# A SURFACE IS TAME IF ITS COMPLEMENT IS 1-ULC[1]

BY

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1. **Introduction.** In this paper we show that a 2-manifold $M$ in Euclidean 3-space $E^3$ is tame if $E^3-M$ is uniformly locally simply connected.

A closed subset $X$ of a triangulated manifold $Y$ is *tame* if there is a homeomorphism of $Y$ onto itself taking $X$ onto a polyhedron (geometric complex) in $Y$. If there is no such homeomorphism, $X$ is called ***wild***. Examples of 2-manifolds wildly embedded in $E^3$ are found in [1; 8; 6].

An *n-manifold* is a separable metric space each of whose points lies in a neighborhood homeomorphic to Euclidean $n$-space. An ***n-manifold-with-boundary*** is a separable metric space each of whose points lies in a neighborhood whose closure is a topological $n$-cell. If $M$ is an $n$-manifold-with-boundary, we use Int $M$ to denote the set of points of $M$ with neighborhoods homeomorphic to Euclidean $n$-space and Bd $D$ to denote $M-\text{Int } M$. For example, if $D$ is a disk, Bd $D$ is a simple closed curve which is the rim of the disk. If we have a manifold embedded in a larger space and treat the manifold as a subset rather than a space, we insist that it be closed. If $S$ is a 2-sphere embedded in $E^3$, we use Int $S$ and Ext $S$ to denote the bounded and unbounded components of $E^3-S$. The double meaning of the symbol Int should not lead to confusion.

A subset $X$ of a manifold-with-boundary is *locally tame* at a point $p$ of $X$ if there is a neighborhood $N$ of $p$ and a homeomorphism of $\overline{N}$ (the closure of $N$) onto a cell that takes $X\cdot\overline{N}$ onto a polyhedron. If the manifold-with-boundary is triangulated, we say that $X$ is locally polyhedral at $p$ if there is a neighborhood $N$ of $p$ such that $X\cdot\overline{N}$ is a polyhedron.

Suppose $D$ is a disk. We say that *a map of* Bd $D$ *into a set* $Y$ *can be shrunk to a constant in* $Y$ if the map can be extended to take $D$ into $Y$. If each map of Bd $D$ into $Y$ can be shrunk to a constant in $Y$, we say that $Y$ is *simply connected*. Also, $Y$ is *locally simply connected* at a point $p$ of $\overline{Y}$ if for each neighborhood $U$ of $p$ there is a neighborhood $V$ of $p$ such that each map of Bd $D$ into $V\cdot Y$ can be shrunk to a point in $U\cdot Y$. A metric space $Y$ is *uniformly locally simply connected* (or 1-ULC) if for each $\epsilon>0$ there is a $\delta>0$ such that each map of Bd $D$ into a $\delta$ subset of $Y$ can be shrunk to a point on an $\epsilon$ subset of $Y$. If $\overline{Y}$ is a compact subset of a metric space, it can be shown that $Y$ is 1-ULC if it is locally simply connected at each point of $\overline{Y}$.

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The surfaces studied in solid geometry are tame. It is useful to have criteria for determining which surfaces are tame. We mention three such criteria.

1. A closed subset of a triangulated 3-manifold-with-boundary is tame if and only if it is locally tame at each of its points.

2. A 2-sphere in $E^3$ is tame if and only if it can be homeomorphically approximated from both sides.

3. A 2-manifold $M$ in a triangulated 3-manifold $M_3$ is tame if and only if $M_3 - M$ is locally simply connected at each point of $M$.

Criterion 1 is proved in [2] and [10]. Criterion 2 is proved in [5] and restated below as Theorem 0. Criterion 3 is Theorem 7 of the present paper.

The distance function is denoted by $\rho$. We shall make use of the following approximation theorem proved in [3].

