ON THE CHARACTERIZATION OF LINEAR AND PROJECTIVE LINEAR GROUPS. I

BY

JOHN H. WALTER

I. Introduction and preliminaries

1. Introduction. In the Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, R. Brauer raised the following question: "Given a group $\mathfrak{N}$ containing an involution J in its center. What are the groups $\mathfrak{G}$ containing $\mathfrak{N}$ as a subgroup such that $\mathfrak{N}$ is the normalizer of J in $\mathfrak{G}$?" He went on to propose essentially the study of the characterization of certain finite groups by primarily specifying the centralizers of their involutions.

In [4] Brauer (followed by Suzuki in [12]) initiated this program with regard to the projective unimodular groups(1) $PSL(3, q)$. The centralizers of the involutions are described in terms of the groups $GL(2, q)$ by Brauer and in terms of related groups by Suzuki. We will continue this program by characterizing the groups $PGL(n, q)$ and its subgroups which contain $PSL(n, q)$ as subgroups of index at least 1 or 2 according as there is or is not an element of determinant -1 in the center of $GL(n, q)$; we restrict $n \geqq 4$ and q to be odd. By assuming $n \geqq 5$, we will obtain this result for the corresponding groups determined over division rings of characteristic not 2. In the study of this more general case, we will use the notation $GL(n)$, $SL(n)$, etc. and omit notational reference to the underlying division ring.

This result will be found actually in a second paper to follow this one. Here we investigate the characterization of the subgroups of $GL(n)$ which contain the subgroup $TL(n)$ of elements of determinant ± 1. Our results for these groups are not as definitive as those we will present for the corresponding subgroups of $PGL(n)$. We will show that the factor groups of the groups that we consider by their centers are isomorphic to certain subgroups of $PGL(n)$. There remains the problem of identifying, for example, that central extension of the group $PGL(n)$ which is the group $GL(n)$. In the case that some of the groups that we consider are direct factors in the centralizers of

Presented to the Society, January 28, 1960 under the title *Characterization of certain finite linear groups;* received by the editors August 19, 1960.

(1) In general, $GL(n)$ denotes the group of all linear transformations acting on a vector space of dimension n over a division ring K; $SL(n)$ denotes the group of linear transformations of determinant 1. By $PGl(n)$ and $PSL(n)$, we mean the corresponding groups of tranformations induced on the projective geometry formed of the subspaces of the given vector space.

In case K is a finite field of q elements, we denote the corresponding groups by $GL(n, q)$, $SL(n, q)$, $PGL(n, q)$, and $PSL(n, q)$.

involutions belonging to groups of the same type (cf. Corollary 2), we obtain this identification. The groups we consider here play an important part in the description of the centralizers of the involutions of the groups that we will characterize in the sequel to this paper.

The main assumption used in characterizing the groups that we consider is an inductive description of the centralizers of their involutions. We also assume certain other conditions which describe the factor commutator subgroup and the center of the commutator subgroup. Along with Brauer [4], we assume that the commutator subgroup is perfect. The results for infinite groups require an artificial assumption (Condition E), which is used at only one point in the proof (cf. Proposition 10.1). However, no extra work is involved in including this case, and we feel that it is interesting to record the status of our characterization in the infinite case.

2. **Notation. Principal definition and theorems.** Let $\mathfrak{G}$ be a group. If $\mathfrak{G}$ together with a set $\mathcal{S}$ is contained in some group, we designate by $C_{\mathfrak{G}}(\mathcal{S})$ the centralizer of $\mathcal{S}$ in $\mathfrak{G}$. By $Z(\mathfrak{G})$ we mean the center of $\mathfrak{G}$. We denote by $P(\mathfrak{G})$ the factor group $\mathfrak{G}/Z(\mathfrak{G})$ and by $D(\mathfrak{G})$ the commutator subgroup of $\mathfrak{G}$. Let $T(\mathfrak{G})$ be the subgroup generated by the involutions in $\mathfrak{G}$. We will call this the *involutory subgroup* of $\mathfrak{G}$; clearly it is a normal subgroup. When it is clear from the context that we are referring to the centralizer in a given group $\mathfrak{G}$, we will write $C(\mathcal{S})$ for $C_{\mathfrak{G}}(\mathcal{S})$. For $T(C(\mathcal{S}))$, we write $C^*(\mathcal{S})$ and call this subgroup the *involutory centralizer* of $\mathcal{S}$. Whenever there is a unique element of order 2 in the center of a group, it will be designated by -1; $-G=(-1)G$ for an element G in the same group. By K we mean a fixed division ring, and by K^* its multiplicative group.

A direct product $\mathfrak{P}$ of two groups $\mathfrak{A}$ and $\mathfrak{B}$ will be denoted by $\mathfrak{A}\times\mathfrak{B}$. If $C\in\mathfrak{A}\times\mathfrak{B}$ has the form AB where $A\in\mathfrak{A}$ and $B\in\mathfrak{B}$,we will write $C=A\times B$ and call this the *representation* of G relative to the decomposition $\mathfrak{P}=\mathfrak{A}\times\mathfrak{B}$. With this convention, there will be no need to identify A as a member of $\mathfrak{A}$ or B as a member of $\mathfrak{B}$.

Fix K to have characteristic not 2. Let $TL(n)$ denote the subgroup of $GL(n)$ consisting of elements of determinant ± 1. This is the involutory subgroup $T(GL(n))$ of $GL(n)$. Of course, $D(GL(n))$ is the group $SL(n)$ of elements of determinant 1. Also $[TL(n):SL(n)]=2$. We have $PGL(n)=P(GL(n))$, $PSL(n)=P(SL(n))$, and we set $PTL(n)=P(TL(n))$. Then $[PTL(n):PSL(n)]$ is 1 or 2 according as there is an element of determinant -1 in the center of $GL(n)$ or not.

A *linear group of rank n* is a group $\mathfrak{G}$ such that

$$TL(n)\subseteq\mathfrak{G}\subseteq GL(n).$$

A *projective linear group of rank n* is a group G such that

$$PTL(n)\subseteq\mathfrak{G}\subseteq PGL(n).$$

We will now define inductively a *quasilinear group G of rank n.* If $n=0$, $\mathfrak{G}$ is to be the identity group. If $n=1$, $\mathfrak{G}$ is to be a subgroup of K^* containing the element -1. Thus $GL(1)\supseteq\mathfrak{G}\supseteq TL(1)$. If $n=2$, we require that $\mathfrak{G}$ be linear of rank 2. These groups may be characterized group-theoretically when they are finite using the results of Brauer, Suzuki, and Wall [6]. For $n=\text{rk}\ \mathfrak{G}\geqq 3$, a group $\mathfrak{G}$ is said to be quasilinear if the following conditions are satisfied.

CONDITION A. *There are exactly $n+1$ classes $\mathfrak{K}_0, \mathfrak{K}_1, \cdots, \mathfrak{K}_n$ of conjugate involutions*(2) *such that $\mathfrak{K}_0$ and $\mathfrak{K}_n$ contain elements of the center of G. If $U\in\mathfrak{K}_p$, $0<p<n$, then $C(U)$ is a subgroup of a direct product $\mathfrak{U}_1\times\mathfrak{U}_2$ of quasilinear groups of ranks p and $n-p$, respectively, which contains the involutory subgroup $T(\mathfrak{U}_1\times\mathfrak{U}_2)=T(\mathfrak{U}_1)\times T(\mathfrak{U}_2)$ as the involutory centralizer of U. Furthermore, assume that the decomposition $C^*(U)=T(\mathfrak{U}_1)\times T(\mathfrak{U}_2)$ may be taken so that neither $T(\mathfrak{U}_1)$ nor $T(\mathfrak{U}_2)$ contains the involution -1 of the center of $\mathfrak{G}$.*

We will always choose $\mathfrak{K}_0$ so that $1\in\mathfrak{K}_0$; then -1 belongs to $\mathfrak{K}_n$ and is the unique element of order 2 in $Z(\mathfrak{G})$. When we consider a quasilinear group $\mathfrak{H}$ other than the particular group G for which we will be proving the Principal Theorem, we will designate its classes by $\mathfrak{K}_0(\mathfrak{H})$, $\mathfrak{K}_1(\mathfrak{H})$, etc.

CONDITION B. *The commutator subgroup $D(\mathfrak{G})$ is perfect. The center $Z(D(\mathfrak{G}))$ is isomorphic to the multiplicative group of the nth-roots of unity in the center of the division ring K.*

Thus $Z(D(\mathfrak{G}))$ is cyclic of order dividing n and contains an element of order 2, which must be -1, if and only if n is even.

CONDITION C. *There exists an involution*(3) *in G which is not in $D(\mathfrak{G})$. The group $\mathfrak{G}/D(\mathfrak{G})$ is isomorphic to a subgroup of $K^*/D(K^*)$.*

Thus there is only one element of order 2 in $G/D(G)$.

CONDITION D. $Z(T(\mathfrak{G}))=Z(\mathfrak{G})\cap T(\mathfrak{G})$.

This condition is superfluous when $\mathfrak{G}$ is involutory.

CONDITION E. *When G is infinite, let $U\in\mathfrak{K}_1$ or $U\in\mathfrak{K}_{n-1}$, and let V be in $\mathfrak{K}_p$, $1<p<n-1$. Then there exists an involution $W\neq\pm 1$ in $C(U, V)$.*

Furthermore, if $n=3$ and also if $n=4$ and $\mathfrak{G}$ is infinite, we assume that $P(\mathfrak{G})$ is a projective linear group.

When $\mathfrak{G}$ is finite and $n=3$, results of Brauer [4] will be applicable for the characterization of $P(\mathfrak{G})$.

It follows directly from standard results in the theory of the classical groups that a linear group of rank $n\geqq 3$ is quasilinear (cf. Dieudonné [8, Chapter I, §3 and Chapter II, §1]). Even in the case where $n=2$, Conditions A, C, D, and E hold. Condition B is valid except for the case where $K=F_3$, the field of three elements. Then $D(\mathfrak{G})$ is not perfect; however, the second statement of Condition B still holds.

We define inductively a *full quasilinear group* of rank n by taking $\mathfrak{G}=K^*$

(2) By an *involution*, we mean an element G such that $G^2=1$.

(3) In the cases where n is odd, it follows from Condition B that the involution $-1\in\mathfrak{K}_n$ is not in $D(\mathfrak{G})$ because this involution would be in the center $Z(D(\mathfrak{G}))$ if it were in $D(\mathfrak{G})$.

if $n=1$ and $\mathfrak{G}=GL(n)$ if $n=2$. For $n\geqq 3$, we assume that for $U\in\mathfrak{K}_p$, $C(U)$ is the direct product $\mathfrak{U}_1\times\mathfrak{U}_2$ of full quasilinear groups of ranks p and $n-p$, respectively. Also if $n=3$ and, when $\mathfrak{G}$ is infinite, if $n=4$, we assume that $P(\mathfrak{G})$ is isomorphic to $PGL(n)$. An *involutory quasilinear group* is one for which $T(\mathfrak{G})=\mathfrak{G}$.

PRINCIPAL THEOREM. *Let $\mathfrak{G}$ be a quasilinear group of rank $n\geqq 4$ if $\mathfrak{G}$ is finite or of rank $n\geqq 5$ if $\mathfrak{G}$ is infinite. Then $P(\mathfrak{G})$ is a projective linear group of rank n. The involutory subgroup $T(\mathfrak{G})$ of $\mathfrak{G}$ is a quasilinear group of rank n and $PT(\mathfrak{G})$ is isomorphic to $PTL(n)$.*

COROLLARY 1. *If $\mathfrak{G}$ is a finite full quasilinear group of rank $n\geqq 4$ and if $Z(\mathfrak{G})$ is isomorphic to K^*, then $P(\mathfrak{G})$ is isomorphic to $PGL(n)$.*

A quasilinear group $\mathfrak{G}$ of rank n is said to be *imbeddable* in a quasilinear group $\mathfrak{G}'$ of rank $n+1$ if $\mathfrak{G}$ is a direct factor in the centralizer of an involution of $\mathfrak{G}'$.

COROLLARY 2. *If $\mathfrak{G}$ is an imbeddable involutory quasilinear group of rank n, it is isomorphic to $TL(n)$.*

For the remainder of this paper, $\mathfrak{G}$ will denote a fixed quasilinear group of rank $n\geqq 4$ for which we will prove the Principal Theorem. Because we will establish this theorem by induction on n, we assume that if $\mathfrak{H}$ is a quasilinear group of rank $m<n$, then $P(\mathfrak{H})$ is projective linear of rank m. Also we assume that $T(\mathfrak{H})$ is an involutory quasilinear group of rank m and that $P(T(\mathfrak{H}))$ is isomorphic to $PTL(m)$.

II. CLASSIFICATION OF INVOLUTIONS(4)

3. Structure of quasilinear groups of rank $m<n$. Because the results of the section are trivial for the case $m=1$, we will assume that $m>1$. We may use the inductive assumption of the Principal Theorem to identify $D(\mathfrak{H})/Z(D(\mathfrak{H}))$. Indeed, if $\Phi:\mathfrak{H}\rightarrow P(\mathfrak{H})$ is the natural homomorphism, $\Phi(D(\mathfrak{H}))=D(P(\mathfrak{H}))$ is isomorphic to $PSL(m)$. Except in the case where $PSL(m)$ is the finite group $PSL(2, 3)$, $PSL(m)$ is a simple nonabelian group (cf. Dieudonné [8, p. 38] or Artin [1, p. 158]). Note that $PSL(2, 3)$ is isomorphic to the alternating group $\mathfrak{A}_4$ on four letters.

We also have that $Z(D(\mathfrak{H}))=Z(\mathfrak{H})\cap D(\mathfrak{H})$. This is true for $m=2$ because we have assumed $\mathfrak{H}$ to be a linear group in this case. But when $m>2$, $\Phi(D(\mathfrak{H}))=D(\mathfrak{H})/(D(\mathfrak{H})\cap Z(\mathfrak{H}))$ is simple and nonabelian. Therefore, $Z(D(\mathfrak{H}))\subseteq Z(\mathfrak{H})\cap D(\mathfrak{H})$. The opposite inequality is trivial. Hence we may identify $P(D(\mathfrak{H}))$ with $\Phi(D(\mathfrak{H}))$. Likewise because of Condition D, we may identify

(4) The results of this chapter are valid for quasilinear groups of rank $n\geqq 2$. The assumption that $n\geqq 4$ is needed throughout Chapter III.

$\Phi(T(\mathfrak{H}))$ with $P(T(\mathfrak{H}))$. It also follows from the above that the proper normal subgroups of $D(\mathfrak{H})$ are contained in $Z(D(\mathfrak{H}))$ since $D(\mathfrak{H})$ is perfect.

PROPOSITION 3.1[5]. *Let $\mathfrak{H}$ be a quasilinear group of rank $m<n$. Then $T(\mathfrak{H})\supseteq D(\mathfrak{H})$ and $D(T(\mathfrak{H}))=D(\mathfrak{H})$, and $D(\mathfrak{H})$ is indecomposable. If* rk $\mathfrak{H}$ *is even, then $T(\mathfrak{H})$ is indecomposable; if* rk $\mathfrak{H}$ *is odd, $T(\mathfrak{H})=D(\mathfrak{H})\times\mathfrak{E}$ where $\mathfrak{E}$ is a subgroup of order* 2. *In every case, $[T(\mathfrak{H}):D(\mathfrak{H})]=2$. The center $Z(T(\mathfrak{H}))$ is cyclic of even order.*

Proof. We first show that $T(\mathfrak{H})\supseteq D(\mathfrak{H})$ and that $D(T(\mathfrak{H}))=D(\mathfrak{H})$. We assume that $m=\text{rk }\mathfrak{H}\geqq 3$; the proposition may be verified directly when $m=2$. Certainly $T(\mathfrak{H})D(\mathfrak{H})/D(\mathfrak{H})$ is an abelian group of exponent 2. Condition C implies that it has order 2 since it is isomorphic to a subgroup of $\mathfrak{H}/D(\mathfrak{H})$. This means that $[T(\mathfrak{H})D(\mathfrak{H}):D(\mathfrak{H})]=[T(\mathfrak{H}):T(\mathfrak{H})\cap D(\mathfrak{H})]=2$. We assert that $T(\mathfrak{H})\supseteq D(\mathfrak{H})$. Indeed, if $T(\mathfrak{H})\cap D(\mathfrak{H})\neq D(\mathfrak{H})$, $T(\mathfrak{H})\cap D(\mathfrak{H})$ is abelian as it is a proper normal subgroup of $D(\mathfrak{H})$. Thus $T(\mathfrak{H})$ is nilpotent. On the other hand, $T(\mathfrak{H})$ possesses an involution U such that $C_{T(\mathfrak{H})}(U)=C^*(U)=\mathfrak{U}_1\times\mathfrak{U}_2$ where $\mathfrak{U}_1$ is a quasilinear group of rank 2. Since $P(\mathfrak{U}_1)$ contains a subgroup isomorphic to $PSL(2)$, which is not nilpotent, $\mathfrak{U}_1$ and hence $\mathfrak{H}$ itself cannot be nilpotent. This is a contradiction; hence $T(\mathfrak{H})\supseteq D(\mathfrak{H})$. Also we may conclude that $[T(\mathfrak{H}):D(\mathfrak{H})]=2$.

Now it is clear that $D(T(\mathfrak{H}))\subseteq D(\mathfrak{H})$. If $D(T(\mathfrak{H}))\neq D(\mathfrak{H})$, then $D(T(\mathfrak{H}))$ is abelian. Because $T(\mathfrak{H})$ also is a quasilinear group of rank $m<n$, $D(T(\mathfrak{H}))$ contains a nonabelian simple factor group. Thus $T(\mathfrak{H})$ cannot be abelian. Consequently $D(T(\mathfrak{H}))=D(\mathfrak{H})$.

Because all the proper normal subgroups of $D(\mathfrak{H})$ belong to $Z(D(\mathfrak{H}))$, $D(\mathfrak{H})$ is indecomposable. Suppose that $T(\mathfrak{H})=\mathfrak{H}_1\times\mathfrak{H}_2$ is decomposable. Then $D(T(\mathfrak{H}))=D(\mathfrak{H}_1)\times D(\mathfrak{H}_2)$. Hence $D(\mathfrak{H})=D(\mathfrak{H}_1)$ or $D(\mathfrak{H})=D(\mathfrak{H}_2)$. This implies that $T(\mathfrak{H})=D(\mathfrak{H})\times\mathfrak{E}$ where $\mathfrak{E}$ is a group of order 2.

That the center $Z(T(\mathfrak{H}))$ is cyclic of even order now follows from Condition B.

Suppose that rk $\mathfrak{H}$ is even and $T(\mathfrak{H})$ is decomposable. Then $Z(D(\mathfrak{H}))$ contains a unique subgroup $\mathfrak{E}_1$ of order 2, which is necessarily normal in $\mathfrak{H}$. This means that $\mathfrak{E}_1$ is in the center of $\mathfrak{H}$ and thus that $-1\in\mathfrak{E}_1\subseteq Z(D(\mathfrak{H}))$. Then the element $E\neq\pm 1$ of order 2 in $\mathfrak{E}$ is an involution which is not in $Z(\mathfrak{H})$ by Condition A. But $C_{\mathfrak{H}}(E)\supseteq T(\mathfrak{H})$. Therefore, $T(C_{\mathfrak{H}}(E))=T(\mathfrak{H})=D(\mathfrak{H})\times\mathfrak{E}$. Thus if rk $\mathfrak{H}=m$, $D(\mathfrak{H})$ must be an involutory quasilinear group of odd rank $m-1$ Condition A. But $D(\mathfrak{H})$ is its own commutator subgroup; by Condition B, it contains no element of order 2 in its center. This is a contradiction. Hence $T(\mathfrak{H})$ is indecomposable.

If rk $\mathfrak{H}$ is odd, the center $Z(D(\mathfrak{H}))$ has odd order by Condition B. This

[5] In the proof of the Principal Theorem, the critical results will be called theorems. If a result has such great utility that it is frequently used without specific reference, it will be called a lemma. The remaining results are called propositions.

means that the element -1 of order 2 in the center of $\mathfrak{H}$ is not in $D(\mathfrak{H})$. Let $\mathfrak{E}$ be the group generated by -1. Then clearly $T(\mathfrak{H})=D(\mathfrak{H})\times\mathfrak{E}$.

It is a consequence of the inductive assumption of the Principal Theorem that the involutory centralizer $C^*(U)=T(C(U))$ of an involution U in a quasilinear group is the direct product of involutory quasilinear groups. An involution $U\neq\pm 1$ in a quasilinear group $\mathfrak{H}$ will be called a *proper involution of* $\mathfrak{H}$, the proper involutions of $\mathfrak{G}$ will be termed simply *proper involutions.* They belong to the classes $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_{n-1}$. A quasilinear group which is a direct factor in the direct decomposition of the involutory centralizer of a proper involution U will be called a *decomposition group belonging to the involution* U. If $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are the two decomposition groups appearing in a decomposition of the involutory centralizer $C^*(U)$ of a proper involution U, $\operatorname{rk}\mathfrak{U}_1+\operatorname{rk}\mathfrak{U}_2=n$. Furthermore, $1\leqq\operatorname{rk}\mathfrak{U}_i\leqq n-1$ for $i=1, 2$.

Let U be a proper involution in the quasilinear group $\mathfrak{G}$. Then $C^*(U)$ has the decomposition

$$C^*(U) = \mathfrak{U}_1\times\mathfrak{U}_2 \tag{3.1}$$

where $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are involutory quasilinear groups. The center $Z(C^*(U))$ and the factor group $P(C^*(U))$ have the decompositions

$$Z(C^*(U)) = Z(\mathfrak{U}_1)\times Z(\mathfrak{U}_2), \tag{3.2}$$

$$P(C^*(U)) = P(\mathfrak{U}_1)\times P(\mathfrak{U}_2) \tag{3.3}$$

where $Z(\mathfrak{U}_1)$ and $Z(\mathfrak{U}_2)$ are cyclic groups of even order while $P(\mathfrak{U}_1)$ and $P(\mathfrak{U}_2)$ are isomorphic to groups of the form $PTL(p)$. In particular, $Z(C^*(U))$ contains exactly four involutions: ± 1 and $\pm U$. Naturally 1 is represented by $1_1\times 1_2$ relative to (3.1) and (3.2) where 1_1 and 1_2 are the identities in the decomposition groups $\mathfrak{U}_1$ and $\mathfrak{U}_2$, respectively. Let -1_1 and -1_2 be the unique elements of order 2 in $\mathfrak{U}_1$ and $\mathfrak{U}_2$, respectively. Then there are three possibilities for the nontrivial involutions in $Z(C^*(U))$: $-1_1\times 1_2$, $1_1\times -1_2$, and $-1_1\times -1_2$. When there will be no confusion, we will abbreviate this notation to write: -1×1, 1×-1, and -1×-1, respectively. Of course, $1=1_1=1_2$; so $1=1\times 1$. In case $-1=-1\times -1$, we term the decomposition (3.1) *seminormal.* Then by at most a change in notation, we have $U\in\mathfrak{U}_1$ and $-U\in U_2$. A *normal decomposition* is one where $\mathfrak{U}_1\cap Z(\mathfrak{G})=\mathfrak{U}_2\cap Z_2(\mathfrak{G})=1$. Clearly a normal decomposition is seminormal. By virtue of Condition A, $C^*(U)$ always has a seminormal decomposition.

4. **Refinements of decompositions.** We now consider the situation where U and V are commuting involutions in the quasilinear group $\mathfrak{G}$ of rank n. Then $V\in C^*(U)$ and $U\in C^*(V)$. Relative to the decomposition

$$C^*(U) = \mathfrak{U}_1\times\mathfrak{U}_2 \tag{4.1}$$

V is represented by $V_1\times V_2$ where V_1 and V_2 are involutions. Then $C^*(U, V) = T(C(U, V)=T(C_{\mathfrak{U}_1}(V_1))\times T(C_{\mathfrak{U}_2}(V_2))$. Condition A when applied to the quasilinear groups $\mathfrak{U}_1$ and $\mathfrak{U}_2$ yields the decomposition

$$C^*(U, V) = \mathfrak{U}_1' \times \mathfrak{U}_1'' \times \mathfrak{U}_2' \times \mathfrak{U}_2'' \tag{4.2}$$

where, as the notation indicates, $T(C_{\mathfrak{U}_i}(V_i)) = \mathfrak{U}_i' \times \mathfrak{U}_i''$, $i = 1, 2$, is the decomposition into involutory quasilinear groups. Also from Condition A, it follows that we may take the decompositions of $\mathfrak{U}_1$ and $\mathfrak{U}_2$ to be seminormal. We call (4.2) a *seminormal refinement* or, in short, a *refinement* of (4.1). The groups $\mathfrak{U}_i'$ and $\mathfrak{U}_i''$, $i = 1, 2$, will also be called *decomposition groups*. In case $V_i \in Z(\mathfrak{U}_i)$, it follows from Condition A that $C_{\mathfrak{U}_i}(V_i) = \mathfrak{U}_i$. However, (4.2) is not contradicted; we may merely take one of the subgroups $\mathfrak{U}_i'$ or $\mathfrak{U}_i''$ to be the identity and the other to be $\mathfrak{U}_i$. In any event, with this decomposition we have

$$\operatorname{rk} \mathfrak{U}_1' + \operatorname{rk} \mathfrak{U}_1'' + \operatorname{rk} \mathfrak{U}_2' + \operatorname{rk} \mathfrak{U}_2'' = n. \tag{4.3}$$

The center $Z(C^*(U, V))$ has the decomposition corresponding to the decomposition (4.2)

$$Z(C^*(U, V)) = Z(\mathfrak{U}_1') \times Z(\mathfrak{U}_1'') \times Z(\mathfrak{U}_2') \times Z(\mathfrak{U}_2''). \tag{4.4}$$

Here the components $Z(\mathfrak{U}_i')$ and $Z(\mathfrak{U}_i'')$ are either identity groups or cyclic groups of even nonzero order according as the corresponding quasilinear groups $\mathfrak{U}_i'$ and $\mathfrak{U}_i''$ have rank 0 or not. In particular, the number of involutions in $Z(C^*(U, V))$ is 4, 8, or 16 according as two, one, or none of the decomposition groups in (4.2) are the identity.

Let $C^*(V)$ possess the decomposition

$$C^*(V) = \mathfrak{V}_1 \times \mathfrak{V}_2. \tag{4.5}$$

Relative to this decomposition U is represented by $U = U_1 \times U_2$ where U_1 and U_2 are involutions. Just as in the case (4.1), we may form a refinement of (4.5) to obtain

$$C^*(V, U) = \mathfrak{V}_1' \times \mathfrak{V}_1'' \times \mathfrak{V}_2' \times \mathfrak{V}_2''. \tag{4.6}$$

Of course, $C^*(V, U) = C^*(U, V)$; but we find it convenient to adopt the convention of writing $C^*(U, V)$ if we form a refinement of $C^*(U)$ and writing $C^*(V, U)$ if we form a refinement of $C^*(V)$.

These concepts readily extend to the case where there is a set of more than two mutually commuting involutions. Indeed, continuing this process, we will always arrive at a set $\mathfrak{M}$ of mutually commuting involutions for which $C^*(\mathfrak{M})$ is the direct product of n quasilinear groups of rank 1. But there are precisely 2^n involutions in such a direct product. Since $C^*(\mathfrak{M})$ is abelian, $\mathfrak{M}$ is precisely the set of involutions in $C^*(\mathfrak{M})$; this also means that $\mathfrak{M}$ is maximal. To summarize, we have the following proposition, which is valid for quasilinear groups of rank m, $1 \leqq m \leqq n$.

