ON THE STRUCTURE OF ORBIT SPACES OF GENERALIZED MANIFOLDS

BY

GLEN E. BREDON[1]

Introduction. In the author's Chapter XV, *Fixed point sets and orbits of complementary dimension*, of [1], it was shown that if a compact Lie group G acts on a locally separable n-dimensional cohomology manifold M over Z (n-cm_Z) of finite covering dimension in such a way that the dimension of the fixed point set $F(G, M)$ is $n-k-1$ near a point $x \in F(G, M)$, where k is the maximal dimension of any orbit, then the orbit space M/G is an $(n-k)$-cm_Z with boundary $F(G, M)$ near x (see Definition 1.3 of [1, Chapter XV]). Moreover it was shown that, near x, there are exactly two types of orbits, these being the fixed points and the principal orbits. The principal orbits are integral cohomology spheres and there is a cross-section near x for the action of G. This theorem (1.4 of [1, Chapter XV]) and its immediate corollary (1.5 of [1, Chapter XV]) will be referred to in the present paper as the CDT (complementary dimension theorem).

The present paper studies the case in which $\dim_{Z_2} F(G, M) = n-k-2$ near x (see Theorem 1.1 for the precise statement), and, although the results are not nearly as complete as those in the CDT, we obtain that M/G is an $(n-k)$-cm_Z with boundary near x. Using slices we obtain a corollary (1.4) which asserts that in the general case of any action of a compact Lie group G on an n-cm_Z M, the set $C^* \subset M/G$ of points in the orbit space around which M/G is not an $(n-k)$-cm_Z (with or without boundary) is a closed set of dimension at most $(n-k-3)$ (in a certain technical sense). If $k = n-2$, we obtain that M/G is a 2-manifold with boundary (Corollary 1.5), which is the main part of Theorem 11 of [3].

We would like to call attention to Lemmas 2.1, 2.2, and 2.5 as well as Corollary 2.4, which, besides being basic for the present paper, have an independent interest. The technique of the proof of Lemma 5.2 is also of some interest.

As to notation we always denote by H an isotropy subgroup of a point on a principal orbit (that is, H is a minimal isotropy group in the sense that H is conjugate to a subgroup of any other isotropy group). We let $k = \dim G/H = \max\{\dim G(y) \mid y \in M\}$ and let $B = \{y \in M \mid \dim G(y) < k\}$ be the set of points on singular orbits, and $E = \{y \in M \mid \dim G(y) = k,\ G_y \not\sim H\}$, the set of points on exceptional orbits. For any set $A \subset M$ we denote by A^* its image in $M^* = M/G$. T will denote a maximal torus of H and T_0 a maximal torus

Received by the editors April 28, 1960.

[1] This research was supported by NSF Grant G-5863.

of G. For a subgroup $K \subset G$, K^0 denotes the identity component of K, $N(K)$ denotes the normalizer of K, $Z(K)$ denotes the centralizer of K, and $F(K, M) = \{y \mid K(y) = y\}$ denotes the set of fixed points of K on M. The natural map $M \to M/G$ is denoted by π. $\dim_L(X, x)$ stands for the infimum of $\dim_L(U)$ for U ranging over all open neighborhoods of x in X, as in [1, Chapter XV, §1.1].

The definitions of the basic notions that we use will be found in [1]: $\dim_L U$ in I, 1.2; n-cm_L, I, 3.3 (also see Chapter XV, 2.2); n-cm_L with boundary, XV, 1.3; $=_{LF}$, XV, 2.2 (which we also use for Y a locally closed subspace of X in general). The cohomology we use is always the Alexander-Spanier cohomology with compact supports.

Familiarity with some of the basic results in [1] (especially Chapters I, IV, V, IX, XIII, and XV) will be assumed. As in the proof of the CDT, we reduce the theorem to the case in which H is finite by studying the action of $N(T)/T$ on $F(T, M)$. Lemma 2.1 then implies that if H is finite then rank $(G) \leqq 2$, and the proof of the theorem is taken up for each of the three cases rank $(G) = 0$, 1, and 2 in §§3, 4, and 5. In the appendix (§6) we state some theorems that will be used in the proof of our main result. The proofs of these theorems essentially appear elsewhere, but the existing statements of the results are not general enough for our purposes.

1. **The main results.** The main object of this paper is to prove the following theorem.

THEOREM 1.1. *Let M be a locally separable n-cm_Z of finite covering dimension. Let G be a compact Lie group acting effectively on M, let H be an isotropy group of some point on a principal orbit of G, and let $k = \dim G/H$.*

If $k > 0$ and $\dim_{Z_2}(F(G, M), x) = n - k - 2$, then M/G is an $(n-k)$-cm_Z with boundary $B^* \cup E^*$ near $\pi(x)$, and $E^* = \emptyset$ near $\pi(x)$ if G is connected.

If $k = 0$, $G \neq G^+$ (where G^+ denotes the subgroup of G preserving the orientation at x), and $\dim_{Z_2}(F(G, M), x) = n - 2$, then M/G is an n-cm_Z with boundary E^* near $\pi(x)$. Moreover G is dihedral and G^+ is cyclic.

If $k = 0$, $G = G^+$, and $\dim_{Z_p}(F(G, M), x) = n - 2$ for all primes p dividing $\mathrm{ord}(G)$, *then M/G is an n-cm_Z near $\pi(x)$. Moreover G is cyclic.*

REMARK. In case rank $(G) >$ rank(H) one can replace the condition $\dim_{Z_2}(F(G, M), x) = n - k - 2$ by the weaker $\dim_Z(F(G, M), x) = n - k - 2$. This will be clear from the proof of 1.1 since the former condition will follow from the latter almost immediately in these cases by using [1, Chapter I, 4.9]. We do not know whether or not the weaker condition suffices in the other cases of 1.1.

In this section we shall state and prove some corollaries to 1.1. We will then prove the case $k = 0$ of 1.1 and show how this case implies that we may assume that G is connected in the proof of 1.1.

Analogously with Corollary 1.6 of the CDT, we get the following global result.

COROLLARY 1.2. *Let G and M be as in* 1.1 *and also assume* $H_c^*(M, Z) = H_c^*(S^n, Z)$. *If* $k>0$, *or* $k=0$ *and* $G \neq G^+$, *and* $\dim_{Z_2} F(G, M) = n-k-2$, *then* M/G *is a cohomology* $(n-k)$*-cell with boundary* $B^* \cup E^*$ (see [I, Chapter XV, Corollary 1.6]) *and* $E^* = \emptyset$ *if* G *is connected. If* $k=0$, $G=G^+$ *and* $\dim_{Z_p} F(G, M) = n-2$ *for all primes* p *dividing* $\operatorname{ord}(G)$, *then* M/G *is a cohomology* n*-sphere, that is,* M/G *is an* n-cm_Z *and* $H_c^*(M/G, Z) = H_c^*(S^n, Z)$. *In the cases* $k=0$, $G=G^+$ *or* $G \neq G^+$, G *is cyclic or dihedral respectively.*

Proof. As in the proof of [1, Chapter XV, Corollary 1.6], we apply 1.1 to the action of G on the cone cM over M. The details will be left to the reader.

As in [1, Chapter IX] let $M_{u,v} = \{y \in M \mid \dim G(y) = u, v = \text{number of components of } G_y\}$.

COROLLARY 1.3. *With the notation of* 1.1 *say that* $z \in M_{u,v}$, $H \subset G_z$ *and* $\dim_L(M_{u,v}^*, \pi(z)) = n-k-2$, *for* $L = Z_2$ *if* $u<k$ (*resp. for all* $L = Z_p$ *if* $u=k$), *then the hypotheses of* 1.1 *hold for the action of* G_z *on a slice at* z, *so that* M/G *is an* $(n-k)$-cm_Z *with boundary* $B^* \cup E^*$ *near* $\pi(z)$ *if* $u<k$ (*resp.* M/G *is an* $(n-k)$-cm_Z *near* $\pi(z)$ *with or without boundary if* $u=k$).

Proof. This follows exactly as does Corollary 1.5 of [1, Chapter XV].

COROLLARY 1.4. *Let* $C^* = \{z^* \in M/G \mid M/G$ *is not an* $(n-k)$-cm_Z (*with or without boundary*) *near* $z^*\}$. *Then* C^* *is a closed set and*

(1) $\dim_Z C^* \leqq n-k-2$,

(2) $\dim_{Z_2}(C^* \cap B^*) \leqq n-k-3$,

(3) *for each* $z^* \in C^* \cap M_{k,v}^*$ *there is a prime* p (*depending on* z) *such that* $\dim_{Z_p}(M_{k,v}^*, z^*) \leqq n-k-3$.

Proof. C^* is clearly closed since its complement is open. (1) follows from Corollary 1.5 of [1, Chapter XV] in a manner analogous to the following proof of (2). To prove (2), say that $\dim_{Z_2}(C^* \cap B^*) \geqq n-k-2$ and let $C_1^* = \{z^* \in C^* \cap B^* \mid \dim_{Z_2}(C^* \cap B^*, z^*) \geqq n-k-2\}$. C_1^* is clearly closed and also $\dim_{Z_2}(C_1^*, z^*) \geqq n-k-2$ for any $z^* \in C_1^*$. Choose $z^* \in C_1^*$ in such a way that for any $y^* \in C_1^*$ sufficiently close to z^* we have $G_y \sim G_z$. Then if $z^* \in M_{u,v}^*$ we see that $C_1^* \cap M_{u,v}^*$ is a neighborhood of z^* in C_1^*. Hence $\dim_{Z_2}(M_{u,v}^*, z^*) \geqq n-k-2$ and it follows from Corollary 1.3 above or from Corollary 1.5 of [1, Chapter XV] that M/G is an $(n-k)$-cm_Z (possibly with boundary) near z^*, contrary to the definition of C^*. (3) follows immediately from 1.3 or from Corollary 1.5 of [1, Chapter XV].

COROLLARY 1.5. *If* $k=n-2$, *then* M/G *is a* 2*-manifold with boundary containing* B^*.

Proof. It follows immediately from Corollary 1.4 that M/G is a 2-cm_Z with or without boundary. Also it is well-known that a locally separable

2-cm is a 2-manifold. That B^* is contained in the boundary follows easily by application of Corollary 1.3.

This corollary is, of course, just Theorem 11 of [3] for G a Lie group and for M an n-cm$_Z$ instead of a manifold. (The proof in [3] also holds for cms, however.) Similarly the case $k=n-1$ of the CDT would yield immediately a corollary similar to Theorem 10 of [3]. The author intends to give more detailed information about the case $k=n-2$ in a subsequent paper.

We shall now take up the proof of the case $k=0$ of 1.1. We first confine our discussion to the case $G=G^+$. In [2, Theorem 7.8], the local groups of M/G are calculated for $G\approx Z_p$. It is easy to see from that result that if $G=G^+\approx Z_p$ and $\dim_{Z_p}(F(G, M), x)=n-2$, then M/G is an n-cm$_Z$ near $\pi(x)$. By an easy induction this follows for all solvable $G=G^+$. Thus to show M/G to be an $n-$cm it suffices to show that G is solvable. Note that if $K\subset G$ is normal and $K=K^+$ is solvable then by the diagram ($Q=$rationals)

$$\begin{array}{ccc} H_c^n(M/K, Q) & \xrightarrow{gK} & H_c^n(M/K, Q) \\ \downarrow \pi^* & & \downarrow \pi^* \\ H_c^n(M, Q) & \xrightarrow{g} & H_c^n(M, Q) \end{array}$$

in which the maps π^* are isomorphisms by [1, Chapter III, 2.3], we have that $g\in G$ preserves orientation on M if and only if $gK\in G/K$ preserves orientation on M/K. Moreover, if G is effective on M then its principal isotropy subgroup is trivial and it follows easily that G/K is effective on M/K.

Noting also that if $G=G^+$ then $G\not\approx Z_p\oplus Z_p$ by dimensional parity and Borel's formula [1, Chapter XIII, 4.3], we see that the class $\mathcal{S}$ of finite groups $G=G^+$ which can act effectively on an n-cm$_Z$ M with $\dim_{Z_p}(F(G, M), x)=n-2$ near some $x\in F(G, M)$ and for all primes p dividing $\operatorname{ord}(G)$ satisfies the following three statements:

(1) $Z_p\oplus Z_p\notin\mathcal{S}$ for any prime p.

(2) If $G\in\mathcal{S}$ and $G'\subset G$, then $G'\in\mathcal{S}$.

(3) If $G\in\mathcal{S}$ and K is a solvable normal subgroup of G, then $G/K\in\mathcal{S}$.

The case $k=0$, $G=G^+$ of 1.1, except for the statement that G is cyclic, now follows from

LEMMA 1.6. *Let $\mathcal{S}$ be a class of finite groups satisfying* (1), (2) *and* (3) *above. Then every $G\in\mathcal{S}$ is solvable and every p-group $G\in\mathcal{S}$ is cyclic.*

Proof. By [8, Chapter V, 10], it suffices to prove that every p-group $G\in\mathcal{S}$ is cyclic. We use induction on $\operatorname{ord}(G)$. Thus assume all p-groups $G'\in\mathcal{S}$ of order less than p^a are cyclic and let $G\in\mathcal{S}$ be a p-group of order p^a. By [8, Chapter IV, 9] the center C of G is nontrivial. Also (1) implies that C is cyclic. (3) and the induction hypothesis imply that G/C is cyclic. But if g is a representative of a generator of G/C and c is a generator of C, then clearly G is generated by the commuting elements g and c. Thus G is abelian and hence $G=C$ is cyclic by (1).

It remains to show that, in general, G is cyclic. According to Lemma 1.6 and [8, Chapter V, 11], if G is not cyclic then G has a cyclic normal subgroup $G' \approx Z_a$ and a subgroup $G'' \approx Z_b$ such that $G'' \approx G/G'$ naturally. Moreover, letting g and h be generators of G' and G'' respectively, $hgh^{-1} = g^r$ for some $1 < r < a$ and also $((r-1)b, a) = 1$ and $r^b \equiv 1$ (a). These relations imply that a is odd. Let $p | a$ be some odd prime. Then $g^{a/p}$ and h generate a subgroup of G which is abelian if and only if a divides $r(a/p) - (a/p) = (r-1)(a/p)$ and hence if and only if $p | (r-1)$ which is impossible, since $(r-1)$ and a are relatively prime. Thus, passing to this subgroup, we may assume that $a = p$ is an odd prime.

A result of R. G. Swan [6, Theorem 3(b)] implies that the p-period of G is $2t \geqq 4$, where t is the least integer for which $r^t \equiv 1$ (p). This means in the present case, since $p | \text{ord}(G)$ and $p^2 \nmid \text{ord}(G)$ and according to [4, Chapter XII, Exercise 11], that

$$H^i(G, Z, p) = \begin{cases} Z_p, & \text{if } 2t \mid i, \\ 0, & \text{otherwise,} \end{cases}$$

where $H^i(G, Z, p)$ denotes the p-primary component of $H^i(G, Z)$. Thus by the universal coefficient theorem

$$H^i(G, Z_p) = \begin{cases} Z_p, & \text{if } 2t \mid i \text{ or } 2t \mid (i+1), \\ 0, & \text{otherwise.} \end{cases}$$

This could also be seen by direct computation using the methods of [4, Chapter XII].