Approximation Theorem. *For each 2-manifold $M$ in a triangulated 3-manifold-with-boundary and each non-negative continuous function $f$ defined on $M$, there is a 2-manifold $M'$ and a homeomorphism $h$ of $M$ onto $M'$ such that*

$$\rho(x, h(x)) \leqq f(x) \qquad (x \in M)$$

*and $M'$ is locally polyhedral at $h(x)$ if $f(x) > 0$.*

Another theorem that we shall use is the Side Approximation Theorem for 2-Spheres. It is proved by the same methods as the Approximation Theorem and its proof will be given in another paper [7].

Side Approximation Theorem for 2-Spheres. *Each 2-sphere $S$ in $E^3$ can be polyhedrally approximated almost from either side—that is for each $\epsilon > 0$ and each component $U$ of $E^3 - S$ there is a homeomorphism $h$ of $S$ onto a polyhedral 2-sphere such that*

*$h$ moves no point more than $\epsilon$ and*

*$h(S)$ contains a finite collection of mutually exclusive disks each of diameter less than $\epsilon$ such that $h(S)$ minus the sum of the disks lies in $U$.*

If $A$ and $B$ are two sets and there is a homeomorphism of $A$ onto $B$ that moves no point by more than $\epsilon$, we write

$$H(A, B) \leqq \epsilon.$$

The following theorem is proved in [5].

Theorem 0. *A 2-sphere $S$ in $E^3$ is tame if it can be homeomorphically approximated from both sides—that is, for each $\epsilon > 0$ and each component $U$ of $E^3 - S$, there is a 2-sphere $S'$ in $U$ such that*

$$H(S, S') < \epsilon.$$

To show that a 2-sphere is tame, a first step might be to show that the hypothesis of Theorem 0 is met. As a step toward proving that a 2-sphere in $E^3$ is tame if its complement is 1-ULC, we prove the following.

THEOREM 1. *If $S$ is a 2-sphere in $E^3$ such that* Int $S$ *is* 1-ULC, *then for each* $\epsilon > 0$, *$S$ can be homeomorphically approximated from* Int $S$—*that is, for each* $\epsilon > 0$ *there is a 2-sphere $S'$ in* Int $S$ *such that*

$$H(S, S') < \epsilon.$$

Our plan for proving Theorem 1 is to get a special cellular decomposition $T$ of $S$, get a homeomorphism of $S$ onto itself that pulls the boundaries of the cells of the decomposition $T$ into Int $S$, and finally pull the cells themselves into Int $S$.

It is hoped that the serious reader will understand the why of the attack as well as the details. Hence, we give our over-all plan of attack first and reserve epsilontics to last so that the reader can see why these particular $\epsilon$'s are used. When we need a close approximation, we let it be an $\epsilon_i$ approximation and decide later how small the $\epsilon_i$ would need to be to make the details work.

2. **Proof of Theorem** 1. a. **Special cellular decomposition** $T$ **of** $S$. We need a cellular decomposition $T$ of $S$ with the following properties.

The mesh of $T$ is less than $\epsilon_1$ ($\epsilon_1$ is a small number whose size is to be described later).

The collection of 2-cells of $T$ is the sum of three subcollections $A_1$, $A_2$, $A_3$ such that no two elements of $A_i$ $(i=1, 2, 3)$ intersect each other.

That for each $\epsilon_1 > 0$ there is such a cellular decomposition $T$ of $S$ follows from a consideration of a triangulation $T'$ of $S$ of mesh less than $\epsilon_1/2$. The vertices of $T'$ are swelled into 2-cells and become the elements of $A_1$. The parts of the 1-simplexes of $T'$ not in elements of $A_1$ are expanded into the elements of $A_2$. The closures of the parts of the 2-simplexes of $T'$ not in elements of $A_1$ or $A_2$ are the elements of $A_3$.

b. **Pulling elements of** $T$ **partially into** $S$. Suppose $T$ is a fixed special cellular decomposition of $S$ such as mentioned in the preceding section. The 1-skeleton of $T$ is the sum of the boundaries of the 2-cells in $T$ and is denoted by $K_1$. We select a small number $\epsilon_2$ whose size is described later. Then there is a polyhedral 2-sphere $S_1$ and a homeomorphism $h_1$ of $S$ onto $S_1$ such that

$h_1$ moves no point more than $\epsilon_2$,
$h_1(K_1) \subset \text{Int } S$, and
$S_1$ contains a finite collection of mutually exclusive $\epsilon_2$ disks such that $S_1$ minus the sum of the interiors of these disks lies in Int $S$.