PROPOSITION 4.1. *In the quasilinear group $\mathfrak{G}$ of rank n, every set $\mathcal{S}$ of mutually commuting involutions is contained in a maximal set $\mathfrak{M}$ of 2^n mutually* *commuting involutions for which* $C^*(\mathfrak{M})$ *is the direct product of n quasilinear groups of rank* 1.

We compare the refinements (4.2) and (4.6) in the following lemma.

LEMMA 4.2. *Let* $\mathcal{S}$ *be a set of mutually commuting involutions in the quasilinear group* $\mathfrak{G}$. *Suppose that there exist two decompositions*

$$C^*(\mathcal{S}) = \mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_k, \tag{4.7}$$

$$C^*(\mathcal{S}) = \mathfrak{Y}_1 \times \mathfrak{Y}_2 \times \cdots \times \mathfrak{Y}_l \tag{4.8}$$

into involutory quasilinear groups. Then $k=l$ *and the factors in* (4.8) *may be arranged so that* $\mathfrak{X}_i$ *and* $\mathfrak{Y}_i$ *have the same rank,* $i=1, 2, \cdots, k$. *In particular, the number of components of a given rank is the same in* (4.7) *as it is in* (4.8).

Proof. By Proposition 3.1, the components $\mathfrak{X}_i$ and $\mathfrak{Y}_i$ are either indecomposable or decomposable so that, for example, $\mathfrak{X}_i = D(\mathfrak{X}_i) \times \mathfrak{C}_i$. Thus we may further decompose (4.7) and (4.8) so that we obtain direct products of indecomposable groups of the form $\mathfrak{X}_i$, $D(\mathfrak{X}_i)$, or $\mathfrak{C}_i$ on the one hand and indecomposable groups $\mathfrak{Y}_i$, $D(\mathfrak{Y}_i)$, and $\mathfrak{C}_i'$ on the other hand. Applying the Krull-Schmidt Theorem to the factor group $D(C^*(\mathcal{S}))/Z(D(C^*(\mathcal{S})))$, we see that the indecomposable factors $\mathfrak{X}_i/Z(\mathfrak{X}_i)$ or $D(\mathfrak{X}_i)/Z(D(\mathfrak{X}_i))$ obtained from (4.7) may be paired with the factors $\mathfrak{Y}_i/Z(\mathfrak{Y}_i)$ or $D(\mathfrak{Y}_i)/Z(D(\mathfrak{Y}_i))$ obtained from (4.8). Then the indecomposable factors of rank greater than one in (4.7) may be paired with the indecomposable factors of rank greater than one in (4.8). Also in comparing the decompositions of $C^*(\mathcal{S})/D(C^*(\mathcal{S}))$ that may be obtained from (4.7) and (4.8), we see that there is the same number of components of order 2 in each of the decompositions (4.7) and (4.8).

Let rk $\mathfrak{X}_i = p_i$ and rk $\mathfrak{Y}_i = p_i'$. The factor groups $P(X_i)$ or $P(D(\mathfrak{X}_i))$ of the indecomposable components in (4.7) contain subgroups of index at most 2 which are isomorphic to $PSL(p_i)$. Similarly, the factor groups $P(\mathfrak{Y}_i)$ or $P(D(\mathfrak{Y}_i))$ of the indecomposable components in (4.8) contain groups of index at most 2 which are isomorphic to $PSL(p_i')$. It follows from results of Dieudonné [7, p. 22], that no two of the groups $PSL(p_i)$ and $PSL(p_i')$ are isomorphic unless $p_i = p_i'$.

This means that either $\mathfrak{X}_i$ and $\mathfrak{Y}_i$ are both indecomposable and isomorphic or that $D(\mathfrak{X}_i)$ and $D(\mathfrak{Y}_i)$ are both indecomposable and isomorphic at least when $p_i = p_i' > 1$. In the latter case both $\mathfrak{X}_i$ and $\mathfrak{Y}_i$ are direct products of a group of order 2 with $D(\mathfrak{X}_i)$ and $D(\mathfrak{Y}_i)$, respectively; thus they are isomorphic. The number of cyclic components in (4.7) is the number of cyclic components in the decomposition of $C^*(\mathcal{S})$ into indecomposable components obtained from the groups $\mathfrak{X}_i$ less the number of decomposable components $\mathfrak{X}_i$ of odd rank $p_i > 1$. A similar statement holds for the cyclic components of (4.8). Because there are as many cyclic components in the decomposition obtained from (4.7) as from that obtained from (4.8) and as many components of odd rank

$p_i > 1$ in (4.7) as there are components of odd rank $p_i' > 1$ in (4.8), the number of cyclic components in (4.7) is the same as in (4.8). The remainder of the lemma now follows easily.

5. **Classification of involutions.** A proper involution U of a quasilinear group will be called an *even involution* if U belongs to a decomposition group of $C^*(U)$ of even rank; otherwise U will be called an *odd involution*. Because of the following proposition, even involutions play an important role in the study of the involutions of $\mathfrak{G}$.

PROPOSITION 5.1. *Let $\mathcal{S}$ be a set of mutually commuting involutions in a quasilinear group of rank n and let*

$$C^*(\mathcal{S}) = \mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_k, \tag{5.1}$$

$$C^*(\mathcal{S}) = \mathfrak{Y}_1 \times \mathfrak{Y}_2 \times \cdots \times \mathfrak{Y}_k \tag{5.2}$$

be direct decompositions of $C^*(\mathcal{S})$ into quasilinear groups. Let rk $\mathfrak{X}_1 = p$ *be even and let U be an involution in the center of $\mathfrak{X}_1$. Then U is in the center of a decomposition group $\mathfrak{Y}_i$ of* (5.2) *of rank p.*

Proof. Form the commutator subgroup $D(C^*(\mathcal{S}))$. From (5.1) we obtain that

$$D(C^*(\mathcal{S})) = D(\mathfrak{X}_1) \times D(\mathfrak{X}_2) \times \cdots \times D(\mathfrak{X}_k). \tag{5.3}$$

$$D(C^*(\mathcal{S})) = D(\mathfrak{Y}_1) \times D(\mathfrak{Y}_2) \times \cdots \times D(\mathfrak{Y}_k). \tag{5.4}$$

The group $D(C^*(\mathcal{S}))$ may not satisfy both chain conditions on its lattice of normal subgroups; so we look at $D(C^*(\mathcal{S}))/Z(D(C^*(\mathcal{S})))$. This group decomposes into simple factors isomorphic to $D(\mathfrak{X}_i)/Z(D(\mathfrak{X}_i))$ and also into simple factors isomorphic to $D(\mathfrak{Y}_i)/Z(D(\mathfrak{Y}_i))$. We may now apply a theorem of Speiser (cf. Zassenhaus [13, Chapter III, §3]) to assert that the nontrivial simple components appearing in a decomposition of $D(C^*(\mathcal{S}))/Z(D(C^*(\mathcal{S})))$ are uniquely determined. This means that after a rearrangement of the components in (5.4), we may set $D(\mathfrak{X}_i)\mathfrak{Z}/\mathfrak{Z} = D(\mathfrak{Y}_i)\mathfrak{Z}/\mathfrak{Z}$ where $\mathfrak{Z} = Z(D(C^*(\mathcal{S})))$. Then if $X \in D(\mathfrak{X}_i)$, there exists $Z_X \in \mathfrak{Z}$ such that XZ_X is in $D(\mathfrak{Y}_i)$. The element Z_X is not uniquely determined but its coset in $\mathfrak{Z}/Z(D(\mathfrak{Y}_i))$ is so determined. Let Z_X^* be this coset. Then $X \rightarrow Z_X^*$ is a homomorphism of $D(\mathfrak{X}_i)$ into $Z/Z(D(\mathfrak{Y}_i))$. Since $D(\mathfrak{X}_i)$ is perfect, Z_X^* is the identity for all $X \in D(\mathfrak{X}_i)$. Thus $Z_X \in Z(D(\mathfrak{Y}_i))$. Since $XZ_X \in D(\mathfrak{Y}_i)$, $X \in D(\mathfrak{Y}_i)$ for all $X \in D(\mathfrak{X}_i)$. Thus $D(\mathfrak{X}_i) \subseteq D(\mathfrak{Y}_i)$. The converse inequality may be similarly obtained. Thus $D(\mathfrak{X}_i) = D(\mathfrak{Y}_i)$, whenever these groups are nontrivial. Certainly this will be true if $D(\mathfrak{X}_i)$ and $D(\mathfrak{Y}_i)$ are trivial. Hence we may take $D(\mathfrak{X}_i) = D(\mathfrak{Y}_i)$, $i = 1, 2, \cdots, k$.

Now if U belongs to the center of $\mathfrak{X}_1$ and rk $\mathfrak{X}_1$ is even, U belongs to the center of $D(\mathfrak{X}_1) = D(\mathfrak{Y}_1)$ by the argument given in the proof of Proposition 3.1. Consequently U belongs to the center of $\mathfrak{Y}_1$. Since $D(\mathfrak{X}_1) = D(\mathfrak{Y}_1)$, $\mathfrak{X}_1$ and $\mathfrak{Y}_1$ have the same ranks.

We now define a *p-involution* for even p, $0<p\leqq n$, to be a proper involution which is contained in the center of a quasilinear group of rank p in a decomposition of its involutory centralizer. Subsequently we will define p-involutions for other values of p.

COROLLARY 5.2. *Let U be an involution in the decomposition group $\mathfrak{X}_1$ of (5.1). Then if $\mathfrak{J}$ is any subset of $\mathfrak{S}$ containing U, U is contained in the center of some decomposition group of rank $p=\operatorname{rk}\mathfrak{X}_1$ in any decomposition of $C^*(\mathfrak{J})$. In particular, if U is a p-involution, it is always contained in the center of a decomposition group of rank p in any decomposition of $C^*(U)$.*

Proof. Suppose that $C^*(\mathfrak{J})$ possesses the decomposition

$$C^*(\mathfrak{J}) = \mathfrak{T}_1 \times \mathfrak{T}_2 \times \cdots \times \mathfrak{T}_l. \tag{5.5}$$

Refine (5.5) to form a decomposition of $C^*(\mathfrak{S})$ such as (5.2). Then it follows from Proposition 5.1 that $U\in Z(\mathfrak{Y}_i)$ for some component $\mathfrak{Y}_i$ such that $\operatorname{rk}\mathfrak{Y}_i=p$. Because (5.2) has now been taken as a refinement of (5.5), $\mathfrak{Y}_i\subseteq\mathfrak{T}_j$ for some component $\mathfrak{T}_j$. The involution $U\in Z(\mathfrak{Y}_i)$ is also contained in $Z(C^*(\mathfrak{J}))$. But $Z(C^*(\mathfrak{J}))\cap\mathfrak{T}_j=Z(\mathfrak{T}_j)$. Hence $U\in Z(\mathfrak{T}_j)$. Since the refinements are seminormal, U cannot belong to both $Z(\mathfrak{Y}_i)$ and $Z(\mathfrak{T}_j)$ unless $Z(\mathfrak{Y}_i)=Z(\mathfrak{T}_j)$. The last statement of the proposition now follows when one takes T to consist of U alone.

It is a consequence of Corollary 5.2 that odd involutions belong to decomposition groups of odd rank. In particular, if n is odd and p is odd, we define a *p-involution* of $\mathfrak{G}$ to be an involution U belongings to the center of a decomposition group of rank p of a seminormal decomposition. Then $-U$ belongs to a decomposition group of even rank $n-p$. So U is a p-involution with p and n odd if and only if $-U$ is an $(n-p)$-involution. We will delay the definition of p-involutions with p odd and n even to a following section. Again we similarly define p-involutions for odd p of decomposition groups $\mathfrak{H}$ of odd rank $m<n$. We do not claim them to be p-involutions of $\mathfrak{G}$ in this case.

PROPOSITION 5.3. *Let U be a p-involution with p even. Then U is conjugate to $-U$ only if $p=n/2$.*

Proof. Let $C^*(U)$ possess the seminormal decomposition

$$C^*(U) = \mathfrak{U}_1 \times \mathfrak{U}_2, \tag{5.7}$$

where $U\in\mathfrak{U}_1$ and $\operatorname{rk}\mathfrak{U}_1=p$. Now if there exists $R\in\mathfrak{G}$ such that $RUR^{-1}=-U$, then $RC(U)R^{-1}=C(U)$. From this it follows that $RC^*(U)R^{-1}=C^*(U)$. But then

$$C^*(U) = R\mathfrak{U}_1R^{-1} \times R\mathfrak{U}_2R^{-1} \tag{5.8}$$

is a seminormal decomposition into quasilinear groups. In particular, rk $R\mathfrak{U}_2R^{-1}=n-p$. However, the center $Z(R\mathfrak{U}_2R^{-1})$ contains $R(-U)R^{-1}=U$. This is in contradiction to Corollary 5.2 and proves the proposition.

PROPOSITION 5.4. *The p-involutions of $\mathfrak{G}$ with p even belong to the same conjugate class $\mathfrak{K}_p$.*

Proof. This is clear if $p=n/2$ for the involutions in the class $\mathfrak{K}_{n/2}$ and only these involutions belong to the centers of quasilinear decomposition groups of rank $n/2$. Therefore, let $p\neq n-p$ and let U and V be two p-involutions. Let

$$C^*(U) = \mathfrak{U}_1\times\mathfrak{U}_2, \tag{5.9}$$

$$C^*(V) = \mathfrak{B}_1\times\mathfrak{B}_2 \tag{5.10}$$

be seminormal decompositions where $U\in\mathfrak{U}_1$ and $V\in\mathfrak{B}_1$. Then because of the form of (5.9) and (5.10), it may be seen that U and V belong to one of the two classes $\mathfrak{K}_p$ and $\mathfrak{K}_{n-p}$. This is because involutions in these classes and only in these classes can have isomorphic centralizers as may be seen by identifying the simple groups in a composition series for the involutory centralizers of the involutions in $\mathfrak{G}$ as we did in §4. Hence, if U and V are not conjugate, it follows from Proposition 5.3 that U and $-V$ are conjugate. Then there exists $R\in\mathfrak{G}$ such that $RUR^{-1}=-V$. This leads to a seminormal decomposition

$$C^*(V) = C^*(-V) = R\mathfrak{U}_1R^{-1}\times R\mathfrak{U}_2R^{-1}. \tag{5.11}$$

But $R\mathfrak{U}_2R^{-1}$ is a quasilinear decomposition group of rank $n-p$ containing the involution $V=R(-U)R^{-1}$ in its center. This is a contradiction to the fact that V is a p-involution. Hence U and V are conjugate.

Now we see that one of the classes $\mathfrak{K}_p$ or $\mathfrak{K}_{n-p}$ consists of p-involutions while the other consists of $(n-p)$-involutions when p is even and $p\neq n/2$. Therefore, we may and will enumerate the classes $\mathfrak{K}_p$ so that $\mathfrak{K}_p$ consists of p-involutions when p is even.

To discuss the situation relative to the odd involutions in a quasilinear group of even rank, we prove the following counterpart of Proposition 5.3.

PROPOSITION 5.5. *Let U be an odd involution. Then U is conjugate to $-U$ only if $U\in\mathfrak{K}_{n/2}$.*

Proof. We need treat only the case where $n=\text{rk }\mathfrak{G}$ is even. Let $C^*(U)$ possess the seminormal decomposition

$$C^*(U) = \mathfrak{U}_1\times\mathfrak{U}_2 \tag{5.12}$$

where $U\in\mathfrak{U}_1$ and rk $\mathfrak{U}_1=p$ is odd. We assume that $p\neq n-p$. Thus we may take $n-p>p$. Let s be an even integer such that $n-p>s>p$. Because s is even, we have defined s-involutions in the quasilinear decomposition group $\mathfrak{U}_2$. Let S be such an involution. Upon refining (5.12), we may obtain the seminormal decomposition

(5.13) $$C^*(U, S) = \mathfrak{U}_1 \times \mathfrak{S} \times \mathfrak{U}_2'$$

where $S \in \mathfrak{S}$ and rk $\mathfrak{S} = s$.

Suppose that R is an element of $\mathfrak{G}$ such that $RUR^{-1} = -U$. Then $RC^*(U)R^{-1} = C^*(-U) = C^*(U)$. Let $T = RSR^{-1}$ and form the seminormal decomposition (5.14) of $C^*(U, T) = C^*(-U, RSR^{-1}) = RC^*(U)R^{-1}$:

(5.14) $$C^*(U, T) = \mathfrak{V}_1 \times \mathfrak{T} \times \mathfrak{V}_2',$$

where $\mathfrak{V}_1 = R\mathfrak{U}_1R^{-1}$, $\mathfrak{T} = R\mathfrak{S}R^{-1}$, and $\mathfrak{V}_2' = R\mathfrak{U}_2' R^{-1}$. Then $-U \in \mathfrak{V}_1$ and $T \in \mathfrak{T}$. Next we form a decomposition (5.15) of $C^*(U, T)$ by refining (5.12); comparison with (5.14) shows that this refinement has only three decomposition groups of ranks p, s, and $n-p-s$. Because of the choice of $s > p$, $\mathfrak{U}_1$ will not decompose in forming (5.15). Thus we obtain

(5.15) $$C^*(U, T) = \mathfrak{U}_1 \times \mathfrak{T}' \times \mathfrak{U}_2'',$$

where we may take $T \in \mathfrak{T}'$ because of Proposition 5.1. Then rk $\mathfrak{T} = s$ and rk $\mathfrak{U}_2'' = n-p-s$.

Thus S and T are s-involutions of the quasilinear group $\mathfrak{U}_2$. We have just argued that they are conjugate in $\mathfrak{U}_2$ in Proposition 5.4. Thus there exists $V \in \mathfrak{U}_2$ such that $VTV^{-1} = S$. Replacing R by VR, we obtain that $RSR^{-1} = S$. Thus both R and U are in $C(S)$. Let $C^*(S) = \mathfrak{X}_1 \times \mathfrak{X}_2$ be a seminormal decomposition. Relative to this decomposition, let R and U be represented by $R_1 \times R_2$ and $U_1 \times U_2$, respectively. Then $RUR^{-1} = -U$ implies that

(5.16) $$R_1U_1R_1^{-1} = (-1_1)U_1 \quad \text{and} \quad R_2U_2R_2^{-1} = (-1_2)U_2.$$

On the other hand, we may refine the decomposition of $C^*(S)$ to form a decomposition of $C^*(S, U)$, which may be seen to contain but three components by virtue of a comparison with (5.13). Then one of these components is $\mathfrak{X}_1$ or $\mathfrak{X}_2$; and the corresponding component U_1 or U_2 of U is, therefore, in the center of $\mathfrak{X}_1$ or $\mathfrak{X}_2$ because it is in the center of $C^*(S, U)$. This contradicts (5.16). So we cannot have $RUR^{-1} = -U$; this proves the proposition.

PROPOSITION 5.6. *Let n be odd and p be odd. Then the p-involutions of $\mathfrak{G}$ belong to the same conjugate class $\mathfrak{K}_p$.*

Proof. From the form of a seminormal decomposition of the involutory centralizer of a p-involution U, it follows that U and $-U$ belong to one of the two classes $\mathfrak{K}_p$ or $\mathfrak{K}_{n-p}$. Also $-U$ is an even involution and thus in $\mathfrak{K}_{n-p}$. This implies that the p-involution U is in $\mathfrak{K}_p$. Thus all p-involutions are conjugate in this case.

6. **Automorphisms of seminormal decompositions.** In this section we study various decompositions of the involutory centralizers of proper involutions. But first we need to study the relations between proper involutions in an involutory quasilinear group $\mathfrak{H}$ of rank m, $2 \leq m \leq n$, and those in the projective linear group $PTL(m)$.

In [8, Chapter I, §3 and §4], Dieudonné distinguishes two kinds of involutions in $PTL(m)$, namely, those of the *first kind* which are images by the natural homomorphism $\Lambda: \overline{\mathfrak{T}} \to P(\overline{\mathfrak{T}}) = PTL(m)$ of involutions in $\overline{\mathfrak{T}} = TL(m)$ and those of the *second kind* which are not images of the elements of $K_p(\overline{\mathfrak{T}})$ (here we set $\mathfrak{L}_p = \mathfrak{L}_{m-p}$ in case $p > m/2$). By comparing the centralizers of involutions in $\mathfrak{H}$ and $\overline{\mathfrak{T}}$, we will arrive at the following proposition.

PROPOSITION 6.1. *Let U be an involution in the class $\mathfrak{K}_p(\mathfrak{H})$ of the quasilinear group $\mathfrak{H}$ of rank m, $2 \leqq m < n$. Let $\Phi: \mathfrak{H} \to PTL(m)$ be an epimorphism. Then ΦU is an involution in the class $\mathfrak{L}_p$ of $PTL(m)$.*

Proof. The proposition is trivial in case $m = 2$; therefore, assume $m > 2$. Let U^* be an involution in $P(\overline{\mathfrak{T}})$. Then the inverse image of the centralizer $C(U^*)$ of U^* in $\overline{\mathfrak{T}}$ consists of the group $C'(\overline{U})$ of elements $\overline{V}$ such that $\overline{V}\overline{U} = \pm \overline{U}\overline{V}$ where $\overline{U}$ is an element of $\overline{T}$ such that $\Lambda\overline{U} = U^*$. If $U^* \in \mathfrak{L}_p$, we may take $\overline{U}$ in $\mathfrak{K}_p(\overline{\mathfrak{T}})$ and conclude that $[C'(\overline{U}): C^*(\overline{U})] \leqq 2$. Thus $C^*(\overline{U}) = C_{\overline{\mathfrak{T}}}(\overline{U})$ is normal in $C'(\overline{U})$. Since $\overline{\mathfrak{T}}$ is quasilinear, we have that $P(D(C^*(\overline{U})))$ is isomorphic to $PSL(p) \times PSL(m-p)$. Thus it is possible to determine a normal series for $C'(U)$ such that the only possible nonabelian factors are the groups $PSL(p)$ and $PSL(m-p)$. Since $m \geqq 3$, one such factor will be nonabelian. Clearly then we can make the same statement about $C(U^*) = \Lambda C'(\overline{U})$. From Dieudonné's results [7, p. 22] and the Schreier refinement theorem for normal series, it follows that for distinct p, $0 < p \leqq m/2$, the centralizers $C(U^*)$ of involutions $U^* \in \mathfrak{L}_p$ are nonisomorphic.

A class $\mathfrak{L}'$ of involutions of the second kind exists only when m is even. Let $\overline{U}'$ be an element of $\overline{\mathfrak{T}}$ such that $\Lambda\overline{U}' = U'^* \in \mathfrak{L}'$. Then it follows from Dieudonné's characterization of the centralizer of such an involution in $GL(m)$ that $C^*(\overline{U}')$ has a normal series which contains as the only possible nonabelian simple factor the projective unimodular group $PSL(m/2, K')$ defined over a quadratic extension K' of K. The same will be true for the groups $C'(\overline{U}')$ and $\Lambda C'(\overline{U}') = C(U'^*)$. Again from Dieudonné's result [7, p. 22], and the Schreier refinement theorem, it follows that $C(U'^*)$ is not isomorphic to any of the groups $C(U^*)$ considered in the preceding paragraph.

Thus if $U \in \mathfrak{K}_p(\mathfrak{H})$, the conjugate class of $U^* = \Phi U$ in $P(\overline{\mathfrak{T}})$ is determined by identifying up to isomorphism the centralizer $C(U^*)$ of U^* in $P(\overline{\mathfrak{T}})$. As above, this may be done by identifying the projective unimodular groups in a particular normal series for $C^*(U)$ in $\mathfrak{H}$. But these groups are $PSL(p)$ and $PSL(m-p)$ as follows from Condition A. Therefore, $U^* \in \mathfrak{L}_p$. This proves the proposition.

COROLLARY 6.2. *An involution in the class $\mathfrak{K}_p(\mathfrak{H})$ of a quasilinear group $\mathfrak{H}$ of rank $m < n$ is a product of p involutions from the classes $\mathfrak{K}_1(\mathfrak{H})$ or $\mathfrak{K}_{n-1}(\mathfrak{H})$.*

Proof. An involution $\overline{U}$ in $\mathfrak{K}_p(\overline{\mathfrak{T}})$ is the product of p mutually commuting involutions in $\mathfrak{K}_1(\overline{\mathfrak{T}})$. Taking $\overline{\mathfrak{T}}$ for the group $\mathfrak{H}$ in Proposition 6.1, we see that an involution U^* in $\mathfrak{L}_p$ is the product of p mutually commuting involutions from the class $\mathfrak{L}_1$. Now let $U \in \mathfrak{K}_p(\mathfrak{H})$. Then ΦU is the product of p mutually commuting involutions from the class $\mathfrak{L}_1$. This means that there are p mutually commuting or anticommuting involutions in the classes $\mathfrak{K}_1(\mathfrak{H})$ or $\mathfrak{K}_{n-1}(\mathfrak{H})$ such that their product is of the form ZU where $Z \in Z(\mathfrak{G})$. Because of Propositions 5.3 and 5.5, these involutions commute. Thus this product is an involution. Therefore, $Z = \pm 1$. By replacing one of the involutions in this product by its negative, if necessary, we arrive at the result.

PROPOSITION 6.3. *Let U be an even involution in a quasilinear group $\mathfrak{H}$. Then U is in $D(\mathfrak{H})$. If U is an odd involution in a quasilinear group $\mathfrak{H}$ of rank $m<n$, U is not in $D(\mathfrak{H})$.*

Proof. If U is an even involution in a quasilinear group $\mathfrak{H}$, then U belongs to the center of a quasilinear group $\mathfrak{U}$ of even rank. But $D(\mathfrak{U})$ contains an element E of order 2 in its center by virtue of Condition B. Since this is the only element of order 2 in $Z(D(\mathfrak{U}))$, it is in $Z(\mathfrak{U})$. But there is only one element of order 2 in $Z(\mathfrak{U})$ by Condition A. Therefore, $U=E$ is in $D(\mathfrak{U}) \subseteq D(\mathfrak{H})$.

Next let U be an odd involution in a quasilinear group $\mathfrak{H}$ of odd rank m. Then $-U$ is an even involution and hence in $D(\mathfrak{H})$. Condition B asserts that -1 is not in $D(\mathfrak{H})$. Therefore, $U=(-1)(-U)$ is not in $D(\mathfrak{H})$.

Finally we treat the case of odd involutions in a quasilinear group $\mathfrak{H}$ of even rank m. These are the involutions in $\mathfrak{K}_p(\mathfrak{H})$ with p odd. If $U \in \mathfrak{K}_p(\mathfrak{H})$, then $-U \in \mathfrak{K}_{m-p}(\mathfrak{H})$ by Proposition 5.6. Since rk $\mathfrak{H}$ is even, either both U and $-U$ are in $D(\mathfrak{H})$ or both are not. Thus both $\mathfrak{K}_p(\mathfrak{H})$ and $\mathfrak{K}_{m-p}(\mathfrak{H})$ consist of involutions in $D(\mathfrak{H})$ or both contain no involution of $D(\mathfrak{H})$. Since the group $T(\mathfrak{H})/D(\mathfrak{H})$ is of order 2, it follows that a product of an odd number of involutions not in $D(\mathfrak{H})$ is also not in $D(\mathfrak{H})$. Because of Corollary 6.2, the involutions in $\mathfrak{K}_p(\mathfrak{H})$ and $\mathfrak{K}_{m-p}(\mathfrak{H})$ belong to $D(\mathfrak{H})$ with p odd if the involutions in $\mathfrak{K}_1(\mathfrak{H})$ and $\mathfrak{K}_{m-1}(\mathfrak{H})$ belong to $D(\mathfrak{H})$. But the involutions in $\mathfrak{K}_p(\mathfrak{H})$ and $\mathfrak{K}_{m-p}(\mathfrak{H})$ with p even already have been shown to be in $D(\mathfrak{H})$. Hence if the involutions of $\mathfrak{K}_1(\mathfrak{H})$ and $\mathfrak{K}_{m-1}(\mathfrak{H})$ are in $D(\mathfrak{H})$, we obtain a contradiction of Condition C. This proves the proposition.