Now let $F = F(G', M)$, which is an $(n-2)$-cm over Z_p, and hence coincides with $F(G, M)$ in some open set near x. In particular G preserves orientation on F. It follows from [1, Chapter IV, 3.6 and Chapter V, 2.1], that if $i > n$ and $V \subset U$ are properly chosen, then in the diagram

$$\begin{array}{ccc} H^i_c(V_G, Z_p) & \to & H^i_c((V \cap F)_G, Z_p) \\ \downarrow j_1 & & \downarrow j_2 \\ H^i_c(U_G, Z_p) & \to & H^i_c((U \cap F)_G, Z_p) \end{array}$$

the horizontal maps are isomorphisms, $\text{Im } j_1 \approx H^{i-n}(G, H^n_c(U, Z_p)) \approx H^{i-n}(G, Z_p)$, and $\text{Im } j_2 \approx H^{i-(n-2)}(G, H^{n-2}_c(U \cap F, Z_p)) \approx H^{i-n+2}(G, Z_p)$. Thus, for all $j > 0$, $H^j(G, Z_p) \approx H^{j+2}(G, Z_p)$, which contradicts our previous result and finishes the proof that $G = G^+$ is cyclic.

We now take up the case $k = 0$, $G \neq G^+$ of 1.1. We first show:

LEMMA 1.7. *If $g \in G - G^+$, then there is an i such that* $\dim_{Z_2}(F(g^i, M), x) = n-1$. *Thus* $\dim_L(F(G, M), x) = n-2$ *for any L.*

Proof. The second statement follows from the first, from the CDT $(k=0)$ (or [7, Lemma 16, Corollary]) which implies that $F(g^i, M)$ is an $(n-1)-$cm over Z near x, and from [1, Chapter I, 4.9].

To prove the first statement let ord $g=2^j a$, a odd. Then g^a also reverses orientation so that we may assume that ord $g=2^j$. Let G' be the group generated by g, ord $G'=2^j$. By the case $k=0$, $G=G^+$ of 1.1 we have that M/G'^+ is an n-cm$_Z$ near x and we also have that g must reverse orientation on M/G'^+. Thus since $g^2\in G'^+$ we have that $\dim_{Z_2}(F(g, M/G'^+), x)=n-1$. However, if $\dim_{Z_2} F(g^i, M)=n-2$ for all $i<2^j$ we would have that G' acts freely outside $F(G', M)$ near x, since $F(g, M)\subset F(g^i, M)$, both being $(n-2)$-cm's over Z_2 near x, are equal near x. It would follow that G'/G'^+ acts freely outside $F(G', M)/G'^+\approx F(G', M)$ on M/G'^+ and hence that $F(g, M/G'^+)\approx F(g, M)$ is of dimension $n-2$ over Z_2. This contradiction finishes the proof of the lemma.

By Lemma 1.7, and the case $k=0$, $G=G^+$ of 1.1, we have that M/G^+ is an n-cm$_Z$ near x. By the CDT for $k=0$ applied to the action of G/G^+ on M/G^+ we have that M/G is an n-cm$_Z$ with boundary. We must show that the boundary of M/G is E^* and that G is dihedral. There is an invariant neighborhood U of a point $y\in F(G, M)$ arbitrarily near x, such that $F(g, U)=F(G, U)=F$ for all $e\neq g\in G^+$. We may assume that U is so small that if $z\in U$, $G_z\neq(e)$, G, then $G_z\approx Z_2$ and $F(G_z, U)$ is an $(n-1)$-cm$_Z$. It is also possible to assume that if C^* is the boundary of U/G, then $F^*\subset C^*$ is connected, C^*-F^* has exactly two components (since F is an $(n-2)$-cm$_{Z_p}$ for any $p|$ord G^+) C_0^* and C_1^*, and U^*-C^* is connected.

It follows from Theorem 6.5 that U^* may be taken to be so small that $\pi^{-1}(U^*-C^*)$ consists of ord(G) components each mapping homeomorphically onto U^*-C^* via π. Let W be one of these components.

Note that since F is an $(n-2)$-cm$_{Z_p}$ for any $p|$ord G^+ we obtain from the cohomology sequence of C^* mod F^* that $H_c^{n-2}(C_i^*, Z_p)={}_{LF}0$ for $i=0, 1$ and any $p|$ord G^+. Also, if $z\in\pi^{-1}(C_i^*)$, then $G^+(z)=G^+G_z(z)=G(z)$ so that

$$\frac{\pi^{-1}(C_i^*)}{G^+}\approx C_i^*.$$

Thus, again by Theorem 6.5, we obtain that $\pi^{-1}(C_i^*)$ consists of ord(G^+) $=(1/2)$ord(G) components. Let C_i be a component of $\pi^{-1}(C_i^*)$ such that $C_i\cap\overline{W}\neq\emptyset$. We see easily, using the connectivity of C_i, that $C_i\subset\overline{W}$. It follows that $\overline{W}=W\cup C_0\cup C_1\cup F$. Note that G_z is constant on C_0 and on C_1 and hence if $z\in C_0$ then $C_0\subset F(G_z, U)$.

We can construct in $\overline{W}$ an arc α: $[0, 1]\to\overline{W}$ such that $\alpha(a)\in W$ for $a\in(0, 1)$, $z_0=\alpha(0)\in C_0$ and $z_1=\alpha(1)\in C_1$. Define g_1 to be the nontrivial element of G_{z_1} and let g_2 be the nontrivial element of $G_{g_1(z_0)}$. In general define g_{2i} to be the nontrivial element of $G_{g_{2i-1}g_{2i-3}\cdots g_3g_1(z_0)}$ and g_{2i+1} to be the nontrivial element of $G_{g_{2i}g_{i-2}\cdots g_4g_2(z_1)}$.

We see easily that $\bigcup_j\{g_jg_{j-1}\cdots g_2g_1(\overline{W}-F)\}$ is connected and is open and closed in $U-F$. Thus it is equal to $U-F$ and this implies that

$\bigcup_j \{g_j g_{j-1} \cdots g_2 g_1\} = G$. Now $\bigcup_j \{(g_j g_{j-1} \cdots g_2 g_1)\alpha([0, 1])\}$ must be a circle (since it is a compact connected one-manifold) and hence $\gamma = G(\alpha([0, 1]))$ is a circle upon which G operates effectively (since the isotropy group is trivial on $\alpha(0, 1)$). It follows that G is dihedral.

Now let V be an arbitrary invariant neighborhood of x such that the complement X in V/G of the boundary (as an n-cm with boundary) of V/G is connected and assume that V is so small that for any sufficiently small neighborhood $V' \subset V$ of x and any element $g \in G$ with $\dim F(g, V) = n-1$, then there are exactly two components of $V - F(g, V)$ which touch V' (and these are permuted by g).

In some such V' we can find a U as above and $\gamma \subset U$. Say that $a, b \in \gamma$ are in different components of $\gamma - F(g, \gamma)$. Then clearly $a = g(b')$ for some b' in the same component as b. By the selection of V and V', a and b' (hence a and b) are in different components of $V - F(g, V)$. Thus we see that $\pi^{-1}(X)$ has at least $\text{ord}(G)$ components, since $\gamma \cap \pi^{-1}(X)$ has this many components and since any two points in different components of $\gamma - \pi^{-1}(X)$ are actually in different components of $\gamma - F(g, \gamma)$ for some $g \in G$.

However, the number of components of $\pi^{-1}(X)$ is at most the number of elements of $G(z)$ for any z for which $\pi(z) \in X$. Thus, for any such z, $G(z)$ has as many points as G, that is, $G(z)$ is principal. This concludes the proof of the case $k=0$, $G \neq G^+$ of 1.1.

The case $k=0$ enables us to reduce the proof of Theorem 1.1 to the connected case as follows. Say that (G, M) is a transformation group satisfying the hypotheses of 1.1. Then (G^0, M) also satisfies these hypotheses (or the hypotheses of the CDT) so that M/G^0 is an $(n-k)$-cm_Z with boundary B/G^0. We now apply the case $k=0$, $G \neq G^+$ to the action of $G/G^0 \oplus K$ on $(M/G^0)^{d(B/G^0)}$ (see [1, Chapter XV, Definition 1.2]), where K is induced by the doubling operation. We obtain that

$$M/G \approx \frac{M/G^0}{G/G^0} \approx \frac{(M/G^0)^{d(B/G^0)}}{(G/G^0) \oplus K}$$

is an $(n-k)$-cm_Z with boundary. The boundary is clearly $B^* \cup E^*$ by the results of the case $k=0$, $G \neq G^+$ of 1.1. Thus we may restrict our attention to the case in which G is connected in the proof of Theorem 1.1.

Remark. A closer examination of the action of G/G^0 on M/G^0 yields that (when $k>0$) effectively G/G^0 is either trivial or Z_2. Also the former case implies that the orbits of G are connected (and, in particular, $E = \emptyset$ near x since this is true for G^0), and the latter case implies that the boundary of M/G is $B^* \cup E^*$ with $B^* \cap \overline{E^*}$ an $(n-k-2)$-cm_Z. To see this, assume G/G^0 does not act trivially on M/G^0, let G' be the effective (factor) group, and let $B' = B/G^0$ be the boundary of $M' = M/G^0$. Then G' is effectively of order two on B' since its fixed point set is of dimension $= n-k-2 = \dim B' - 1$. Since any group leaving B' stationary must also leave M' stationary (see [1, Chapter XV, remark below 2.1]) we see that $G' \approx Z_2$. The rest follows from the fact that $\overline{E^*} \approx F(G', M') = F(G/G^0, M/G^0)$ and $\overline{E^*} \cap B^* \approx F(G/G^0, B/G^0)$. (Also compare Lemma 2.5.)

2. **Preliminary results.** Following the method of [1, Chapter XV], we shall reduce the proof of Theorem 1.1 to the case in which H is finite by studying the action of $N(T)/T$ on $F(T, M)$, where T is a maximal torus of H. Our first lemma gives a bound for rank $(N(T)/T) = \text{rank } (G) - \text{rank } (H)$ and is an improvement of the technique used in [1, Chapter XV]. The lemma is also an improvement of the known inequalities $\dim_Z F(G, M) \leqq n - k - 1$ and $\dim_Z F(T_0, M) \leqq n - 2[\text{rank } (T_0)]$ where T_0 is the maximal torus of G.

LEMMA 2.1. *Let G act effectively on a locally separable n-cm_Z M with principal isotropy group H and $k = \dim G/H$. Then near a point $x \in F(G, M)$ we have* $\dim_Z F(G, M) \leqq \dim_Z F(T_0, M) \leqq n - k - (\text{rank}(G) - \text{rank}(H)) \leqq n - 2 \text{ rank}(G)$.

Proof. Let $r = \text{rank } (G) - \text{rank } (H)$. The first inequality is clear. We now prove the second inequality. It suffices to consider the case in which H is finite, since if T is a maximal torus of H we have that near x (see the proof of Lemma 2.7 of [1, Chapter XV])

$$\dim_Z F(T, M) = n - k + \dim \frac{N(T)}{N(T) \cap H}$$

and the lemma for H finite says that near x

$$\begin{aligned}\dim_Z F(T_0, M) &\leqq \left(n - k + \dim \frac{N(T)}{N(T) \cap H}\right) - \dim \frac{N(T)}{N(T) \cap H} - r \\ &= n - k - r.\end{aligned}$$

Note also that it suffices to prove the inequality for G connected.

We shall assume from now on that H is finite and hence that $r = \text{rank } (G)$. If $r = 0$ the inequality is clear. If $r = 1$ then the natural map

$$\frac{F(T_0, M)}{N(T_0)} \to M/G$$

is a homeomorphism into B^*. Thus by [1, Chapter IX, 2.2 Corollary],

$$\dim F(T_0, M) = \dim \frac{F(T_0, M)}{N(T_0)} \leqq \dim B^* \leqq n - k - 1$$

as was to be shown. Thus assume $r > 1$.

We shall prove the inequality by induction on $n = \dim M$. If $n = 0$ the inequality is trivial. Say that the inequality always holds for $n < n_0$ and let $\dim M = n_0$. First suppose that $F(T_0, M)$ does not coincide with $F(G, M)$ in any open set near x. Then let y be a point of the component of $F(T_0, M)$

containing x, such that $y \notin F(G, M)$ and such that for $y' \in F(T_0, M)$ near y, $G_y \sim G_{y'}$. Let S be a slice at y. Then by the inductive hypothesis $\dim F(T_0, S) \leqq (n_0 - \dim G/G_y) - \dim G_y - r = n_0 - k - r$. If $z \in F(T_0, G(S))$ and $z' \in S \cap G(z)$ then $G_{z'} \subset G_y$ is of maximal rank so that there is an element $g \in G_y$ such that $G_{g(z')} = gG_{z'}g^{-1}$ contains T_0. Hence $g(z') \in F(T_0, S) \cap G(z)$ and it follows that $F(T_0, S)^* = F(T_0, G(S))^*$. Thus, since y is near x, we have $\dim F(T_0, S) = \dim F(T_0, S)^* = \dim F(T_0, G(S))^* = \dim F(T_0, G(S)) = F(T_0, M)$ and hence

$$\dim F(T_0, M) \leqq n_0 - k - r$$

as was to be shown.

Now say that $F(T_0, M)$ coincides with $F(G, M)$ in some open set near x. Then in order to compute $\dim F(T_0, M)$ we may assume that $F(T_0, M) = F(G, M)$ near x. By Borel's formula [1, Chapter XIII, 4.3] there is a $T_1 \subset T_0$, rank $T_1 = r - 1$, such that near x, $\dim F(T_1, M) > \dim F(T_0, M)$. Let $y \in F(T_1, M) - F(T_0, M)$ be in the component of $F(T_1, M)$ containing x and assume that for $y' \in F(T_1, M)$ close to y; $G_{y'} \sim G_y$. Since $F(T_0, M) = F(G, M)$ we may assume that rank $G_y = r - 1$, that is, that T_1 is a maximal torus of G_y. Let S be a slice at y. Then by the inductive assumption

$$\begin{aligned} \dim F(T_1, S) &\leqq (n_0 - \dim G/G_y) - \dim G_y - r + 1 \\ &= n_0 - k - r + 1. \end{aligned}$$

Since rank $G_y = r - 1$ we have that $F(T_1, S)/(N(T_1) \cap G_y) \approx F(T_1, S)^*$ so that $\dim F(T_1, S) = \dim F(T_1, S)^*$. Just as above we see easily that

$$F(T_1, S)^* = F(T_1, G(S))^* \approx \frac{F(T_1, G(S))}{N(T_1)}.$$

But $N(T_1)$ acting on $F(T_1, M)$ or $F(T_1, G(S))$ is effectively of dimension equal to $\dim N(T_1)/T_1$ so that

$$\begin{aligned} \dim F(T_1, M) = \dim F(T_1, G(S)) &= \dim \frac{F(T_1, G(S))}{N(T_1)} + \dim \frac{N(T_1)}{T_1} \\ &\leqq n_0 - k - r + 1 + \dim \frac{N(T_1)}{T_1}. \end{aligned}$$

Also $N(T_1)$ is effectively of rank one on $F(T_1, M)$ and hence by the inequality for $r = 1$ which has already been proved we have

$$\begin{aligned} \dim F(T_0, M) = \dim F(T_0, F(T_1, M)) &\leqq \left(n_0 - k - r + 1 + \dim \frac{N(T_1)}{T_1}\right) \\ &\quad - \dim \frac{N(T_1)}{T_1} - 1 = n_0 - k - r \end{aligned}$$

as was to be shown.