That for each $\epsilon_2$ there are such an $S_1$ and an $h_1$ follows from the Side Approximation Theorem of 2-Spheres. We let $\epsilon_2/2$ be the $\epsilon$ in the statement of that theorem and $S_1$ be the $S'$ guaranteed by the conclusion of that theorem. The homeomorphism $h_1$ is the homeomorphism $h$ guaranteed by that theorem followed by a homeomorphism of $S_1$ onto itself that moves no point by more than $\epsilon_2/2$ but pulls the image of $K_1$ off of the disks.

c. **The next approximations to elements of** $T$. For each 2-cell $D$ of $T$, $h_1(D)$ is a first approximation to $D$. We note that $h_1(D)$ is homeomorphically close to $D$ and $h_1(\text{Int } D)$ contains a finite collection $E_1, E_2, \cdots, E_n$ of mutually exclusive $\epsilon_2$ disks such that $h_1(D) - \sum \text{Int } E_i \subset \text{Int } S$. We suppose that $h_1(\text{Bd } D)$ is a polygon.

The second approximation $h_2(D)$ to $D$ may not be quite as close homeomorphically to $D$ as is $h_1(D)$ but it will still be close. However, it will have the advantage that the components of $S \cdot h_2(D)$ will have diameters much smaller than $\epsilon_2$ (which is an upper bound on the diameters of $E_1, E_2, \cdots, E_n$) and simple closed curves in Int $S$ near the components of $S \cdot h_2(D)$ can be shrunk to points in Int $S$ without hitting $h_1(K_1)$.

Let $\epsilon_3$ be a very small positive number selected in a fashion to be described later and $S'$ be a polyhedral 2-sphere which is homeomorphically within $\epsilon_3$ of $S$ and which contains a finite collection of mutually exclusive $\epsilon_3$ disks such that $S'$ minus the sum of the interiors of these disks is contained in Int $S$. We select $\epsilon_3$ so that $S' \cdot h_1(D) \subset \sum \text{Int } E_i$ and suppose that $S' \cdot h_1(D)$ is the sum of a finite collection of mutually exclusive simple closed curves $J_1, J_2, \cdots, J_r$.

The $\epsilon_2$ and $\epsilon_3$ were selected so that each $J_i$ bounds a disk $F_i$ on $S'$ of small diameter. We suppose that these disks $F_i$ are ordered by size with the small ones first so that no $F_i$ contains an $F_{i+j}$.

The disk in $h_1(D)$ bounded by $J_1$ is first replaced by $F_1$ and then pushed slightly to one side of $S'$ so as to reduce the number of components with the intersection with $S'$. The process is continued by replacing disks in the adjusted $h_1(D)$ by $F_i$'s and then pushing slightly so as to get a polyhedral disk $h_2(D)$ which is close to $D$ homeomorphically and which lies on Int $S'$. We select $h_2$ so that it agrees with $h_1$ in a neighborhood of Bd $D$ and such that the components of $S \cdot h_2(D)$ are not much bigger than those of $S' \cdot h_1(D)$. By selecting $\epsilon_3$ very small, we can insure that the components of $h_2(D) \cdot S$ are very small—in fact of diameter less than some preselected positive number $\epsilon_4$.