Let U be a proper involution and let

$$C^*(U) = \mathfrak{U}_1 \times \mathfrak{U}_2 \tag{6.1}$$

be a seminormal decomposition. Then the subgroups $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are involutory quasilinear subgroups with unique subgroups $\mathfrak{E}_1$ and $\mathfrak{E}_2$ of order 2 in their respective centers. We may assume that $U=-1_1 \in \mathfrak{E}_1$ and that $-U=-1_2 \in \mathfrak{E}_2$. By Proposition 3.1, there exist unique epimorphisms $\phi_i\colon \mathfrak{U}_i \to \mathfrak{E}_i$ and $\psi_i\colon \mathfrak{U}_i \to \mathfrak{E}_j$ for $i=1, 2$ and $i \neq j$. Define homomorphisms Φ_i, Ψ_i, and Ω_i of $\mathfrak{U}_i$ into $C^*(U)$ by setting $\Phi_i(X) = \phi_i(X)X$, $\Psi_i(X) = \psi_i(X)X$, and $\Omega_i(X) = \phi_i(X)\psi_i(X)X$ for $X \in U_i$. One may verify that these are normal homomorphisms. In general, designate by $\Lambda_1 \times \Lambda_2$ the normal endomorphism of $C^*(U)$ whose restriction to $\mathfrak{U}_i$ is a normal homomorphism Λ_i, $i=1, 2$.

PROPOSITION 6.4. *Let U be an odd involution and let $n = \text{rk}\,\mathfrak{G}$ be even. Then the endomorphism $\Omega = \Omega_1 \times \Omega_2$ is an automorphism such that*

$$C^*(U) = \mathfrak{U}_1' \times \mathfrak{U}_2' \tag{6.2}$$

is a seminormal decomposition with $\mathfrak{U}_1' = \Omega\mathfrak{U}_1$ and $\mathfrak{U}_2' = \Omega\mathfrak{U}_2$. Furthermore, if V is an involution of $\mathfrak{U}_i$, $\Omega V = \pm V$ according as $V \in D(\mathfrak{U}_i)$ or not. In particular, $\Omega_1 U = -U$ is in $\mathfrak{U}_1'$ and $\Omega(-U) = U$ is in $\mathfrak{U}_2'$.

Proof. We must show that the homomorphism Ω is an automorphism. Let $X_1 \times X_2$ be the representation of X in $C^*(U)$ relative to (6.1). If $\Omega(X) = 1$, then we obtain that $\phi_1(X_1)\psi_2(X_2)X_1 = 1_1$ and $\phi_2(X_2)\psi_1(X_1)X_2 = 1_2$. This implies that $X_i = \pm 1_i$, $i = 1, 2$. But the commutator subgroups $D(\mathfrak{U}_i)$ do not contain involutions in their centers by Condition B. Thus, if $X_1 = -1_1 = U$, we obtain that $\phi_1(U)\psi_2(X_2)U = \psi_2(X_2) = 1_1$. But $-1_2 \notin D(\mathfrak{U}_2)$; thus $X_2 = 1_2$. But then $\phi_2(1_2)\psi_1(X_1)1_2 = 1_2$ and $\psi_1(X_1) = 1_2$, which implies that $X_1 = 1_1$. This is a contradiction. Hence $X_1 = 1_1$. Similarly, $X_2 = 1_2$ and $X = 1$. Thus Ω is a monomorphism. Since Ω induces the identity homomorphism on $C^*(U)/D(C^*(U))$ and is the identity on $D(C^*(U))$, it is an automorphism.

To prove the second statement, observe merely that $\phi_i(V) = \pm 1_i$ and $\psi_i(V) = \pm 1_j$ for $i = 1, 2$, and $i \neq j$, according as V is in $D(\mathfrak{U}_i)$ or not. The last statement is a consequence of Proposition 6.3.

PROPOSITION 6.5. *Let U be an even involution of $\mathfrak{G}$. Form* (6.1) *with $U \in U_1$. Let Ω, Φ, and Ψ be the respective endomorphisms $\Omega_1 \times 1$, $\Phi_1 \times 1$, and $\Psi_1 \times 1$ of $C^*(U)$. Then Ω, Φ, and Ψ are automorphisms and*

$$C^*(U) = \mathfrak{U}_1' \times \mathfrak{U}_2' \tag{6.3}$$

where $\mathfrak{U}_i' = \Omega\mathfrak{U}_i$, $\Phi\mathfrak{U}_i$, or $\Psi\mathfrak{U}_i'$, as the case may be. Furthermore, if $V \in \mathfrak{U}_1$ is an involution, then $\Omega(V) = V$ or $-V$, $\Phi(V) = V$ or UV and $\Psi(V) = V$ or $-UV$ according as V is in $D(\mathfrak{U}_1)$ or not.

In each case the proof is a simplification of the proof of Proposition 6.4.

7. **1-involutions.** In the case where $n = \text{rk}\,\mathfrak{G}$ is odd, 1-involutions of $\mathfrak{G}$. were defined in §5 and the classes $\mathfrak{K}_p$ were enumerated so that they belonged to $\mathfrak{K}_1$ and for such an involution U

$$C^*(U) = \mathfrak{U}_1 \times \mathfrak{U}_2 \tag{7.1}$$

where $U \in \mathfrak{U}_1$ and $\text{rk}\,\mathfrak{U}_1 = 1$. When n is even, we take an arbitrary involution U satisfying (7.1) and call it and its conjugates 1-*involutions.* Then we may again enumerate the classes so that $U \in \mathfrak{K}_1$. This arbitrary way of choosing 1-involutions in $\mathfrak{G}$ leaves ambiguous the definition of 1-involutions in quasilinear decomposition groups of even rank. Eventually we will define 1-involutions in these groups. However, in any quasilinear group $\mathfrak{H}$ of rank $m \leqq n$, we will designate the involutions of the classes $\mathfrak{K}_1(\mathfrak{H})$ and $\mathfrak{K}_{m-1}(\mathfrak{H})$ as *extremal involutions.*

Proposition 7.1. *Let U be a 1-involution in the quasilinear group $\mathfrak{G}$ and suppose that $C^*(U)$ possesses the seminormal decomposition* (7.1). *Let V be an extremal involution of the quasilinear group $\mathfrak{U}_2$. Then one of the involutions $\pm V$ and $\pm UV$ is a 1-involution V' of $\mathfrak{G}$. Furthermore, V' is an extremal involution of a quasilinear decomposition group $\mathfrak{U}_2'$ of rank $n-1$ belonging to a seminormal decomposition*

$$C^*(U) = \mathfrak{U}_1 \times \mathfrak{U}_2'. \tag{7.2}$$

By refining (7.2), *we may obtain a decomposition*

$$C^*(U, V) = \mathfrak{U}_1 \times \mathfrak{W}_1' \times \mathfrak{W}_2' \tag{7.3}$$

where rk $\mathfrak{W}_1' = 1$ *and* $V' \in \mathfrak{W}_1'$. *When n is even, $V' = V$ or $V' = -UV$ and* $\mathfrak{U}_2' = \mathfrak{U}_2$.

Note that in case rk $\mathfrak{U}_2$ is odd, this proposition implies that the 1-involution V' of $\mathfrak{G}$ is a 1-involution in $\mathfrak{U}_2$. Thus 1-involutions in $\mathfrak{G}$ are 1-involutions in $\mathfrak{U}_2$, which is as it should be.

Proof. Because V is an extremal involution in $\mathfrak{U}_2$, we obtain the refinement

$$C^*(U, V) = \mathfrak{U}_1 \times \mathfrak{W}_1 \times \mathfrak{W}_2 \tag{7.4}$$

where rk $\mathfrak{W}_1 = 1$. Now $-U$ is the element of order 2 in the center of $\mathfrak{U}_2$. Hence either V or $-UV$ belongs to $\mathfrak{W}_1$.

We first dispense with the case that n is even. Choose V' to be that one of V or $-UV$ which belongs to $\mathfrak{W}_1$. Then relative to the decomposition (7.3), U and V' are represented by $-1\times 1\times 1$ and $1\times -1\times 1$, respectively. Thus $-UV' = 1\times 1\times -1$ and is in the center of $\mathfrak{W}_2$ and in $\mathfrak{U}_2$. This means that $-UV'$ is an $(n-2)$-involution and that UV' is a 2-involution. The proposition will be established for this case by proving the following lemma and setting $UV' = T$, $U = R$, and $V' = S$.

Lemma 7.2. *Let T be a 2-involution such that $T = RS$ where R and S are commuting involutions. Suppose, furthermore, that $C^*(T)$ has the seminormal decomposition*

$$C^*(T) = \mathfrak{T}_1 \times \mathfrak{T}_2 \tag{7.5}$$

with $T \in \mathfrak{T}_1$ and the refinement

$$C^*(T, R) = \mathfrak{T}_1' \times \mathfrak{T}_1'' \times \mathfrak{T}_2 \tag{7.6}$$

with rk $\mathfrak{T}_1' =$ rk $\mathfrak{T}_1'' = 1$. *Then R and S are conjugate by an involution in $\mathfrak{T}_1$.*

Proof. Relative to (7.5), $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Because of the form of the refinement (7.6) and the fact that $R \in Z(C^*(T, R))$, it follows that R_2 is in $Z(\mathfrak{T}_2)$; that is, $R_2 = \pm 1_2$. But relative to (7.5), $T = T_1 \times T_2$ with $T_2 = 1_2$. Since $RS = T$, $S_2 = R_2$. Thus $R_1 S_1 = T$ and R_1 and S_1 are proper involutions of the quasilinear group $\mathfrak{T}_1$ of rank 2. By Condition A, they are conjugate in $\mathfrak{T}_1$.

We now continue with the proof of Proposition 7.1 and treat the remaining case where n is odd. Replacing V by $-UV$, if necessary, we may assume that $V \in \mathfrak{W}_1$. Let $C^*(V)$ possess the seminormal decomposition (7.7)

$$C^*(V) = \mathfrak{V}_1 \times \mathfrak{V}_2 \tag{7.7}$$

where $V \in \mathfrak{V}_1$. Next form a decomposition for $C^*(V, U)$ by refining (7.7). By comparison with (7.4), we see that there are but three quasilinear groups in a decomposition of $C^*(V, U) = \mathfrak{Y}_1 \times \mathfrak{Y}_2 \times \mathfrak{Y}_3$; thus we obtain

$$C^*(V, U) = \mathfrak{Y}_1 \times \mathfrak{Y}_2 \times \mathfrak{Y}_3 \tag{7.8}$$

where $\operatorname{rk} \mathfrak{Y}_1 = \operatorname{rk} \mathfrak{Y}_2 = 1$ and $\operatorname{rk} \mathfrak{Y}_3 = n-2$ by virtue of Proposition 5.1. There are four possibilities:

$$\mathfrak{Y}_1 = \mathfrak{V}_1 \quad \text{and} \quad \mathfrak{Y}_2, \mathfrak{Y}_3 \subseteq \mathfrak{V}_2, \tag{7.9}$$

$$\mathfrak{Y}_2, \mathfrak{Y}_3 \subseteq \mathfrak{V}_1 \quad \text{and} \quad \mathfrak{Y}_1 = \mathfrak{V}_2, \tag{7.10}$$

$$\mathfrak{Y}_1, \mathfrak{Y}_2 \subseteq \mathfrak{V}_1 \quad \text{and} \quad \mathfrak{Y}_3 = \mathfrak{V}_2, \tag{7.11}$$

$$\mathfrak{Y}_3 = \mathfrak{V}_1 \quad \text{and} \quad \mathfrak{Y}_1, \mathfrak{Y}_2 \subseteq \mathfrak{V}_2. \tag{7.12}$$

Because n is odd, V is a p-involution if $\operatorname{rk} \mathfrak{V}_1 = p$. Thus in (7.9), V is a 1-involution, and we may take $V' = V$ and $\mathfrak{U}_2' = \mathfrak{U}_2$ in this case.

In the case (7.10), V is an $(n-1)$-involution. Hence $-V$ is a 1-involution and conjugate to U. But $-V$ is represented relative to (7.4) by $-1 \times 1 \times -1$ and thus is not in U_2. However, $-UV = 1 \times 1 \times -1$ relative to (7.4) and is thus in the center of $\mathfrak{W}_2$. But $\operatorname{rk} \mathfrak{W}_2 = n-2$ and hence is odd. Thus $-UV$ is an odd involution of the quasilinear group $\mathfrak{U}_2$. Define Ψ to be the automorphism defined in Proposition 6.5 and set $\mathfrak{U}_2' = \Psi\mathfrak{U}_2$ (here, of course, we interchange the roles of U and $-U$ and $\mathfrak{U}_1$ and $\mathfrak{U}_2$). Then $-V = U(-UV) = \Psi(-UV)$ is in $\mathfrak{U}_2'$ because $-UV \notin D(\mathfrak{U}_2)$ by Proposition 6.3. Furthermore, upon setting $\mathfrak{W}_1'' = \Psi\mathfrak{W}_1$ and $\mathfrak{W}_2'' = \Psi\mathfrak{W}_2$, we obtain

$$C^*(U, -V) = \mathfrak{U}_1 \times \mathfrak{W}_1'' \times \mathfrak{W}_2'' \tag{7.13}$$

in which $-V \in \mathfrak{W}_2''$ and $\operatorname{rk} \mathfrak{W}_2'' = n-2$. But now apply Proposition 6.4 to the seminormal decomposition $T(C_{\mathfrak{U}_2'}(-V)) = \mathfrak{W}_1'' \times \mathfrak{W}_2''$ to obtain a decomposition $T(C_{\mathfrak{U}_2'}(-V)) = \mathfrak{W}_1' \times \mathfrak{W}_2'$ where $-V \in \mathfrak{W}_1'$ and $\operatorname{rk} \mathfrak{W}_1' = 1$. Then upon setting $V' = -V$, we may obtain (7.3).

In the case (7.11), V is a 2-involution. Now $U(UV) = V$. We may apply Lemma 7.2 because (7.8) and (7.11) imply (7.6). Thus UV is conjugate to U and is a 1-involution of $\mathfrak{G}$. On the other hand, V is an odd involution of the quasilinear group $\mathfrak{U}_2$ and so is not in $D(\mathfrak{U}_2)$ by Proposition 6.3. Let Ψ be defined as in Proposition 6.5 (with an interchange of the roles of $\mathfrak{U}_1$ and $\mathfrak{U}_2$). Then if $\mathfrak{U}_2' = \Psi\mathfrak{U}_2$, $UV = \Psi V$ is a 1-involution of $\mathfrak{G}$ which belongs to $\mathfrak{U}_2'$. Now $C^*(U, UV)$ has the decomposition of $C^*(U, -V)$ in (7.13) and UV

$\in \mathfrak{W}_1''$, which has rank 1. Thus we may obtain (7.3) by taking $V' = UV$ and $\mathfrak{W}_1' = \mathfrak{W}_1''$ and $\mathfrak{W}_2' = \mathfrak{W}_2''$ in (7.13).

In the case (7.12), V is an $(n-2)$-involution. Then $-V$ is a 2-involution. Furthermore, $-V = U(-UV)$. As in the previous paragraph, we may apply Lemma 7.2 to obtain that $-UV$ is conjugate to U and is a 1-involution of $\mathfrak{G}$. Relative to the decomposition (7.4), $-UV$ is represented by $1 \times 1 \times -1$ and, therefore, is in the center of $\mathfrak{W}_2$, which has odd rank $n-2$. Proposition 6.3 implies that $-UV$ is not in $D(\mathfrak{U}_2)$. Thus we may apply Proposition 6.4 to the decomposition of $T(C_{\mathfrak{U}_2}(V))$ to obtain a decomposition (7.3) where $\mathfrak{W}_1' = \Omega\mathfrak{W}_1$ and $\mathfrak{W}_2' = \Omega\mathfrak{W}_2$. Here $\mathfrak{W}_1'$ contains $\Omega V = -UV$ in its center. Hence we take $\mathfrak{U}_2' = \mathfrak{U}_2$ and $V' = -UV$ in this case.

PROPOSITION 7.3. *Let U and V' be as in Proposition* 7.1. *Then UV' is a* 2-*involution in* $\mathfrak{G}$.

Proof. In proving Proposition 7.1, we chose $V' = V$ when n was even and showed that $UV' = UV$ is a 2-involution. Therefore, assume that n is odd. Refer to the decomposition (7.3); here $\mathfrak{W}_1' \times \mathfrak{W}_2'$ is a seminormal decomposition of the involutory centralizer $T(C_{\mathfrak{U}_2'}(V'))$ in $\mathfrak{U}_2'$. Taking, for the moment, V' to be a 1-involution of $\mathfrak{U}_2'$ and of $\mathfrak{G}$ and using Proposition 7.1, we may find an involution X in $\mathfrak{W}_2'$ which is conjugate to V' in $\mathfrak{U}_2'$ and for which we have the decomposition

$$T(C_{\mathfrak{U}_2'}(V', X)) = \mathfrak{W}_1' \times \mathfrak{X}_1 \times \mathfrak{X}_2, \tag{7.14}$$

where rk $\mathfrak{X}_1 = 1$ and rk $\mathfrak{X}_2 = n-3$ and $X \in \mathfrak{X}_1$. From (7.14) we have

$$C^*(U, V', X) = \mathfrak{U}_1 \times \mathfrak{W}_1' \times \mathfrak{X}_1 \times \mathfrak{X}_2. \tag{7.15}$$

Here $\mathfrak{U}_1$, $\mathfrak{W}_1'$, and $\mathfrak{X}_1$ are the unique groups of order 2 containing the 1-involutions U, V', and X, respectively. Because X is conjugate to V', it is a 1-involution. Thus in forming a decomposition of $C^*(X, UV')$ by refining a decomposition of the involutory centralizer of the 1-involution X, only three groups appear in the decomposition, one of which must be X_1; we obtain

$$C^*(X, UV') = \mathfrak{X}_1 \times \mathfrak{Y}_1 \times \mathfrak{Y}_2. \tag{7.16}$$

Now form $C^*(X, UV', U) = C^*(U, V', X)$ by refining (7.16). Comparison with (7.15) shows that only four groups appear in the decomposition given by this refinement; that is, only one of the decomposition groups in (7.16) decomposes. We may suppose that this group is $\mathfrak{Y}_2$. Thus we obtain

$$C^*(X, UV', U) = \mathfrak{X}_1 \times \mathfrak{Y}_1 \times \mathfrak{Z}_1 \times \mathfrak{Z}_2. \tag{7.17}$$

Furthermore, because (7.17) is a refinement of (7.16), it follows that the involutions in the center of $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are not in the center of $C^*(X, UV')$. But $-UV'X$ is an even involution in the center of $\mathfrak{X}_2$ in (7.15). By Proposition 5.1, it is in the center of a decomposition group in (7.17). Since $-UV'X$

also belongs to the center of $C^*(X, UV')$, it does not belong to the center of $\mathfrak{Z}_1$ or of $\mathfrak{Z}_2$. Since $X \in \mathfrak{X}_1$, $-UV'X$ is in the center of $\mathfrak{Y}_1$. Now comparing the decompositions (7.15) and (7.17), we see that the decomposition groups not containing $-UV'X$ have rank 1. Thus rk $\mathfrak{Z}_1 =$ rk $\mathfrak{Z}_2 = 1$ and rk $\mathfrak{Y}_2 = 2$. Since the decomposition (7.16) is seminormal, Y_2 contains $-X(-UV'X) = UV'$. Therefore, UV' is a 2-involution; the proposition is proved.

Proposition 7.4. *Let U be a 2-involution and let*

$$C^*(U) = \mathfrak{U}_1 \times \mathfrak{U}_2 \tag{7.18}$$

be a seminormal decomposition with $U \in \mathfrak{U}_1$. Let V be a proper involution of the group $\mathfrak{U}_1$. Then V is an extremal involution of $\mathfrak{G}$ and U is a product of two 1-involutions, which may be taken to be V and UV or $-V$ or $-UV$.

Proof. To prove the proposition, we assume that V is not an extremal involution and arrive at a contradiction. The proposition is trivial when $m=2$, and only the last statement need be proved when $n=3$. Therefore, assume that $n>3$ to prove the first statement of the proposition. Form the refinement of (7.18)

$$C^*(U, V) = \mathfrak{U}_1' \times \mathfrak{U}_1'' \times \mathfrak{U}_2 \tag{7.19}$$

where $V \in \mathfrak{U}_1'$ and rk $\mathfrak{U}_1' =$ rk $\mathfrak{U}_1'' = 1$. By Lemma 7.2, V and UV are conjugate. Form the seminormal decomposition

$$C^*(V) = \mathfrak{V}_1 \times \mathfrak{V}_2. \tag{7.20}$$

Since V is not an extremal involution and $n>3$, rk $\mathfrak{V}_1 > 1$ and rk $\mathfrak{V}_2 > 1$. Because there are but three decomposition groups in (7.19), we obtain only three decomposition groups in forming $C^*(V, U)$ by refining (7.20) and two of these groups have rank 1. Since rk $\mathfrak{V}_1 > 1$ and rk $\mathfrak{V}_2 > 1$, these are subgroups either of $\mathfrak{V}_1$ or of $\mathfrak{V}_2$; we may assume that they are subgroups of $\mathfrak{V}_1$. Thus we obtain

$$C^*(V, U) = \mathfrak{V}_1' \times \mathfrak{V}_1'' \times \mathfrak{V}_2 \tag{7.21}$$

where rk $\mathfrak{V}_1' =$ rk $\mathfrak{V}_1'' = 1$. Replacing V by $-V$, if necessary, we have that $V \in \mathfrak{V}_1$ and rk $\mathfrak{V}_1 = 2$. Hence V will be a 2-involution.

Now in (7.19), $-U$ is in the center of $\mathfrak{U}_2$; and in (7.21), $-V$ is in the center of $\mathfrak{V}_2$. If n is even, rk $\mathfrak{U}_2 =$ rk $\mathfrak{V}_2 = n-2$ is even, and it follows from Proposition 5.1 that $-U$ is in the center of $\mathfrak{V}_2$. This means that $-U = -V$, which is impossible.

Thus we may assume that n is odd. Then $n \geqq 5$, and rk $\mathfrak{U}_2 =$ rk $\mathfrak{V}_2 \geqq 3$. We choose a 1-involution W in the decomposition group $\mathfrak{U}_2$ of odd rank. Then we obtain from (7.19) the decomposition

$$C^*(U, V, W) = \mathfrak{U}_1' \times \mathfrak{U}_1'' \times \mathfrak{U}_2' \times \mathfrak{U}_2'' \tag{7.22}$$

where rk $\mathfrak{U}_2'=1$, rk $\mathfrak{U}_2''=n-3$ is even, and $W\in\mathfrak{U}_2'$. In forming the decomposition of $C^*(V, U, W)$ by refining (7.21), there is only one possibility by comparison with (7.22). Thus we obtain

$$C^*(V, U, W) = \mathfrak{V}_1' \times \mathfrak{V}_1'' \times \mathfrak{V}_2' \times \mathfrak{V}_2'' \tag{7.23}$$

where rk $\mathfrak{V}_2'=1$ and rk $\mathfrak{V}_2''=n-3$. Proposition 5.1 implies that $-UW$ is in the center of $\mathfrak{V}_2''$ as it is in the center of $\mathfrak{U}_2''$.

When we form the decomposition $C^*(U, W)$ obtained from refining (7.18), we obtain

$$C^*(U, W) = \mathfrak{U}_1 \times \mathfrak{U}_2' \times \mathfrak{U}_2''. \tag{7.24}$$

Then the decomposition of $C^*(W, U)$ obtained from refining a seminormal decomposition $C^*(W)=\mathfrak{W}_1\times\mathfrak{W}_2$ may be taken to be of the form

$$C^*(W, U) = \mathfrak{W}_1 \times \mathfrak{W}_2' \times \mathfrak{W}_2''. \tag{7.25}$$

Here, by virtue of Proposition 5.1, U and $-UW$ are in the centers of decomposition groups of rank 2 and $n-3$, respectively. Since neither of these involutions is in the center of $C^*(W)$, U and $-UW$ belong to the centers of $\mathfrak{W}_2'$ and $\mathfrak{W}_2''$. Therefore, $W\in\mathfrak{W}_1$, which must have rank 1. Hence W is a 1-involution in $\mathfrak{G}$.

Next form the decomposition of $C^*(W, V)$ by refining a decomposition of $C^*(W)$ to obtain

$$C^*(W, V) = \mathfrak{W}_1 \times \mathfrak{X}_1 \times \mathfrak{X}_2. \tag{7.26}$$

In forming the decomposition of $C^*(W, V, U)$ by refining (7.26), we see by comparison with (7.22) that only one of the decomposition groups in (7.26) may decompose and we must obtain three decomposition groups of ranks 1 and one of rank $n-3$. There are two possibilities in (7.26): either one of the groups has rank 1 and the other has rank $n-2\geqq 3$ or that one has rank 2. We may suppose that rk $\mathfrak{X}_1=1$ or 2. In the former case, there is only one way of refining (7.20) so that we may obtain two decomposition groups of rank 1; namely,

$$C^*(V, W) = \mathfrak{Y}_1' \times \mathfrak{Y}_1'' \times \mathfrak{V}_2. \tag{7.27}$$

But $U\in C^*(V, W)$ and (7.21) shows that the component of U in $\mathfrak{V}_2$ relative to the decomposition (7.20) is in the center of $\mathfrak{V}_2$. Thus in (7.27), U has a representation $U=U_1'\times U_1''\times U_2$ where U_2 is in the center of $\mathfrak{V}_2$. Clearly U_1' and U_1'' are in the centers of $\mathfrak{Y}_1'$ and $\mathfrak{Y}_1''$, respectively. This means that $C^*(V, W)=C^*(V, W, U)=C^*(U, V, W)$; comparing (7.27) and (7.22), we obtain a contradiction. Thus we may assume that rk $\mathfrak{X}_1=2$ in (7.26). Then in forming a decomposition of $C^*(V, W)$ from (7.20), we obtain

$$C^*(V, W) = \mathfrak{V}_1 \times \mathfrak{Y}_2' \times \mathfrak{Y}_2'' \tag{7.28}$$

where rk $\mathfrak{B}_1=2$, rk $\mathfrak{Y}_2'=1$ and rk $\mathfrak{Y}_2''=n-3$. Also V is in the center of $\mathfrak{B}_1$; hence by Proposition 5.1, V is in the center of $\mathfrak{X}_1$. Then $-VW$ is in the center of $\mathfrak{X}_2$. In refining (7.26) to form $C^*(W, V, U)$, we see by comparison with (7.22) that only one decomposition group decomposes and that this must be $\mathfrak{B}_1$. We will then obtain that the only decomposition group of even rank in this decomposition is $\mathfrak{Y}_2''$. By Proposition 5.1, $\mathfrak{Y}_2''$ and $\mathfrak{U}_2''$ contain the same element of order 2 in their centers. Thus $-UW=-VW$ and $U=V$. This is a contradiction to the assumption that V is not an extremal involution. The remainder of the proposition follows directly from Lemma 7.2.

PROPOSITION 7.5. *Every maximal set M of mutually commuting involutions is a group of order 2^n generated by a subset $\mathfrak{N}$ of 1-involutions $V_1, V_2, \cdots, V_n$. Furthermore,*

$$\mathfrak{M} = C^*(\mathfrak{M}) = C^*(\mathfrak{N}) = \mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_n \tag{7.29}$$

is a direct product of involutory quasilinear groups $\mathfrak{X}_i$ of rank 1 which contain, respectively, the involutions V_i, $i=1, 2, \cdots, n$.