The last inequality can be derived from the middle one as follows. Note that it is equivalent to $k \geqq \text{rank } (G) + \text{rank } (H)$. We consider the action of T on G/H. This action is effective since G is effective on G/H (for otherwise H would contain a subgroup K normal in G and we would have that K acts trivially on M). Thus $\text{rank } (G) - \text{rank } (H) \leqq \dim N(T)/(N(T) \cap H) = \dim N(T)H/H = \dim F(T, G/H) \leqq k - \dim (T) - \text{rank } (T) = k - 2 \text{ rank } (H)$, by the middle inequality applied to the action of T on G/H. Thus $k \geqq \text{rank } (G) + \text{rank } (H)$ as was to be shown.

REMARK. Notice that the proof of the middle inequality implies that if we have equality for the first two inequalities and H is finite, then, since $F(T_0, M) = F(G, M)$ in some open set near x, we would have that for any $T_1 \subset T_0$ of codimension one, either $\dim(F(T_1, M), x) = \dim(F(T_0, M), x) = n - k - r$ or $\dim(F(T_1, M), x) = n - k - r + 1 + \dim N(T_1)/T_1$.

REMARK. In the situation of the main theorem (1.1), Lemma 2.1 allows us to obtain the bound $(\text{rank } (G) - \text{rank } (H)) \leqq n - k - \dim F(G, M) = (n-k) - (n-k-2) = 2$, and hence $\text{rank } (G) \leqq 2$ when H is finite.

The next lemma is an improvement of Lemma 2.5 of [1, Chapter XV]. It was implicit in some of the proofs there, but was not explicitly stated.

LEMMA 2.2. *Let G act on the n-cm$_Z$ M and assume that M is such that the map $F(T, M)/N(T) \to M/G$ is a homeomorphism* (see [1, Chapter XV, Lemma 2.7]). *Let $z \in F(T, M)$. If $N(T)(z)$ is principal for $N(T)$ on $F(T, M)$ then $G(z)$ is principal for G on M.*

Proof. There is an element $g \in G$ such that $g^{-1}Hg \subset G_z$. Thus $H \subset gG_zg^{-1} = G_{g(z)}$ so that $g(z) \in F(H, M)$ and $g(z)$ and z are on the same orbit of $N(T)$ on $F(T, M)$ since they are on the same orbit of G and are contained in $F(T, M)$. Thus we may assume that $H \subset G_z$ so that $z \in F(H, M)$.

The hypothesis implies that $N(T) \cap G_z = N(T) \cap H$ since the latter group is a principal isotropy group for $N(T)$ on $F(T, M)$. Let S be a slice at z for G on M so that by Lemma 2.7 of [1, Chapter XV] any sufficiently small neighborhood S' of z in $F(T, S)$ is a slice for the action of $N(T)$ on $F(T, M)$. Let $z' \in S'$. Then since z is on a principal orbit of $N(T)$ we must have

$$N(T) \cap G_{z'} = N(T) \cap G_z = N(T) \cap H.$$

Hence

$$\emptyset \neq F(H, G_z/G_{z'}) \subset F(T, G_z/G_{z'}) \approx \frac{N(T) \cap G_z}{N(T) \cap G_{z'}} = \frac{N(T) \cap H}{N(T) \cap H} = \text{a point}.$$

Thus $F(H, G_z(z')) = F(T, G_z(z')) = z'$, since $T \subset G_{z'}$, for any $z' \in S'$. This implies that $F(H, S') = F(T, S') = S'$ is a cross-section near z for the orbit map $F(T, M) \to F(T, M)/N(T)$ and hence also for the orbit map $M \to M/G$. The lemma now follows from [1, Chapter XV, 2.6].

REMARK. Note that by Lemma 2.7 of [1, Chapter XV] we may assume, in the proof of Theorem 1.1, that the natural map $F(T, M)/N(T) \to M/G$ is a homeomorphism. Moreover, since the finite group $(N(T) \cap H)/T$ is a principal isotropy group for the action of $N(T)/T$ on $F(T, M)$, it follows from Lemma 2.2 that if rank $(G) >$ rank (H) (that is, if $\dim(N(T)/T) > 0$), then it suffices to prove Theorem 1.1 for the action of $N(T)/T$ on $F(T, M)$ (made effective) and hence for the case in which H is finite. (The fact that $E^* = \emptyset$ near $\pi(x)$ when G is connected does not follow directly, but is easily obtained by the method used in the last paragraph of §3.) In case rank(G) $=$ rank(H) there are some minor complications due to the essential difference between the cases $k>0$ and $k=0$ of Theorem 1.1. These complications are dealt with in §3.

LEMMA 2.3. *If G acts on a locally separable n-cm$_Z$ M of finite covering dimension in such a way that there are no exceptional orbits and such that* $\dim_Z(F(G^0, M), x) = n-k-1$, $x \in F(G, M)$, *then the orbits of G on M are connected. In particular $F(G, M) = F(G^0, M)$.*

Proof. By hypothesis the transformation group (G^0, M) satisfies the hypotheses of the CDT so that M/G^0 is an $(n-k)$-cm with boundary $F(G^0, M)$. Also the points of $F(G^0, M)$ are the only singular orbits of G^0 and hence of G. Thus $F(G^0, M) = B$. Let G/K, $G^0 \subset K \subset G$, be the effective group acting on M/G^0. Since there are no exceptional orbits of G on M, the set $M-B$ of principal orbits is connected. Since the isotropy groups of two nearby points of $M-B$ are conjugate by an element of G^0 we see that this is true for any two points of $M-B$. If $G_y = H$ it is easy to see that the isotropy subgroup of G of the point $G^0(y) \in M/G^0$ is G_0H. It follows that G_0H leaves pointwise fixed the dense set $(M-B)/G^0 \subset M/G^0$ and hence, in fact, $K = G_0H$. Since we also have that if $G_{y'} \sim H$ (by an element of G^0) then $G^0G_{y'} = G^0H$ we see that the points of $(M-B)/G^0$ are all on principal orbits of G/K on M/G^0.

Let $g \in G$ be such that $\operatorname{ord}(gK) = p$ in G/K, p prime. Since the points of $(M - F(G^0, M))/G^0 = (M-B)/G^0$ are principal for G/K (and hence have trivial isotropy groups) we see that $F(g, M/G^0) \subset F(G^0, M) =$ Boundary of M/G^0. By the cohomology diagram

$$\begin{array}{ccccccc}
H_c^{n-k-1}(M/G^0) & \to & H_c^{n-k-1}(B/G^0) & \to & H_c^{n-k}\left(\dfrac{M-B}{G^0}\right) & \to & H_c^{n-k}(M/G^0) \\
\downarrow g^* & & \downarrow g^* & & \downarrow g^* & & \downarrow g^* \\
H_c^{n-k-1}(M/G^0) & \to & H_c^{n-k-1}(B/G^0) & \to & H_c^{n-k}\left(\dfrac{M-B}{G^0}\right) & \to & H_c^{n-k}(M/G^0)
\end{array}$$

in which the groups in the first and last columns are $=_{L_Z} 0$ (see [1, Chapter XV, Lemma 2.3]) we see easily that g preserves orientation on B/G^0 $= F(G^0, M)$ if and only if it does so on $(M-B)/G^0$ (and hence on $(M/G^0)^{d(B/G^0)}$).

Thus by dimensional parity we see that if $r=\dim_{Z_p}(F(g, M/G^0))$, then $(n-k-1)-r$ is even if and only if $(n-k)-r$ is even.

This contradiction shows that G/G^0 must act trivially on M/G^0 and hence that the orbits of G are connected as claimed.

COROLLARY 2.4. *Say that G acts on a locally separable n-cm$_Z$ M of finite covering dimension. Assume that $x\in B$ and that all orbits of points in B sufficiently close to x are of the same dimension and that* $\dim_Z(B^*, \pi(x))=n-k-1$. *Then if there are no exceptional orbits near x, the transformation group (G, M) satisfies the hypotheses of the CDT* [1, Chapter XV, Corollary 1.5] *near x so that all the orbits in B near x are of the same type.*

Proof. This follows immediately by application of Lemma 2.3 to the action of G_x on a slice at x. (See, for example, the proof of [1, Chapter XV, 1.5].)

LEMMA 2.5. *Let X be an m-cm over Z_2 with boundary B. Say that $G\approx Z_2$ acts on X with $x\in B\cap F(G, X)$. Suppose* $\dim_{Z_2}(F(G, B), x)=r$. *Then locally at x, F(G, X) is an $(r+1)$-cm$_{Z_2}$ with boundary F(G, B). If $r=m-2$ then locally at x, X is the union of two subsets X_1 and X_2 with $X_1\cap X_2=F(G, X)$, each of which is a cross-section for the action of G and for which X_i is an m-cm$_{Z_2}$ with boundary $(X_i\cap B)\cup F(G, X)$.*

Proof. Let $M=X^{dB}$ and let h be the doubling operation. Let g be the nontrivial element of G and let it act on M in the natural manner. The transformations e, g, h, gh form a group K isomorphic to $Z_2\oplus Z_2$ acting on M. Note that gh interchanges the components of $M-B$ so that $F(gh, M)\subset B=F(h, M)$ and it follows that $F(gh, M)=F(K, M)=F(g, B)$. By Borel's formula [1, Chapter XIII, 4.3]

$$m - r = ((m - 1) - r) + (r - r) + (\dim_{Z_2} F(g, M) - r)$$

so that $F(g, M)$ is an $(r+1)$-cm$_{Z_2}$ near x. Clearly h acts on $F(g, M)$ with $F(h, F(g, M))=F(g, B)$ and $F(g, X)\approx F(g, M)/(e, h)$. From known facts it follows that $F(g, X)$ is an $(r+1)$-cm$_{Z_2}$ with boundary $F(g, B)=F(h, F(g, M))$.

If $r=m-2$, then from known facts M is the union, near x, of subsets M_1, M_2 with g interchanging M_1 and M_2, and with $M_1\cap M_2=F(g, M)$. It follows that $X=M/h$ is the union $X_1\cup X_2$, $X_i=M_i/h$, $X_1\cap X_2=F(g, X)=F$. Let $B_i=X_i\cap B$ which are $(m-1)$-cm's with boundary $F(g, B)=F\cap B$. The Mayer-Vietoris sequence with coefficients in Z_2,

$$\to H_c^j(X) \to (H_c^j(X_1) \oplus H_c^j(X_2)) \to H_c^j(X_1 \cap X_2) \to$$

implies that $H_c^j(X_i)=_{L_y}0$ for $y\in F(g, B)$ and this is also true for $y\in(B_i\cup F)-(F\cap B_i)$ since X_i is an m-cm$_{Z_2}$ with boundary near such points. Thus $H_c^j(X_i)_{=(LB_i\cup F)}0$ for all j. Moreover $B_i\cup F$ is the union of the two $(m-1)$-cm's B_i and F with intersection $B_i\cap F=F(g, B)$ the boundary of each. Thus $B_i\cup F$ is an $(m-1)$-cm$_{Z_2}$ and the conclusion of the lemma follows from [1, Chapter XV, 2.3] (which is also valid for coefficients in a field).

We will need the following lemma on cm's with boundary. We understand that F. Raymond has been able to prove a more general result in this direction.

LEMMA 2.6. *Let X be a locally compact space and $F\subset B\subset X$, closed subspaces. Suppose that the following conditions hold:*

(1) *$X-F$ is an m-cm_Z with boundary $B-F$.*

(2) *B is an $(m-1)$-cm_Z.*

(3) $\dim_Z F\leqq m-2$.

(4) *For all $x\in F$, $H_c^i(X-B, Z) =_{Lx} \begin{cases} Z, & i=m,\\ 0, & i\neq m.\end{cases}$*

Then X is an m-cm_Z with boundary B.

Proof. According to Lemma 2.3 of [1, Chapter XV] we must show that $H_c^i(X, Z) =_{LB} 0$ for all i and by (1) this reduces to showing that $H_c^i(X, Z) =_{Lx} 0$ for any $x\in F$ and all i. By the cohomology sequences of X mod B (locally about x) it clearly suffices to show that for a fundamental system of neighborhoods U of x, the connecting homomorphism $H_c^{m-1}(U\cap B, Z) \to H_c^m(U-B, Z)$ is an isomorphism onto.

By (2) and (4), since $X-B$ is an m-cm, we see that it can be assumed that $U\cap B$ and $U-B$ are connected and orientable. Let $y\in U\cap B-F$ (which exists by (3)) and let $V\subset U$ be a neighborhood of y such that $V\cap B\subset B-F$ is connected and $V-B\subset U-B$ is connected. Then, since $H_c^m(V)=0$ by (1) (V is a proper closed subset of the m-cm $V^{d(B\cap V)}$), we have that $Z\approx H_c^{m-1}(V\cap B)\to H_c^m(V-B)\approx Z$ is onto and hence is an isomorphism. But $H_c^{m-1}(V\cap B)\to H_c^{m-1}(U\cap B)$ and $H_c^m(V-B)\to H_c^m(U-B)$ are also isomorphisms and it follows that $H_c^{m-1}(U\cap B)\to H_c^m(U-B)$ is an isomorphism as was to be shown.

3. **Case** I. rank $(G)=$rank (H). In this case $N(T)$ acts on $F(T, M)$ effectively as a finite group and $\dim F(T, M)=n-k$.

We will first show that there is an element $g\in N(T)$ which reverses the orientation of $F(T, M)$ (near x). If not, then we see that $\dim F(g, F(T, M)) \leqq n-k-2$ for all $g\in N(T)$ (by dimensional parity). It follows that, near x, the set U of points on principal orbits of $N(T)$ on $F(T, M)$ forms a dense connected set in $F(T, M)$. At least one of these points has H as an isotropy group (for G on M) and since $F(H, U)\subset F(T, U)$ must touch all the nearby orbits it follows from the connectivity of U that $F(H, U)=F(T, U)$ (and also that $F(H, M)=F(T, M)$) near x.

We now have that $N(H)/H=F(H, G/H)=F(T, G/H)=N(T)H/H$ so that $N(H)=N(T)H$ and hence $N(H)\supset N(T)$. Thus $G/N(H)$ is acyclic over the rationals and is, in particular, nonorientable. Considering the fibering $U\to^{F(H,U)}G/N(H)$ (see [1, Chapter XII, 1.3(2)]) we see that there must be an element of the structural group which reverses the orientation of $F(H, U) = F(T, U)$. But this structural group is just $N(H)/H = N(T)H/H$. This implies, of course, that there is an element $g \in N(T)$ reversing the orientation of $F(T, U)$ and hence of $F(T, M)$ as was to be shown.

By the case $k=0$, $G \neq G^+$ of Theorem 1.1, and by Lemma 2.2, we now have that M/G is an $(n-k)$-cm$_Z$ with boundary $B^* \cup E^*$. It remains to show that if G is connected, then $E^* = \emptyset$ near x^*. But if $z^* \in E^*$, then, since E^* is open in $B^* \cup E^*$, we have $\dim_Z E^* = n-k-1$ and hence we would have $\dim_Z E = n-1$ contrary to Theorem 4 of [5]. This completes the proof of Case I.