Although we could have chosen the sum of the $h_2(D)$'s to be a 2-sphere, we did not insist on this since at the third approximation of $D$, there seems to be no easy way to prevent the approximating disks from intersecting at interior points of each.

d. **Third approximation to** $D$. Since each component of $S \cdot h_2(D)$ is of diameter less than $\epsilon_4$, $h_2(\text{Int } D)$ contains a collection of mutually exclusive disks $E_1', E_2', \cdots, E_m'$ which cover $S \cdot h_2(D)$ such that each Bd $E_i'$ lies close to $S$ and is of diameter less than $\epsilon_4$. Each Bd $E_i'$ lies in Int $S$ since Bd $D$ does.

Each Bd $E_i'$ is of such small diameter that it can be shrunk to a point on a small subset of Int $S$ where $\epsilon_4$ and $\epsilon_3$ have been selected so that this subset will not intersect $h_1(K_1) = h_2(K_1)$. Hence there is a map $g$ of $h_2(D)$ into Int $S$

such that $gh_2(D)$ lies close to $h_2(D)$, and $g$ is the identity in a neighborhood of $h_2(\text{Bd } D)$, $gh_2(D)$ intersects $h_2(K_1)$ only in $h_2(\text{Bd } D)$, and $gh_2(D)$ has no singular points near $gh_2(\text{Bd } D)$. It follows from Dehn's lemma as proved by Papakyriokopoulos [11] that for each neighborhood $U$ of the set of singular points of $gh_2(D)$, there is a homeomorphism $h_3$ of $D$ onto a polyhedral disk $h_3(D)$ in $gh_2(D)+U$ that agrees with $h_2$ in a neighborhood of Bd $D$. The third approximation to $D$ is $h_3(D)$. The advantage it has over $h_2(D)$ is that it lies in Int $S$. The only thing that makes $h_3(D)$ and $D$ homeomorphically close is that each is of small diameter and their boundaries are homeomorphically close.

e. **The fourth approximation to** $D$. Our task is now to untangle the $h_3(D)$'s so that their sum forms a 2-sphere in Int $S$. We recall that the collection of 2-cells of $T$ is the sum of three subcollections $A_1$, $A_2$, $A_3$ so that no two elements of any $A_i$ have a point in common. We will have enlarged the $h_1(D)$'s so little as we changed them to $h_2(D)$'s and then to $h_3(D)$'s that two $h_3(D)$'s will not intersect if the corresponding $D$'s do not intersect.

For each element $D$ of $A_1$, the fourth approximation $h(D)$ to $D$ is $h_3(D)$.

We may suppose that for each element $D$ of $A_1$, the intersection of $h_3(\text{Int } D)=h(\text{Int } D)$ and the sum of the images under $h_3$ of the elements of $A_2$ is the sum of a finite collection of mutually exclusive simple closed curves $J_1, J_2, \cdots, J_s$ such that $J_i$ bounds a disk $G_i$ in Int $h(D)$ and the $J_i$'s are ordered according to the size of the disks $G_i$ they bound in $h(D)$, with the small ones coming first.

If $J_1$ lies in an $h_3(D')$ for an element $D'$ of $A_2$, we replace the disk in $h_3(D')$ by $G_1$ and shove this replaced disk slightly to one side of $h_3(D)$. This process is continued until for each element $D'$ of $A_2$, there is an $h$ defined on $D'$ so that $h(D)$ and the $h(D')$'s fit together only along their boundaries as they should.

Finally we turn to the elements of $A_3$. We suppose that for each element $D$ of $A_1+A_2$, $h(\text{Int } D)$ intersects the sum of the images under $h_3$ of the elements of $A_3$ in the sum of a collection of mutually exclusive simple closed curves. These simple closed curves are eliminated one by one, starting at the inside in a manner already described. For each element $D'$ of $A_3$, the resulting adjustment of $h_3(D')$ is called $h(D')$.