Of course, involutory quasilinear groups of rank 1 are groups of order 2. A set $\mathfrak{N}$ of 1-involutions for which (7.29) holds is always a set of generators of $\mathfrak{M}$ as an abelian group. We will call such sets *complete sets of mutually commuting 1-involutions.* Later we will show that they are unambiguously determined by the maximal sets of involutions which they generate.

Proof. We first show that $\mathfrak{M}$ contains a 2-involution. Let W_1 be a proper involution in $\mathfrak{M}$ and form the seminormal decomposition $C^*(W_1)=\mathfrak{W}_1'\times\mathfrak{W}_1''$. For one of these groups, say $\mathfrak{W}_1'$, rk $\mathfrak{W}_1'\geqq 2$. If rk $\mathfrak{W}_1'>2$, let W_2 be in $\mathfrak{M}$ and form the seminormal decomposition $T(C_{W_1'}(W_2))=\mathfrak{W}_2'\times\mathfrak{W}_2''$. Again for one of these groups, say $\mathfrak{W}_2'$, rk $\mathfrak{W}_2'\geqq 2$. If rk $\mathfrak{W}_2'>2$, continue on in this manner. Clearly after m steps we will arrive at a decomposition $T(C_{\mathfrak{W}_{m-1}'}(W_m))$ $=\mathfrak{W}_m'\times\mathfrak{W}_m''$ where now one of these groups, say $\mathfrak{W}_m'$, has rank 2. Then $\mathfrak{W}_m'$ is a decomposition group in a decomposition for $C^*(W_1, W_2, \cdots, W_m)$ formed by first forming the seminormal decomposition for $C^*(W_1)$ and then refining this successively to decompositions of $C^*(W_1, W_2)$, $C^*(W_1, W_2, W_3)$, $\cdots$, $C^*(W_1, W_2, \cdots, W_m)$. Let T be the involution in the center of $\mathfrak{W}_m'$. Then T is in the center of $C^*(W_1, W_2, \cdots, W_m)$. Form by refining $C^*(W_1, W_2, \cdots, W_m)$ and using Proposition 4.1

$$C^*(\mathfrak{M}) = C^*(W_1, W_2, \cdots, W_m, \mathfrak{M}) = \mathfrak{Y}_1 \times \mathfrak{Y}_2 \times \cdots \times \mathfrak{Y}_n, \tag{7.30}$$

where rk $\mathfrak{Y}_i=1$. The elements of the groups $\mathfrak{Y}_i$ certainly generate the abelian group $\mathfrak{M}$ of order 2^n and $T\in\mathfrak{M}$.

Next form a seminormal decomposition

$$C^*(T) = \mathfrak{T}_1 \times \mathfrak{T}_2 \tag{7.31}$$

where $T\in\mathfrak{T}_1$. Refine (7.31) to a decomposition of $C^*(\mathfrak{M})=C^*(T, \mathfrak{M})$. It is clear that this decomposition contains quasilinear subgroups of rank 1 of $\mathfrak{T}_1$.

Let V_1 be the element of order 2 in one of these subgroups. By Proposition 7.4, V_1 is an extremal involution in $\mathfrak{G}$. Then either V_1 or $-V_1$ is a 1-involution. But $-V_1$ is also in $\mathfrak{M}$ as -1 certainly is. Therefore, by replacing V_1 by $-V_1$, if necessary, we may suppose that V_1 is a 1-involution.

Form next a seminormal decomposition

$$C^*(V_1) = \mathfrak{X}_1 \times \mathfrak{Y}_1 \tag{7.32}$$

where $V_1 \in \mathfrak{X}_1$ and $\mathrm{rk}\, \mathfrak{X}_1 = 1$. Then $\mathfrak{Y}_1$ is a quasilinear group of rank $n-1$. Forming a decomposition of $C^*(\mathfrak{M}) = C^*(V_1, \mathfrak{M})$ by refining (7.32), we obtain

$$C^*(\mathfrak{M}) = \mathfrak{X}_1 \times \mathfrak{X}_2' \times \mathfrak{X}_3' \times \cdots \times \mathfrak{X}_n', \tag{7.33}$$

where $\mathfrak{X}_i'$, $i = 2, 3, \cdots, n$, are linear subgroups of $\mathfrak{Y}_1$ of rank 1. The elements of order 2 in these groups generated a maximal set $\mathfrak{M}'$ of mutually commuting involutions of $\mathfrak{Y}_1$ containing 2^{n-1} elements. The argument given above for $\mathfrak{M}$ now tells us that $\mathfrak{M}'$ contains an extremal involution V_2' of $\mathfrak{Y}_1$. Proposition 7.1 implies that one of $\pm V_2'$ or $\pm V_1 V_2'$ is a 1-involution V_2 of $\mathfrak{Y}_1$ and that there exists a decomposition

$$C^*(V_1, V_2) = \mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{Y}_2 \tag{7.34}$$

where $\mathrm{rk}\, \mathfrak{X}_2 = 1$ and $\mathfrak{X}_2$ contains V_2. We may repeat the above process until we obtain a decomposition group $\mathfrak{Y}_{n-2}$ of rank 2.

Then by Proposition 7.4, V_{n-1}' or $-V_{n-1}'$ is a 1-involution of $\mathfrak{G}$ in $\mathfrak{Y}_{n-1}$. Here if the latter case occurs, replace $\mathfrak{Y}_{n-1}$ by $\mathfrak{Y}_{n-1}' = \Omega \mathfrak{Y}_{n-1}$ where Ω is the automorphism of $C^*(V_1 V_2)$ described[6] in Proposition 6.5. Then V_{n-1}' is not in the commutator group of the quasilinear group $\mathfrak{Y}_{n-1}$ of rank 2. Therefore, as in Proposition 6.5, $\Omega V_{n-1}' = -V_{n-1}' = V_{n-1}$ is in $\mathfrak{Y}_{n-1}$. If T_{n-1} is the element of order 2 in the center of $\mathfrak{Y}_{n-1}$, $T_{n-1} = V_{n-1} V_n$ where V_n is an involution in $\mathfrak{Y}_{n-1}$ which commutes with V_{n-1} and which is a 1-involution in $\mathfrak{G}$ by Lemma 7.2. Therefore, $T(C_{\mathfrak{Y}_{n-1}}(V_{n-1}, V_n)) = \mathfrak{X}_{n-1} \times \mathfrak{X}_n$ and we may obtain (7.29). This proves the proposition.

We remark that the involution V_1 has been chosen so that $W_i = V_1 V_i$, $i = 1, 2, \cdots, n$, is a 2-involution by virtue of Proposition 7.3.

8. **Standard decompositions. Classification of involutions.** Let U be a proper involution of $\mathfrak{G}$. Then U is contained in a maximal set of mutually commuting involutions, which contains in turn a complete set $\mathfrak{N}$ of mutually commuting 1-involutions. We may suppose that (7.29) holds. Then relative to (7.29),

$$U = U_1 \times U_2 \times \cdots \times U_n.$$

By rearranging the components of (7.29), we may suppose that $U_i = -1_i = V_i$, $i = 1, 2, \cdots, p$ and $U_i = 1_i$, $i = p+1, p+2, \cdots, n$. Then $U = V_1 V_2 \cdots$

(6) Of course, now Ω must be defined to be the identity on the factors $\mathfrak{X}_i$, $1 \leq i \leq n-2$, and to be Ω_{n-1} on $\mathfrak{Y}_{n-1}$.

V_p and $-U=V_{p+1}V_{p+2}\cdots V_n$. If necessary, for the convenience of notation we replace U by $-U$ in the following considerations so that $\mathfrak{X}_1$ is not altered in the rearrangement of the components of (7.29).

A *standard decomposition of* $C^*(U)$ *relative to a complete set* $\mathfrak{N}$ *of mutually commuting* 1-*involutions which belong to* $C^*(U)$ is a decomposition $C^*(U)=\mathfrak{U}_1\times\mathfrak{U}_2$ in which the elements of $\mathfrak{N}$ belong to $\mathfrak{U}_1$ or $\mathfrak{U}_2$.

PROPOSITION 8.1. *Let U be a proper involution and $\mathfrak{N}$ a complete set of mutually commuting* 1-*involutions in $C^*(U)$ arranged as above. Then there exists a standard decomposition $C^*(U)=\mathfrak{U}_1\times\mathfrak{U}_2$ where $V_1, V_2, \cdots, V_p$ are conjugate extremal involutions of $\mathfrak{U}_1$ and $V_{p+1}, V_{p+2}, \cdots, V_n$ are conjugate extremal involutions of $\mathfrak{U}_2$.*

Proof. The proof will be developed by presenting several auxiliary propositions.

(8.1a). *Any two of the involutions V_h and V_i are conjugate by an involution T_{hi} which commutes with the other involutions of $\mathfrak{N}$. If $h, i\leqq p$ or if $h, i>p$, $T_{hi}\in C^*(U)$.* Indeed, we mentioned at the end of the proof of Proposition 7.5 that $W_i=V_1V_i$, $i=2, 3, \cdots, n$, is a 2-involution. Form the seminormal decomposition

$$C^*(W_i) = \mathfrak{W}_i' \times \mathfrak{W}_i'' \tag{8.1}$$

where $W_i\in\mathfrak{W}_i'$. Let $k\neq 1, i$. Relative to (8.1), we have the representations $V_1=V_1'\times V_1''$, $V_i=V_i'\times V_i''$, $V_k=V_k'\times V_k''$. Form the decomposition of $C^*(V_k, W_i)$ by refining a seminormal decomposition of $C^*(V_k)$; because V_k is a 1-involution, this decomposition can have but three factors. The same is then true for the decomposition of $C(W_i, V_k)$ obtained by refining (8.1). This means that either V_k' or V_k'' is in the center of the corresponding group $\mathfrak{W}_i'$ or $\mathfrak{W}_i''$. On the other hand, as we showed in the proof of Lemma 7.2, V_1' and V_i' are proper involutions of the quasilinear group $\mathfrak{W}_i'$ of rank 2. Therefore, $T(C_{\mathfrak{W}_i'}(V_1'))=\mathfrak{Y}_1\times\mathfrak{Y}_2$ and contains but four involutions in its center. Suppose that V_k' is a proper involution in $\mathfrak{W}_i'$; then $V_k'\in T(C_{\mathfrak{W}_i'}(V_1))$ as $V_k\in C(V_1)$. Thus $V_k'=V_1'$ or $V_k'=V_i'$. One may see similarly that V_1'' is in the center of $\mathfrak{W}_i''$. Hence we obtain that $W_k=V_1V_k$ is in the center of $C^*(W_i)$. Clearly $W_k\neq\pm 1$ and $W_k\neq W_i$. But if $W_k=-W_i$, $V_k=-V_i$, which is a contradiction because the 1-involution V_k would be conjugate to the $(n-1)$-involution $-V_i$, contrary to Proposition 5.6.

Therefore, V_k' must be in the center of $\mathfrak{W}_i'$. But by Lemma 7.2, V_1 and V_i are conjugate by an involution T_i in $\mathfrak{W}_i'$. Hence relative to (8.1), $T_i=T_i'\times 1$ and $T_iV_kT_i^{-1}=V_k$. Set $T_{1i}=T_i$. For $h\neq 1, i$, set $T_{hi}=T_hT_iT_h$. It may be verified that V_i and V_h are conjugate by the involution T_{hi} and that T_{hi} commutes with V_k, $k\neq h, i$.

Finally let $h, i\leqq p$. Since $U=V_1V_2\cdots V_p$ contains the factors V_i and V_h, $T_{hi}UT_{hi}^{-1}=U$ and $T_{hi}\in C^*(U)$. If $h, i>p$, then $T_{hi}(-U)T_{hi}^{-1}=-U$ for the same reason. Hence $T_{hi} \in C^*(-U) = C^*(U)$.

(8.1b). *Let*

$$C^*(U) = \mathfrak{U}_1 \times \mathfrak{U}_2 \tag{8.2}$$

be a seminormal decomposition. Then either the involutions V_i, $i \leqq p$, *or the involutions* $-V_i$, $i \leqq p$, *are conjugate extremal involutions of one of the decomposition groups* $\mathfrak{U}_l$ *while either the involutions* V_j, $j > p$, *or the involutions* $-V_j$, $j > p$, *are conjugate extremal involutions in the other decomposition group* $\mathfrak{U}_m$. We first consider that $V_1 \neq U$. Because V_1 is a 1-involution, there are but three decomposition groups in a decomposition of $C^*(U, V_1)$. By interchanging the decomposition groups $\mathfrak{U}_1$ and $\mathfrak{U}_2$ in (8.2), if necessary, we may suppose that

$$C^*(U, V_1) = \mathfrak{U}_1' \times \mathfrak{U}_1'' \times \mathfrak{U}_2 \tag{8.3}$$

is a decomposition of $C^*(U, V_1)$ obtained by refining (8.2). Then let $V_1 = V_1' \times V_1''$ be the representation of V_1 relative to (8.2). As V_1 is in the center of $C^*(U, V_1)$, it follows from (8.3) that V_1'' is in the center of $\mathfrak{U}_2$. Thus $V_1'' = \pm 1_2$. This means that either $\pm V_1$ is an involution X_1 in $\mathfrak{U}_1$. Let $i \leqq p$, then $T_{1i} \in C^*(U)$. Hence $T_{1i} = T_{1i}' \times T_{1i}''$ relative to (8.2). This means that $X_i = T_{1i} X_1 T_{1i}^{-1} = T_{1i}' X_i T_{1i}'^{-1}$ has the representation $X_i = X_i' \times 1$ relative to (8.2) and is in $\mathfrak{U}_1$. Furthermore, $X_i = V_i$ or $-V_i$ according as $X_1 = V_1$ or $-V_i$. Also X_1 and X_i are conjugate by an involution T_{1i}' in $\mathfrak{U}_1$. Because of (8.3), they are extremal involutions of $\mathfrak{U}_1$.

If $V_1 = U$, we take $X_1 = V_1$.

A similar argument shows that $X_j = V_j$ or $-V_j$, $j > p$, is in one of the decomposition groups $\mathfrak{U}_1$ or $\mathfrak{U}_2$ according as $X_n = V_n$ or $-V_n$ is in the same decomposition group. Also we may conclude that they are conjugate extremal involutions of that group. We claim that X_n is in $\mathfrak{U}_2$. Indeed, if this is not the case, either $W_n = V_1 V_n = (-V_1)(-V_n)$ or $-W_n = (-V_1) V_n = V_1(-V_n)$ is in $\mathfrak{U}_1$; that is, either $W_n \in \mathfrak{U}_1$ or $W_n \in \mathfrak{U}_2$. In either case we obtain upon refining (8.2)

$$C^*(U, W_n) = \mathfrak{U}_1' \times \mathfrak{U}_1'' \times \mathfrak{U}_2 \quad \text{or} \quad C^*(U, W_n) = \mathfrak{U}_1 \times \mathfrak{U}_2' \times \mathfrak{U}_2'' \tag{8.4}$$

where $\mathfrak{U}_1'$ or $\mathfrak{U}_2'$ has rank 2 and contains W_n in its center. Form now a seminormal decomposition of $C^*(W_n, U)$ by refining a seminormal decomposition (8.1) of $C^*(W_n)$ to obtain from Lemma 4.2 and Proposition 5.1

$$C^*(W_n, U) = \mathfrak{W}_n' \times \mathfrak{Y}_1 \times \mathfrak{Y}_2. \tag{8.5}$$

From (8.5), the involution U has a component relative to (8.1) which is in the center of $\mathfrak{W}_n'$. Furthermore, V_1 and V_n are conjugate by an involution $T_{1n} \in \mathfrak{W}_n'$ which commutes with the involutions V_k, $k \neq 1, n$. Since $T_{1n} \in \mathfrak{W}_n'$, T_{1n} also commutes with U. On the other hand, $U = V_1 V_2 \cdots V_p$ and $-U = V_{p+1} V_{p+2} \cdots V_n$. This shows that $T_{1n} \notin C^*(U)$, which is a contradiction. Thus X_n is in $\mathfrak{U}_2$ along with X_j, $j > p$.

Throughout the remainder of the proof, we will preserve our choice of U_1 as the decomposition group containing $X_1, X_2, \cdots, X_p$ and U_2 as the decomposition group containing $X_{p+1}, X_{p+2}, \cdots, X_n$.

(8.1c). *Let n be even and let U be an odd involution. Then $X_1, X_2, \cdots, X_p$ are 1-involutions of $\mathfrak{U}_1$ and $X_{p+1}, X_{p+2}, \cdots, X_n$ are 1-involutions of $\mathfrak{U}_2$.* We need only show the first statement for the involution X_1. If U is a 1-involution, we have taken $X_1 = V_1 = U$; so there is nothing more to be shown. Assume, therefore, that rk $\mathfrak{U}_1 > 1$. Again refer to (8.3). We may assume that rk $\mathfrak{U}_1' = 1$. Then rk $\mathfrak{U}_1''$ is even. Hence either X_1 or $(-1_1)X_1$ is in $\mathfrak{U}_1''$ where -1_1 is in the center of $\mathfrak{U}_1$. If the former case holds, then X_1 is an even involution in $\mathfrak{G}$ as well as in $\mathfrak{U}_1$. This is impossible as n being even implies that both V_1 and $-V_1$ are odd involutions. Hence $X \in \mathfrak{U}_1'$ and X_1 is a 1-involution of the group $\mathfrak{U}_1$ of odd rank.

(8.1d). *For the decomposition groups* $\mathfrak{U}_1$ *and* $\mathfrak{U}_2$, rk $\mathfrak{U}_1 = p$ *and* rk $\mathfrak{U}_2 = n - p$. Now the group $\mathfrak{U}_1$ contains p mutually commuting conjugate extremal involutions of $\mathfrak{G}$. We may take these extremal involutions or their negative in $\mathfrak{U}_1$ to be 1-involutions in $\mathfrak{U}_1$. They generate an abelian group of order 2^p, for any relation among these involutions would imply a corresponding relation among the involutions $V_1, V_2, \cdots, V_p$ in the complete set $\mathfrak{N}$, which is impossible. But then Proposition 4.1 implies that rk $\mathfrak{U}_1 \geqq p$. Similarly, rk $\mathfrak{U}_2 \geqq n-p$. But rk $\mathfrak{U}_1$+rk $\mathfrak{U}_2 = n$. Hence rk $\mathfrak{U}_1 = p$ and rk $\mathfrak{U}_2 = n-p$.

Now we are ready to prove the proposition. We first treat the case where n is odd. Without loss of generality, we may suppose that $p =$ rk $\mathfrak{U}_1$ is even and that $n-p =$ rk $\mathfrak{U}_2$ is odd. Then $\mathfrak{U}_1$ contains $X_1X_2 \cdots X_p = (\pm 1)^p U = U$. This implies that $X_j = V_j, j > p$, for otherwise $\mathfrak{U}_2$ would contain $X_{p+1}X_{p+2} \cdots X_n = (-1)^{n-p}(-U) = U$, which is in $\mathfrak{U}_1$. If $X_i = V_i, i \leqq p$, (8.2) is the standard decomposition which we seek. If $X_i = -V_i, i \leqq p$, form the decomposition $C^*(U) = \mathfrak{U}_1' \times \mathfrak{U}_2$ using the automorphism Ω of Proposition 6.5. As X_i is an extremal involution in the quasilinear group $\mathfrak{U}_1$ of even rank, it does not belong to its commutator subgroup by Proposition 6.3. Then $\Omega X_i = -X_i = V_i$ is in $\mathfrak{U}_1' = \Omega\mathfrak{U}_1$ for $i = 1, 2, \cdots, p$. Thus again we have obtained a standard decomposition.

Next consider the case where n is even and take U to be an even involution. Then p and $n-p$ are even. Here $\mathfrak{U}_1$ contains $X_1X_2 \cdots X_p = U$ and $\mathfrak{U}_2$ contains $X_{p+1}X_{p+2} \cdots X_n = -U$. We apply the automorphism Ω of Proposition 6.5 to $\mathfrak{U}_1$ if $X_i = -V_i, i \leqq p$ and the automorphism $1 \times \Omega_2$ to $\mathfrak{U}_2$ if $X_j = -V_j$, $j > p$. As above we obtain a standard decomposition.

Secondly take U to be an odd involution and n to be even. Then $\mathfrak{U}_1$ contains $\pm U$ according as $X_i = \pm V_i, i \leqq p$ and $\mathfrak{U}_2$ contains $-U$ or U according as $X_j = V_j$ or $-V_j, j > p$. As (8.2) is seminormal, either $X_i = V_i, i \leqq p$, and $X_j = V_j, j > p$, or $X_i = -V_i, i \leqq p$, and $X_j = -V_j, j > p$. If the former case occurs, then (8.2) is the standard decomposition which we seek. If the latter case occurs, then we apply the automorphism Ω of Proposition 6.4 to obtain a decomposition $C^*(U) = \mathfrak{U}_1' \times \mathfrak{U}_2'$. By (8.1c), $X_i, i \leqq p$, and $X_j, j > p$, are

1-involutions in $\mathfrak{U}_1$ and $\mathfrak{U}_2$, respectively. Therefore, they are not in $D(\mathfrak{U}_1)$ or $D(\mathfrak{U}_2)$ by Proposition 6.3. Then $\Omega X_i = -X_i = V_i \in \mathfrak{U}_1'$ for $i \leqq p$ and $\Omega X_j = -X_j = V_j \in \mathfrak{U}_2'$ for $j > p$. Thus we have obtained a standard decomposition in this final case.

The involution U in Proposition 8.1 is the product of p 1-involutions from the set $\mathfrak{N}$. Statement (8.1d) shows that $U \in \mathfrak{K}_p$ or $U \in \mathfrak{K}_{n-p}$. When $p=1$ or $n-1$, it is clear that U is a 1-involution or an $(n-1)$-involution. In general, if p is even or if n is odd, U is a p-involution. Under these circumstances, p determines the conjugate class to which U belongs. We will show that this is true in general.

Corollary 8.2. *Let $\mathfrak{M}$ be a maximal set of mutually commuting involutions. Let $\mathfrak{N}$ be a complete set of mutually commuting* 1*-involutions in $\mathfrak{M}$. Then $\mathfrak{N}$ contains all the* 1*-involutions of $\mathfrak{G}$ in $\mathfrak{M}$.*

Proof. Let $\mathfrak{N}$ consist of the elements $V_1, V_2, \cdots, V_n$. Then any involution of $\mathfrak{M}$ not in $\mathfrak{N}$ is a product of $p > 1$ of these involutions and thus belongs to $\mathfrak{K}_p$ or $\mathfrak{K}_{n-p}$. Hence U cannot be a 1-involution unless possibly $p = n-1$. But in this case, U is an $(n-1)$-involution since $-U$ will be one of the involutions $V_i \in \mathfrak{N}$.

Proposition 8.3. *Let $\mathfrak{N}$ be a complete set of mutually commuting* 1*-involutions $V_1, V_2, \cdots, V_n$ which commute with a proper involution U. Let*

$$C^*(U) = \mathfrak{U}_1 \times \mathfrak{U}_2 \tag{8.6}$$

be a standard decomposition where $V_1, V_2, \cdots, V_p$ are in $\mathfrak{U}_1$ and $V_{p+1}, V_{p+2}, \cdots, V_n$ are in $\mathfrak{U}_2$. Then if $\mathfrak{N}'$ is a second complete set of mutually commuting 1*-involutions $V_1', V_2', \cdots, V_n'$ which commute with U, we may arrange the elements of this set so that $V_1', V_2', \cdots, V_p'$ are in $\mathfrak{U}_1$ and $V_{p+1}', V_{p+2}', \cdots, V_n'$ are in $\mathfrak{U}_2$.*

Proof. By the argument of the preceding proposition, we may arrange the involutions of $\mathfrak{N}'$ so that either V_i', $i \leqq r$, or $-V_i'$, $i \leqq r$, are extremal involutions X_i' in $\mathfrak{U}_1$. By (8.1d), $r = \text{rk } \mathfrak{U}_1 = p$. We thus obtain the statement of the proposition for the involutions V_i', $i \leqq p$, if $X_i' = V_i'$. If $X_i' = -V_i'$, $i \leqq p$, are extremal involutions in $\mathfrak{U}_1$, then either V_1 or $(-1_1)V_1 = UV_1$ is conjugate in $\mathfrak{U}_1$ to $X_i' = -V_i'$. But V_1 and $-V_i'$ cannot be conjugate for V_1 is a 1-involution and $-V_i'$ is an $(n-1)$-involution. If UV_1 is conjugate to $-V_i'$, then $-UV_1 = V_1V_{p+1}V_{p+2} \cdots V_n$ is a 1-involution in $\mathfrak{M}$. This contradicts Corollary 8.2. Hence $X_i' \neq V_i'$. A similar argument shows that V_j', $j > p$, belong to $\mathfrak{U}_2$ and proves the proposition.

Corollary 8.4. *Let U be a proper involution and let $C^*(U)$ possess the standard decomposition* (8.6). *Then* (8.6) *is a standard decomposition relative to any complete set of mutually commuting* 1*-involutions which commute with U.*

The 1-*involutions in* $C^*(U)$ *belong to* $\mathfrak{U}_1$ *or* $\mathfrak{U}_2$. *The decomposition groups of a standard decomposition are uniquely determined.*

Proof. We need only mention that the 1-involutions in $C^*(U)$ belong to either $\mathfrak{U}_1$ or $\mathfrak{U}_2$ because such involutions can always be included in a maximal set of mutually commuting involutions of $\mathfrak{G}$ contained in $C^*(U)$ and to this set we may apply Corollary 8.2 and Proposition 8.3. Also the groups $\mathfrak{U}_1$ and $\mathfrak{U}_2$ in a standard decomposition are uniquely determined because they are involutory groups which can now be seen to be generated by the 1-involutions which they must contain.

Because of this corollary, we henceforth speak merely of a *standard decomposition* rather than of a standard decomposition relative to a particular complete set of mutually commuting 1-involutions. Because of the uniqueness of the decomposition groups in a standard decomposition, we will always choose the notation $\mathfrak{U}^+$ and $\mathfrak{U}^-$ for the decomposition groups of a standard decomposition with the convention that $U\in\mathfrak{U}^+$ and $-U\in\mathfrak{U}^-$. Such decomposition groups will be called *standard decomposition groups* of $C^*(U)$ or of U. In particular, the group $\mathfrak{U}^+$ will be called the *positive decomposition group* and the group $\mathfrak{U}^-$ the *negative decomposition group*.

Let U be an involution. Then Proposition 8.3 implies that U is always the product of a certain number p of involutions from any complete set of mutually commuting 1-involutions which commute with U. If $p=1$, $n-1$, or is even, or if n is odd, then we have mentioned that U is actually a p-involution. In the remaining case, we define U to be a *p-involution* if it is the product of p distinct mutually commuting 1-involutions. If $\mathfrak{H}$ is a standard decomposition group of odd rank, it is a consequence of (8.1c) that a 1-involution of $\mathfrak{G}$ in $\mathfrak{H}$ is also a 1-involution of $\mathfrak{H}$. Even in case rk $\mathfrak{H}$ is even, such an involution is always an extremal involution of $\mathfrak{H}$ by Proposition 8.1. We, therefore, distinguish the extremal involutions of $\mathfrak{H}$ which are 1-involutions as 1-involutions in $\mathfrak{H}$; in general, we define a *p-involution* of *a standard decomposition group* $\mathfrak{H}$ to be an involution in $\mathfrak{H}$ which is a p-involution of $\mathfrak{G}$.