4. **Case** II. rank (G) = rank $(H)+1$. It will suffice to consider the case in which H is finite and G is connected, and hence for which rank$(G)=1$. If $G=\mathrm{SO}(2)$ then dim $F(G, M)$ could not be $n-k-2=n-3$, by dimensional parity [1, Chapter V, 3.2]. Thus $G=\mathrm{SO}(3)$ or Sp(1) effectively and $k=3$, $\dim_{Z_2} F(G, M) = n-5$ and $\dim_Z F(T_0, M) = n-4$, since it is of the same parity as n and is at most $n-k-1=n-4$ by Lemma 2.1. We will sometimes denote $F(G, M)$ by F.

Lemma 4.1. *$E = \emptyset$ near x.*

Proof. As in the proof of [1, Chapter XV, 6.1], we see that if there were exceptional orbits arbitrarily near x, then there would be a p-Sylow subgroup $P \subset G_z$ for some $z \in E$ such that ord $P \nmid$ ord H, and, noticing that $P \subset N(T_0)$ for some T_0, such that $\dim_{Z_p} F(P, M) \geqq \dim_{Z_p} F(N(T_0), F(T_0, M)) + 1 \geqq n-4$ (since $F(N(T_0), F(T_0, M))$ is either the $(n-4)$-cm$_Z$ $F(T_0, M)$ or is an $(n-5)$-cm$_Z$), and furthermore if $P \subset T_0$ for some T_0, then it may be assumed that $\dim_{Z_p} F(P, M) \geqq \dim_{Z_p} F(T_0, M) + 1 = n-3$.

Now, if $P \not\subset T_0$, then $N(P)$ is finite and it follows easily that $\dim_{Z_p} E^* \geqq \dim_{Z_p} F(P, M) \geqq n-4$, and therefore $\dim_{Z_p} E \geqq \dim_{Z_p} E^* + 3 \geqq n-1$, contrary to a theorem of Montgomery and Yang [5, Theorem 4].

Similarly if $P \subset T_0$ and $P \neq$ center G, then $\dim_{Z_p} E^* \geqq \dim_{Z_p} F(P, M) - 1 \geqq n-4$, which implies that $\dim_{Z_p} E \geqq n-4+3$ which is impossible. On the other hand if P = center G ($G = \mathrm{Sp}(1)$), then $\dim_{Z_p} F(P, M) = n-2$ by dimensional parity. Consider the action of $G/P \approx \mathrm{SO}(3)$ on the $(n-2)$-cm$_{Z_2}$ $F(P, M)$. Since $F(P, M)$ is assumed to contain points in E, the maximum dimension of the orbits of this action is three, but since $\dim_{Z_2} F(G, M) = n-5$ and $F(G/P, F(P, M)) = F(G, M) \approx (F(G, M))^*$ this is seen to contradict Corollary 6.3.

Lemma 4.2. *$N(T_0)$ acts trivially on $F(T_0, M)$.*

Proof. Assume the contrary. Then $N(T_0)/T_0 \approx Z_2$ acts effectively on $F(T_0, M)$ and hence $F(N(T_0), M)$ is an $(n-5)$-cm$_Z$ by the CDT ($k=0$) (or by [7, Corollary to Lemma 16]). Thus it coincides with $F(G, M)$ near some point $y \in F(G, M)$ arbitrarily near x. Near y there are exactly three orbit types: fixed points; $G/T_0 \approx S^2$; and G/H. Note also that, near y, $B^* \approx F(T_0, M)/N(T_0)$ is an $(n-4)$-cm_Z with boundary $F(G, M)$ so that $H_c^i(B^*, Z) =_{LF} 0$ for all i and

$$H_c^i(B^* - F, Z) =_{LF} \begin{cases} Z, & i = n - 4, \\ 0, & \text{otherwise,} \end{cases}$$

near y. We shall restrict the discussion in the rest of this proof to small neighborhoods of y.

Note also that $F(N(T_0), M)$ separates $F(T_0, M)$ into two subsets, near y, each of which is a cross-section for $B-F \to B^*-F$. Thus $B-F \approx (B^*-F) \times S^2$.

Consider the Leray spectral sequence of the map $B \to B^*$. We see that

$$E_2^{p,q} =_{LF} \begin{cases} Z, & p = n - 4 \text{ and } q = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$H_c^i(B, Z) =_{LF} \begin{cases} Z, & i = n - 2, \\ 0, & \text{otherwise} \end{cases}$$

(and, in fact, this would imply that B is an $(n-2)$-cm over Z).

The cohomology sequence of M mod B, then yields that

$$H_c^i(M - B, Z) =_{LF} \begin{cases} Z, & i = n, n - 1, \\ 0, & \text{otherwise,} \end{cases}$$

and hence also $H_c^i(M-B, Z_2) =_{LF} 0$ for $i \leqq n-2$. Now consider the fibre map

$$M - B \xrightarrow{G/H} \frac{(M - B)}{G}.$$

Since $H \subset T_0$, it is cyclic and hence

$$H^1(G/H, Z_2) = H^2(G/H, Z_2) = \begin{cases} 0, & G = \mathrm{Sp}(1), \operatorname{ord}(H) \text{ odd}, \\ Z_2, & \text{otherwise,} \end{cases}$$

and it follows that the structural group of $M-B \to (M-B)/G$ acts trivially on $H^*(G/H, Z_2)$. By Theorem 6.6, this implies that $H_c^i((M-B)/G, Z_2) =_{L_y} 0$ for $i \leqq n-2$. However $(M-B)/G$ is an $(n-3)$-cm over Z and hence over Z_2, so that $H_c^{n-3}((M-B)/G, Z_2) =_{L_y} Z_2$. This contradiction finishes the proof of the lemma.

Remark. Note that in deriving the local groups of B above, we have essentially proved a special case of the converse of the CDT. (That is, if X is an $(n-k)$-cm_Z with boundary F and G/H is a cohomology k-sphere, then $F \cup ((X-F) \times G/H)$ is an n-cm_Z.) That the converse of the CDT is true in general may be seen in exactly the same manner.

It follows from Lemma 4.2 and the CDT that $N(T_0)/H \approx S^1$. If $G = \mathrm{Sp}(1)$, then this implies that H contains the center of G, and hence that center (G) acts trivially on M. Thus $G \approx \mathrm{SO}(3)$ effectively. Note that we now have that near x there are exactly three types of orbits: fixed points; $G/N(T_0) \approx P^2$; and G/H.

LEMMA 4.3. $H \approx Z_2 \oplus Z_2$.

Proof. Let $Z_2 \oplus Z_2 \approx K \subset N(T_0)$. We first show that a conjugate of K is contained in H. Note that $F(K, G/N(T_0))$ consists of three points, so that if K is not conjugate to a subgroup of H, then $\dim_{Z_2} F(K, M) = \dim_{Z_2}(F(K, M))^* = \dim_{Z_2} B^* = n-4$. Since the three subgroups of K isomorphic to Z_2 are conjugate in G, their fixed point sets have the same dimension, j, over Z_2. Borel's formula [1, Chapter XIII, 4.3] then reads

$$4 = n - (n - 4) = 3(j - (n - 4))$$

and this contradiction shows that $K \subset H$ (up to conjugacy).

Thus H is dihedral of order $4a$, $a \geqq 1$. Say that $a > 1$, and note that in that case $F(H, G/H) = N(H)/H \approx Z_2$ consists of two points and, in fact, these points are in $F(H, N(T_0)/H)$. Using the CDT we easily see from this that $F(H, M - F(G, M))$ is an $(n-3)$-cm over Z. But $F(H, M-F) \subset F(K, M-F)$ which is a connected $(n-3)$-cm over Z_2. Since $F(H, M-F)$ is closed in $F(K, M-F)$ this implies that $F(H, M-F) = F(K, M-F)$ and this is inconsistent with the fact that $F(K, G/N(T_0))$ consists of three points while $F(H, G/N(T_0))$ is one point. Thus $a=1$ and $H = K \approx Z_2 \oplus Z_2$ as was to be shown.

Let $X = F(H, M)$. Then X is an $(n-3)$-cm over Z_2. We may assume, by taking M smaller (about x), that M contains no exceptional orbits and hence that the isotropy groups in M are conjugates of G, $N(T_0)$, and H.

Put $B' = B \cap X$. $X - B'$ is then a principal $N(H)/H$-bundle over $(M-B)/G \approx (X-B')/N(H)$, and $N(H)/H \approx \sigma_3$, the group of permutations of three objects. We wish to show that this bundle is trivial if M is sufficiently small. Let $J \approx Z_3$ be the subgroup of $N(H)/H \approx \sigma_3$ of order 3. We then have the bundle maps

$$X - B' \xrightarrow{\quad Z_3 \quad} \frac{X - B'}{J} \xrightarrow{\quad Z_2 \quad} \frac{X - B'}{N(H)} \approx \frac{M - B}{G} \cdot$$

Since X is an $(n-3)$-cm_{Z_2} it follows easily from [1, Chapter III, 2.3] that X/J is also an $(n-3)$-cm_{Z_2}. Since $F((N(H)/H)/J,\ X/J) = B'/J$ is $(n-4)$-dimensional we have that the second bundle map above is trivial (near x) by [7, Lemma 16]. (Note that it also follows that $X/N(H) \approx M/G$ is an $(n-3)$-cm_{Z_2} with boundary $B'/N(H) \approx B/G$.) Thus $X - B'$ breaks into two parts each of which is a J-bundle over $(M-B)/G$.

Note that, since $G/N(T_0) \approx P^2$ we have that $\pi^*: H_c^t(B^*, Z_3) \to H_c^t(B, Z_3)$ is an isomorphism (onto) and, since $B^* \approx F(T_0, M)$ is an $(n-4)$-cm$_Z$, we have that $H_c^i(B, Z_3) = 0$ for $i \geqq n-3$. This implies that $H_c^{n-1}(M-B, Z_3) =_{Lx} 0$. Thus for the spectral sequence $E_r^{p,q}$, with coefficients in Z_3, of the fibering $M-B \to (M-B)/G$ we have (see [1, Chapter XV, 6.2])

$$0 =_{Lx} H_c^{n-1}(M - B, Z_3) \approx E_\infty^{n-4,3} \approx E_2^{n-4,3} \approx H_c^{n-4}\left(\frac{M - B}{G}, Z_3\right),$$

since $H^*(G/H, Z_3) = H^*(S^3, Z_3)$. Also note that

$$H_c^{n-3}\left(\frac{M - B}{G}, Z_3\right) \approx E_2^{n-3,3} \approx E_\infty^{n-3,3} \approx H_c^n(M - B, Z_3) \approx Z_3$$

so that $(M-B)/G$ is orientable. Thus, by Theorem 6.5, we have that (near x) $X-B'$ splits into six parts each homeomorphic via π to $(M-B)/G$.

LEMMA 4.4. *$H_c^i(F, Z_2) =_{LF} 0$ for $i \leqq n-6$, and $H_c^i((B-F)/G, Z_2) =_{LF} 0$ for $i \leqq n-5$.*

Proof. It has been shown above that M/G is an $(n-3)$-cm over Z_2 with boundary B/G and that there is a cross-section in $F(H, M)$ for the action of G on $M-B$. Thus $H_c^i((M-B)/G, Z_2) =_{Ly} 0$ for $i \leqq n-4$ and any $y \in F(G, M)$. Also $M-B \approx (M-B)/G \times G/H$ so that $H_c^i(M-B, Z_2) =_{Ly} 0$ for $i \leqq n-4$.

Since $X-B'$ is the union of six disjoint copies of $(M-B)/G$ we have $H_c^i(X-B', Z_2) =_{Ly} 0$ for $i \leqq n-4$. Since X is an $(n-3)$-cm$_{Z_2}$ this implies that $H_c^i(B', Z_2) =_{Ly} 0$ for $i \leqq n-5$.

Note that if T_0, T_1, and T_2 are the three circle subgroups of G with normalizers containing H, then, since $F(T_i, M) = F(N(T_i), M) \subset F(H, M) = X$, we have $F(T_0, M) \cup F(T_1, M) \cup F(T_2, M) = B'$ and $B' - F$ is the disjoint union of the $(F(T_i, M) - F)$. Also π is a homeomorphism on each $(F(T_i, M) - F)$ to $(B-F)/G$. It follows that $\pi^*: H_c^i((B-F)/G, Z_2) \to H_c^i(B'-F)$ cannot be surjective unless these groups are trivial and this also holds in the local sense. Thus, by the diagram (coefficients in Z_2)

$$\begin{array}{ccccccc} H_c^{i-1}(B/G) & \to & H_c^{i-1}(F) & \to & H_c^i\left(\dfrac{B - F}{G}\right) & \to & H_c^i(B/G) \\ & & \downarrow \approx & & \downarrow \pi^* & & \downarrow \pi^* \\ & & H_c^{i-1}(F) & \to & H_c^i(B' - F) & \longrightarrow & H_c^i(B') \end{array}$$

and the fact, shown above, that $H_c^i(B', Z_2) =_{Ly} 0$ for $i \leqq n-5$, it follows easily that $H_c^i((B-F)/G, Z_2) =_{Ly} 0$ for $i \leqq n-5$.

Since $B/G \approx F(T_0, M)$ is an $(n-4)$-cm over Z and hence over Z_2, it also follows that $H_c^{i-1}(F, Z_2) =_{Ly} 0$ for $i \leqq n-5$, that is, for $i-1 \leqq n-6$, as was to be shown.

We will now complete the proof of Case II. Let $y \in F = F(G, M)$ be arbitrary. Note that, since $F(T_0, G/N(T_0))$ is a point, we have that $F(T_0, B-F) = F(N(T_0), B-F)$ is a cross-section for the orbits in $B-F$ so that

$$B - F \approx \frac{B - F}{G} \times P^2.$$

Consider the Leray spectral sequence $E_r^{p,q}$ (coefficients in Z) of the map $B \to^{\pi} B/G$. We have

$$E_2^{p,q} = H_c^p(B/G, \mathcal{H}^q(G/G_w, Z) = 0 \qquad \text{for } q \neq 0, 2,$$

since G/G_w is a point for $w \in F$, and is a projective plane for $w \in B-F$. Since the sheaf $\mathcal{H}^2(G/G_w, Z)$ is zero on F and constant on $(B-F)/G$ with stalks $H^2(P^2, Z) \approx Z_2$, we have: $E_2^{p,q}=0$, $q \neq 0$, 2; $E_2^{p,0} \approx H_c^p(B/G, Z) =_{Ly} 0$, for $p \neq n-4$; and $E_2^{p,2} \approx H_c^p((B-F)/G, Z_2) =_{Ly} 0$, for $p \neq n-4$. It follows that $H_c^i(B,Z) =_{Ly} 0$, for $i \neq n-4$, $n-2$. Thus also $H_c^i(M-B, Z) =_{Ly} 0$, for $i \leqq n-4$, and since $M-B \approx (M-B)/G \times G/H$, this implies that $H_c^i((M-B)/G, Z) =_{Ly} 0$, for all $i \leqq n-4$.

Since $(M-B)/G$ is an orientable $(n-3)$-cm over Z and is connected (even locally near y) because $M-B$ is connected, and since $M/G-F$ is an $(n-3)$-cm$_Z$ with boundary $B/G-F$ by the CDT, it follows from Lemma 2.6 that M/G is an $(n-3)$-cm$_Z$ with boundary B/G, which completes the proof of Case II.

We shall now show that F is an $(n-5)$-cm over Z_2, since this will be needed in the treatment of Case III.