The sum of the $h(D)$'s is a 2-sphere in Int $S$. We note that $h=h_1=h_2=h_3$ on $K_1$. We will cause $h$ to be near the identity by picking the $\epsilon_i$'s so that the $D$'s and the $h(D)$'s are small and $h$ is near the identity on $K_1$.

f. **Epsilontics.** In this section we explain the sizes for $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, $\epsilon_4$ and the reasons for these selections. We recall that

$\epsilon_1$ limits the mesh of $T$,

$\epsilon_2$ limits $h_1$ to be near the identity and limits the sizes of the $E_i$'s,

$\epsilon_3$ limits $S'$ to be near $S$ and limits the sizes of the disks in $S'$ such that $S'$ minus these disks lies in Int $S$, and

$\epsilon_4$ limits the sizes of the components of $h_2(D)\cdot S$.

We choose

$$\epsilon_1 < \epsilon/8,$$

where $\epsilon$ is the number mentioned in the statement of Theorem 1. This is the only restriction we place on $\epsilon_1$.

Let $\delta_1$ be a positive number so small that the distance between two elements of $T$ without a common point is more than $\delta_1$.

Our goal is to choose $\epsilon_2$, $\epsilon_3$, $\epsilon_4$ so that for each 2-cell $D$ of $T$, the following conditions are satisfied.

$$h_1(D) \text{ lies in a } \delta_1/6 \text{ neighborhood of } D.$$

$$h_2(D) \text{ lies in a } \delta_1/6 \text{ neighborhood of } h_1(D).$$

$$h_3(D) \text{ lies in a } \delta_1/6 \text{ neighborhood of } h_2(D).$$

This will insure that if $D_1$, $D_2$ are two 2-cells of $T$ without a common point, then $h_3(D_1)\cdot h_3(D_2)=0$.

Since each point of $h_3(D)$ lies within $3\delta_1/6$ of a point of $D$ and $\delta_1<\epsilon_1$,

$$\text{diameter } h_3(D) < \epsilon_1 + \delta_1 < 2\epsilon_1.$$

Since $h=h_3$ on elements of $A_1$,

$$\text{diameter } h(D) < 2\epsilon_1 \text{ if } D \text{ is an element of } A_1.$$

In changing from $h_3$ to $h$ on elements $D$ of $A_2$, the adjustment was made so that $h(D)$ lies within $2\epsilon_1$ of $h_3(D)$. Hence,

$$\text{diameter } h(D) < 6\epsilon_1 \text{ if } D \text{ is an element of } A_2.$$

For each element $D$ of $A_3$,

$$h(D) \text{ lies within } 6\epsilon_1 \text{ of } h_3(D).$$

Since each point of $h(D)$ lies within $6\epsilon_1$ of a point of $h_3(D)$, this point in turn lies within $3\delta_1/6$ of a point of $D$, and diameter $D<\epsilon_1$, we find that $h$ moves no point as much as $8\epsilon_1$. This is the reason we selected $\epsilon_1<\epsilon/8$.

Let $\epsilon_2$ be a positive number so small that each subset of $S$ of diameter $3\epsilon_2$ lies in a disk on $S$ of diameter less than $\delta_1/10$. We note that

$$\epsilon_2 < \delta_1/30 < \delta_1/6.$$

Since $\epsilon_2<\delta_1/6$, for each 2-cell $D$ of $T$, $h_1(D)$ lies in a $\delta_1/6$ neighborhood of $D$. The more stringent condition that $\epsilon_2<\delta_1/30$ is used later to help insure that $h_2(D)$ lies in a $\delta_1/6$ neighborhood of $h_1(D)$.

Let $\delta_2$ be a positive number so small that for each 2-cell $D$ of $T$, $\delta_2$ is less than the distance between $S$ and $h_1(D)-\sum E_i$. We note that

$$\delta_2 < \epsilon_2.$$

Although we shall place more stringent conditions on $\epsilon_3$, we first consider the sizes of the components of $S' \cdot h_1(D)$ if we merely suppose