THEOREM 8.5. *Two involutions U and U' of $\mathfrak{G}$ are conjugate if and only if each is the product of the same number of* 1-*involutions from a complete set of mutually commuting* 1-*involutions which contains it.*

Proof. The necessity is obvious. To prove the sufficiency let $\mathfrak{N}$ be a set of mutually commuting 1-involutions $V_1, V_2, \cdots, V_n$ such that $U=V_1V_2\cdots V_p$. Let $\mathfrak{N}'$ be a second complete set of mutually commuting 1-involutions $V_1', V_2', \cdots, V_n'$ such that $U=V_1'V_2'\cdots V_p'$. By the remarks preceding this proposition, it follows that U and U' are in the same class $\mathfrak{K}_p$ when $p=1$, $n-1$, or is even and also when n is odd. Therefore, assume that U is a nonextremal involution and that p is odd and n is even.

There exists $R\in\mathfrak{G}$ such that $RV_1'R^{-1}=V_1$. Let $U''=RUR^{-1}$ and $V_i''=RV_iR^{-1}$. It suffices to show that U and U'' are conjugate. Note that

$V_1'', V_2'', \cdots, V_n''$ form a complete set of mutually commuting 1-involutions and $U'' = V_1'' V_2'' \cdots V_p''$. Also $V_1'' = V_1$.

Next form the standard decomposition

$$C^*(V_1) = C^*(V_1'') = \overset{+}{V_1} \times \overset{-}{V_1}. \tag{8.7}$$

By Proposition 8.3, $V_2, V_3, \cdots, V_n$ as well as $V_2'', V_3'', \cdots, V_n''$ are in $\mathfrak{V}_1$. Hence $V_1U = V_2V_3 \cdots V_p$ and $V_1''U'' = V_2''V_3'' \cdots V_p''$ are $(p-1)$-involutions of the quasilinear group $\mathfrak{V}_1$ of odd rank $n-1$. Thus V_1U and $V_1''U'' = V_1U''$ are conjugate by an element S in $\mathfrak{V}_1$. Then $S \in C^*(V_1)$ and U and U'' are conjugate by S also. This proves the theorem.

If $\mathfrak{H}$ is a standard decomposition group, we may form standard decompositions of the involutory centralizers of involutions in H. In particular, in forming refinements of standard decompositions, we will choose standard decompositions for the decompositions of the involutory centralizers of involutions in the components. We will also call such decompositions *standard decompositions* or *standard refinements.* Such decomposition groups will also be called *standard decomposition groups.* They are uniquely determined because of the following proposition.

PROPOSITION 8.6. *Let $\mathcal{S}$ be a set of mutually commuting involutions such that $C^*(\mathcal{S})$ has a standard decomposition*

$$C^*(\mathcal{S}) = \mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_k. \tag{8.8}$$

Let U be the involution in the center of $\mathfrak{X}_1$. Then $\mathfrak{X}_1 = \mathfrak{U}^+$, the positive decomposition group of U.

Proof. We may assume that $\mathcal{S}$ contains proper involutions so that $0 < \operatorname{rk} \mathfrak{X}_i < n$. Let $\mathfrak{N}_i$, $i = 1, 2, \cdots, k$, be arbitrary complete sets of mutually commuting 1-involutions in the corresponding decomposition groups $\mathfrak{X}_i$. Then $\mathfrak{N} = \bigcup_{i=1}^k \mathfrak{N}_i$ is a complete set of mutually commuting 1-involutions in $C^*(\mathcal{S})$. The product of the involutions in $\mathfrak{N}_1$ is U and of the involutions in $\mathfrak{N}' = \bigcup_{i=2} \mathfrak{N}_i$ is $-U$.

Form the standard decomposition

$$C^*(U) = \mathfrak{U}^+ \times \mathfrak{U}^-. \tag{8.9}$$

By Corollary 8.4, the elements of $\mathfrak{N}_1$ and the elements of $\mathfrak{N}'$ belong to either $\mathfrak{U}^+$ or $\mathfrak{U}^-$. But as their product is U and $-U$, respectively, the elements of $\mathfrak{N}_1$ belong to $\mathfrak{U}^+$ and the elements of $\mathfrak{N}'$ belong to $\mathfrak{U}^-$. In particular, since $\mathfrak{N}_1$ was chosen arbitrarily, all the 1-involutions of $\mathfrak{X}_1$ are contained in $\mathfrak{U}^+$. Hence the involutory quasilinear group $\mathfrak{X}_1$ is contained in $\mathfrak{U}^+$.

Now the abelian group generated by the elements of $\mathfrak{N}'$ together with the involution -1 contains the center

$$Z(C^*(\mathcal{S})) = Z(\mathfrak{X}_1) \times Z(\mathfrak{X}_2) \times \cdots \times Z(\mathfrak{X}_k) \tag{8.10}$$

and hence the set $\mathcal{S}$ itself. But then $C^*(\mathcal{S}, \mathfrak{N}') = C^*(\mathfrak{N}')$. Similarly $C^*(U, \mathfrak{N}') = C^*(\mathfrak{N}')$. Thus we may form $C^*(\mathfrak{N}')$ by refining (8.8) and (8.9). In the first case, the components $\mathfrak{X}_i$, $i>1$, decompose so that we obtain a direct product of $\mathfrak{X}_1$ and quasilinear groups of order 1. In the second case, only the component $\mathfrak{U}^-$ decomposes, and we obtain the direct product of $\mathfrak{U}^+$ and quasilinear groups of rank 1. Using Lemma 4.2 to compare these two refinements, we see that $\mathfrak{U}^+$ and $\mathfrak{X}_1$ are isomorphic. Since $\mathfrak{X}_1 \subseteq \mathfrak{U}^+$, they can be seen to be generated by the same set of involutions and $\mathfrak{X}_1 = \mathfrak{U}^+$, which proves the proposition.

Proposition 8.7. *Let U and V be commuting involutions with positive and negative decomposition groups $\mathfrak{U}^\pm$ and $\mathfrak{V}^\pm$. Then the standard decomposition groups belonging to a standard decomposition of $C^*(U, V)$ are the groups $\mathfrak{U}^\pm \cap \mathfrak{V}^\pm$.*

Proof. We may assume that $U \neq V$ and that U and V are proper involutions. Form the standard decomposition

$$C^*(U, V) = \mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_3 \times \mathfrak{X}_4. \tag{8.11}$$

Each of the groups $\mathfrak{X}_i$ is uniquely determined by the involution in its center according to Proposition 8.6. Therefore, whether we obtain (8.11) from standard refinements of $C^*(U)$ or of $C^*(V)$, we will obtain the same decomposition groups. This means that the decomposition groups $\mathfrak{U}^+$, $\mathfrak{U}^-$, $\mathfrak{V}^+$, and $\mathfrak{V}^-$ each contain two of the groups $\mathfrak{X}_i$ and none contains the same two groups. Therefore, we may suppose that $\mathfrak{U}^+ \supseteq \mathfrak{X}_1, \mathfrak{X}_2$, $\mathfrak{U}^- \supseteq \mathfrak{X}_3, \mathfrak{X}_4$, $\mathfrak{V}^+ \supseteq \mathfrak{X}_1, \mathfrak{X}_3$, and $\mathfrak{V}^- \supseteq \mathfrak{X}_2, \mathfrak{X}_4$. Then $\mathfrak{X}_1 \subseteq \mathfrak{U}^+ \cap \mathfrak{V}^+$, $\mathfrak{X}_2 \subseteq \mathfrak{U}^+ \cap \mathfrak{V}^-$, $\mathfrak{X}_3 \subseteq \mathfrak{U}^- \cap \mathfrak{V}^+$, and $\mathfrak{X}_4 \subseteq \mathfrak{U}^- \cap \mathfrak{V}^-$.

On the other hand, $\mathfrak{U}^\pm \cap \mathfrak{V}^\pm \subseteq C^*(U, V)$. Now if an element $X \in \mathfrak{U}^+ \cap \mathfrak{V}^+$ has the representation $X = X_1 \times X_2 \times X_3 \times X_4$ relative to (8.11), it follows that $X_2 = X_3 = X_4 = 1$, for otherwise $X_1^{-1}X$ would be a nontrivial element in $\mathfrak{U}^+ \cap \mathfrak{V}^+$, which is impossible. Hence $\mathfrak{X}_1 = \mathfrak{U}^+ \cap \mathfrak{V}^+$; the other equalities are similarly obtained.

9. **Characterization of decomposition groups.** Let $\mathfrak{H}$ be a quasilinear decomposition group of rank $m<n$. In §6, we introduced two epimorphisms; namely Φ: $\mathfrak{H} \to PTL(m)$, and Λ: $\overline{\mathfrak{T}} \to P(\overline{\mathfrak{T}}) = PTL(m)$. We will compare these epimorphisms in this section. We will use the notation and results developed in §6.

Considered as transformations(7) of the vector space $\mathcal{E}$ on which $\overline{\mathfrak{T}}$ acts, the involutions in the conjugate class which we term p-involutions, $K_p(\overline{\mathfrak{T}})$, have two complementary eigenspaces $\mathcal{U}^-$ and $\mathcal{U}^+$ associated with the eigenvalues -1 and $+1$, respectively, such that $\dim \mathcal{U}^- = p$ and $\dim \mathcal{U}^+ = m - p$. This characterizes the involutions of this class. It is not difficult to see that

$$C^*(\overline{U}) = \overline{\mathfrak{U}}^+ \times \overline{\mathfrak{U}}^- \tag{9.1}$$

(7) The reader is referred to Dieudonné's treatise [8, Chapter I], for a more complete discussion of the involutions of linear groups and their centralizers.

where $\bar{\mathfrak{U}}^+$ is the subgroup of $\bar{\mathfrak{T}}$ consisting of transformations leaving fixed $\mathcal{U}^+$ and $\bar{\mathfrak{U}}^-$ is the subgroup of transformations leaving fixed $\mathcal{U}^-$. Then $\bar{\mathfrak{U}}^+$ contains $\bar{U}$ and is isomorphic to $TL(p)$ while $\bar{\mathfrak{U}}^-$ contains $-\bar{U}$ and is isomorphic to $TL(m-p)$. Also $\bar{\mathfrak{U}}^+$ and $\bar{\mathfrak{U}}^-$ contain involutions with 1-dimensional negative eigenspaces. Consequently (9.1) is a standard decomposition.

PROPOSITION 9.1. *Let $\mathfrak{H}$ be a quasilinear decomposition group of rank m, $2<m<n$. To each involution $U\in\mathfrak{K}_p(\mathfrak{H})$ with standard decomposition groups $\mathfrak{U}^+$ and $\mathfrak{U}^-$, there is determined an unique involution $\bar{U}\in\mathfrak{K}_p(\bar{\mathfrak{T}})$ such that $\Phi U=\Lambda\bar{U}$, $\Phi\mathfrak{U}^+=\Lambda\bar{\mathfrak{U}}^+$, and $\Phi\mathfrak{U}^-=\Lambda\bar{\mathfrak{U}}^-$, and conversely. The groups $\mathfrak{U}^+$ and $\mathfrak{U}^-$ are involutory linear groups and $C^*(U)=\mathfrak{U}^+\times\mathfrak{U}^-$ is a normal decomposition.*

Proof. Let $U\in\mathfrak{K}_p(\mathfrak{H})$. By Proposition 6.1, the involution ΦU is in $\mathfrak{L}_p$. But the elements of $\mathfrak{L}_p$ are the images of involutions in $\mathfrak{K}_p(\bar{\mathfrak{T}})$ by the natural homomorphism; thus $\Phi U=\Lambda\bar{U}$ for some involution $\bar{U}\in\mathfrak{K}_p(\bar{\mathfrak{T}})$. But in the coset $\bar{U}(Z(\bar{\mathfrak{T}}))$, there are but two involutions, namely, $\bar{U}$ and $-\bar{U}$. Here $-\bar{U}\in\mathfrak{K}_{m-p}(\bar{\mathfrak{T}})$. Thus unless $p=m/2$, there is only one involution $\bar{U}\in\mathfrak{K}_p(\bar{\mathfrak{T}})$ satisfying $\Phi U=\Lambda\bar{U}$.

Conversely, let $U\in\mathfrak{K}_p(\bar{\mathfrak{T}})$. Then $\Lambda\bar{U}=U^*\in\mathfrak{L}_p$. On the other hand there exists $V\in\mathfrak{K}_p(\mathfrak{H})$ such that $\Phi V\in\mathfrak{L}_p$. Choose $R\in\mathfrak{H}$ such that $(\Phi R)(\Phi V)(\Phi R)^{-1}=U^*$ and set $U=RVR^{-1}$. Then $\Phi U=\Lambda\bar{U}$. As above, when $p\neq m/2$, U is the only involution of $\mathfrak{K}_p(\mathfrak{H})$ such that $\Phi U=\Lambda\bar{U}$.

Now let V be a 1-involution in the positive decomposition group $\mathfrak{U}^+$ of an involution U. Then $V^*=\Phi V\in\Phi\mathfrak{U}^+$ commutes with $U^*=\Phi U$. Let $\bar{U}$ be such that $\Lambda\bar{U}=U^*$. Then if $\bar{V}$ is a 1-involution such that $\Lambda\bar{V}=V^*$, $\bar{V}\bar{U}\bar{V}=\pm\bar{U}$. But then $\bar{V}$ cannot anticommute with $\bar{U}$ for this would imply that $\bar{V}$ and $-\bar{V}$ were conjugate, which is not the case as $m>2$. Hence $\bar{V}\in C(\bar{U})$. Thus corresponding to each 1-involution $V\in\mathfrak{U}^+$, there is a 1-involution $\bar{V}\in C^*(U)$. Let $\bar{\mathfrak{U}}_1$ be the subgroup of $C^*(\bar{U})$ generated by these 1-involutions. The correspondence between 1-involutions of $C^*(U)$ and those of $C^*(\bar{U})$ maps commuting involutions onto commuting involutions. Thus if $U=V_1V_2\cdots V_p$ with $V_i\in\mathfrak{U}^+$, $\Lambda(\bar{V}_1\bar{V}_2\cdots\bar{V}_p)=\Phi U=\Lambda(\pm\bar{U})$ where $\Lambda\bar{V}_i=\Phi V_i$. Replacing $\bar{U}$ by $-\bar{U}$, if necessary, we have that $\bar{V}_1\bar{V}_2\cdots\bar{V}_p=\bar{U}$. Thus applying the theory of quasilinear groups and, in particular, Corollary 8.4 to $\bar{\mathfrak{T}}$, we see that the involutions $\bar{V}_i$ belong to $\bar{\mathfrak{U}}^+$ or $\mathfrak{U}^-$. Since their product is $\bar{U}$, they belong to $\bar{\mathfrak{U}}^+$. This implies that $\bar{\mathfrak{U}}_1\subseteq\bar{\mathfrak{U}}^+$. Also it is clear that $\Phi\mathfrak{U}^+\subseteq\bar{\mathfrak{U}}^+$ since $\Phi\mathfrak{U}^+=\Lambda\bar{\mathfrak{U}}_1$. Similarly, $\Phi\mathfrak{U}^-\subseteq\Lambda\bar{\mathfrak{U}}^-$. However, it is possible to interchange the roles of $\mathfrak{H}$ and $\bar{\mathfrak{T}}$ in the argument of this paragraph. Thus we obtain the opposite inequality and that $\Phi\mathfrak{U}^+=\Lambda\bar{\mathfrak{U}}^+$ and that $\Phi\mathfrak{U}^-=\Lambda\bar{\mathfrak{U}}^-$. Note that for only one choice of the involutions $\bar{U}$ or $-\bar{U}$ are these relations possible. This proves the first statements of the proposition. The converse may be obtained by interchanging the roles of $\mathfrak{H}$ and $\bar{\mathfrak{T}}$.

To prove the last statements, observe first that (9.1) is a normal decomposition. Since the kernel of Λ is $Z(\bar{\mathfrak{T}})$, the restriction of Λ to the groups $\bar{\mathfrak{U}}^+$ or

$\bar{\mathfrak{U}}^-$ is an isomorphism. Thus if U is a p-involution $\Phi\mathfrak{U}^+=\Lambda\bar{\mathfrak{U}}^+$ is isomorphic to $TL(p)$ and $\Phi\mathfrak{U}^-$ is isomorphic to $TL(m-p)$. We wish to show that Φ induces an isomorphism of $\mathfrak{U}^+$ and $\mathfrak{U}^-$. By the inductive assumption $P(\mathfrak{U}^+)$ is isomorphic to $PTL(p)$. Hence Φ induces an isomorphism of $P(\mathfrak{U}^+)$ onto $P(\Phi\mathfrak{U}^+)$. This means that the kernel of the restriction of Φ to $\mathfrak{U}^+$ lies in $Z(\mathfrak{U}^+)$. By Condition B, $Z(D(\mathfrak{U}^+))$ has the same order as $Z(D(\Phi\mathfrak{U}^+))$. In §3, we showed that these groups were contained in the centers of the corresponding involutory quasilinear groups $\mathfrak{U}^+$ and $\Phi\mathfrak{U}^+$. Thus the restriction of Φ to $D(\mathfrak{U}^+)$ is an isomorphism. However, by Proposition 3.1, $[\mathfrak{U}^+: D(\mathfrak{U}^+)]=2$. Thus the restriction to $\mathfrak{U}^+$ of Φ has a kernel of order at most 2. However, no proper involution in $\mathfrak{H}$ is mapped onto 1 by Φ; therefore, the kernel cannot contain a proper involution. Since $-1\notin\mathfrak{U}^+$, the kernel is trivial. Hence the restriction of Φ to $\mathfrak{U}^+$ is an isomorphism. A similar statement holds for $\mathfrak{U}^-$. This shows that $\mathfrak{U}^+$ is isomorphic to the linear group $\Phi\mathfrak{U}^+=\Lambda\mathfrak{U}^+$ and $\mathfrak{U}^-$ is isomorphic to the linear group $\Phi\mathfrak{U}^-$. Since the kernel of Φ is $Z(\mathfrak{H})$, $\mathfrak{U}^+\cap Z(\mathfrak{H})=\mathfrak{U}^-\cap Z(\mathfrak{H})=1$. Consequently, the standard decomposition of $C^*(U)$ is normal.

PROPOSITION 9.2. *Let $\mathfrak{U}$ be a standard decomposition group of rank at most $n-2$. Then $\mathfrak{U}$ is an involutory linear group.*

Proof. Let U be in the center of $\mathfrak{U}$; then U is in the positive decomposition group $\mathfrak{U}^+$ of U and rk $\mathfrak{U}^+\leqq n-2$. Let V be a 1-involution in $\mathfrak{U}^-$; then $\mathfrak{V}^-\supseteq\mathfrak{U}^+$. Since rk $\mathfrak{V}^->$rk $\mathfrak{U}^+$, $\mathfrak{V}^-\neq\mathfrak{U}^+$. Because of Proposition 8.6, $\mathfrak{U}^+$ is a standard decomposition group of $\mathfrak{V}^-$. The result now follows from Proposition 9.1.

III. THE CHARACTERIZATION OF THE QUASILINEAR GROUP $\mathfrak{G}$

10. A fundamental lemma.

LEMMA 10.1. *Let U be a 1-involution and V a p-involution, $1<p<n-1$. Then there exists a 1-involution in $C^*(U, V)$.*

Proof. If $\mathfrak{G}$ is an infinite group, it follows from Condition E that there is an involution $S\neq\pm1$ in $C^*(U, V)$. When $\mathfrak{G}$ is a finite group, we shall still show that this is true(8). Clearly, we may suppose that U and V do not commute. Then $D=\{U, V\}$ is a dihedral group of order $2k$ where k is the order of UV. If k were odd, then U and V would be conjugate. Hence some power S of UV is an involution in the center of D. Furthermore, in D, U and US are conjugate. This means that $S\neq\pm1$.

Hence we may form

$$(10.2)\quad C^*(S, V) = (\mathfrak{S}^+\cap\mathfrak{V}^+)\times(\mathfrak{S}^+\cap\mathfrak{V}^-)\times(\mathfrak{S}^-\cap\mathfrak{V}^+)\times(\mathfrak{S}^-\cap\mathfrak{V}^-).$$

There are at least three nontrivial decomposition groups in (10.2) as $S\neq V$. We have that $U\in C(S)$. Hence $U\in\mathfrak{S}^+$ or $U\in\mathfrak{S}^-$; say, $U\in\mathfrak{S}^+$. As one of

(8) This result is essentially a consequence of a lemma of Brauer and Fowler [4, Lemma 3A]. The approach we use here was suggested by the referee, and shortens our proof.

$\mathfrak{S}^- \cap \mathfrak{V}^+$ or $\mathfrak{S}^- \cap \mathfrak{V}^-$ is nontrivial, there exists a 1-involution W in one of these groups. Then $W \in C^*(U, V)$ is the involution we seek. Obviously we can obtain the same conclusion when $U \in \mathfrak{S}^-$.

Proposition 10.2. *Let U and V be 1-involutions in $\mathfrak{G}$. Then U and V are conjugate in $T(\mathfrak{G})$.*

Proof. If U and V commute, we may apply statement (8.1a) in the proof of Proposition 8.1 as U and V belong to a complete set of mutually commuting 1-involutions.

Hence suppose that $UV \neq VU$. Then choose a 1-involution $V' \in C^*(V)$ and set $W = VV'$. Then W is a 2-involution, and by Lemma 10.1 there exists a 1-involution $T \in C^*(U, W)$. Form

$$C^*(T, W) = \mathfrak{T}^+ \times (\mathfrak{T}^- \cap \mathfrak{W}^+) \times (\mathfrak{T}^- \cap \mathfrak{W}^-). \tag{10.3}$$

Here $\mathfrak{T}^+ \subseteq \mathfrak{W}^-$ or $\mathfrak{T}^+ \subseteq \mathfrak{W}^+$. Also $V \in \mathfrak{W}^+$ as $VV' = W$.

If $\mathfrak{T}^+ \subseteq \mathfrak{W}^-$, then $T \in \mathfrak{T}^+ \subseteq C^*(V)$ and $T \in C^*(U, V)$. Hence we have that U and V are conjugate 1-involutions in the negative decomposition group $\mathfrak{T}^- \subseteq T(\mathfrak{G})$. This is our result in this case.

Thus consider that $\mathfrak{T}^+ \subseteq \mathfrak{W}^+$. Then $W = T(TW)$ and T and TW belong to $\mathfrak{W}^+$. Furthermore, they are conjugate by Lemma 7.2. In particular, TW is a 1-involution in $\mathfrak{T}^- \cap \mathfrak{W}^+$. Then V and TW are conjugate in $\mathfrak{W}^+ \subseteq T(\mathfrak{G})$, and TW and U are conjugate in $\mathfrak{T}^- \subseteq T(\mathfrak{G})$. Thus V and U are conjugate in $T(\mathfrak{G})$. This proves the proposition.

11. **Congruent involutions.** In this section, we lay the foundation for the construction of a projective geometry on which we will determine $P(\mathfrak{G})$ as a group of automorphisms.

Let W be an arbitrary 1-involution in $\mathfrak{G}$ and let $\mathfrak{W} = \mathfrak{W}^-$ be its negative decomposition group. Then[9] rk $\mathfrak{W} = m = n-1$. By the inductive assumption of the Principal Theorem, there exists a homomorphism $\Phi_{\mathfrak{W}}$ of $\mathfrak{W}$ onto $PTL(m)$. As in §9, we will set $\overline{\mathfrak{T}} = TL(m)$ and $P(\overline{\mathfrak{T}}) = PTL(m)$. If $R \in C^*(W)$, $R = R^+ \times R^-$ where $R^+ \in \mathfrak{W}^+$ and $R^- \in \mathfrak{W}^-$. Then the inner automorphism I_R induced by R on $\mathfrak{G}$ leaves $\mathfrak{W}$ invariant and coincides with the inner automorphism I_{R^-} induced by the component R^- of R in $\mathfrak{W}^-$ on $\mathfrak{W} = \mathfrak{W}^-$. We will term a *particularization* of $\mathfrak{W}$ a homomorphism of the form $\Phi_{\mathfrak{W}} I_R$ where I_R is the inner automorphism induced by an element $R \in C^*(W)$. Then $\Phi_{\mathfrak{W}} I_R = I_{\Phi_{\mathfrak{W}}(R^-)} \Phi_{\mathfrak{W}}$ where $I_{\Phi_{\mathfrak{W}}(R^-)}$ is the inner automorphism of $P(\overline{\mathfrak{T}})$ induced by the image $\Phi_{\mathfrak{W}}(R^-)$ of the component R^- of R in $\mathfrak{W} = \mathfrak{W}^-$. If W' is a second 1-involution in $\mathfrak{G}$ with negative decomposition group $\mathfrak{W}' = \mathfrak{W}'^-$, there exists an inner automorphism I_S of $\mathfrak{G}$ induced by an element $S \in T(\mathfrak{G})$ such that $S\mathfrak{W}'S^{-1} = \mathfrak{W}$ by virtue of Proposition 10.2. Set $\Phi_{\mathfrak{W}'} = \Phi_{\mathfrak{W}} I_S$; then $\Phi_{\mathfrak{W}'}$ is an homomorphism of $\mathfrak{W}'$ onto $P(\overline{\mathfrak{T}})$. If S' is a second element of $T(\mathfrak{G})$ such that $I_{S'}(\mathfrak{W}') = \mathfrak{W}$, then $S' = RS = ST$ where R is in $C^*(W)$ and T is in $C^*(W')$.

(9) For the remainder of this paper, m will always represent the integer $n-1$.

Then $\Phi_{\mathfrak{W}} I_{S'} = \Phi_{\mathfrak{W}'} I_T$. We will call any such homomorphism a *particularization* of $\mathfrak{W}'$.

Let $\mathcal{E}$ be the vector space on which $\overline{\mathfrak{T}}$ acts; then $\mathcal{E}$ has dimension $m = n-1$. Let $\Phi_{\mathfrak{W}}$ be a particularization of a standard decomposition group $\mathfrak{W}$ of rank m. If U is an involution in $\mathfrak{W}$, we will conventionally denote by $\overline{U}$ the unique involution in $\overline{\mathfrak{T}}$ determined in accordance with Proposition 9.1 so that $\Phi_{\mathfrak{W}} U = \Lambda \overline{U}$. Let $\mathcal{U}^+$ and $\mathcal{U}^-$ designate the eigenspaces of $\overline{\mathfrak{U}}$; these are complementary subspaces of $\mathcal{E}$. On the space $\mathcal{U}^-$ the standard decomposition group $\overline{\mathfrak{U}}^+$ may be considered to act since it leaves $\mathcal{U}^-$ invariant and $\mathcal{U}^+$ fixed. Correspondingly, $\overline{\mathfrak{U}}^-$ acts on $\mathcal{U}^+$. If U is a p-involution, so is $\overline{U}$ and dim $\mathcal{U}^- = p$. We will make consistent use of these symbolic relationships between an involution $U \in \mathfrak{H}$ and the corresponding involution $\overline{U} \in \overline{\mathfrak{T}}$ and their standard decomposition groups. In a discussion where only one particularization is introduced, we will not further specify these relations.