By the fact that $M-B \approx (M-B)/G \times G/H$ and that $H^1(G/H, Z_2) \approx H^2(G/H, Z_2) \approx Z_2 \oplus Z_2$, we have that $H_c^{n-2}(M-B, Z_2) =_{LF} Z_2 \oplus Z_2$ and hence that $H_c^{n-3}(B, Z_2) =_{LF} Z_2 \oplus Z_2$. We again consider the Leray spectral sequence of $B \to B/G$, but with coefficients in Z_2. We have that $E_2^{p,q} =_{LF} 0$ for $q > 2$ or for $p \neq n-4$. It follows readily that

$$H_c^{n-4}\left(\frac{B-F}{G}, Z_2\right) \approx E_2^{n-4,1} =_{LF} E_\infty^{n-4,1} =_{LF} H_c^{n-3}(B, Z_2) =_{LF} Z_2 \oplus Z_2.$$

Now the cohomology sequence

$$H_c^{n-5}(B/G, Z_2) \to H_c^{n-5}(F, Z_2) \to H_c^{n-4}\left(\frac{B-F}{G}, Z_2\right) \to H_c^{n-4}(B/G, Z_2) \to 0$$

is locally about F

$$0 \to H_c^{n-5}(F, Z_2) \to Z_2 \oplus Z_2 \to Z_2 \to 0$$

and it can be seen easily that this implies that $H_c^{n-5}(F, Z_2) =_{LF} Z_2$ and our contention follows from Lemma 4.4 (see [1, Chapter XV, 2.2]).

5. **Case** III. rank $(G)=$rank $(H)+2$. It will suffice to treat the case in which H is finite and G is connected. Hence rank $(G)=2$. Note that by Lemma 2.1, $\dim_Z F(T_0, M)=n-k-2$ so that $F(G, M)$ and $F(T_0, M)$ coincide in some open set near x (but not necessarily containing x). It follows that $\dim_L(F(G, M), x)=n-k-2$ over any coefficient ring L. We first prove a lemma useful for the treatment of E.

LEMMA 5.1. *Let P be a p-group with* $\dim N(P)>0$; *then either P is conjugate to a subgroup of H or $F(P, M)\cap E=\emptyset$ near x.*

Proof. Assume that P is not conjugate to a subgroup of H but that $F(P, M)\cap E\neq\emptyset$ near x, that is, the component of the $\text{cm}_{Z_p} F(P, M)$ containing x also contains points of E. Let z be such a point. For convenience of notation we shall assume that $F(P, M)$ is connected. We may assume that if $z'\in F(P, M)\cap E$ is sufficiently close to z, then $G_z\sim G_{z'}$. Now $F(P, G/G_z)$ is easily seen to have the same dimension as $N(P)$ (since G_z is a finite group), and moreover, near z, the map $F(P, M)/N(P)\to F(P, M)^*$ is finite to one and hence cannot lower dimension. Thus

$$\dim_{Z_p}\frac{F(P, M)}{N(P)}\leqq \dim_{Z_p}(F(P, M)^*, z^*)\leqq \dim_{Z_p}(E^*, z^*)\leqq n-k-2,$$

since if it were $\geqq n-k-1$, then $\dim_{Z_p}E\geqq n-1$ contrary to [5, Theorem 4]. But $F(G, M)\approx F(G, M)/N(P)$ is contained in the singular set of the action of $N(P)$ on $F(P, M)$ so that by Corollary 6.3, $\dim_{Z_p}F(G, M)\leqq n-k-3$ (even near x), which contradicts the fact that $\dim_L(F(G, M), x)=n-k-2$.

The following lemma is the main tool of this section.

LEMMA 5.2. *G is not a simple group. Furthermore, there are exactly two mutually nonconjugate circle groups S_1 and S_2 contained in T_0 such that, for any circle group $S\subset T_0$, we have that* $\dim_Z(F(S, M), x)>\dim_Z(F(T_0, M), x)=n-k-2$ *if and only if S is conjugate to one of the S_i, $i=1, 2$.*

Proof. Any rank two group is locally isomorphic to one of the following groups: $D_1\times D_1$; $D_1\times A_1$; $A_1\times A_1$; A_2; B_2; or G_2 (where D_1 denotes the circle group SO(2) and the other notation is standard). These have dimensions 2, 4, 6, 8, 10, and 14 respectively. The corresponding Weyl groups $W(G)$ are (e), $D(2)\approx Z_2$, $D(4)\approx Z_2\oplus Z_2$, $D(6)$, $D(8)$, and $D(12)$, where $D(m)$ denotes the dihedral group of order m.

If $S\subset T_0$ is a circle group, then let $n(S)$ be the number of conjugates of S which are contained in T_0. As may be seen by consideration of the action of $W(G)$ on $\tilde{T}_0\approx E^2$, the situation is as shown in Table 1.

Since in Lemma 2.1 we have equality for the first two inequalities, it follows from the remark below 2.1 that for any S with $\dim(F(S, M), x)>n-k-2$ we have

TABLE 1

G	S	$n(S)$	Number of nonconjugate $S\subset T_0$ having this value of $n(S)$
$D_1\times D_1$	(—)	1	∞
$D_1\times A_1$	singular	1	1
	regular	1	1
		2	∞
$A_1\times A_1$	singular	1	2
	regular	2	∞
A_2	singular	3	1
	regular	3	1
		6	∞
B_2	singular	2	2
	regular	4	∞
G_2	singular	3	2
	regular	6	∞

$$\dim_Z (F(S, M), x) = n - k - 1 + \dim \frac{N(S)}{S} = \begin{cases} n - k, & S \text{ regular}, \\ n - k + 2, & S \text{ singular}. \end{cases}$$

[Also note that, by the proof of Lemma 2.1, we have that for such an $S\subset T_0$, $\dim_Z F(S, M)^* = n-k-1$.]

Let p_i be the number of mutually nonconjugate $S\subset T_0$ such that $\dim_Z(F(S, M), x) > n-k-2$, for S singular and $n(S)=i$. Similarly, let q_i be the analogous number for S regular. Then the formula of Borel [1, Chapter XIII, 4.3] implies that

$$k + 2 = n - (n - k - 2) = \sum_i (4ip_i) + \sum_i (2iq_i).$$

With the restrictions imposed by Table 1 this formula is as in Table 2 (in which any p_i or q_i is zero if it does not appear explicitly).

We see immediately that the formula cannot be solved in the last three cases, so that G is not simple, and in the first three cases we must have $\sum p_i + \sum q_i = 2$ which is exactly the remainder of the statement of Lemma 5.2.

TABLE 2

G	Formula of Borel
$D_1 \times D_1$	$4 = 2q_1$.
$D_1 \times A_1$	$6 = 4p_1 + 2q_1 + 2\cdot 2q_2$; $p_1 = 0, 1$; $q_1 = 0, 1$.
$A_1 \times A_1$	$8 = 4p_1 + 2\cdot 2q_2$; $p_1 = 0, 1, 2$.
A_2	$10 = 4\cdot 3p_3 + 2\cdot 3q_3 + 2\cdot 6q_6$; $p_3 = 0, 1$; $q_3 = 0, 1$.
B_2	$12 = 4\cdot 2p_2 + 2\cdot 4q_4$; $p_2 = 0, 1, 2$.
G_2	$16 = 4\cdot 3p_3 + 2\cdot 6q_6$; $p_3 = 0, 1, 2$.

REMARK. Note that in case G is locally isomorphic to $D_1 \times A_1$, we must have $q_1 = 1$, which means that one of the S_i is contained in the 3-dimensional factor A_1.

Now consider the action of $N(S_i)$ on $F(S_i, M)$ (near x). As can be seen from the proof of Lemma 2.1, $\dim_Z F(S_i, M)/N(S_i) = n-k-1$ and hence the action of $N(S_i)$ on $F(S_i, M)$ satisfies the hypotheses of the CDT. Hence $N(S_i)$ has exactly two types of orbits near x on $F(S_i, M)$, principal orbits and fixed points and the fixed point set $F(N(S_i), M)$ must be an $(n-k-2)$-cm$_Z$. Since $F(T_0, M) \subset F(N(S_i), M)$ is an $(n-k-2)$-cm$_Z$ we have that $F(T_0, M) = F(N(S_i), M)$, near x, and its complement in $F(S_i, M)$ is the set of principal orbits of $N(S_i)$ on $F(S_i, M)$. It follows that $F(S_i, M) - F(T_0, M)$ consists of points with isotropy groups (in G) of rank one (and hence with S_i as a maximal torus). Also by the CDT we must have that the principal orbits of $N(S_i)$ on $F(S_i, M)$ are cohomology spheres (of odd dimension). Thus if $x_i \in F(S_i, M) - F(T_0, M)$ and $K_i = G_{x_i}$ then

$$\frac{N(S_i)}{N(S_i) \cap K_i} \text{ is an integral cohomology sphere.}$$

Now choose $x_i \in F(S_i, M) - F(T_0, M)$ in such a way that the K_i are locally minimal isotropy groups for points in $F(S_i, M)$ (that is, if $x_i' \in F(S_i, M)$ is sufficiently close to x_i then $G_{x_i'} \sim G_{x_i} = K_i$). Some conjugate of H, say H_i, is a principal isotropy group of the action of K_i on a slice at x_i. Note that this action of K_i on a slice at x_i satisfies the hypotheses of the CDT so that

$$\frac{K_i}{H_i} \text{ is an integral cohomology sphere.}$$

Using these facts we shall attempt to classify the groups (G, K_1, K_2, H). We will obtain, as the only possibilities, the cases in Table 3. (I denotes an icosahedral subgroup of SO(3) and I' denotes an icosahedral subgroup of Sp(1).)

TABLE 3

Case	G	K_1	K_2	H
1	$SO(2)\times SO(2)$	$SO(2)\times(e)$	$(e)\times SO(2)$	(e)
2	$SO(3)\times SO(2)$	$SO(3)\times(e)$	$I\times SO(2)$	$I\times(e)$
3	$SO(3)\times SO(2)$	S_1H ($S_1\subset SO(3)$) (K_1 not abelian)	S_2 (regular) ($K_2\cap SO(3)=(e)$)	Z_2 ($H\cap SO(3)=(e)$)
4	$Sp(1)\times SO(2)$	$Sp(1)\times(e)$	$(e)\times SO(2)$	(e)
5	$Sp(1)\times SO(2)$	$Sp(1)H$	S_2 (regular) ($K_2\cap Sp(1)=(e)$)	cyclic ($H\cap Sp(1)=(e)$)
6	$SO(3)\times SO(3)$	$SO(3)\times I$	$I\times SO(3)$	$I\times I$
7	$SO(3)\times Sp(1)$	$SO(3)\times(e)$	$I\times Sp(1)$	$I\times(e)$
8	$Sp(1)\times Sp(1)$	$Sp(1)\times(e)$	$(e)\times Sp(1)$	(e)
9	$Sp(1)\times Sp(1)$	$Sp(1)\times(e)$	$Sp(1)$ (diagonal)	(e)
10	$\dfrac{Sp(1)\times Sp(1)}{Z_2}$	$\dfrac{Sp(1)\times I'}{Z_2}$	$SO(3)$ (diagonal)	I ($\subset K_2$)
11	$\dfrac{Sp(1)\times Sp(1)}{Z_2}$	$S_1H\approx O(2)$	$S_2H\approx O(2)$	Z_2+Z_2 ($H\subset SO(3)$, the "diagonal")

In all cases in the table except (5) the action is assumed to be effective, while in case (5), for convenience, we assume only that G is almost effective, so that the effective group is $G/(H\cap \text{center } G)$. In cases (10) and (11) the group G is $Sp(1)\times Sp(1)$ divided by the "diagonal" central element, that is $G\approx SO(4)$.

REMARK. Examples can be given for all the cases (1) through (10). For cases (1), (3), (4), (5), (8) and (9) linear examples can be constructed and this can be seen to be impossible in the other cases. We strongly suspect that case (11) cannot arise, but we know of no way to see this. The exact situation in cases (3) and (5) is also in doubt.

For the proof of the "classification" we shall assume, unless otherwise specified, that $G=G_1\times G_2$, $G_i\approx SO(2)$, or $Sp(1)$ and that G is only *almost* effective, so that the effective group is obtained by dividing out by

$$H\cap \text{center }(G).$$

Note that $\mathrm{Sp}(1)/I' \approx \mathrm{SO}(3)/I$ is an integral cohomology 3-sphere. If $G_i \approx \mathrm{SO}(2)$ then we can (and shall) assume, without loss of generality, that $H \cap G_i = (e)$.

In case $G_i = \mathrm{SO}(2)$, $i = 1, 2$, H may be assumed to be trivial, since it is central. Thus $K_i/H_i = K_i$, to be a cohomology sphere, must be S_i. It remains to show that $S_1 \cap S_2 = (e)$. If not then let $P \subset S_1 \cap S_2$, $P \approx Z_p$ for some prime p. Then $F(P, M)$ is a cm_{Z_p} properly containing the $(n-2)$-cm_Z $F(S_1, M)$. Thus $\dim_{Z_p} F(P, M) = n-1$, which is impossible since P preserves orientation (see [1, Chapter V, 2.3]).

Now in the other cases, if $S_i \subset G_i$, $i = 1, 2$, then every conjugate of S_1 commutes with every conjugate of S_2. From this we see easily that it may be assumed in this case that $H_1 = H = H_2$. We also see that this can be assumed if either K_i contains one of the G_i, since such a K_i must intersect every maximal torus of G. (That is, if $K_1 \supset G_1$ say, then we could replace K_1 by any conjugate gK_1g^{-1} and still have $gK_1g^{-1} \cap T_0 \supset S_1$. Thus g could be chosen so that H_2 is a principal isotropy group of the action of $gK_1g^{-1} = gG_{x_1}g^{-1} = G_{g(x_1)}$ on a slice at $g(x_1)$.)

Say that $G = G_1 \times G_2$, $G_1 \approx \mathrm{Sp}(1)$ and $G_2 \approx \mathrm{SO}(2)$. First assume that $S_1 \subset G_1$, $S_2 \subset G_2$. Then, since $G/K_2 = N(S_2)/(N(S_2) \cap K_2)$ is a cohomology sphere, we see that $K_2 = S_2$ or $K_2 = I' \times S_2$. Thus $H = (e)$ or $H = I' \times (e)$ respectively. If $H = (e)$ then, since K_1/H is a cohomology sphere, we have $K_1 = S_1$ or $K_1 = G_1$. But, if $K_1 = S_1$, then $N(S_1)/(N(S_1) \cap K_1)$ is not connected. Thus $K_1 = G_1$ and this yields case (4). If $H = I' \times (e)$, then clearly $K_1 = G_1$ in order for K_1/H to be a cohomology sphere. This yields case (2).

