f^{-1}(c, P) \notin S$ which, in turn, implies $f^{-1}(P, \infty) \in S$ (Lemma 2.4) and thus $f^{-1}(P) = (X - [f^{-1}(-\infty, c) \cup f^{-1}(c) \cup f^{-1}(P, \infty)]) - f^{-1}(c, P) \notin S$ (Lemma 2.1-1). If $P = c$, let e be a number less than P. Then $f^{-1}(e, d) \in S$ (Lemma 2.5) which implies $f^{-1}(e, P) = f^{-1}(e, d) - f^{-1}[P, d) \notin S$ ($P = c$, Lemma 2.1-1) which, in turn, implies $f^{-1}(P, d) \in S$ (Lemma 2.4-1) and thus $f^{-1}(P) = f^{-1}[P, d) - f^{-1}(P, d) \notin S$. The lemma now follows from Lemma 2.4 and Lemma 2.5.

REMARK. Since $f \in Rm(X, S)$ if and only if $-f \in Lm(X, S)$ we have a dual result for $f \in C(X, S) - Rm(X, S)$ (i.e., there exists uniquely a point P such that $f^{-1}(P) \notin S$ and $b > P$ implies $f^{-1}(P, b) \notin S$).

THEOREM 2.3. *Let (X, S) be a set algebra. Then $C(X, S) = Lm(X, S) \cup Rm(X, S)$ if and only if S has property Q.*

Proof. If S has property Q and $f \in C(X, S) - Lm(X, S)$ then there exists uniquely a point P satisfying the conditions of Lemma 2.6. Suppose $f \notin Rm(X, S)$. Then there exists uniquely a point P' satisfying the conditions of the remark following Lemma 2.6; moreover, $P' = P$ (Lemma 2.4-3). This contradicts the supposition that $f \notin Rm(X, S)$ (Lemma 2.6 and the remark following Lemma 2.6, Lemma 2.4-1). If S does not have property Q then there exist sequences $\{E_i\} \uparrow$ in S and $\{F_i\} \uparrow$ in S such that $(\bigcup E_i) \cap (\bigcup F_i) = \theta$ and each of $\bigcup E_i$ and $\bigcup F_i \notin S$. Let $f(x) = 1$ if $x \in E_1$, $f(x) = 2^{-i}$ if $i > 1$ and $x \in E_i - \bigcup_{j<i} E_j$, $f(x) = -1$ if $x \in F_1$, $f(x) = -(2^{-i})$ if $i > 1$ and $x \in F_i - \bigcup_{j<i} F_j$, and $f(x) = 0$ if $x \in X - \bigcup (E_i \cup F_i)$; $f \in C(X, S) - [Lm(X, S) \cup Rm(X, S)]$.

We conclude with an example to show that the property of being a sigma algebra (while sufficient) is not necessary in order that S have property Q. Let I be the set of positive integers, let $E \in T$ if and only if $E \subset I$, let J be a maximal proper ideal in T such that if E is a finite subset of I then $E \in J$, let $X = I + [0]$, and let $E \in S$ if and only if one of E and $X - E \in J$ (i.e., we add 0 to the elements of $T-J$). (X, S) is a set algebra which is not a sigma algebra and such that S does have property Q.

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