Now let U and V be 1-involutions in $\mathfrak{G}$. Suppose, furthermore, that there exists a 1-involution $W \neq U, V$ which is in $C^*(U, V)$. By Corollary 8.4, U and V are in the negative decomposition group $\mathfrak{W} = \mathfrak{W}^-$ of W. Let $\Phi_{\mathfrak{W}}$ be a particularization of $\mathfrak{W}$. If $\overline{U}$ and $\overline{V}$ have a common 1-dimensional eigenspace $\mathfrak{U}^- = \mathfrak{V}^-$, then we say that U and V are *congruent in* $\mathfrak{W}$. This concept is independent of the particularization $\Phi_{\mathfrak{W}}$ in the sense that if $\Phi'_{\mathfrak{W}} = \Phi_{\mathfrak{W}} I_R$ is a second particularization with $R \in \mathfrak{W}$, $\Phi'_{\mathfrak{W}} U = I_{\Phi_{\mathfrak{W}}(R)} \Phi_{\mathfrak{W}} U$ has fixed subspaces $\overline{R}\mathcal{U}^{\pm}$ and $\Phi'_{\mathfrak{W}} V$ has fixed subspaces $\overline{R}\mathcal{V}^{\pm}$ where $\Phi_{\mathfrak{W}} R = \Lambda \overline{R}$. Thus $\Phi'_{\mathfrak{W}} U$ and $\Phi'_{\mathfrak{W}} V$ have the common fixed subspace $\overline{R}\mathcal{U}^- = \overline{R}\mathcal{V}^-$. We will show that the relation of congruence is independent of the standard decomposition group $\mathfrak{W}$ determined by the particular 1-involution W chosen from $C^*(U, V)$.

We remark that two involutions $\overline{U}$ and $\overline{V}$ commute if and only if their associated eigenspaces satisfy[10] $\mathcal{U}^+ = (\mathcal{U}^+ \cap \mathcal{V}^+) \oplus (\mathcal{U}^+ \cap \mathcal{V}^-)$ and $\mathcal{U}^- = (\mathcal{U}^- \cap \mathcal{V}^+) \oplus (\mathcal{U}^- \cap \mathcal{V}^-)$. If $\overline{U}$ is a 1-involution and $\overline{U} \neq \overline{V}$, then must $\mathcal{U}^- \subseteq \mathcal{V}^+$ and $\mathcal{V}^- \subseteq \mathcal{U}^+$. In particular, if U and V are commuting 1-involutions congruent in a standard decomposition group $\mathfrak{W}$, this implies that $\overline{U} = \overline{V}$; hence $U = V$.

Proposition 11.1. *Let U and V be 1-involutions in $\mathfrak{G}$ which are congruent in a standard decomposition group $\mathfrak{W} = \mathfrak{W}^-$ belonging to a 1-involution $W \in C^*(U, V)$. Then there exists a 2-involution $P \in C^*(U, V, W)$ with standard decomposition groups $\mathfrak{P}^+$ and $\mathfrak{P}^-$ such that U and V are in $\mathfrak{P}^+$ while $W \in \mathfrak{P}^-$.*

A 2-involution P satisfying the conditions in this proposition will be said to *enclose* U, V, and W or to *enclose* U and V.

Proof. We may assume that U, V, and W are distinct. Let $\mathfrak{W} = \mathfrak{W}^-$ be the negative decomposition group of W, and let $\Phi_{\mathfrak{W}}$ be a particularization of $\mathfrak{W}$. Then the eigenspaces $\mathcal{U}^+$ and $\mathcal{V}^+$ are hyperplanes. Since $\mathcal{U}^- = \mathcal{V}^-$, we may obtain a direct decomposition

[10] For example, cf. Rickart [9, p. 454], or Dieudonné [7, p. 4].

(11.1) $$\mathcal{E} = \mathcal{U}^- \oplus \mathcal{U}^+ \cap \mathfrak{X}$$

by choosing a 1-dimensional complement $\mathfrak{X}$ to $\mathcal{U}^- \oplus \mathcal{U}^+ \cap \mathcal{V}^+$. Let $\overline{P}$ be the 2-involution in $\overline{\mathfrak{T}}$ with eigenspaces $\mathcal{P}^- = \mathcal{U}^- \oplus \mathfrak{X} = \mathcal{V}^- \oplus \mathfrak{X}$ and $\mathcal{P}^+ = \mathcal{U}^+ \cap \mathcal{V}^+$. Then it may be verified that $\overline{U}$ and $\overline{V}$ belong to $\overline{\mathfrak{P}}^+$. By Proposition 9.1, there exists a 2-involution $P \in \mathfrak{W}$ corresponding to $\overline{P}$ such that U and V belong to the standard decomposition group $\mathfrak{P}^+$. As $P \in \mathfrak{W}^-$, $C^*(P, W) = \mathfrak{P}^+ \times \mathfrak{P}^- \cap \mathfrak{W}^+ \times \mathfrak{P}^- \cap \mathfrak{W}^-$. Hence $W \in \mathfrak{P}^-$. Thus P is the 2-involution we seek.

Proposition 11.2. *Let U and V be congruent 1-involutions in a standard decomposition group $\mathfrak{W}$ of rank $n-1$. Let $\mathfrak{W}'$ be a second standard decomposition group of rank $n-1$ containing U and V. Then U and V are congruent in $\mathfrak{W}'$.*

Proof. We may assume that $U \neq V$. Let W and W' be the 1-involutions whose negative decomposition groups are $\mathfrak{W}$ and $\mathfrak{W}'$, respectively. Then both W and W' belong to $\mathfrak{U}^- \cap \mathfrak{V}^-$. We will show that they are conjugate in $\mathfrak{U}^- \cap \mathfrak{V}^-$.

First consider the case that $n \geqq 5$ and set $m = n-1$. Let P and P' be the 2-involutions enclosing U, V, and W and U, V, and W', respectively. Then both $Q = -P$ and $Q' = -P'$ are $(m-1)$-involutions which are associated with involutions $\overline{Q}$ and $\overline{Q}'$ of $TL(m)$, respectively, by means of a particularization $\Phi_{\mathfrak{U}}$ of $\mathfrak{U}^-$. As the eigenspaces $\mathcal{Q}^+$ and $\mathcal{Q}^-$ as well as the eigenspaces $\mathcal{Q}'^+$ and $\mathcal{Q}'^-$ are complementary and as $\dim \mathcal{Q}^+ \cap \mathcal{Q}'^+ \geqq m-2 \geqq 2$, there exists a 1-dimensional subspace $\mathfrak{X}^-$ contained in $\mathcal{Q}^+ \cap \mathcal{Q}'^+$ but not in $\mathcal{Q}^- + \mathcal{Q}'^-$, which has dimension at most 2. Let $\mathfrak{X}^+$ be an $(m-1)$-dimensional subspace complementary to $\mathfrak{X}^-$ and containing $\mathcal{Q}^- + \mathcal{Q}'^-$. Let $\overline{X}$ be the 1-involution in $\overline{T}$ with the eigenspaces $\mathfrak{X}^+$ and $\mathfrak{X}^-$. Then both the 1-involutions $-\overline{Q}$ and $-\overline{Q}'$ commute with $\overline{X}$ since $\mathfrak{X}^+ \supseteq \mathcal{Q}^- + \mathcal{Q}'^-$ while $\mathfrak{X}^- \subseteq \mathcal{Q}^+ \cap \mathcal{Q}'^+$. Let X be the 1-involution in $\mathfrak{U}^-$ corresponding to $\overline{X}$ by means of the particularization $\Phi_{\mathfrak{U}}$. Then X is in the standard decomposition groups $\mathfrak{Q}^+$ and $\mathfrak{Q}'^+$ of Q and Q', respectively, inasmuch as $\overline{X}$ is in the standard decomposition groups $\overline{\mathfrak{Q}}^+$ and $\overline{\mathfrak{Q}}'^+$. But $\mathfrak{Q}^+ = \mathfrak{P}^-$ and $\mathfrak{Q}'^+ = \mathfrak{P}'^-$ where $\mathfrak{P}^\pm$ and $\mathfrak{P}'^\pm$ are the standard decomposition groups of the involutions P and P', respectively. Now since U and V are 1-involutions in $\mathfrak{P}^+$ and $\mathfrak{P}'^+$, $\mathfrak{U}^- \cap \mathfrak{V}^-$ contains both $\mathfrak{P}^-$ and $\mathfrak{P}'^-$. But W and X are conjugate in $\mathfrak{P}^-$ and X and W' are conjugate in $\mathfrak{P}'^-$. Hence W and W' are conjugate in $\mathfrak{U}^- \cap \mathfrak{V}^-$. This argument suffices for infinite groups as we assume $n \geqq 5$ in this case.

Suppose that G is finite and that $n = 4$. Assume that W and W' are not conjugate in $U^- \cap V^-$. By the argument of Lemma 10.1, there exists an involution $X \neq 1$ in the center of the group $D = \{W, W'\}$, and W and WX are conjugate in D. Thus X is not a 1-involution as WX cannot be a 2-involution. Likewise X is not a 3-involution since then $-WX = W(-X)$ and also WX itself would be 2-involutions.

Hence take X to be a 2-involution with decomposition groups $\mathfrak{X}^+$ and $\mathfrak{X}^-$. But U, V, W, and W' are all in $C^*(X)$. Because U and V do not commute, they belong to the same standard decomposition group of X, say $\mathfrak{X}^-$. Now if $W \in \mathfrak{X}^-$, then either $W=U$ or WU is a 2-involution and $WU=-X$. Also either $W=V$ or $WV=-X$. Certainly $W \neq U$ and $W \neq V$, and $WU \neq WV$. Hence $W \in \mathfrak{X}^+$. Similarly W' is in $\mathfrak{X}^+$. Thus again W and W' are 1-involutions in a standard decomposition group $\mathfrak{X}^+$ of rank 2. Hence they are conjugate in $\mathfrak{X}^+$. As U and V are in $\mathfrak{X}^-$, $\mathfrak{X}^+ \subseteq \mathfrak{U}^- \cap \mathfrak{V}^-$. This again obtains a contradiction.

Thus we may conclude that W and W' are conjugate by an element R in $\mathfrak{U}^- \cap \mathfrak{V}^- \subseteq C^*(U, V)$. Thus $I_R W = W'$, $I_R U = U$, and $I_R V = V$. Let $\Phi_{\mathfrak{W}}$ be a particularization of the negative decomposition group $\mathfrak{W} = \mathfrak{W}^-$ of W and define a particularization $\Phi_{\mathfrak{W}'}$ of the negative decomposition group $\mathfrak{W}' = \mathfrak{W}'^-$ of W' by $\Phi_{\mathfrak{W}'} = \Phi_{\mathfrak{W}} I_R$. Then $\Phi_{\mathfrak{W}'} U = \Phi_{\mathfrak{W}} U$ and $\Phi_{\mathfrak{W}'} V = \Phi_{\mathfrak{W}} V$. Thus U and V are congruent in $\mathfrak{W}'$ inasmuch as they are congruent in $\mathfrak{W}$.

Consequently, we will say that two 1-involutions U and V are *congruent* if they are congruent in any standard decomposition group of rank $n-1$ in which they are contained. It is obvious that this is a symmetric and reflexive relation defined on the set of 1-involutions.

THEOREM 11.3. *The relation of congruence is an equivalence relation.*

Proof. We need only show that congruence is a transitive relation. Therefore, let U_1 and U_3 be 1-involutions which are congruent to a 1-involution U_2. We may suppose that U_1, U_2, and U_3 are all distinct. Let P be a 2-involution enclosing U_1 and U_2 and let P' be a 2-involution enclosing U_2 and U_3. Then both $Q=-P$ and $Q'=-P'$ are $(m-1)$-involutions in the negative decomposition group $\mathfrak{U}_2^-$ of U_2. Relative to a particularization $\Phi_{\mathfrak{U}}$ of $\mathfrak{U}_2^-$, there are associated with Q and Q' $(m-1)$-involutions $\overline{Q}$ and $\overline{Q}'$ in $\overline{\mathfrak{T}}$.

We have seen in the proof of Proposition 11.2 that if $n \geqq 5$ and $m \geqq 4$, there is determined a 1-involution $\overline{X}$ with complementary eigenspaces $\mathfrak{x}^+$ and $\mathfrak{x}^-$ such that $\mathfrak{x}^+ \supseteq \mathfrak{q}^+ + \mathfrak{q}'^+$ and $\mathfrak{x}^- \subseteq \mathfrak{q}^- \cap \mathfrak{q}'^-$. Then $\overline{X}$ commutes with $\overline{Q}$ and $\overline{Q}'$. The corresponding 1-involution X in $\mathfrak{G}$ will belong to the decomposition groups $\mathfrak{U}_2^-$, $\mathfrak{P}^- = \mathfrak{Q}^+$, and $\mathfrak{P}'^- = \mathfrak{Q}'^+$. This means that X commutes with U_1, U_2, and U_3. Thus U_1, U_2, and U_3 are in the negative decomposition group $\mathfrak{X}^-$ of X, and, by Proposition 11.2, U_2 and U_1 as well as U_2 and U_3 are congruent in $\mathfrak{X}^-$. Then, if $\Phi_{\mathfrak{X}}$ is a particularization of $\mathfrak{X}^-$, $\Phi_{\mathfrak{X}} U_1$ and $\Phi_{\mathfrak{X}} U_3$ will have the same 1-dimensional fixed subspace as $\Phi_{\mathfrak{X}} U_2$. Hence when $n \geqq 5$, U_1 and U_3 are congruent.

We now suppose that $n=4$. Then it may not be possible to find complementary subspaces $\mathfrak{x}^+$ and $\mathfrak{x}^-$ such that $\mathfrak{x}^- \subseteq \mathfrak{q}^- \cap \mathfrak{q}'^-$ and $\mathfrak{x}^+ \supseteq \mathfrak{q}^+ + \mathfrak{q}'^+$ because $\mathfrak{q}^- + \mathfrak{q}'^+$ may contain $\mathfrak{q}^- \cap \mathfrak{q}'^-$. Indeed, suppose that this is the case. Then we will replace Q' and hence P' by different involutions for which this difficulty does not arise. To do this, let $\mathfrak{y}_1^-$ be a 1-dimensional subspace of $\mathfrak{q}'^-$ distance from $\mathfrak{q}^- \cap \mathfrak{q}'^-$. Let $\mathfrak{y}_1^+$ be the subspace $(\mathfrak{q}^- \cap \mathfrak{q}'^-) \oplus \mathfrak{q}'^+$. Let $\overline{Y}_1$ be the 1-involution in $\overline{\mathfrak{T}}$ with eigenspaces $\mathfrak{y}_1^+$ and $\mathfrak{y}_1^-$. Let $\mathfrak{y}_2^-$ be the subspace $\mathfrak{q}^- \cap \mathfrak{q}'^-$ and let $\mathfrak{y}_2^+ = \mathfrak{q}'^+ \oplus \mathfrak{y}_1^-$. Let $\overline{Y}_2$ be the 1-involution with eigenspaces $\mathfrak{y}_2^+$ and $\mathfrak{y}_2^-$. Then

$$\mathcal{E} = \mathcal{Y}_1^- \oplus \mathcal{Y}_2^- \oplus (\mathcal{Y}_1^+ \cap \mathcal{Y}_2^+). \tag{11.2}$$

Since $\mathcal{Y}_2^- \subseteq \mathcal{Y}_1^+$ and $\mathcal{Y}_1^- \subseteq \mathcal{Y}_2^+$, the involutions Y_1 and Y_2 commute and $Y_1Y_2 = Q'$.

Choose $\overline{Y}_2'$ to be the 1-involution with eigenspaces $\mathcal{Y}_2' = \mathcal{Y}_2^- = \mathcal{Q}^- \cap \mathcal{Q}'^-$ and $\mathcal{Y}_2'^+ = \mathcal{Q}^+ + \mathcal{Y}_1^-$. Since $\mathcal{Y}_1^- \subseteq \mathcal{Y}_2'^+$ and $\mathcal{Y}_2'^- \subseteq \mathcal{Y}_1^+$, $\overline{Y}_1$ and $\overline{Y}_2'$ commute. Let Y_1, Y_2, and Y_2' be the involutions in $\mathfrak{U}_2^-$ corresponding to $\overline{Y}_1$, $\overline{Y}_2$, and $\overline{Y}_2'$, respectively, by means of the particularization $\Phi_{\mathfrak{U}}$. Then Y_1 and Y_2 commute with U_2 and U_3 because they are in the decomposition group $\mathfrak{P}'^-$ of the enclosing involution P'. Also Y_2' commutes with Y_1 and U_2 and is congruent to Y_2. We claim that $P'' = -Y_1Y_2'$ is a second 2-involution enclosing U_2, U_3, and Y_1.

To see this, take a particularization $\Phi_{\mathfrak{Y}_1}$ of the negative decomposition group $\mathfrak{Y}_1^-$ of Y_1. Designate by $\overline{U}_2^*$, $\overline{U}_3^*$, $\overline{Y}_2^*$ and $\overline{Y}_2'^*$ the involutions of $\overline{\mathfrak{T}}$ corresponding to the involutions U_2, U_3, Y_2, and Y_2' in $\mathfrak{Y}_1^-$, respectively. Designate by $\mathcal{U}_2^{+*}$ and $\mathcal{U}_2^{-*}$, etc., the eigenspaces of $\overline{U}_2^*$, etc. Because Y_2' commutes with U_2, it follows that $\mathfrak{Y}_2'^{+*} \supseteq \mathcal{U}_2^{-*} = \mathcal{U}_3^{-*}$. Because Y_2' is congruent to Y_2, it follows that $\mathcal{Y}_2'^{-*} = \mathcal{Y}_2^{-*} \subseteq U_3^{+*}$. Therefore, $\overline{Y}_2'^*$ and $\overline{U}_3^*$ commute. This implies that Y_2' commutes with U_2, U_3, and Y_1 along with P''.

Now we study the 2-involution $Q'' = -P''$ relative to the particularization $\Phi_{\mathfrak{U}}$. Then $\overline{Q}'' = \overline{Y}_1\overline{Y}_2'$ has eigenspaces $\mathcal{Q}''^- = \mathcal{Y}_1^- \oplus \mathcal{Y}_2'^- = \mathcal{Y}_1^- \oplus \mathcal{Y}_2^- = \mathcal{Q}'^-$ and $\mathcal{Q}''^+ = \mathcal{Y}_1^+ \cap \mathcal{Y}_2'^+$. But $\mathcal{Q}^+ \subseteq \mathcal{Y}_1^+ = (\mathcal{Q}^- \cap \mathcal{Q}'^-) \oplus \mathcal{Q}'^+$ since the assumption $\mathcal{Q}^- \cap \mathcal{Q}'^- \subseteq \mathcal{Q}^+ \oplus \mathcal{Q}'^+$ implies $(\mathcal{Q}^- \cap \mathcal{Q}'^-) \oplus \mathcal{Q}'^+ = \mathcal{Q}^+ \oplus \mathcal{Q}'^+$. Also we have seen that $\mathcal{Q}^+ \subseteq \mathcal{Y}_2'^+$. Therefore, $\mathcal{Q}''^+ = \mathcal{Q}^+$ and hence $\mathcal{Q}^+ + \mathcal{Q}''^+$ does not contain $\mathcal{Q}^- \cap \mathcal{Q}''^-$. In this situation, we may find a 1-involution commuting with P and P'' by the method we used in the case where $n \geqq 5$ and we may conclude that U_1 and U_3 are congruent. This proves the theorem.

12. **Construction of the projective geometry.** Let U be a 1-involution in $\mathfrak{G}$. Denote by $a = a(U)$, the equivalence class of 1-involutions congruent to U. For $G \in P(\mathfrak{G})$, define $G^*a = a(GUG^{-1})$ where G is in the coset G^*. Let $\boldsymbol{A}$ be the set of equivalence classes of congruent 1-involutions; we will call its elements *points*. We have made $\mathfrak{G}$ into a transformation group acting on $\boldsymbol{A}$. By showing that $\boldsymbol{A}$ is the set of points in a projective geometry, we will be in a position to show that $P(\mathfrak{G})$ is a projective linear group.

We say that a set of points $\alpha = \{a_1, a_2, \cdots, a_p\}$ of $\boldsymbol{A}$ is an *independent* set if there exists a mutually commuting set of involutions $U_1, U_2, \cdots, U_p$ such that $U_i \in a_i$, $i = 1, 2, \cdots, p$. If U is an involution in an equivalence class $a \in \boldsymbol{A}$, we say that U *represents* a. Also for the above set α, we say that the p-involution $A = U_1U_2 \cdots U_p$ *represents* α. If a set of points is not independent, it is said to be dependent. If $\alpha = \{a_1, a_2, \cdots, a_p\}$ is an independent set of points while $\{\alpha, b\} = \{a_1, a_2, \cdots, a_p, b\}$ is a dependent set, then b is said to be *dependent* on the set α.

Let $\alpha = \{a_1, a_2, \cdots, a_p\}$ be an independent set of points. Then the set

$[\alpha]$ of points b dependent upon α will be called the *linear set* generated by α. In this case an involution A representing α will also be said to *represent* the linear set $[\alpha]$. The set $\boldsymbol{P}$ of all linear sets formed from the elements of $\boldsymbol{A}$ together with the set $[\ ]$ containing no points will be shown to be a projective geometry. That is, we will show that $\boldsymbol{P}$ may be made into a complemented modular lattice of lattice dimension n, all of whose "lines" contain at least three points. The main difficulty is found in characterizing the linear sets so that the lattice operations may be defined.

PROPOSITION 12.1. *Let $\alpha=\{a_1, a_2, \cdots, a_m\}$ be an independent set of points, $1\leqq m\leqq n$. Then there exists a maximal independent set of n points $a_1, a_2, \cdots, a_n$ containing α. All maximal independent sets of points contain n elements.*

Proof. Let $U_i\in a_i$, $i=1, 2, \cdots, m$ be chosen so that $U_1, U_2, \cdots, U_m$ form a set of mutually commuting involutions. Then there exists a maximal set of mutually commuting involutions containing the involutions U_i, $i=1, 2, \cdots, m$. This and every maximal set of mutually commuting involutions contains n elements by virtue of Corollary 8.3.

PROPOSITION 12.2. *Let U_1 and U_2 be commuting 1-involutions and let U_1' be a 1-involution congruent to U_1. Then there exists a 1-involution W such that both $A=U_1U_2$ and U_1' belong to the negative decomposition group $\mathfrak{W}^-$ of W.*

Proof. We may assume that $U_1\neq U_2$. By Lemma 10.1, there exists a 1-involution W in $C(A, U_1')$. Then $U_1'\in\mathfrak{W}^-$, the negative decomposition group of W, but A may or may not be in $\mathfrak{W}^-$. We wish to eliminate the latter case. Therefore, we assume that A is not in $\mathfrak{W}^-$. Under this assumption, WA is in the center of the standard decomposition group $\mathfrak{W}^-\cap\mathfrak{A}^+\subseteq\mathfrak{W}^-$. Form a particularization $\Phi_{\mathfrak{W}}$ of $\mathfrak{W}^-$. Then let $\overline{A}$ in $\overline{\mathfrak{T}}$ correspond to WA by means of $\Phi_{\mathfrak{W}}$ and let $\mathcal{A}^+$ and $\mathcal{A}^-$ be its eigenspaces. As WA is a 1-involution, dim $\mathcal{A}^-=1$ and dim $\mathcal{A}^+=m-1=n-2\geqq 2$. Let $\mathcal{U}_1'^+$ and $\mathcal{U}_1'^-$ be the eigenspaces of the involution $\overline{U}_1'$ corresponding to U_1' by means of $\Phi_{\mathfrak{W}}$. Then dim $\mathcal{U}_1'^+=m-1$ and dim $\mathcal{U}_1'^+\cap\mathcal{A}^+\geqq m-2\geqq 1$. If dim $\mathcal{U}_1'^+\cap\mathcal{A}^+\geqq 2$, it is possible to choose a 1-dimensional subspace $\mathfrak{W}'^-\subseteq\mathcal{U}_1'^+\cap\mathcal{A}^+$ and a complementary subspace $\mathcal{W}'^+$ containing $\mathcal{U}_1'^-\oplus\mathcal{A}^-$ because if $\mathcal{U}_1'^-+\mathcal{A}^-\supseteq\mathcal{U}_1'^+\cap\mathcal{A}^+$, then dim $\mathcal{A}^-\cap(\mathcal{U}_1'^+\cap\mathcal{A}^+)$ $=$dim $\mathcal{U}_1'^+\cap\mathcal{A}^+-1$ on the one hand and $\mathcal{A}^-\cap\mathcal{U}_1'^+\cap\mathcal{A}^+=0$ on the other hand. The 1-involution $\overline{W}'$ with these eigenspaces is in $C(\overline{U}_1', \overline{A})$. Hence the corresponding involution $W'\in\mathfrak{W}^-$ is in $\mathfrak{A}^-$ and also commutes with U_1'. Thus we replace W by W' in this case where $\mathcal{A}^-+\mathcal{U}_1'^-$ does not contain $\mathcal{U}_1'^+\cap\mathcal{A}^+$.

There still remains the case where dim $\mathcal{U}_1'^+\cap\mathcal{A}^+=1$ and $\mathcal{A}^-+\mathcal{U}_1'^-\supseteq$ $\mathcal{U}_1'^+\cap\mathcal{A}^+$. We will eliminate this case by arriving at a contradiction. Since dim $\mathcal{U}_1'^+\cap\mathcal{A}^+=1$, dim $\mathcal{A}^+=2$. Hence $m=3$ and $n=4$. Let $\mathcal{Y}=\mathcal{U}_1'^+\cap\mathcal{A}^+$ and let $\mathcal{X}$ be a 1-dimensional subspace of $\mathcal{A}^+$ such that $\mathcal{A}^+=\mathcal{X}\oplus\mathcal{Y}$. Then

$$\mathcal{E}=\mathcal{X}\oplus\mathcal{Y}\oplus\mathcal{A}^-.$$

But certainly $\mathfrak{A}^- \oplus \mathfrak{U}_1'^- = \mathfrak{A}^- \oplus \mathfrak{Y}$ since it contains $\mathfrak{Y}$. Thus we also have $\mathfrak{E} = \mathfrak{X} \oplus \mathfrak{U}_1'^- \oplus \mathfrak{A}^-$. Let $\overline{U}_1''$ be the 1-involution in $\overline{\mathfrak{T}}$ with eigenspaces $\mathfrak{U}_1''^- = \mathfrak{U}_1'^-$ and $\mathfrak{U}_1''^+ = \mathfrak{X} \oplus \mathfrak{A}^-$. Then the corresponding involution U_1'' in $\mathfrak{W}^-$ is congruent to U_1'. Let W' be the 1-involution in $\mathfrak{W}^-$ such that the involution $\overline{W}$ given by $\Lambda \overline{W}' = \Phi_{\mathfrak{W}} W'$ has the eigenspaces $\mathfrak{W}'^- = \mathfrak{X}$ and $\mathfrak{W}'^+ = \mathfrak{Y} \oplus \mathfrak{A}^- = \mathfrak{U}_1''^- \oplus \mathfrak{A}^-$. Then $\overline{W}'$ commutes with $\overline{U}_1''$ and lies in the group $\overline{\mathfrak{A}}^-$ of transformations commuting with $\overline{A}$ and leaving fixed $\mathfrak{A}^-$. Thus $V = WA$, W, and U_1'' belong to the negative decomposition group $\mathfrak{W}'^-$ of W'.

Let $\Phi_{\mathfrak{W}'}$ be a particularization of $\mathfrak{W}'^-$. Let $\overline{W}^*$ be the 1-involution in $\overline{\mathfrak{T}}$ corresponding to W by means of $\Phi_{\mathfrak{W}'}$; let $\mathfrak{W}^{*+}$ and $\mathfrak{W}^{*-}$ be its eigenspaces. Let $\overline{A}^*$ be the 2-involution with eigenspaces $\mathfrak{A}^{*+}$ and $\mathfrak{A}^{*-}$ that corresponds to A. Use a similar notation for the involutions corresponding to U_1, U_1'', and V. Because $U_1 \in \mathfrak{A}^+ \subseteq \mathfrak{W}'^-$, $\mathfrak{U}_1^{*-} \subseteq \mathfrak{A}^{*-}$. Since U_1 and U_1'' are congruent, $\mathfrak{U}_1''^{*-} = \mathfrak{U}_1^{*-} \subseteq \mathfrak{A}^{*-}$. On the other hand, U_1'' commutes with W but certainly $U_1'' \neq W$. Therefore, $\mathfrak{U}_1''^{*-} \subseteq \mathfrak{A}^{*-} \cap \mathfrak{W}^{*+} = \mathfrak{V}^{*-}$. This implies that U_1'' and $V = WA$ are congruent. Then they are congruent in the group $\mathfrak{W}^-$. This means that $\mathfrak{A}^- = \mathfrak{U}_1''^- = \mathfrak{U}_1'^-$. This contradicts $\mathfrak{A}^- + \mathfrak{U}_1'^- \supseteq \mathfrak{U}_1'^+ \cap \mathfrak{A}^+$.