Now say that $S_1 \subset G_1$, S_2 is regular, and $\dim K_1 = 1$. First assume that $G_1 \approx \mathrm{SO}(3)$ effectively so that we may assume that $G \approx \mathrm{SO}(3) \times \mathrm{SO}(2)$ is effective. We have that $N(S_2) = T_0$ and, since $K_2^0 = S_2$, we have $H_2 \subset K_2 \subset N(S_2) = T_0$. Also $N(S_1) = N_{G_1}(S_1) \times G_2 = N(T_0)$ and, since

$$\frac{N(S_1)}{N(S_1) \cap K_1} = \frac{N(S_1)}{K_1}$$

must be a circle, we have that K_1 contains an element of the form $g = g_1g_2$, where $g_1 \in N_{G_1}(S_1) - S_1$ and $g_2 \in G_2$. Since K_1/H_1 is also a circle, H_1 must contain an element of the coset gS_1 and this element may be assumed to be g. Then, since H_1 contains the central element $g^2 = g_1^2g_2^2 = g_2^2$, and since G is effective, we have $g_2^2 = (e)$. We now claim that $K_2 \cap G_1 = (e)$. If not there is a p-group $P \subset K_2 \cap S_1$, for some prime p, and hence for which $F(P, M)$ contains the $(n-4)$-cm $F(S_1, M)$ properly. Then we see that P must be conjugate to a subgroup of H, since if not we would have

$$\dim_{Z_p} F(P, M) = \dim_{Z_p} [F(S_1, M)]^* + \dim F(P, G/K_1) = (n-5) + 1 = n-4$$

and $F(P, M)$ could not contain the $(n-4)$-cm $F(S_1, M)$ properly. But if P is conjugate to a subgroup of H (and hence H_1) we must have $P \approx Z_2$ (by the facts that $g \in H_1$ and $H_1 \sim H_2 \subset T_0$) and hence by dimensional parity

$\dim_{Z_2} F(P, M) = n-2$. Consider the subgroup $J \approx Z_2 \oplus Z_2$ of $G_1 \approx \mathrm{SO}(3)$. Since $H_2 \subset T_0$ we see that J is not conjugate to a subgroup of H and by Lemma 5.1 we have that $F(J, M) \subset B$. But all the subgroups of order two of J are conjugate in G to P so that Borel's formula reads

$$n - \dim_{Z_2} F(J, M) = 3(\dim_{Z_2} F(J, M) - (n - 2))$$

and hence $\dim_{Z_2} F(J, M) = n-3$. But the map $F(J, M)/N(J) \to F(J, M)^* \subset B^*$ is clearly a homeomorphism, and, since $\dim B^* = n-5$ and $\dim N(J) = 1$, this is a contradiction. Thus $K_2 \cap G_1 = (e)$ as claimed. It follows that, in fact, $H_2 \subset S_2$, $S_1 \cap S_2 = (e)$, and $g_2 \neq (e)$. It now follows immediately that $H_2 = \{e, g\} \approx Z_2$, $K_1 = S_1H_1$, and $K_2 = S_2$. This yields case (3).

In order to treat the case in which $G_1 \approx \mathrm{Sp}(1)$ effectively we must first investigate case (3) more fully. We see immediately from Lemma 5.1 that there are no exceptional orbits. Moreover, by Corollary 2.4 all isotropy groups of points in $F(S_2, M) - F(T_0, M)$ are conjugate to K_2. Thus if $J \approx Z_2 \oplus Z_2$ is a subgroup of G_1 we see that $F(J, M) \subset B_1 = GF(S_1, M)$. By Borel's formula $n - \dim_{Z_2} F(J, M)$ must be divisible by 3 and hence

$$\dim_{Z_2} F(J, M) = n - 3 \text{ or } n - 6.$$

But just as above, it cannot be $n-3$ and it follows that $F(J, M)$ is precisely the $(n-6)$-cm$_Z$ $F(N(T_0), M)$. (Note that it also follows from this that there are no points in M with a 3-dimensional isotropy group.)

Now consider the case in which $G_1 \approx \mathrm{Sp}(1)$ effectively, $G_2 \approx \mathrm{SO}(2)$, $S_1 \subset G_1$, S_2 regular, and $\dim K_1 = 1$. Consider the element $e \neq g \in$ center G_1. Then $F(g, M) \supset GF(S_1, M) = B_1$ is of dimension $n-2$ (since G_1 is effective). Thus $M/(e, g)$ is an n-cm on which $G/(e, g)$ acts, which is case (3). Also note that $G/(e, g)$ acts on the $(n-2)$-cm$_{Z_2}$ $F(g, M)$. As shown above $\dim_{Z_2} F(J, M/(e, g)) = n-6$, and also we have shown that

$$F(J, M/(e, g)) \subset B_1 = GF(S_1, M/(e, g)) \subset \frac{F(g, M)}{(e, g)} \approx F(g, M)$$

and hence $F(J, M/(e, g)) \approx F(J, F(g, M))$ is an $(n-6)$-cm$_{Z_2}$. But since $G_1/(e, g)$ acts on $F(g, M)$, Borel's formula implies that $(n - 2) - \dim_{Z_2} F(J, F(g, M))$ is divisible by 3. This contradiction eliminates this case.

The next case to be considered is that for which $G_1 \approx \mathrm{Sp}(1)$, $G_2 \approx \mathrm{SO}(2)$, $S_1 \subset G_1$, S_2 regular and $\dim K_1 = 3$. Here $K_1^0 = G_1$ and thus by the remarks above we may assume $H_1 = H = H_2$. We have that $K_1 = G_1 \times J_2$ for some subgroup $J_2 \subset G_2$. Since K_1/H is an integral cohomology sphere we must have that $G_1 \cap H = (e)$ or I'. However $H \subset N(S_2) = T_0$ so that we must have $G_1 \cap H = (e)$. We now claim that $G_1 \cap K_2 = (e)$. Note that since $K_2 \subset N(S_2) = T_0$ we have $G_1 \cap K_2 = S_1 \cap K_2$. Let P be some p-group in $S_1 \cap K_2$. Then $F(P, M)$ contains then $(n-4)$-cm $F(S_1, M)$ properly. However P is not conjugate to a subgroup of H and it follows that

$$\dim_{Z_p} F(P, M) = n - 5 + \dim F(P, G/K_1) = n - 4,$$

and hence $F(P, M)$ cannot contain $F(S_1, M)$ properly. This is a contradiction which shows that $G_1 \cap K_2 = (e)$ as claimed. It follows that $K_2 = S_2$ and hence that $H \subset S_2$. This yields case (5).

We now consider the cases in which $G_i \approx \mathrm{Sp}(1)$, $i=1, 2$, $G = G_1 \times G_2$ is almost effective. We first take up the case in which $S_1 \subset G_1$ and $\dim K_1 = 1$. We claim that this situation cannot arise. To see this consider the nontrivial element $g \in$ center G_1. Since $g \in S_1$ we see that $F(g, M) \supset GF(S_1, M)$ which is of dimension $n-2$. Hence $\dim F(g, M) = n-2$ or n and this implies that $M/(e, g)$ is an n-cm$_Z$ on which $G/(e, g)$ acts. Thus we may assume for the present consideration that $G = G_1 \times G_2$ is almost effective and $G_1 \approx \mathrm{SO}(3)$, $G_2 \approx \mathrm{Sp}(1)$. Let $K \approx Z_2 \oplus Z_2$ be a subgroup of G_1 and consider a point $y \in F(K, M)$ such that G_y is locally minimal in this set. Now Borel's formula applied to the action of K implies that $n - \dim_{Z_2} F(K, M)$ is divisible by 3. First say that G_y is of rank 2. Then we may assume that $y \in F(T_0, M) = F(N(T_0), M)$ and hence $F(K, M) = F(T_0, M)$ (for otherwise G_y would not be locally minimal) is of dimension $n-k-2 = n-8$. But $8 = n-(n-8)$ is not divisible by 3 so that we must have rank $(G_y) \leqq 1$. If $\mathrm{rank}(G_y) = 1$ then it may be assumed that $S_i \subset G_y$ for some $i = 1, 2$. The CDT applied to the action of $N(S_i)$ on $F(S_i, M)$ implies that $N(S_i) \cap G_y$ is conjugate in $N(S_i)$ to $N(S_i) \cap K_i$. Since K is conjugate to a subgroup of $N(S_i)$ this implies that in fact K is conjugate to a subgroup of K_i (so that in fact we may assume $G_y = K_i$). Say that $K \subset K_1$. Then we can compute

$$\dim_{Z_2} F(K, M) = n - k - 1 + \dim F(K, G/K_1) = n - 4$$

contrary to the above. Thus $K \subset K_2$, which is only possible when $S_2 \subset G_2$ (otherwise $K \not\subset N(K_2^0)$). We again compute

$$\dim_{Z_2} F(K, M) = n - k - 1 + \dim F(K, G/K_2) = n - 7 \text{ or } n - 5,$$

according as $\dim K_2 = 3$ or 1, which again is a contradiction. Thus we must have G_y finite and by Lemma 5.1 we see that K must be conjugate to a subgroup of H. This implies that $S_2 \subset G_2$ (since K can be assumed to be in K_2 and if S_2 is regular then $K \not\subset N(K_2^0)$). Thus by a remark above we may assume $H_1 = H = H_2$. However $N(S_2)/(N(S_2) \cap K_2)$ must be a cohomology sphere and, since $K \subset K_2$, this can be so only if $K_2 \cap G_1 \approx I$. However $H \subset K_1 \subset N(S_1) = N_{G_1}(S_1) \times G_2$ and hence K_2/H could not possibly be connected, contrary to its being a cohomology sphere. This finishes the proof of our contention that if $S_1 \subset G_1$ then $\dim K_1 = 3$.

Thus in the present case in which $S_1 \subset G_1$ we have $K_1^0 = G_1$, and by a remark above we may assume that $H_1 = H = H_2$. K_1/H is a cohomology sphere and it follows that since $K_1/H \approx K_1^0/(H \cap K_1^0)$, we have $H \cap G_1 = (e)$ or I'. Assume for the moment that $H \cap G_1 = I'$. Then S_2 must be in G_2 in order that $H \subset N(K_2^0)$. Thus by symmetry we have $K_2^0 = G_2$ and $H \cap G_2 = (e)$ or I'. If $H \cap G_2 = (e)$ then we must have $H = I' \times (e) \subset G_1$ which yields case (7). If $H \cap G_2 = I'$, then $H = I' \times I'$ which yields case (6). Now assume that $H \cap G_1 = (e)$. Note that $K_1 = G_1 \times (K_1 \cap G_2)$ and hence

$$\frac{N(S_1)}{N(S_1) \cap K_1} \approx \frac{G_2}{K_1 \cap G_2}.$$

Since this a cohomology sphere we must have $K_1 \cap G_2 = (e)$ or I'. If $K_1 \cap G_2 = (e)$ then $H \subset K_1 = G_1$ implies $H = (e)$. If $S_2 \subset G_2$ then by symmetry we see that $K_2 \supset G_2$ and hence $K_2 = G_2$ yielding case (8). If S_2 is regular then dim $K_2 = 3$ in order for $N(S_2)/(N(S_2) \cap K_2)$ to be a cohomology sphere (since K_2 is connected, by $H = (e)$). This yields case (9). If, on the other hand, $K_1 \cap G_2 = I'$ then we must have $H \approx I'$, either in the "diagonal" or in G_2. If $H \subset G_2$ then we must have $K_2 = G_2$ and this yields case (7). If H is in the "diagonal" then since K_2/H is a cohomology sphere, we must have dim $K_2 = 3$ and, in fact, either $K_2 \approx \mathrm{Sp}(1)$ is diagonal (yielding case (10)) or $K_2 = I' \times \mathrm{Sp}(1)$. We must rule out this latter case. Let c be a central element of G not contained in $H \approx I'$. Then we see that $c \in K_1 \cap K_2$ so that $F(c, M) \supset GF(S_1, M) \cup GF(S_2, M) = B$. But also Lemma 5.1 implies that $B \supset F(c, M)$ since $c \notin H$. Hence $F(c, M) = B$. Note that dim $B = n - 7 + 3 = n - 4$ and B is separated by $GF(T_0, M)$. But since it may be assumed that $F(T_0, M) = F(G, M)$, we see that $F(c, M)$ is an $(n-4)$-cm$_{Z_2}$ which is separated by the $(n-8)$-cm$_Z$ $F(T_0, M)$. This is impossible, and hence this finishes the case $S_1 \subset G_1$.

The only remaining cases are those for which $G = G_1 \times G_2$, $G_i \approx \mathrm{Sp}(1)$ and both S_i are regular. First suppose that G is effective, so that $H \cap \mathrm{center}(G) = (e)$. Then $N(S_1) = T_0 \cup sT_0$ where $s = s_1 s_2$, and s_i is an element of $(N_{G_i}(T_0 \cap G_i) - T_0)$. At most one K_i can be 3-dimensional, so say that dim $K_1 = 1$. Then the fact that $N(S_1)/(N(S_1) \cap K_1)$ is a circle implies that we may assume $K_1 = N(S_1) \cap K_1 = (T_0 \cap K_1) \cup s(T_0 \cap K_1)$, since s is determined only up to an element of T_0. Thus K_1 contains the group $S_1 \cup sS_1$ (which is isomorphic to the nornalizer of a circle subgroup of Sp(1)). Thus, since K_1/H_1 is a circle, we must have that H_1 contains some element st, $t \in S_1$. But then H_1 contains $(st)^2 = st\, st = s^2 t^{-1} t = s^2 = s_1^2 s_2^2$ which is a nontrivial central element of G and consequently G would not be effective.

Now suppose that effectively $G = G_1 \times G_2$ and one of the G_i, say G_1, is SO(3). Let $K \approx Z_2 \oplus Z_2$ be contained in $N_{G_1}(T_0 \cap G_1) \subset G_1 \approx \mathrm{SO}(3)$. Then, since K cannot normalize any connected rank one group which is not contained in one of the G_i, we see that K is not conjugate to a subgroup of either K_i. We may confine our attention to the neighborhood of a point y near which $F(T_0, M) = F(G, M)$. By Lemma 5.1 we see that $F(K, M) \subset B$. Let $z \in F(K, M)$, so that $K \subset G_z$, dim $G_z > 0$. By Lemma 5.2 we may assume that a conjugate gS_ig^{-1} of one of the S_i is contained in G_z. But then $G_z = [KG_z] \supset [KgS_ig^{-1}]$

must be of rank two, and since $F(T_0, M)=F(G, M)$ near y, we must have that $z\in F(G, M)$. Hence $F(K, M)=F(G, M)$ is an $(n-8)$-cm$_{Z_2}$ near y. But by Borel's formula [1, Chapter XIII, 4.3] and the fact that the proper sub-subgroups of K are pairwise conjugate in G, we see that $8=n-(n-8)$ must be divisible by 3 which is a contradiction.

This completes the proof of our partial classification except for the case in which

$$G \approx \frac{\mathrm{Sp}(1)\times \mathrm{Sp}(1)}{Z_2} \qquad (Z_2 \text{ diagonal})$$

effectively and both S_i are regular, which will take up most of the remainder of this section. We will now restrict our attention, for the time being, to this case, which we shall call case (11) although it will be some time before it is narrowed down to the situation denoted by case (11) in Table 3. G will be assumed to be effective.

LEMMA 5.3. *In case* (11), $F(G, M)=F(T_0, M)=F(c, M)$ *near* (x), *where* c *is the nontrivial central element of* G.