$$\epsilon_3 < \delta_2.$$

Let $X$ be a component of $S' \cdot h_1(D)$ and $h'$ be a homeomorphism of $S$ onto $S'$ that moves no point more than $\epsilon_3$. Then $X$ lies in an $E_i$ (which is of diameter less than $\epsilon_2$), $h'^{-1}(X)$ is of diameter less than $\epsilon_2+2\epsilon_3<3\epsilon_2$, $h'^{-1}(X)$ lies in a disk of diameter less than $\delta_1/10$, and the image of this disk under $h'$ is a disk in $S'$ of diameter less than $\delta_1/10+2\epsilon_3<\delta_1/10+2\delta_1/30=\delta_1/6$. Hence, the restrictions we have placed on $\epsilon_2$, $\epsilon_3$ are enough to insure that we can select $h_2$ so that $h_2(D)$ lies in a $\delta_1/6$ neighborhood of $h_1(D)$.

Let $\epsilon_4$ be a positive number so small that each simple closed curve in $E^3-S$ of diameter less than $\epsilon_4$ can be shrunk to a point on a subset of $E^3-S$ of diameter less than the minimum of $\delta_1/6$ and $\delta_2/2$. Note that $\epsilon_4$ does not depend on $\epsilon_3$. The Bd $E_i'$'s are selected to lie within $\epsilon_4<\delta_2/2$ of $S$. Also, the Bd $E_i'$'s have diameters less than $\epsilon_4$ so that $gh_2(D)$ intersects $h_2(K_1)$ only in $h_2(\text{Bd } D)$. Furthermore, $h_3(D)$ lies in a $\delta_1/6$ neighborhood of $h_2(D)$.

The final restriction we place on $\epsilon_3$ is to insure that each component of $S \cdot h_2(D)$ is of diameter less than $\epsilon_4$. Let $\delta_3$ be a number so small that each $\delta_3$ subset of $S$ lies in a disk in $S$ of diameter less than $\epsilon_4$. We suppose

$$\epsilon_3 < \delta_3.$$

Each component $X$ of $S \cdot h_2(D)$ lies on Int $S'$ and is separated from the big component of $S-S'$ by a disk in $S'$ of diameter less than $\epsilon_3<\delta_3$. Hence, diameter $X<\epsilon_4$.

3. **Conditions under which a 2-sphere is tame.** Tame 2-spheres in $E^3$ have uniformly locally simply connected complements. A converse is as follows.

THEOREM 2. *A 2-sphere $S$ in $E^3$ is tame if each component of $E^3-S$ is* 1-ULC.

**Proof.** It follows from Theorem 1 that for each positive number $\epsilon$ there is a 2-sphere $S'$ in Int $S$ such that

$$H(S, S') < \epsilon.$$

Similarly, it follows that there is a 2-sphere $S''$ in Ext $S$ such that

$$H(S, S'') < \epsilon.$$

That $S$ is tame then follows from Theorem 0.

COROLLARY. *A 2-sphere $S$ in $E^3$ is tame if $E^3-S$ is locally simply connected at each point of $S$.*

4. **Retaining simple connectivity.** We shall be applying the Approximation Theorem to a 2-manifold to make part of the 2-manifold locally poly-

hedral. We are interested in seeing what this does to the local simple connectivity of the complement. First we examine the effect of throwing away part of the 2-manifold.

THEOREM 3. *Suppose $M$ is a 2-manifold in $E^3$, $D$ is a disk in $M$, and $p$ is a point of $D$ at which $E^3-M$ is locally simply connected. Then $E^3-D$ is locally simply connected at $p$.*

**Proof.** We only consider the case where $p \in \text{Bd } D$. Suppose $U$ is a given neighborhood of $p$. Let $V'$ be a neighborhood of $p$ such that each map of a circle into $V' \cdot (E^3-M)$ can be shrunk to a point in $U \cdot (E^3-M)$ and $V$ be a neighborhood of $p$ such that each pair of points of $V \cdot (M-D)$ lie in an arc in $V' \cdot (M-D)$. We show that if $E$ is a plane circular disk and $f$ is a map of Bd $E$ into $V \cdot (E^3-D)$, then there is a map of $E$ into $U \cdot (E^3-D)$ that agrees with $f$ on Bd $E$.