LEMMA 12.3. *Let A and A' be the products of distinct mutually commuting 1-involutions: $A = U_1 U_2 \cdots U_p$ and $A' = U_1' U_2' \cdots U_p'$ where U_i and U_i' are congruent, $i = 1, 2, \cdots, p$. Let $A = V_1 V_2 \cdots V_p$ be a second representation of A as a product of distinct mutually commuting 1-involutions. Then there exists a family of distinct mutually commuting 1-involutions $V_1', V_2', \cdots, V_p'$ such that $A' = V_1', V_2', \cdots, V_p'$ and V_i and V_i' are congruent, $i = 1, 2, \cdots, p$.*

Proof. We will first establish the lemma for $p = 2$ and then prove it by induction for $2 \leqq p \leqq n-1$. The lemma is trivial if $p = 1$ or if $p = n$.

From Proposition 12.2, it follows that there exists a 1-involution $W_j \in C(A, U_j')$, $j = 1, 2$, such that both U_j' and $A = U_1 U_2$ belong to the negative decomposition group $\mathfrak{W}_j^-$. Let $\Phi_{\mathfrak{W}_j}$ be a particularization of $\mathfrak{W}_j^-$ and use the same notation as in the preceding proposition with the exception that now $\overline{A}$ corresponds to A under the particularization $\Phi_{\mathfrak{W}_j}$. We then obtain that $\mathfrak{E} = \mathfrak{U}_j'^- \oplus \mathfrak{U}_j'^+$ and that $\mathfrak{U}_j'^- = \mathfrak{U}_j^- \subseteq \mathfrak{A}^-$. Let $\mathfrak{X}_j' = \mathfrak{A}^- \cap \mathfrak{U}_j'^+$ and $\mathfrak{Y}_j'$ be subspaces such that $\mathfrak{U}_j'^+ = \mathfrak{X}_j' \oplus \mathfrak{Y}_j'$. Also for the involutions U_i, $i = 1, 2$, we have that $\mathfrak{E} = \mathfrak{U}_i^- \oplus \mathfrak{U}_i^+$. Because $U_i \in \mathfrak{A}^+$, we have that $\mathfrak{U}_i^+ \supseteq \mathfrak{U}_k^- \oplus \mathfrak{A}^+$ where $k \neq i$. Comparing dimensions, we see that $\mathfrak{U}_i^+ = \mathfrak{U}_k^- \oplus \mathfrak{A}^+$. Let A'' be the involution in $\mathfrak{W}_j^-$ such that $\Phi_{\mathfrak{W}_j} A''$ has the fixed subspaces $\mathfrak{A}''^- = \mathfrak{A}^-$ and $\mathfrak{A}''^+ = \mathfrak{Y}_j'$. Correspondingly let U_i'', $i = 1, 2$, be the 1-involutions in $\mathfrak{W}_j^-$ such that the corresponding involution $\overline{U}_i'' \in \overline{\mathfrak{T}}$ has the eigenspaces $\mathfrak{U}_i''^- = \mathfrak{U}_i^-$ and $\mathfrak{U}_i''^+ = \mathfrak{U}_k^- \oplus \mathfrak{Y}_j'$ where $k \neq i$. Then U_1'' and U_2'' are congruent to U_1 and U_2, respectively, and $A'' = U_1'' U_2'' = U_2'' U_1''$. In addition, $U_j' \in \mathfrak{A}''^+$, the positive decomposition group of A''.

We claim that it suffices to prove the theorem with U_1, U_2, and A replaced by U_1'', U_2'', and A'', respectively. To see this, let V_1 and V_2 be distinct mutually commuting 1-involutions in $\mathfrak{A}^+ \subseteq \mathfrak{W}_j^-$. Then $\mathcal{A}^- = \mathcal{V}_1^- \oplus \mathcal{V}_2^-$ where $\mathcal{V}_1^-$ and $\mathcal{V}_2^-$ are the negative eigenspaces of $\overline{V}_1$ and $\overline{V}_2$, respectively. Let V_i'', $i=1, 2$, be the 1-involution in $\mathfrak{W}_j^-$ such that $\overline{V}_i''$ has the eigenspaces $\mathcal{V}_i''^- = \mathcal{V}_i^-$ and $\mathcal{V}_i''^+ = \mathcal{V}_k^- \oplus \mathcal{Y}_j'$, $k \neq i$. Then V_i'' is congruent to V_i, $i=1, 2$, and $A'' = V_1'' V_2'' = V_2'' V_1''$. Because congruence is a transitive relation, we may obtain the desired reduction.

This means that by applying the above argument with $j=2$, we may assume that U_2' belongs to $\mathfrak{A}^+$. Apply the above process with $j=1$. Then $U_2' \in \mathfrak{W}_1^-$ as $\mathfrak{W}_1^- \supseteq \mathfrak{A}^+$. The corresponding involution $\overline{U}_2' \in \overline{\mathfrak{T}}$ has the negative eigenspace $\mathcal{U}_2^-$. As $\overline{U}_1^-$ has the negative eigenspace $\mathcal{U}_1^-$, $\overline{A}' = \overline{U}_1' \overline{U}_2'$ has the negative eigenspace $\mathcal{A}'^- = \mathcal{U}_1^- \oplus \mathcal{U}_2^-$. But $\mathcal{A}'^- = \mathcal{A}^- = \mathcal{V}_1^- \oplus \mathcal{V}_2^-$. Applying the argument of the preceding paragraph, we obtain involutions V_1' and V_2' congruent to V_1 and V_2, respectively, such that $A = V_1' V_2'$. This proves the lemma in the case $p=2$.

Now assume the validity of the lemma for p'-involutions where $2 \leqq p' < p < n$. Consider the situation stated in the hypothesis of the lemma. Let W be a 1-involution in $\mathfrak{A}^-$ and W' a 1-involution in $\mathfrak{A}'^-$. Then U_i and V_i, $i=1, 2, \cdots, p$, are in $\mathfrak{W}^-$ and U_i', $i=1, 2, \cdots, p$, are in $\mathfrak{W}'^-$. Let $\Phi_{\mathfrak{W}}$ and $\Phi_{\mathfrak{W}'}$ be particularizations of $\mathfrak{W}^-$ and $\mathfrak{W}'^-$, respectively. Designate by $\overline{A}$, $\overline{U}_i$, and $\overline{V}_i$ the involutions of $\overline{\mathfrak{T}}$ corresponding to A, U_i, and V_i, respectively, by means of $\Phi_{\mathfrak{W}}$: let $\mathcal{A}^\pm$, $\mathcal{U}_i^\pm$, and $\mathcal{V}_i^\pm$ be their eigenspaces. Likewise designate by $\overline{A}'$ and $\overline{U}_i'$ the involutions of $\overline{\mathfrak{T}}$ corresponding to A' and U_i', respectively, by means of the particularization $\Phi_{\mathfrak{W}'}$; let $A'^\pm$ and $\mathcal{U}_i'^\pm$ be their eigenspaces.

We now introduce new commuting sets of 1-involutions in $\mathfrak{W}^-$ by defining appropriate eigenspaces for the corresponding involutions in $\overline{\mathfrak{T}}$. Not every subspace $\mathcal{U}_i^- \subseteq \mathcal{V}_1^+$ because $\mathcal{E} = \oplus_{i=1}^p \mathcal{U}_i^-$. Thus we may assume that $\mathcal{U}_1^- \cap \mathcal{V}_1^+ = 0$. Then note that $\dim \mathcal{U}_1^+ \cap \mathcal{V}_1^+ \geqq m-2$. Therefore, choose 1-dimensional eigenspaces $\mathcal{R}_3^-, \mathcal{R}_4^-, \cdots, \mathcal{R}_p^- \subseteq \mathcal{U}_1^+ \cap \mathcal{V}_1^+$ and $\mathcal{R}_2^- \subseteq \mathcal{U}_1^+$ such that

$$\mathcal{E} = \mathcal{U}_1^- \oplus \mathcal{U}_1^+ = \mathcal{U}_1^- \oplus \mathcal{R}_2^- \oplus \mathcal{R}_3^- \oplus \cdots \oplus \mathcal{R}_p^- \oplus \mathcal{A}^+. \tag{12.1}$$

In case $\mathcal{U}_1^- \neq \mathcal{V}_1^-$, we require that $\mathcal{R}_2^-$ or $\mathcal{R}_3^-$ be the subspace $(\mathcal{U}_1^- \oplus \mathcal{V}_1^-) \cap \mathcal{U}_1^+$ in accordance as

$$(\mathcal{U}_1^- \oplus \mathcal{V}_1^-) \cap (\mathcal{U}_1^+ \cap \mathcal{V}_1^+) = 0 \tag{12.2}$$

holds or not. Thus we obtain that $\mathcal{V}_1^- \subseteq \mathcal{U}_1^- \oplus \mathcal{R}_2^-$ or that

$$\mathcal{R}_3^- = (\mathcal{U}_1^- \oplus \mathcal{V}_1^-) \cap \mathcal{U}_1^+.$$

Thus we obtain that $\mathcal{V}_1^- \subseteq \mathcal{U}_1^- \oplus \mathcal{R}_2^-$ or $\mathcal{V}_1^- \subseteq \mathcal{U}_1^- \oplus \mathcal{R}_3^-$. Let R_i, $i = 2, 3, \cdots, p$, be the 1-involutions of $\mathfrak{W}^-$ corresponding to the involutions $\overline{R}_i \in \overline{\mathfrak{T}}$ by means of $\Phi_{\mathfrak{W}}$ where $\overline{R}_i$ has eigenspaces $\mathcal{R}_i^-$ and $\mathcal{R}_i^+ = \oplus_{j \neq i} \mathcal{R}_j^- \oplus \mathcal{U}_1^- \oplus \mathcal{A}^+$. Then we obtain a family of mutually commuting 1-involutions U_1, R_2, R_3,

$\cdots, R_p$ such that $B=U_1A=R_2R_3\cdots R_p=U_2U_3\cdots U_p$ and $B'=U_1'A'=U_2'U_3'\cdots U_p'$ are $(p-1)$-involutions. We apply the lemma inductively to B and B' to obtain a family $R_2', R_3', \cdots, R_p'$ of mutually commuting 1-involutions which are congruent, respectively, to $R_2, R_3, \cdots, R_p$ and for which $B'=R_2'R_3'\cdots R_p'$.

Now we specialize to the case where $\mathfrak{V}_1^-\subseteq\mathfrak{U}_1^-\oplus\mathfrak{R}_2^-$. Then V_1 commutes with the 2-involution U_1R_2 and $U_1R_2=V_1X_1$ where X_1 is the 1-involution $V_1U_1R_2$. Again applying the lemma inductively to the involutions U_1R_2 and $U_1'R_2'$, we obtain 1-involutions V_1' and X_1' which are congruent to V_1 and X_1, respectively, and for which $U_1'R_2'=V_1'X_1'$ is a 2-involution. But then V_1' is in the positive decomposition group of $U_1'R_2'$ while $R_3', R_4', \cdots, R_p'$ are in the negative decomposition group. Thus V_1' commutes with R_j', $j=3, 4, \cdots, p$, and $C'=V_1'A'=X_1'R_3'R_4'\cdots R_p'$ is a $(p-1)$-involution. But also $C=V_1A=X_1R_3R_4\cdots R_p=V_2V_3\cdots V_p$. Again apply the lemma inductively to the involutions C and C' to obtain the involutions $V_2', V_3', \cdots, V_p'$, which together with V_1' are the involutions described in the conclusion of the lemma.

There still remains the case where $\mathfrak{V}_1^-\subseteq\mathfrak{U}_1^-\oplus\mathfrak{R}_3^-$. Note that $A=R_2U_1R_3R_4\cdots R_p$ and $A'=R_2'U_1'R_3'R_4'\cdots R_p'$. Therefore, to treat this case, we interchange the roles of R_2 and U_1 in the preceding argument. Now

$$\mathfrak{R}_2^+=\mathfrak{U}_1^-\oplus\mathfrak{R}_3^-\oplus\mathfrak{R}_4^-\oplus\cdots\oplus\mathfrak{R}_p^--\mathfrak{A}^+. \tag{12.3}$$

We have taken $\mathfrak{U}_1^-\cap\mathfrak{V}_1^+=0$. Hence $\mathfrak{V}_1^+\cap\mathfrak{R}_2^+=\mathfrak{V}_1^+\cap\mathfrak{U}_1^+=\mathfrak{R}_3^-\oplus\mathfrak{R}_4^-\oplus\cdots\oplus\mathfrak{R}_p^-\oplus\mathfrak{A}^+$. This confirms (12.1) with the roles of R_2 and U_1 interchanged. Also $(\mathfrak{R}_2^-\oplus\mathfrak{V}_1^-)\cap\mathfrak{R}_2^+=\mathfrak{V}_1^-$; hence $(\mathfrak{R}_2^-\oplus\mathfrak{V}_1^-)\cap(\mathfrak{R}_2^+\cap\mathfrak{V}_1^+)=0$. Thus we have obtained the counterpart of (12.2). Since this case has already been considered, we have completed the proof of the lemma.

We will term the involutions A and A' as described in Lemma 12.3 as *congruent p-involutions*. In particular, if $\Phi_{\mathfrak{B}}$ is a particularization of a standard decomposition group containing A and A', then the corresponding involutions $\overline{A}$ and $\overline{A}'$ have a common negative eigenspace when A and A' are congruent. It follows from Theorem 11.3 that this extended definition of congruence is also an equivalence relation.

PROPOSITION 12.4. *Let $a_1, a_2, \cdots, a_n$ be a maximal independent set of points. Then there is only one maximal set of mutually commuting 1-involutions $U_1, U_2, \cdots, U_n$ such that U_i represents a_i, $i=1, 2, \cdots, n$.*

Proof. Let $U_1', U_2', \cdots, U_n'$ be a second set of mutually commuting 1-involutions such that U_i and U_i' are congruent. We wish to show that they are equal; we need only do this for U_1 and U_1'.

Let P be a 2-involution enclosing U_1 and U_1' and let P^- be its negative decomposition group. Let $V_3, V_4, \cdots, V_n$ be a family of distinct mutually commuting 1-involutions in $\mathfrak{P}^-$. Then $V_3V_4\cdots V_n=-P$. Set $V_2=U_1P$.

Then $-U_1 = U_2U_3 \cdots U_n = V_2V_3 \cdots V_n$. Also $-U_1' = U_2' U_3' \cdots U_n'$. Therefore, from Lemma 12.3, it follows that $-U_1' = V_2' V_3' \cdots V_n'$ where the involutions V_i', $i=2, 3, \cdots, n$, form a mutually commuting family of 1-involutions such that V_i and V_i' are congruent.

Now the negative decomposition group $\mathfrak{U}_1'^-$ contains $\mathfrak{P}^-$ and hence V_i, $i=3, 4, \cdots, n$; it also contains V_i', $i=2, 3, \cdots, n$. Let $\Phi_{\mathfrak{U}}$ be a particularization of $\mathfrak{U}_1'^-$ and use the usual notation to denote the eigenspaces of involutions in $\overline{\mathfrak{T}}$ corresponding to involutions in $\mathfrak{U}_1'^-$. Thus

$$(12.4)\qquad \mathcal{E} = \mathcal{X}_2^- \oplus \mathcal{V}_3^- \oplus \cdots \oplus \mathcal{V}_n^- = \mathcal{V}_2'^- \oplus \mathcal{V}_3'^- \oplus \cdots \oplus \mathcal{V}_n'^-$$

where $\mathcal{X}_2^-$ is the eigenspace of the involution $\overline{X}_2$ corresponding to $U_1' U_1 V_2 = -U_1' V_3 V_4 \cdots V_n$. Because V_i and V_i' are congruent, $\mathcal{V}_i^- = \mathcal{V}_i'^-$, $i=3, 4, \cdots, n$. Now if $\mathcal{X}_2^- = \mathcal{V}_2'^-$, it follows that $\mathcal{V}_i^+ = \mathcal{V}_i'^-$, $i=3, 4, \cdots, n$. Hence $\overline{V}_i = \overline{V}_i'$ and $V_i = V_i'$, $i=3, 4, \cdots, n$. This implies that $U_1V_2 = U_1' V_2'$.

Let $\mathcal{Q}^-$ be the negative eigenspace of the involution $\overline{Q}$ corresponding to $Q = -P$ by means of $\Phi_{\mathfrak{U}}$. Then $\dim \mathcal{Q}^- = m-1$; indeed, $\mathcal{Q}^- = \mathcal{V}_3^- \oplus \mathcal{V}_4^- \oplus \cdots \oplus \mathcal{V}_n^-$. We treat the case where $\mathcal{X}_2^- \neq \mathcal{V}_2'^-$ by replacing $\mathcal{V}_3^-$ by the subspace $(\mathcal{X}_2^- \oplus \mathcal{V}_2'^-) \cap \mathcal{Q}^-$ to obtain a new set of mutually commuting 1-involutions in $\mathfrak{P}^- = \mathfrak{Q}^+$, which we will again designate by $V_3, V_4, \cdots, V_n$. For the involutions V_i, $i \geqq 4$, $\mathcal{V}_i^+ \supseteq \mathcal{V}_2'^-$ inasmuch as $\mathcal{V}_i^+ \supseteq \mathcal{V}_3^- \oplus \mathcal{X}_2^- = \mathcal{V}_3^- \oplus \mathcal{V}_2'^-$. But this means that $\mathcal{V}_i^+ = \oplus_{j\neq i} \mathcal{V}_j^- = \oplus_{j\neq i} \mathcal{V}_j'^- = \mathcal{V}_i'^+$, $i=4, 5, \cdots, n$. Hence $V_i' = V_i$, $i=4, 5, \cdots, n$, which implies that $U_1V_2V_3 = U_1' V_2' V_3'$.

Now let W be a 1-involution in the negative decomposition group of $A = U_1V_2 = U_1' V_2'$ or $B = U_1V_2V_3 = U_1' V_2' V_3'$, as the case may be. Then the negative decomposition group W^- of W contains the 1-involutions U_1, V_2, U_1', etc. Let $\Phi_{\mathfrak{W}}$ be a particularization of $\mathfrak{W}^-$. Relative to this particularization adopt the usual system of notation for the designation of eigenspaces of involutions. We then obtain that $\mathcal{E} = \mathcal{A}^- \oplus \mathcal{A}^+$ or $\mathcal{E} = \mathcal{B}^- \oplus \mathcal{B}^+$ and that $\mathcal{A}^- = \mathcal{U}_1^- \oplus \mathcal{V}_2^- = \mathcal{U}_1'^- \oplus \mathcal{V}_2'^-$ or $\mathcal{B}^- = \mathcal{U}_1^- \oplus \mathcal{V}_2^- \oplus \mathcal{V}_3^- = \mathcal{U}_1'^- \oplus \mathcal{V}_2'^- \oplus \mathcal{V}_3'^-$. Since the involutions U_1 and U_1' and also V_i and V_i' are congruent, it follows that $\mathcal{U}_1^- = \mathcal{U}_1'^-$ and $\mathcal{V}_i^- = \mathcal{V}_i'^-$, $i=2, 3$. But in both cases, this implies that $\overline{U}_1 = \overline{U}_1'$ and finally that $U_1' = U_1$. This proves the proposition.

Proposition 12.5. *Let $\alpha = \{a_1, a_2, \cdots, a_p\}$ be an independent set of points and let b be a point. Then there exist involutions A and B representing α and b such that $AB = BA$ is a $(p-1)$-involution or a $(p+1)$-involution. Furthermore, $\{\alpha, b\}$ is dependent or independent according as AB is a $(p-1)$-involution or a $(p+1)$-involution.*

Proof. Let U_i' represent a_i, $i=1, 2, \cdots, p$, so that they form a mutually commuting set. Set $A' = U_1' U_2' \cdots U_p'$. Let B' represent b. Let $W \neq B'$ be a 1-involution in $C(A', B')$. Clearly $B' \in \mathfrak{W}^-$, the negative decomposition group of W. If $A' \in \mathfrak{W}^-$, set $V_i' = U_i'$, $i=1, 2, \cdots, p$. On the other hand if $A' \notin \mathfrak{W}^-$, then $W \in \mathfrak{A}'^-$, the negative decomposition group of A'. In $\mathfrak{A}'^+$, there exists a maximal set of mutually commuting 1-involutions $V_1' = W$, $V_2', \cdots, V_p'$ such that $A' = V_1' V_2' \cdots V_p'$ and $WA' = V_2' V_3' \cdots V_p'$. Let $\Phi_{\mathfrak{W}}$ be a particularization of $\mathfrak{W}^-$ and use the usual notation to denote the eigenspaces of involutions in $\overline{\mathfrak{T}}$ corresponding to involutions in $\mathfrak{W}^-$ by means of $\Phi_{\mathfrak{W}}$. There are two cases to consider.

The first is that $\mathcal{B}'^- \subseteq \mathcal{A}'^-$. Then set $\mathcal{B}^- = \mathcal{B}'^-$ and $\mathcal{B}^+ = (\mathcal{B}'^+ \cap \mathcal{A}'^-) \oplus \mathcal{A}'^+$ to obtain an involution $B \in \mathfrak{W}^-$ which is congruent to B' and which commutes with $A = A'$. We then obtain the first part of the proposition for this case.

The second case is that $\mathcal{B}'^- \cap \mathcal{A}'^- = 0$. Choose subspaces $\mathcal{A}^+$ containing $\mathcal{B}'^-$ and $\mathcal{B}^+$ containing $\mathcal{A}'^-$ which are complementary, respectively, to $\mathcal{A}^- = \mathcal{A}'^-$ and $\mathcal{B}^- = \mathcal{B}'^-$. Again we obtain commuting involutions A and B, and B is congruent to B'. We also choose 1-involutions V_i congruent to V_i', $i = 1, 2, \cdots, p$, as follows. For $i \geqq 2$, set $\mathcal{V}_i^+ = (\mathcal{V}_i'^+ \cap \mathcal{A}'^-) \oplus \mathcal{A}^+$ and $\mathcal{V}_i^- = \mathcal{V}_i'^-$. If $A' \in \mathfrak{W}^-$, also set $\mathcal{V}_1^+ = (\mathcal{V}_1'^+ \cap \mathcal{A}'^-) \oplus \mathcal{A}^+$. If $A' \notin \mathfrak{W}^-$ and $V_1' = W$, set $V_1 = V_1'$. As $\mathcal{V}'^+ \cap \mathcal{A}'^-$ is the direct sum of the subspaces $\mathcal{V}_j'^-$, $j \neq i$, which are contained in $\mathcal{A}'^-$, we see that $\mathcal{V}_i^- \subseteq \mathcal{V}_j^+$ and $\mathcal{V}_j^- \subseteq \mathcal{V}_i^+$. This means that the involutions V_i, $i = 1, 2, \cdots, p$, form a mutually commuting family of involutions and $A = V_1 V_2 \cdots V_p$. Since $A' = V_1' V_2' \cdots V_p' = U_1' U_2' \cdots U_p'$, using Lemma 12.3 we may find a family of mutually commuting 1-involutions $U_1, U_2, \cdots, U_p$ such that $A = U_1 U_2 \cdots U_p$ and U_i and U_i' are congruent. Thus A represents α.

Now A is a p-involution. If $B \in \mathfrak{A}^-$, AB is a $(p+1)$-involution since B will then commute with each involution U_i, $i = 1, 2, \cdots, p$ and $AB = U_1 U_2 \cdots U_p B$. If $B \in \mathfrak{A}^+$, then AB is a $(p-1)$-involution because it is in the center of $\mathfrak{A}^+ \cap \mathfrak{B}^-$.

Suppose now that AB is a $(p-1)$-involution. Should $\{\alpha, b\}$ be independent, there would exist a family of mutually commuting 1-involutions $U_1', U_2', \cdots, U_p'$, and B' such that U_i' and U_i are congruent, $i = 1, 2, \cdots, p$, B and B' are congruent, and $A'B' = U_1' U_2' \cdots U_p' B'$ is a $(p+1)$-involution. Because AB is a $(p-1)$-involution, there exists a mutually commuting family of 1-involutions V_i such that $A = V_1 V_2 \cdots V_p$ with $V_1 = B$. Applying Lemma 12.3, we determine a family of mutually commuting 1-involutions V_i' such that $A' = V_1' V_2' \cdots V_p'$ and V_1' and B are congruent. This means that V_1' is in $\mathfrak{A}'^+$ and is distinct from and commutes with $B' \in \mathfrak{A}'^-$. Since V_1' and B' are congruent, we obtain a contradiction. Thus $\{\alpha, b\}$ is a dependent set. It is obvious that if $AB = U_1 U_2 \cdots U_p B$ is a $(p+1)$-involution, then $\{\alpha, b\}$ is independent. This proves the proposition.

THEOREM 12.6. *Two linear sets $[\alpha]$ and $[\beta]$ are identical if and only if they are represented by congruent involutions.*

Proof. Let B be an involution representing β and let A be an involution representing α which is congruent to B. It follows from Lemma 12.3 that A also represents β. Then it is a corollary to Proposition 12.5 that $[\alpha] = [\beta]$. The necessity of the theorem is obvious.

If a linear set $[\alpha]$ is represented by a p-involution, we will write dim $[\alpha] = p$.

PROPOSITION 12.7. *Let* $\alpha = \{a_1, a_2, \cdots, a_p\}$ *be an independent set of points. Let b and c be points such that* $\{\alpha, b\}$ *and* $\{\alpha, c\}$ *are independent sets. Then if* $c \in [\alpha, b]$, $[\alpha, b] = [\alpha, c]$.

Proof. Let A be a p-involution representing α and let B and C be 1-involutions representing b and c, respectively. We may further require that C commutes with a $(p+1)$-involution D representing $\{\alpha, b\}$ and, by replacing A and B by congruent involutions, that $D = AB$ where $AB = BA$ because of Proposition 12.5 and Theorem 12.6. Then by Proposition 12.5, CD is a p-involution. Form the standard decomposition

$$C^*(D) = \mathfrak{D}^+ \times \mathfrak{D}^-. \tag{12.5}$$

Now $D \in \mathfrak{D}^+$. If $C \in \mathfrak{D}^-$, the CD is a $(p+1)$-involution, which is not the case. Hence $C \in \mathfrak{D}^+$. Because $AB = D$, both A and B are in $\mathfrak{D}^+$. Let W be a 1-involution in $\mathfrak{D}^-$. Let $\Phi_{\mathfrak{W}}$ be a particularization of the negative decomposition group $\mathfrak{W}^-$. Since A and C are in $\mathfrak{D}^+ \subseteq \mathfrak{W}^-$, the eigenspaces $\mathfrak{A}^-$ and $\mathfrak{C}^-$ of the respective involutions $\overline{A}$ and $\overline{C}$ corresponding to A and C by means of $\Phi_{\mathfrak{W}}$ are both contained in the negative eigenspace $\mathfrak{D}$ of the involution $\overline{D}$ corresponding to D. One may determine involutions A' and C' in $\mathfrak{W}^-$ such that the corresponding involutions $\overline{A}'$ and $\overline{C}'$ in $\overline{T}$ have the same negative eigenspaces as do $\overline{A}$ and $\overline{C}$, respectively, and $\overline{A}'\overline{C}' = \overline{C}'\overline{A}'$ just as we did in the proof of Proposition 12.5. Then A' is congruent to A and C' is congruent to C and $A'C' = C'A'$. By Proposition 12.5 and Theorem 12.6, $A'C'$ is a $(p+1)$-involution since it represents an independent set $\{\alpha, c\}$. Since the positive decomposition group $\mathfrak{D}^+$ has rank $p+1$ and contains $A'C'$, $A'C' = D$. Then by Theorem 12.6, $[\alpha, b] = [\alpha, c]$.