Proof. Since c must preserve orientation on M and since G is effective, $r=\dim_{Z_2} F(c, M)=n-2, n-4, n-6$, or $n-8$. To determine r we may restrict our attention to the neighborhood of a point y near which $F(G, M)=F(T_0, M)$. If $r=n-2$ then $G/c\approx \mathrm{SO}(3)\times\mathrm{SO}(3)$ acts on the n-cm$_Z$ M/c which was shown above to be impossible since the S_i are regular. If $r=n-4$, consider the action of G/c on $F(c, M)$. By Corollary 6.3 we have that the maximum dimension of an orbit is at most three. But the principal isotropy group could not be of maximal rank, since $F(T_0, M)=F(G, M)$ near y. Moreover the principal isotropy group must contain a conjugate of one of the S_i and hence it must be a "diagonal" SO(3) in $G/c\approx\mathrm{SO}(3)\times\mathrm{SO}(3)$, since this is the only such group. Moreover G/c has only two types of orbits on $F(c, M)$, fixed points and those with isotropy SO(3) in the diagonal, since there are no proper subgroups between the diagonal SO(3) and $G/c\approx\mathrm{SO}(3)\times\mathrm{SO}(3)$. This implies that the subgroup $Z_2\oplus Z_2$ of one factor of G/c must act freely outside $F(G, M)$ in $F(c, M)$ contrary to Borel's formula. If $r=n-6$ then Corollary 6.3 implies that the maximum dimension of any orbit of G/c on $F(c, M)$ is one. Thus, in fact, $F(c, M)=F(G, M)$ in this case, a contradiction. But we see in the same way that in the only remaining case $r=n-8$ we must have $F(c, M)=F(G, M)$ (even near x). Since $c\in T_0\subset G$ the conclusion follows.

LEMMA 5.4. *In case* (11), $E=\emptyset$ *near* x.

Proof. Let $z\in E$ and let P be a p-Sylow subgroup of G_z such that $\mathrm{ord}(P)\nmid\mathrm{ord}(H)$. By [1, Chapter VII] there are at most a finite number of nonconjugate P possible, and, since $F(P, M)$ is a cm$_{Z_p}$, it is clear that we may assume $\dim_{Z_p} F(P, M)>\dim F(T_0, M)=n-8$. We may also assume that z is in the component of $F(P, M)$ containing x and that if $z'\in F(P, M)$ is close to z then $G_{z'}\sim G_z$. By Lemma 5.1, $N(P)$ is finite, and it follows that, since $F(P, G/G_z)=N(P)G_z/G_z$, and is of dimension zero, $\dim_{Z_p} F(P, M)^* =\dim_{Z_p} F(P, M)/N(P)=\dim_{Z_p} F(P, M)\geqq n-7$ and hence $\dim_{Z_p} E\geqq n-7 +\dim G/G_z=n-1$, contrary to Theorem 4 of [5]. Thus $E=\emptyset$ near x as claimed.

Let $G=G_1G_2$, $G_i\approx \mathrm{Sp}(1)$, be effective, let q_i be the nontrivial element in center G_i and put $q=q_1q_2$. Then $G\approx(G_1\times G_2)/(e, q)$. Let $T_i=G_i\cap T_0$ and let $s_i\in(N_{G_i}(T_i)-T_i)$. Put $s=s_1s_2$. Then $s^2=s_1^2s_2^2=q_1q_2=q=e$ in G. Clearly $N(S_i) =T_0\cup sT_0$. At least one of the K_i, say K_1, is of dimension one, and therefore $H_1\subset K_1\subset N(S_1)=T_0\cup sT_0$. Now if dim $K_2=3$ then we see that, in order for K_2/H_2 to be a cohomology sphere, H_2 must contain an icosahedral group. Since $H_1\sim H_2$ does not contain such a group we see that both K_i are of dimension one. Since $N(S_i)/(N(S_i)\cap K_i)$ is a circle we must have

$$K_i = N(S_i) \cap K_i = J_i \cup st_iJ_i$$

where $J_i=K_i\cap T_0$ and t_i is some element of T_0 defined up to an element of J_i. Since s is defined only up to an element of T_0 we see easily that s can be so chosen that

$$K_i = J_i \cup sJ_i.$$

Moreover, since $K_i/H_i\approx S^1$, we have $H_i=H_i'\cup sj_iH_i'$, where $H_i' =H_i\cap T_0 \subset J_i$, $J_i/H_i' \approx S^1$ and $j_i\in J_i$. Now all the elements st, $t\in T_0$ are conjugate in G. Hence considering the group $K\approx Z_2\oplus Z_2\subset \mathrm{SO}(3)\subset G$ (diagonally), we have that each $e\neq h\in K$ is conjugate to $sj_i\in H_i$. Thus for $e\neq h\in K$ we have

$$\dim_{Z_2} F(h, M) = n - 6 + \dim F(sj_i, G/H_i) = n - 6 + \dim N(sj_i) = n - 4.$$

Thus by Borel's formula, if $r=\dim_{Z_2} F(K, M)$, we have

$$n - r = 3((n - 4) - r) = 3n - 3r - 12$$

and hence $r=n-6$. Now, since $N(K)$ is finite, it follows easily that $F(K, G/G_z)$ is finite for any z. Thus it follows that $\dim_{Z_2}(F(K, M))^*=\dim_{Z_2} F(K, M) =n-6=\dim M/G$ and hence $F(K, M)$ must touch principal orbits. Therefore we may assume that $K\subset H$.

It is easy to see from the structure of H known so far that, since $H_1\sim H_2$ in G, $H_1\sim H_2$ in $N(T_0)$. The verification of this will be left to the reader. Since the only requirements on the S_i are that they lie in T_0 we may, therefore, assume that $H_1=H=H_2$.

LEMMA 5.5. *In case* (11), *if* $z\in F(S_i, M)-F(T_0, M)$, *then either* $G_z\sim \mathrm{SO}(3)$ (*the "diagonal" of* G) *or* $G_z\sim K_i$.

Proof. Let $i=1$. G_z is of rank one and it is easy to see that if dim $G_z=3$ then, since S_1 is regular, G_z must be "diagonal" and it is also clear that such groups must be isomorphic to SO(3) since they cannot contain center (G) by Lemma 5.3. Assume dim $G_z=1$, so that $G_z^0=S_1$ and $G_z\subset N(S_1)$. By the CDT applied to the action of $N(S_1)$ on $F(S_1, M)$ we see that

$$G_z = N(S_1) \cap G_z \sim N(S_1) \cap K_1 = K_1$$

as was to be shown.

Let C be the set of points z for which dim $G_z=3$. Let $z\in C$ and let U be a slice at z, dim $U=n-3$. If $\dim_Z F(G_z, U)=n-7$, then by the CDT we would have that G_z is locally constant near z on B and this has been shown to be impossible for dim $G_z=3$ in the present case (that is G_z could not be a K_i). Thus $\dim_Z F(G_z, U)\leqq n-8$ and also we must have $\dim_Z C\leqq n-5$. Note that if $G_z\supset K$ for some $z\in C$ then, since there are only a finite number of subgroups of G isomorphic to SO(3) and containing K, we see that, near z, $F(K, M)\cap C$ is contained in a slice at z. Hence $\dim_Z(F(K, M)\cap C)\leqq n-8$.

It is easily seen that $F(H, K_i/H)$ consists of two points and it follows from the CDT that near x (with $F=F(G, M)=F(T_0, M)$ near x), $F(H, M-F-C)$ is an $(n-6)$-cm_Z. Also $F(K, M)$ is an $(n-6)$-cm over Z_2 and it follows that $F(K, M)-F-C=F(H, M)-F-C$ since the former is connected. Thus, in particular, $F(K, G/H)=F(H, G/H)$.

Say that $K'\sim K$ in G and $K'\subset H$. Let P, P' be 2-Sylow subgroups of H, $P\supset K$, $P'\supset K'$. Then there is an element $h\in H$ such that $hPh^{-1}=P'$. However, since center $(G)\not\subset H$ we see that K is precisely the maximal abelian subgroup of P with elements all of order two. It follows that $hKh^{-1}=K'$, that is, $K\sim K'$ in H. Thus we can compute

$$F(K,G/H) = \frac{N(K)H}{H} \quad \text{and} \quad F(H,G/H) = \frac{N(H)}{H} \cdot$$

But $N(K)H\supset N(H)$ so that $N(K)H=N(H)$ and hence $N(K)\subset N(H)$. But there is an element $a\in N(K)$ such that $T_0\cap aT_0a^{-1}=$center (G). Noting that any element of H of order greater than two must be in T_0, it follows that H contains only elements of order two, and since center $(G)\not\subset H$ it follows from the next lemma that

$$H = K \approx Z_2 \oplus Z_2.$$

LEMMA 5.6. *The only subgroup of G containing K, not containing center (G) and all of whose elements have order two is K itself.*

Proof. Let K' satisfy the hypotheses. Since center $G\not\subset K'$ the map $f\colon G\to G/$center $G\approx L_1\times L_2=L$ (which defines the L's), $L_i\approx$SO(3) is an isomorphism on K'. We also know that $K'\cap G_i=(e)$ and it follows easily that that $f(K')\cap L_i=(e)$ also. Thus the map $f'\colon L\to L_1$ is also an isomorphism on $f(K')$ and it follows from known facts about SO(3) that K' is either $\approx Z_2$ or $\approx Z_2\oplus Z_2$. Since $K'\supset K\approx Z_2\oplus Z_2$ we must have $K'=K$ as claimed.

REMARK. It can be shown that if the hypothesis $K' \supset K$ is removed from Lemma 5.6 then the group K' is conjugate to a subgroup of K.

Note that we now have that case (11) is as shown in Table 3, since the fact that $K_i = S_iH$ follows from the fact that $K_i/H \approx S^1$. The fact that $K_i \approx O(2)$ follows from the fact that $S_i \cap H = T_0 \cap H$. (If $S_i \cap H \neq T_0 \cap H$ then $K_i \cap T_0$ contains $Z_2 \oplus Z_2 \subset T_0$ which contains c contrary to Lemma 5.3.)

LEMMA 5.7. *In case* (11), *the natural map* $F(H, M)/N(H) \to M/G$ *is a homeomorphism near* x.

Proof. This is equivalent to $F(H, G/G_z) = N(H)G_z/G_z$ for any $G_z \supset H$, and this would follow if we knew that whenever $H' \sim H$ in G, $H' \subset G_z$ then $H' \sim H$ in G_z. To see this we may assume that rank $(G_z) = 1$, for otherwise $G_z = H$ or G. In this case we know that either $G_z \sim K_i \approx O(2)$, or $G_z \approx SO(3)$ by Lemma 5.5. But in these cases the result is clear since all groups isomorphic to $Z_2 \oplus Z_2$ are conjugate in these subgroups of G.

We shall now show that there is a cross-section in $F(H, M)$ for the fibre bundle

$$M - B \xrightarrow{G/H} \frac{M - B}{G}\,.$$

Note that $N(H)/H \approx Z_2 \oplus \sigma_3$. Let $J \subset N(H)/H$, $J \approx Z_3$, and $K = (N(H)/H)/J \approx Z_2 \oplus Z_2$ (not to be confused with our previous use of the letter K). Consider the orbit maps

$$F(H, M) \xrightarrow{J} \frac{F(H, M)}{J} \xrightarrow{K} \frac{F(H, M)}{N(H)} \approx M/G.$$

We know, by [1, Chapter III, 2.3], that $F(H, M)/J$ is an $(n-6)$-cm$_{Z_2}$. Also, since K acts effectively on $F(H, G/H)/J$ and hence on $F(H, M)/J$, we must have, by Borel's formula, that $\dim_{Z_2}(F(K, F(H, M)/J)) = n-8$ and that there is an element $g \in K$ with a fixed point set of dimension $n-7$. Lemma 2.5 applied to the action of K/g on $(F(H, M)/J)/g$ yields that M/G is an $(n-6)$-cm$_{Z_2}$ with boundary consisting of the exceptional orbits of the above action of K.

Note that if $G_z \supset H$ is an isotropy group of a singular orbit of G on M, then $((G_z \cap N(H))/H)/J$ is not trivial so that we see that the boundary of $M/G \approx F(H, M)/N(H)$ is precisely B^*. In particular, $B^* = B/G$ is an $(n-7)$-cm over Z_2.

Lemma 2.5 also implies that $F(H, M-B)/J$ is the union of four disjoint copies of $F(H, M-B)/N(H) \approx (M-B)/G$. Since $(M-B)/G$ is an $(n-6)$-cm over Z we have only to show (by 6.5) that it is orientable over Z_3 and that $H_c^{n-7}((M-B)/G, Z_3) = {}_{Lx} 0$, in order to conclude that the map $F(H, M-B) \to (M-B)/G$ has a cross-section near x.

Let $F=F(G, M)$ and $B=F\cup C\cup D_1\cup D_2$, where D_i consists of points with isotropy groups conjugate to K_i. We know dim $F\cup C\leqq n-5$ and G/K_i is easily seen to be nonorientable (since conjugation by an element of K_i-S_i reverses the orientation of G/S_i). Thus for the spectral sequence with coefficients in Z_3 of the fibering $D_i\to D_i^*$ we have

$$E_2^{n-7,5} \approx H_c^{n-7}(D_i^*, H^5(G/K_i, Z_3)) = 0$$

since the coefficients are trivial. Thus $H_c^{n-2}(D_i, Z_3)=E_\infty^{n-7,5}=0$, and consequently $H_c^{n-2}(B, Z_3)=0$. This implies that $H_c^{n-1}(M-B, Z_3)={}_{Lz}\,0$. But for the spectral sequence (coefficients in Z_3) of $M-B\to(M-B)/G$ we have (by [1, Chapter XV, Lemma 6.2])

$$0 ={}_{Lz} H_c^{n-1}(M - B, Z_3) \approx E_\infty^{n-7,6} \approx E_2^{n-7,6} \approx H_c^{n-7}\left(\frac{M - B}{G}, Z_3\right)$$

since $H^6(G/H, Z_3)\approx H_1(G/H, Z_3)=0$. Moreover,

$$H_c^{n-6}\left(\frac{M - B}{G}, Z_3\right) \approx E_2^{n-6,6} \approx E_\infty^{n-6,6} \approx H_c^n(M - B, Z_3) = Z_3$$

so that $(M-B)/G$ is orientable as was to be shown.

We will now investigate the structure of C in order to show eventually that $C=\varnothing$. Let $z\in C$ and assume, as we may, that $K_1\subset G_z$. We will first show that $\dim_L(C, z)=n-5$ for any L. Let U be a slice at z, dim $U=n-3$. Then, since $\dim(S_1, G_z/K_1)=0$ we have that $\dim_Z F(S_1, U)=n-7$. Note that it will suffice to show $\dim_{Z_2} F(G_z, U)=n-8$.