We want to simplify $f$ so that $M \cdot f(\text{Bd } E)$ does not have infinitely many components. Suppose $aa'$ is an arc on Bd $E$ and $bb'$ is an arc on $V \cdot (M-D)$ such that $f(a)=b$, $f(a')=b'$, and $bb'+f(aa')$ lies in a convex subset of $V-D$. Then there is a homotopy $F_t$ on $aa'$ such that $F_0=f$, $F_t$ is constant on $a$ and $a'$, $F_1(aa')=bb'$, and each $F_t(aa')$ lies in $V-D$. Hence, we suppose with no loss of generality that $f^{-1}(M \cdot f(\text{Bd } E))$ is the sum of a finite number of arcs $a_1a_2, a_3a_4, \cdots, a_{2n-1}a_{2n}$ on Bd $E$ ordered so that there is no $a_j$ between any $a_i$ and $a_{i+1}$.

Let $q$ be the center of $E$ and $f(q)$ be any point of $V \cdot (M-D)$. Extend $f$ to map the radius $qa_i$ of $E$ onto an arc in $V' \cdot (M-D)$ from $f(q)$ to $f(a_i)$. Since each component of $M$ divides $E^3$ into two pieces and any arc in $M$ can be approximated by arcs on either side of $M$, the map $f$ on the boundary of the sector $a_iqa_{i+1}$ of $E$ can be extended to map an annulus ring in the sector one of whose boundary components is the boundary of the sector into $V' \cdot (E^3-D)$ so that the image of the other boundary component of the annulus misses $M$. The map $f$ can be further extended to take the rest of the sector into $U \cdot (E^3-M)$.

THEOREM 4. *Suppose $M$ is a 2-manifold in $E^3$, $D$ is a disk in $M$, $p$ is a point of $M$ at which $E^3-M$ is locally simply connected, and $h$ is a homeomorphism of $M$ into $E^3$ such that $h$ is the identity on $D$ and $h(M)$ is locally polyhedral at each point of $h(M)-D$. Then $E^3-h(M)$ is locally simply connected at $p$.*

**Proof.** The only case we consider is the one in which $p$ is a point of Bd $D$. Suppose $U$ is a given neighborhood of $p$. Let $V'$ be a neighborhood of $p$ such that each closed set in $V' \cdot (h(M)-D)$ lies in a disk in $U \cdot (h(M)-D)$. It follows from Theorem 3 that there is a neighborhood $V$ of $p$ such that each map of a circle into $V \cdot (E^3-D)$ can be shrunk to a point in $V' \cdot (E^3-D)$. We show that if $E$ is a disk and $f$ is a map of Bd $E$ into $V\cdot(E^3-h(M))$, then $f$ can be extended to map $E$ into $U\cdot(E^3-h(M))$.

Let $f'$ be a map of $E$ into $V'\cdot(E^3-D)$ that is an extension of $f$ on Bd $E$. Let $E'$ be the component of $E-f'^{-1}(h(M)\cdot f(E))$ containing Bd $E$. With no loss of generality we suppose that $E'$ is $E$ minus a finite collection of mutually exclusive subdisks $E_1, E_2, \cdots, E_n$ of $E$.

Since $f'(\text{Bd } E_i)$ lies on the interior of a polyhedral disk in $U\cdot(h(M)-D)$, it is possible to adjust $f'$ on disks in $E$ slightly larger than the $E_i$'s so as to take the larger disks slightly to one side of $h(M)$. The adjusted $f'$ is $f$ and takes $E$ into $U\cdot(E^3-h(M))$.

QUESTION. Would Theorem 4 be true if we supposed that $D$ were merely a closed subset of $M$ with only nondegenerate components rather than actually a disk in $M$?