COROLLARY 12.8. *Let* $\alpha = \{a_1, a_2, \cdots, a_p\}$ *be a set of points. Then any maximal independent subset determines the same linear set.*

This is a direct consequence of Proposition 12.7 obtained by using the standard argument in the theory of vector spaces. Henceforth, we extend our notation to designate by $[\alpha]$ the unique linear set determined by a set α of points, not necessarily independent. Also $[\alpha, \beta]$ will denote the linear set determined by the points of α and β.

COROLLARY 12.9. *Let* $[\alpha]$ *and* $[\beta]$ *be linear sets. Then if* α *is a subset of* β, $[\alpha] \subseteq [\beta]$. *In general,* $[\alpha] \cap [\beta]$ *is a linear set.*

Proof. If α is a subset of β, we may choose a maximal independent set α' from α and extend it to a maximal independent set β' of β. Then if B is an involution representing β', $B = AC$ where A is an involution representing α'. The first statement of the corollary now follows from Proposition 12.5.

If $[\alpha]$ and $[\beta]$ are linear sets, choose a maximal independent set γ from $[\alpha]\cap[\beta]$. From the first statement of this corollary, it follows that $[\gamma]\subseteq[\alpha]\cap[\beta]$. Then it is clear that $[\alpha]\cap[\beta]=[\gamma]$ from the fact that γ is a maximal independent set.

We have now defined two operations on the set $\boldsymbol{P}$ of linear sets.

THEOREM 12.10. *The set $\boldsymbol{P}$ is a projective geometry over the field K.*

Proof. We first show that $\boldsymbol{P}$ is a modular lattice by showing that it is both upper and lower semimodular. Because $\boldsymbol{P}$ satisfies the chain conditions, we must show that if $[\alpha]$ and $[\beta]$ are in $\boldsymbol{P}$ and if $[\alpha]$ and $[\beta]$ cover $[\alpha]\cap[\beta]$, then $[\alpha, \beta]$ covers $[\alpha]$ and $[\beta]$ to show upper semimodularity. To prove lower semimodularity, we show the converse (cf. [3, p. 100]).

To show upper semimodularity, let $[\gamma]=[\alpha]\cap[\beta]$ and suppose that $[\alpha]=[\gamma, a]$ and $[\beta]=[\gamma, b]$ where $a\notin[\beta]$ and $b\notin[\alpha]$. Then $[\alpha, \beta]=[\gamma, a, b]=[\alpha, b]=[\beta, a]$ by Corollary 12.8. Since $b\notin[\alpha]$, $\dim[\alpha, b]=\dim[\alpha]+1$ and $[\alpha, \beta]$ covers $[\alpha]$. Similarly $[\alpha, \beta]$ covers $[\beta]$. Thus P is upper semimodular.

To show lower semimodularity, let $[\alpha, \beta]=[\alpha, b_1]=[\beta, a_1]$. We may clearly suppose that $a_1\in[\alpha]$ and $b_1\in[\beta]$. Using Theorem 12.6, we see that there exist commuting involutions B and A_1 representing β and a_1, respectively, and commuting involutions A and B_1 representing α and b_1, respectively, such that $R=AB_1=BA_1$. Suppose first that $R\neq-1$; then there exists a 1-involution $W\in\mathfrak{R}^-$, the negative decomposition group of R. Hence A, B, A_1, and B_1 belong to $\mathfrak{R}^-\subseteq\mathfrak{W}^-$. Let $\Phi_{\mathfrak{W}}$ be a particularization of $\mathfrak{W}^-$. Use the usual convention of denoting the eigenspaces of involutions in $\overline{\mathfrak{T}}$ corresponding to involutions in $\mathfrak{W}^-$ by means of $\Phi_{\mathfrak{W}}$. We have $R^-=B^-\oplus A_1^-=A^-\oplus B_1^-$. Because $b_1\in[\alpha]$, B_1 is congruent to an involution $B_1'^-$ which commutes with B, and BB_1' is a $(p-1)$-involution by Proposition 12.5. This means that $B_1^-=B_1'^-\subseteq B^-$. Then $B_1^-\subseteq B^-\subseteq A_1^+$ since $\overline{B}$ and $\overline{A}_1$ commute. Similarly $A_1^-\subseteq B_1$. Hence A_1 and B_1 commute. This means that B and B_1 commute. Proposition 12.5 implies that BB_1 is a $(p-1)$-involution. Likewise A and A_1 commute and AA_1 is a $(p-1)$-involution. But now $R=(BB_1)B_1A_1=(AA_1)A_1B_1$. Hence $BA_1=AB_1$ represents a linear set $[\gamma]$ contained in both $[\alpha]$ and $[\beta]$. Here $[\alpha]=[\gamma, a_1]$ and $[\beta]=[\gamma, b_1]$. Hence $[\gamma]=[\alpha]\cap[\beta]$ and $[\alpha]$ and $[\beta]$ cover $[\gamma]$.

If $R=-1$, then $A_1=-B$ and $B_1=-A$. In this case, choose W to be a 1-involution in $C^*(A_1, B_1)$. Then WA, A_1, WB, B_1, and $RW=-W$ belong to W^-. We argue with these involutions just as we did with the involutions A, A_1, B, B_1, and R in the above paragraph to obtain the same result. Therefore, $\boldsymbol{P}$ is a lower semimodular lattice and thus is modular.

It is clear that $\boldsymbol{P}$ has lattice dimension n. To see that $\boldsymbol{P}$ is complemented, note merely that the complement of a linear set $[\alpha]$ represented by an involution A is the linear set $[\alpha']$ represented by $-A$. Finally we note that every linear set $[a, b]$ of dimension 2 contains a point $c\neq a, b$; indeed, choose c to be representeed by a 1-involution in the negative decomposition group of a 2-involution representing $[a, b]$ which is not congruent to the 1-involutions representing a and b. This proves that $\boldsymbol{P}$ is a projective geometry (cf. [3, p. 118]).

To see that $\boldsymbol{P}$ is a projective geometry over K, let U be a 1-involution with negative decomposition group $\mathfrak{U}^-$. Let $\Phi_{\mathfrak{U}}$ be a particularization of $\mathfrak{U}^-$. Then $\Phi_{\mathfrak{U}} U^-$ is a projective linear group acting on a projective geometry over K of dimension $n-1$. This implies that $\boldsymbol{P}$ itself is a projective geometry over K.

13. **Characterization of the group $\mathfrak{G}$. Proof of the Principal Theorem.** In the beginning of §12, we defined $P(\mathfrak{G})$ to be a transformation group of the set $\boldsymbol{A}$ of points of $\boldsymbol{P}$. Then $P(\mathfrak{G})$ may be extended to a transformation group of $\boldsymbol{P}$ in the obvious manner.

PROPOSITION 13.1. *The group $\mathfrak{G}$ acts effectively on $\boldsymbol{P}$.*

Proof. To show this, we must show that if GUG^{-1} is congruent to U for all 1-involutions U of $\mathfrak{G}$, then $G \in Z(\mathfrak{G})$. Consider a maximal independent set of points $a(U_1), a(U_2), \cdots, a(U_n)$ determined by a mutually commuting set of 1-involutions U_i, $i=1, 2, \cdots, n$. Since GU_iG^{-1} is congruent to U_i, it follows from Proposition 12.4 that $GU_iG^{-1}=U_i$, $i=1, 2, \cdots, n$. But every 1-involution U in $\mathfrak{G}$ belongs to a maximal set of commuting involutions. This means that $G \in Z(T(\mathfrak{G}))$ since the 1-involutions generate $T(\mathfrak{G})$. Then Condition D gives our result.

PROPOSITION 13.2. *The mapping Φ: $G \to G^*$ determines a homomorphism of $\mathfrak{G}$ into the group $P\Gamma L(n)$ of collineations of $\boldsymbol{P}$.*

Proof. We must show that, for each $G \in G$, G^* determines a lattice isomorphism of $\boldsymbol{P}$. Let $[\alpha] \subseteq [\beta]$ be linear sets. Then there exists a maximal independent set of points $b_1, b_2, \cdots, b_q$ such that $[\alpha]=[b_1, b_2, \cdots, b_p]$ and $[\beta]=[b_1, b_2, \cdots, b_q]$ with $p<q$. Let $U_1, U_2, \cdots, U_q$ be a mutually commuting set of 1-involutions such that U_i represents b_i, $i=1, 2, \cdots, q$. Then $A=U_1U_2 \cdots U_p$ represents $[\alpha]$ and $B=U_1U_2 \cdots U_q$ represents $[\beta]$ and $B=AC$. Let now $V_i=GU_iG^{-1}$, $i=1, 2, \cdots, q$. Then V_i represents G^*b_i; hence GAG^{-1} represents $G^*[\alpha]$ and GBG^{-1} represents $G^*[\beta]$. Since $GAG^{-1} = V_1V_2 \cdots V_p$ and $GBG^{-1}=V_1V_2 \cdots V_q$, it follows that $G^*[\alpha] \subseteq G^*[\beta]$. This shows that G^* is an order preserving homomorphism of $\boldsymbol{P}$. The same is true of G^{*-1}. Hence G^* is a lattice isomorphism of $\boldsymbol{P}$.

PROPOSITION 13.3. *$T(\mathfrak{G})$ is an involutory quasilinear group.*

Proof. We must verify that Conditions A, B, C, D, and E hold for $T(\mathfrak{G})$. First of all, note that $P(T(\mathfrak{G}))$ is isomorphic to $PTL(n)$. Indeed, under the mapping Φ: $\mathfrak{G} \to P\Gamma L(n)$ determined by $\Phi G=G^*$, the involutions in the class $\mathfrak{K}_p(\mathfrak{G})$ map onto involutions of $\boldsymbol{P}$ with fixed complementary linear sets of dimension p and $n-p$. These are the involutions in $PGL(n)$ which are determined by p-involutions in $GL(n)$. They generate the group $TL(n)$. Consequently, $P(T(\mathfrak{G}))\subseteq PTL(n)$. Conversely, to each involution $U\in GL(n)$, there corresponds an involution U^* in $PTL(n)$ with fixed complementary linear sets $[\alpha]$ and $[\alpha']$ of dimension p and $n-p$, respectively. We have seen that there exists a p-involution A in $TL(n)$ such that A represents $[\alpha]$ and $-A$ represents $[\alpha']$. Then $\Phi A = A^*$ has both $[\alpha]$ and $[\alpha']$ as its fixed linear sets. Thus $\Phi A = U^*$. This shows that $\Phi T(\mathfrak{G}) = PTL(n)$.

Because Φ maps $\mathfrak{G}$ into $P\Gamma L(n)$, we may apply the argument of Proposition 6.1 to conclude that $\Phi\mathfrak{K}_p\subseteq\mathfrak{L}_p$ and $\Phi\mathfrak{K}_{n-p}\subseteq\mathfrak{L}_{n-p}=\mathfrak{L}_p$ where $\mathfrak{L}_p$ is the conjugate class of involutions in $PGL(n)$ consisting of involutions which are the images of involutions in the classes $K_p(\overline{\mathfrak{G}})$ and $K_{n-p}(\overline{\mathfrak{G}})$ of the group $\overline{\mathfrak{G}}=GL(n)$ under the natural homomorphism Λ of $\overline{\mathfrak{G}}$ onto $P(\overline{\mathfrak{G}})=PGL(n)$. However, the classes $\mathfrak{K}_p(\overline{\mathfrak{G}})$ and $\mathfrak{K}_{n-p}(\overline{\mathfrak{G}})$ are conjugate classes in $TL(n)$ as well as in $GL(n)$. This may be seen from the fact that there exists an element in $GL(n)$ with arbitrary determinant commuting with a given involution and the fact that $TL(n)$ is the subgroup of $GL(n)$ consisting of elements of determinant ± 1. Thus $\mathfrak{L}_p$ is a conjugate class of involutions in $PTL(n)$.

This means that if U and V are in $\mathfrak{K}_p$, there exists $T\in T(\mathfrak{G})$ such that $TUT^{-1}=ZV$ where $Z\in Z(\mathfrak{G})$. We have seen that this implies that $Z=\pm 1$. If $Z=1$, then U and V are conjugate in $T(\mathfrak{G})$. In case $Z=-1$, we obtain that U and $-V$ are conjugate in $T(\mathfrak{G})$. This implies that V and $-V$ are conjugate in $\mathfrak{G}$. Propositions 5.4 and 5.6 imply that $p=n/2$. Therefore, in a maximal set of commuting involutions containing V, there exists a complete set of mutually commuting 1-involutions chosen in accordance with Proposition 8.1 so that $V_1, V_2, \cdots, V_p$ lie in the positive decomposition group $\mathfrak{V}^+$ and $V_{p+1}, V_{p+2}, \cdots, V_n$ lie in the negative decomposition group $\mathfrak{V}^-$. By the argument of the proof of statement (8.1a) of Proposition 8.1, it follows that there exists in the positive decomposition group of the 2-involution V_iV_{p+i}, $i=1, 2, \cdots, p$, an involution W_i such that $W_iV_iW_i=V_{p+i}$. Certainly the involutions W_i, $i=1, 2, \cdots, p$, all commute. Set $W=W_1W_2\cdots W_p$; we have $V=V_1V_2\cdots V_p$ and $-V=V_{p+1}V_{p+2}\cdots V_n$. Therefore, $WVW^{-1} = -V$. Thus in this case we have that U and V are conjugate in $T(\mathfrak{G})$. This shows that there are $n+1$ classes $\mathfrak{K}_0, \mathfrak{K}_1, \cdots, \mathfrak{K}_n$ of conjugate involutions in $T(\mathfrak{G})$.

To verify the remaining part of Condition A for $T(\mathfrak{G})$, let $U\in\mathfrak{K}_p$. Then $C^*(U)=\mathfrak{U}^+\times\mathfrak{U}^-$ where $\mathfrak{U}^+$ and $\mathfrak{U}^-$ are the quasilinear decomposition groups. But certainly $\mathfrak{U}^+=T(\mathfrak{U}^+)$ and $\mathfrak{U}^-=T(\mathfrak{U}^-)$ are contained in $T(\mathfrak{G})$. Hence as $C^*(U)=T(C_{\mathfrak{G}}(U))\supseteq T(C_{T(\mathfrak{G})}(U))$, $T(C^*(U))$, $T(C_{T(\mathfrak{G})}(U))=T(C_{\mathfrak{G}}(U)) =\mathfrak{U}^+\times\mathfrak{U}^-$, which is as required.

To verify Condition B, the argument of Proposition 3.1 may be applied to obtain that $D(T(\mathfrak{G}))=D(\mathfrak{G})$. Then the condition follows directly. Conditions C, D, and E follow immediately from the same conditions on the group

$\mathfrak{G}$. Hence $T(\mathfrak{G})$ is a quasilinear group and the proposition is proved. This enables us to conclude by induction that the involutory subgroup of a quasilinear group is quasilinear.

Proof of the Principal Theorem. We have argued that $PTL(n)\subseteq\Phi\mathfrak{G}\subseteq P\Gamma L(n)$. We must first show that $\Phi\mathfrak{G}\subseteq PGL(n)$. Suppose that this is not the case. Then there exists $R\in\mathfrak{G}$ such that $R^*=\Phi R$ is not in $PGL(n)$. Let U be a 1-involution in $\mathfrak{G}$. We claim that we can take $R\in C(U)$. Indeed, there exists $T\in T(\mathfrak{G})$ such that $TVT^{-1}=U$ where $V=RUR^{-1}$ by Proposition 13.3. Thus $TR\in C(U)$; replace TR by R.

Now R induces an inner automorphism I_R of $\mathfrak{G}$ which certainly leaves $C(U)$ and $C^*(U)$ invariant. Because of Corollary 8.4, we have that $I_R\mathfrak{U}^-=\mathfrak{U}^-$ where $\mathfrak{U}^-$ is the negative decomposition group of U.

We wish to study the restriction of I_R to $\mathfrak{U}^-$. To do this, let $C(U)\subseteq\mathfrak{U}_1\times\mathfrak{U}_2$ where $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are quasilinear groups of ranks 1 and $n-1$, respectively, determined in accordance with Condition A. Then $R=R_1\times R_2$ where $R_i\in\mathfrak{U}_i$, $i=1, 2$. Next note that $C^*(U)=T(\mathfrak{U}_1)\times T(\mathfrak{U}_2)$. Because $\mathfrak{U}_1\subseteq Z(\mathfrak{U}_1\times\mathfrak{U}_2)$, $R\in Z(\mathfrak{U}_1\times\mathfrak{U}_2)$. Hence both R and R_2 induce the same automorphism of $C^*(U)$. Now because $P(\mathfrak{U}_2)$ is projective linear, it follows that the automorphism induced on $P(T(\mathfrak{U}_2))$ by the image of R_2 in $P(\mathfrak{U}_2)$ is that induced by an element of $PGL(n-1)$ when $P(U_2)$ is identified with a subgroup of $PGL(n-1)$ containing $PTL(n-1)$. But $P(C^*(U))=P(T(U_2))$. Thus it follows that the automorphism of $P(C^*(U))$ induced by the image of the element R is also that induced by an element of $PGL(n-1)$ when $P(C^*(U))$ is identified with $PTL(n-1)$.

Consider now the normal decomposition

$$C^*(U) = \mathfrak{U}^+ \times \mathfrak{U}^-.$$

Then Φ: $\mathfrak{G}\to P\Gamma L(n)$ determines an isomorphism of both $\mathfrak{U}^+$ and $\mathfrak{U}^-$. However, $\mathfrak{U}^+$ contains only 1 and U. Hence $\Phi C^*(U)=\Phi\mathfrak{U}^-$. Because $\Phi T(\mathfrak{G})=PTL(n)$, we may apply the argument of Proposition 9.1 to obtain a 1-involution $\overline{U}\in GL(n)$ such that $\Phi U=\Lambda\overline{U}$ and $\Phi\mathfrak{U}^-=\Lambda\overline{\mathfrak{U}}^-$ where $\mathfrak{U}^-$ is the negative decomposition group of $\overline{U}$ in $GL(n)$. It follows directly from the above paragraph that the automorphism of $\Phi C^*(U)=\Phi\mathfrak{U}^-$ induced by the element $\Phi R=R^*$ is such that the automorphism of $P(\Phi C^*(U))$ is that which is induced by an element of $PGL(n-1)$ when $P(\Phi C^*(U))$ is identified with $PTL(n-1)$.

According to Dieudonné [8, §4], $C_\Gamma(\overline{U})$ is a subgroup of $\overline{\mathfrak{U}}_\Gamma^+\times\overline{\mathfrak{U}}_\Gamma^-$ where $\overline{\mathfrak{U}}_\Gamma^+$ and $\overline{\mathfrak{U}}_\Gamma^-$ are the subgroups of $\Gamma=\Gamma L(n)$ which consist of all semilinear transformations leaving fixed, respectively, the positive and negative eigenspaces of $\overline{U}$. Then $\overline{\mathfrak{U}}_\Gamma^+$ is isomorphic to $\Gamma L(1)$ and $\overline{\mathfrak{U}}_\Gamma^-$ is isomorphic to $\Gamma L(n-1)$. Every element of $C_\Gamma(\overline{U})$ has components in the groups of semilinear transformations $\overline{\mathfrak{U}}_\Gamma^+$ and $\overline{\mathfrak{U}}_\Gamma^-$ which are relative to the same automorphism of K; this characterizes the group $C_\Gamma(\overline{U})$. Then $C_{\overline{\mathfrak{G}}}(\overline{U})=\overline{\mathfrak{U}}_{\overline{\mathfrak{G}}}^+\times\overline{\mathfrak{U}}_{\overline{\mathfrak{G}}}^-$ where $\overline{\mathfrak{U}}_{\overline{\mathfrak{G}}}^+=\overline{\mathfrak{U}}_\Gamma^+\cap\overline{\mathfrak{G}}$

and $\bar{\mathfrak{U}}_{\bar{\mathfrak{G}}}^{-}=\bar{\mathfrak{U}}_{\Gamma}^{-}\cap\bar{\mathfrak{G}}$. Also $T(C_{\bar{\mathfrak{G}}}(\bar{U}))=\bar{\mathfrak{U}}^{+}\times\bar{\mathfrak{U}}^{-}$ where $\bar{\mathfrak{U}}^{+}=T(\mathfrak{U}_{\bar{\mathfrak{G}}}^{+})$ and $\bar{\mathfrak{U}}^{-}=T(\bar{\mathfrak{U}}_{\bar{\mathfrak{G}}}^{-})$. Furthermore, as $\bar{U}$ is a 1-involution in $GL(n)$ and thus not an $(n/2)$-involution, it follows that $\Lambda C_{\Gamma}(\bar{U})$ is the centralizer of $U^*=\Lambda\bar{U}$ in $P\Gamma L(n)$.

Now R^* is in the centralizer of U^* but not in $PGL(n)$. This means that there is an element $\bar{R}\in C_{\Gamma}(\bar{U})$ which is not in $C_{\bar{\mathfrak{G}}}(\bar{U})$ for which $\Lambda\bar{R}=R^*$. Then $\bar{R}=\bar{R}^*\bar{R}^-$ where $\bar{R}^+$ and $\bar{R}^-$ are elements of $\bar{\mathfrak{U}}_{\Gamma}^{+}$ and $\bar{\mathfrak{U}}_{\Gamma}^{-}$, respectively, that may be identified as semilinear transformations in $C_{\Gamma}(\bar{U})$ which are relative to automorphisms of K which are not the identity. Now the automorphism of $\bar{\mathfrak{U}}^-$ that is induced by $\bar{R}$ is the same as that induced by the component $\bar{R}^-$. As $\bar{R}^-\notin\bar{\mathfrak{U}}_{\bar{\mathfrak{G}}}^{-}$, the automorphism of $P(\bar{\mathfrak{U}}^-)$ induced by $\bar{R}$ is not induced by an element of $PGL(n-1)$ when $P(\bar{\mathfrak{U}}^-)$ is identified with $PTL(n-1)$.

Just as with the homomorphism Φ, we have that $\Lambda T(C_{\bar{\mathfrak{G}}}(\bar{U}))=\Lambda\bar{\mathfrak{U}}^-=\Phi\mathfrak{U}^-$ and Λ is an isomorphism of $\bar{\mathfrak{U}}^-$. Thus the automorphism of $P(\Lambda\bar{\mathfrak{U}}^-)$ induced by $\Lambda\bar{R}=R^*$ is not that induced by an element of $PGL(n-1)$ when $P(\Lambda\bar{\mathfrak{U}}^-)$ is identified with $PTL(n-1)$. This yields a contradiction to our previous conclusion concerning this automorphism. Hence we must have $\Phi\mathfrak{G}\subseteq PGL(n)$. This proves that $P(\mathfrak{G})$ is projective linear.

That $T(\mathfrak{G})$ is an involutory quasilinear group is Proposition 13.3. In the proof of this proposition, we also showed that $P(T(\mathfrak{G}))$ is isomorphic to $PTL(n)$. This proves the Principal Theorem.

Proof of Corollary 1. Let $\mathfrak{G}$ be a finite full quasilinear group of rank $n\geq 4$. Then K^* is a cyclic group of order $q-1$ where q is an odd prime power. If $\mathfrak{M}$ is a maximal set of commuting involutions in $\mathfrak{G}$, it follows from the definition of a full quasilinear group that $C(U)$ is the direct product of n copies of K^*. Let $\bar{\mathfrak{M}}$ be the set of mutually commuting involutions in $\bar{\mathfrak{G}}=GL(n)$ which is determined in accordance with Proposition 9.1 so that $\Phi\mathfrak{M}=\Lambda\bar{\mathfrak{M}}$. It also follows that $C(\bar{\mathfrak{M}})$ is the direct product of n copies of K^*. Now by comparing the orders of $\Phi C(\mathfrak{M})$ and $\Lambda C(\bar{\mathfrak{M}})$, we see that $\Phi C(\mathfrak{M})=\Lambda C(\bar{\mathfrak{M}})$.

Let $\bar{U}$ be a 1-involution in $\bar{\mathfrak{M}}$. Consider $\bar{\mathfrak{G}}$ to act on the vector space $\mathcal{E}$. Then $C(\bar{U})$ contains in its center the linear transformations of $\mathcal{E}$ which leave invariant the 1-dimensional eigenspace of $\bar{U}$ and leave fixed the $(n-1)$-dimensional eigenspace of $\bar{U}$; these elements are also in $C(\bar{\mathfrak{M}})$. They are called dilatations (cf. [8, p. 5]); and together with the elements of $D(\bar{\mathfrak{G}})$, they generate $\bar{\mathfrak{G}}$. Thus the elements of $\Lambda C(\bar{\mathfrak{M}})$ together with the elements of $\Lambda T(\bar{\mathfrak{G}})$ generate $\Lambda\bar{\mathfrak{G}}$, which is isomorphic to $PGL(n)$. As $\Phi C(\mathfrak{M})=\Lambda C(\bar{\mathfrak{M}})$ and $\Phi T(\mathfrak{G})=\Lambda T(\bar{\mathfrak{G}})$, $\Phi\mathfrak{G}=\Lambda\bar{\mathfrak{G}}$. This proves the corollary.

Proof of Corollary 2. This is a direct consequence of Proposition 9.1 now that the Principal Theorem has been established for quasilinear groups of rank $n+1$.

REFERENCES

1. E. Artin, *Geometric algebra*, New York, Interscience, 1957.

2. F. Bachmann, *Eine Kennzeichnung der Gruppe der gebrochen-linearen Transformationen*, Math. Ann. vol. 126 (1953) pp. 79–92.

3. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, 1948.

4. R. Brauer, *On the structure of groups of finite order*, Proceedings International Congress of Mathematicians, 1954, Amsterdam, vol. I, pp. 209–218.

5. R. Brauer and K. Fowler, *On groups of even order*, Ann. of Math. vol. 62 (1955) pp. 565–583.

6. R. Brauer, M. Suzuki, and G. E. Wall, *A characterization of the one-dimensional unimodular projective groups over finite fields*, Illinois J. Math. vol. 2 (1958) pp. 718–746.

7. J. Dieudonné, *On the automorphisms of the classical groups*, Memoirs Amer. Math. Soc. no. 2, 1951, pp. 1–95.

8. ———, *La géométrie des groupes classiques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.) Heft 5, Berlin, Springer-Verlag, 1955.

9. C. E. Rickart, *Isomorphic groups of linear transformations*, Amer. J. Math. vol. 72 (1950) pp. 451–464.

10. ———, *Isomorphisms of infinite-dimensional analogues of the classical groups*, Bull. Amer. Math. Soc. vol. 57 (1951) pp. 435–448.

11. O. Schreier and B. L. van der Waerden, *Die Automorphismen der Projectiven Gruppen*, Abh. Math. Sem. Univ. Hamburg vol. 6 (1928) pp. 303–323.

12. M. Suzuki, *On characterizations of linear groups*. II, Trans. Amer. Math. Soc. vol. 92 (1959) pp. 205–220.

13. H. Zassenhaus, *The theory of groups*, New York, Chelsea, 1949.

University of Washington,
Seattle, Washington[11]

[11] Now at the University of Illinois, Urbana, Illinois.