There are exactly three types of orbits of G_z on U, namely fixed points, principal orbits (isotropy group $H\approx Z_2\oplus Z_2$), and projective planes (isotropy group $N_{G_z}(S_1)$). We will use the notation $G'=G_z$, $B'=B\cap U$, $F'=F(G', U)$ $=C\cap U$. Note that $F(S_1, U)=F(N_{G'}(S_1), U)$ so that $B'/G'\approx F(S_1, U)$ is an $(n-7)$-cm over Z and also the set $B'-F'$ of points on projective plane orbits has a cross-section $F(S_1, B'-F')$. As seen above there is also a cross-section in $F(H, M-B)$ for the principal orbits of G and it follows that there is a cross-section in $F(H, U-B')$ for the principal orbits of G' on U so that

$$U - B' \approx \frac{U - B'}{G} \times G'/H; \qquad B' - F' \approx \frac{B' - F'}{G'} \times G'/N_{G'}(S_1).$$

Now say that $\dim_{Z_2}(F')<n-8$ near z. Then F' cannot separate the $(n-7)$-cm_{Z_2} B'/G' locally. Thus $H_c^{n-7}((B'-F')/G', Z_2)={}_{Lz}Z_2$. Considering the Leray spectral sequence of the map $B'\to B'/G'$ we see that $E_2^{n-7,2}$ $\approx H_c^{n-7}(B'/G', \mathcal{H}^2(G'/G'_y, Z_2)) = H_c^{n-7}((B' - F')/G', H^2(P^2, Z_2))$ $=H_c^{n-7}((B'-F')/G', Z_2)={}_{Lz}Z_2$. Thus $H_c^{n-5}(B', Z_2)={}_{Lz}Z_2$ which implies that $Z_2={}_{Lz}H_c^{n-4}(U-B', Z_2)={}_{Lz}H_c^{n-6}((U-B')/G', Z_2)\otimes H_c^2(G'/H, Z_2) \approx Z_2\oplus Z_2$

and this contradiction implies that we must have had $\dim_{Z_2}(F', z) = n-8$ as claimed.

It now follows from Case II applied to the action of G' on U that $F' = F(G_z, U)$ is an $(n-8)$-cm_{Z_2}. Since C is locally a product of $F(G_z, U)$ by a 3-cell, we have that C is an $(n-5)$-cm over Z_2. Also C/G is an $(n-8)$-cm over Z_2.

Consider $H_c^{n-7}((B-F-C)/G, Z_2)$, whose rank is the number of components of the $(n-7)$-cm$(B-F-C)/G$. We have $H_c^{n-7}((B-F-C)/G, Z_2) \approx H_c^{n-2}(B-X-C, Z_2) \approx H_c^{n-2}(B, Z_2) = {}_{Lx}H_c^{n-1}(M-B, Z_2) = {}_{Lx}H^5(G/H, Z_2) \approx H_1(G/H, Z_2) \approx Z_2 \oplus Z_2$, since $M-B \approx (M-B)/G \times G/H$ and M/G is an $(n-6)$-cm_{Z_2} with boundary B/G.

Moreover we know that $H_c^{n-8}((B-F)/G, Z_2) = {}_{Lx}0$ and $H_c^{n-7}((B-F)/G, Z_2) = {}_{Lx}Z_2 \oplus Z_2$, since

$$\frac{B-F}{G} \approx \bigcup_i \frac{F(S_i, M) - F(T_0, M)}{N(S_i)}.$$

Thus there are neighborhoods $V_1 \subset V_2 \subset V_3$ of x such that, with $B_i = B \cap V_i$, $C_i = C \cap V_i$, we have that in the diagram (coefficients in Z_2)

$$\begin{array}{ccccccccc}
H_c^{n-8}\left(\dfrac{B_1-F}{G}\right) & \xrightarrow{j_i} & H_c^{n-8}\left(\dfrac{C_1}{G}\right) & \xrightarrow{d_1} & H_c^{n-7}\left(\dfrac{B_1-F-C}{G}\right) & \xrightarrow{i_1} & H_c^{n-7}\left(\dfrac{B_1-F}{G}\right) & \to & 0 \\
\downarrow h_1' & & \downarrow f_1 & & \downarrow g_1 & & \downarrow h_1 & & \\
H_c^{n-8}\left(\dfrac{B_2-F}{G}\right) & \xrightarrow{j_2} & H_c^{n-8}\left(\dfrac{C_2}{G}\right) & \xrightarrow{d_2} & H_c^{n-7}\left(\dfrac{B_2-F-C}{G}\right) & \xrightarrow{i_2} & H_c^{n-7}\left(\dfrac{B_2-F}{G}\right) & \to & 0 \\
\downarrow h_2' & & \downarrow f_2 & & \downarrow g_2 & & \downarrow h_2 & & \\
H_c^{n-8}\left(\dfrac{B_3-F}{G}\right) & \xrightarrow{j_3} & H_c^{n-8}\left(\dfrac{C_3}{G}\right) & \xrightarrow{d_3} & H_c^{n-7}\left(\dfrac{B_3-F-C}{G}\right) & \xrightarrow{i_3} & H_c^{n-7}\left(\dfrac{B_2-F}{G}\right) & \to & 0
\end{array}$$

$h_1' = 0 = h_2'$, $\mathrm{Im}\, g_1 \approx Z_2 \oplus Z_2$ and is mapped isomorphically by g_2 onto $\mathrm{Im}\, g_2$, and similarly with h_1 and h_2.

Thus $Z_2 \oplus Z_2 \approx \mathrm{Im}\, h_1 = \mathrm{Im}\, h_1 i_1 = \mathrm{Im}\, i_2 g_1$ and, since $\mathrm{Im}\, g_1 \approx Z_2 \oplus Z_2$, i_2 must map $\mathrm{Im}\, g_1$ isomorphically. Hence if $\alpha \in H_c^{n-8}(C_1/G)$ we have that $i_2 g_1 d_1(\alpha) = i_2 d_2 f_1(\alpha) = 0$ implies that $0 = g_1 d_1(\alpha) = d_2 f_1(\alpha)$. Thus $f_1(\alpha) = j_2(\beta)$ for some β and $f_2 f_1(\alpha) = f_2 j_2(\beta) = j_3 h_2'(\beta) = 0$ since $h_2' = 0$. Thus $H_c^{n-8}(C/G, Z_2) = {}_{Lx}0$, and, since C/G is an $(n-8)$-cm over Z_2, it follows that $C = \emptyset$ near x as was to be shown.

We shall now finish the proof of Case III. Using Lemma 5.1 we see that $E = \emptyset$ near x in cases (1), (2), (3), (4), (5), (8), and (9). This has also been seen in case (11). This also follows in case (6) since there are no groups contained properly between I and SO(3), and using this fact together with Lemma 5.1 we see that this is true in case (7). In case (10) we see that any subgroup properly containing H must intersect a factor G_i of G and using Lemma 5.1 we see that $E=\emptyset$ near x in this case also, so that this is true in general.

In case (11) we have seen that the principal orbits have a cross-section in $F(H, M)$. In the other cases, except for (3) and (10) we see that $N(H)/H$ is connected and hence acts trivially on $H^*(G/H)$. In case (10), $N(H)/H\approx Z_2$ and is represented by the central element of G, so that, as may easily be seen, its action on $H^*(G/H)$ is trivial. Considering case (3), we note that $G/H \approx SO(3)\times_{Z_2}S^1$ (in the notation of [1, Chapter IV, 1.3]) is an SO(3)-bundle over S^1 and hence the groups $H^i(G/H, Z)$ are Z, Z, Z_2, $(Z\oplus Z_2)$, and Z, for $i=0$, 1, 2, 3, and 4 respectively. Hence $N(H)$ acts trivially on this if and only if it does so on $H^i(G/H, Q)$ (Q=rationals). However $\pi^*: H^*(G/H)\to H^*(G)$ is a rational isomorphism and the action of $N(H)$ on $H^*(G)$ is trivial, since it is just right translation. Thus, in general, the structural group of $M-B \to(M-B)/G$ acts trivially on $H^*(G/H, Z)$.

We have that $F(T_0, M)=F(G, M)$ in all cases except possibly (3), since if $z\in F(T_0, M)$ then $G_z\supset K_1\cup K_2$ (up to conjugation of the K_i) and the only such group is G in these cases (except (11) for which this has already been shown). In case (3), $F(G_2, M)=F(T_0, M)$ by Lemma 5.2, so that G_1 acts on the cm$_Z$ $F(G_2, M)$ with an open set of fixed points. Thus $F(G, M) =F(G_1, F(G_2, M))=F(G_2, M)=F(T_0, M)$ in this case also.

Moreover, Corollary 2.4 implies easily that if $G_z\supset K_i$, $\dim G_z=\dim K_i$, then $G_z=K_i$. But then if $G_z \not\sim G$ or H then rank $(G_z)=1$ and, using special facts already shown in cases (3) and (11), it follows that G_z must be conjugate to one of the K_i. Thus there are exactly four orbit types near x.

Since B/G is the union

$$\frac{F(S_1, M)}{N(S_1)}\cup\frac{F(S_2, M)}{N(S_2)}$$

of two $(n-k-1)$-cms with common boundary $F(G, M)=F(T_0, M)$, an $(n-k-2)$-cm over Z, it follows that B/G is an $(n-k-1)$-cm$_Z$. Considering the maps

$$F(K_i, M) - F(G, M)\to\frac{F(K_i, M) - F(G, M)}{N(K_i)/K_i}\approx\frac{F(S_i, M) - F(G, M)}{N(S_i)}$$

we obtain, by using Theorems 6.4 and 6.5, that these maps have cross-sections. That is, we note that in all cases $(N(K_i)/K_i)/(N(K_i)/K_i)^0$ is solvable and we apply 6.5 to the map $F(K_i, M-F)/(N(K_i)/K_i)^0 \to F(K_i, M-F)/N(K_i)/K_i$ and then apply 6.4 to the map $F(K_i, M-F) \to F(K_i, M-F)/(N(K_i)/K_i)^0$ (or rather to the part over a component of the latter space).

Letting $B-F=D_1\cup D_2$ where $D_1=\{z\mid G_z\sim K_i\}$ we now have that $D_i\approx D_i/G\times G/K_i$ and $D_i/G\approx F(S_i, M)/N(S_i)-F(G, M)$. Now, considering the Leray spectral sequence of the map $B\rightarrow B/G$, we see that $E_2^{p,0} \approx H_c^p(B/G, Z) =_{Lx} 0$ for $p\leqq n-k-2$. Also for $q>0$, $E_2^{p,q}\approx H_c^p(D_1/G, H^q(G/K_1)) \oplus H_c^p(D_2/G, H^q(G/K_2)) =_{Lx} 0$ for $p\leqq n-k-2$, since the coefficients are constant.

It follows that $H_c^i(B, Z) =_{Lx} 0$ for $i\leqq n-k-2$ and hence also $H_c^i(M-B, Z) =_{Lx} 0$ for $i\leqq n-k-1$. Since, as seen above, the structural group of the fibering $M-B\rightarrow(M-B)/G$ acts trivially on $H^*(G/H, Z)$ we have by Theorem 6.6 that $H_c^i((M-B)/G, Z) =_{Lx} 0$ for $i\leqq n-k-1$. Also $H_c^{n-k}((M-B)/G, Z) = E_2^{n-k,k} = E_\infty^{n-k,k} = H_c^n(M-B, Z)\approx Z$. Moreover $M/G-F$ is an $(n-k)$-cm$_Z$ with boundary $B/G-F$ by the CDT and thus, since B/G is an $(n-k-1)$-cm$_Z$, we have by Lemma 2.6 that M/G is an $(n-k)$-cm$_Z$ with boundary B/G, which completes the proof of Case III.

6. **Appendix.** We shall give here the statements of some theorems for which proofs appear (essentially) in the literature but the existing statements of which are not sufficiently general for our purposes. The notation is as in the present paper.

THEOREM 6.1 (MONTGOMERY [1, Chapter IX, 2.1]; YANG [7, Theorem 2]). *Let G be a compact Lie group acting on an n-cm M over K_p (where K_p denotes a field of characteristic p, possibly zero). Then* $\dim_{K_p} B\leqq n-2$.

THEOREM 6.2 (MONTGOMERY [1, Chapter IX, 2.2]; YANG [7, Theorem 2]). *Let G be a compact Lie group acting on an n-cm M over K_p, and let t be an integer $0\leqq t<k$. Then the union of all orbits of dimension $\leqq t$ is a closed set of dimension $\leqq n-k+t-1$ over K_p.*

COROLLARY 6.3 (MONTGOMERY [1, Chapter IX, 2.2 Corollary]). *Let G and M be as above, then* $\dim_{K_p} M/G=n-k$ *(near any point) and* $\dim_{K_p} B/G \leqq n-k-1$.

These three results are essentially the same as those cited except that K_p replaces Z as coefficients. The proofs are almost the same, with K_p in place of Z, and, in the proof of [1, Chapter IX, 2.1], the group H should be taken to be a circle group such that $H\cap G_y$ is finite, the reference to Chapter V, 2.6 being replaced by a reference to [1, Chapter V, 3.2]. The equality $\dim_{K_p} M/G =n-k$ follows from the fact that locally near some orbit of dimension k, M is the product of the orbit with M/G. (Hence $\dim_{K_p} M/G\geqq n-k$ and the reverse inequality follows from the proof of [1, Chapter IX, 2.2].)

THEOREM 6.4. *Let G be a compact connected Lie group acting on a locally separable, locally compact space M of finite covering dimension. Assume that G acts freely outside the closed set A, that $(M-A)/G$ is an m-cm over Z, and that*

$$H_c^i\left(\frac{M-A}{G}, Z\right) =_{Lx^*} \begin{cases} Z, & i=m, \\ 0, & i\neq m, \end{cases} \qquad \text{for some } x\in A.$$

Then there is a local cross-section for the orbits of G on $M-A$ near x.

This follows from the proof of [1, Chapter XV, 3.3].

THEOREM 6.5. *Let G be a finite solvable group acting on a space X freely outside the closed set $A \subset X$. Say that $x^* \in A^*$ has a fundamental system of neighborhoods Y_α such that each $Y_\alpha - A/G$ is connected. Assume further that $(X - A)/G$ is an orientable n-cm over Z_p for all $p \mid \mathrm{ord}(G)$ and that $H_c^{n-1}((X-A)/G, Z_p) = {}_{Lx}0$ for all $p \mid \mathrm{ord}(G)$. Then there is a local cross-section near x for the orbits of G on $X - A$.*

This follows from the proof of [1, Chapter XV, 3.4].

THEOREM 6.6. *Let G be a compact connected Lie group acting on a space X and let A be a closed invariant subset of X such that the orbits of G in $X - A$ are all of the same type G/H. Let L be a principal ideal ring and suppose that the structural group of the fibering*

$$X - A \xrightarrow{G/H} \frac{X - A}{G}$$

acts trivially on $H^*(G/H, L)$. Let $x \in F(G, X) \subset A$ and say that $H_c^i(X - A, L) = {}_{Lx}0$ for all $i \leqq i_0$. Then also $H_c^i((X-A)/G, L) = {}_{Lx^*}0$ for all $i \leqq i_0$.

This essentially follows from the proof of [1, Chapter XV, 6.3].

REFERENCES

1. A. Borel, *Seminar on transformation groups*, Annals of Mathematics Studies, no. 46, 1960.

2. G. E. Bredon, *Orientation in generalized manifolds and applications to the theory of transformation groups*, Michigan Math. J. vol. 7 (1960) pp. 35–64.

3. ———, *Some theorems on transformation groups*, Ann. of Math. vol. 67 (1958) pp. 104–118.

4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1956.

5. D. Montgomery and C. T. Yang, *Orbits of highest dimension*, Trans. Amer. Math. Soc. vol. 87 (1958) pp. 284–293.

6. R. G. Swan, *Groups with periodic cohomology*, Bull. Amer. Math. Soc. vol. 65 (1959) pp. 368–370.

7. C. T. Yang, *Transformation groups on a homological manifold*, Trans. Amer. Math. Soc. vol. 87 (1958) pp. 261–283.

8. H. J. Zassenhaus, *The theory of groups*, New York, Chelsea Publishing Co., 1958.

INSTITUTE FOR ADVANCED STUDY,
PRINCETON, NEW JERSEY