5. **Enlarging a disk to a 2-sphere.** Not each disk in $E^3$ lies on a 2-sphere. An example of such a disk is obtained by taking a horizontal disk $D$ in $E^3$; removing two circular holes from Int $D$; adding tubes from the holes, one tube going up and the other down and around; and finally adding hooked disks as shown in [1]. The disk does not lie on a 2-sphere since its boundary cannot be shrunk to a point in the complement of the disk. Although it does not lie on a 2-sphere, it does lie on a torus as was pointed out to me by David Gillman. If instead of removing a pair of holes from the horizontal disk and replacing the holes with hooked wild disks, one had removed an infinite collection of pairs of holes converging to a boundary point of $D$ and replaced each pair of holes with wild disks hooked over the boundary of $D$, there would have resulted a wild disk in $E^3$ that does not lie on any 2-manifold in $E^3$.

The following result shows that each disk contains many small disks each of which lies on a 2-sphere.

THEOREM 5. *Suppose $M$ is a 2-manifold in $E^3$, $p$ is a point of $M$, and $U$ is a neighborhood of $p$. Then there is a disk $D$ in $M\cdot U$ and a 2-sphere $S$ in $U$ such that $p\in \text{Int } D\subset S$ and $S$ is locally polyhedral at each point of $S-D$.*

**Proof.** Let $E$ be a disk in $M$ such that $p\in\text{Int } E$ and $C$ be a cube in $U$ whose interior contains $p$ and whose exterior contains Bd $E$. Let $D$ be a disk in $M\cdot\text{Int } C$ such that $p\in\text{Int } D$. It follows from the Approximation Theorem that there is a homeomorphism $h$ of $E$ into $E^3$ such that $h$ is the identity on $D$, $h$ takes Bd $E$ into Ext $C$, and $h(E)$ is locally polyhedral at each point of $h(E)-D$. We suppose with no loss of generality that Bd $C\cdot h(E)$ is the sum of a finite collection of mutually exclusive polygons.

Let $E'$ be the component of $h(E)-\text{Bd } C$ containing $D$. It is topologically equivalent to a 2-sphere minus the sum of a finite collection of mutually exclusive disks. By adding polygonal disks in $U\cdot(C+\text{Ext } C)$ to $E'$, one obtains the required 2-sphere $S$.

6. **Locally tame subsets of 2-manifolds.** A 2-sphere $S$ in $E^3$ may not be

locally tame at a point $p$ even if $E^3-S$ is locally simply connected at $p$ as shown by the following example.

EXAMPLE. Consider a spherical 2-sphere $S'$ and a sequence of mutually exclusive spherical disks $E_1, E_2, \cdots$ in $S'$ converging to a point $p$ of $S'$. Fox and Artin have described [8] a wild arc which is locally polyhedral except at one end point. For each $E_i$ let $A_i$ be such an arc reaching out from the center of $E_i$ such that the arc is of diameter less than the radius of $E_i$ and such that $A_i$ intersects $S$ only at the polyhedral end of $A_i$. By replacing each $E_i$ in $S$ by a disk obtained by swelling up $A_i$ as done in [7] one can obtain a 2-sphere $S$ such that $E^3-S$ is locally simply connected at $p$ even though $S$ is not locally tame at $p$.

THEOREM 6. *Suppose $M_2$ is a 2-manifold embedded in a 3-manifold $M_3$ and $U$ is an open subset of $M_2$ such that $M_3-M_2$ is locally simply connected at each point of $U$. Then $M_2$ is locally tame at each point of $U$.*

**Proof.** Since local tameness is only a local property, we suppose that $M_3$ is $E^3$ and $U$ is all of $M_2$. If this were not the case already we would take a homeomorphism $h$ of a neighborhood of $p$ in $M_3$ into $E^3$ such that this neighborhood did not intersect $M_2-U$.

It follows from Theorem 5 that there is a disk $D$ in $M_2$ and a 2-sphere $S$ such that $p\in \operatorname{Int} D\subset S$ and $S$ is locally polyhedral at each point of $S-D$. It follows from Theorem 4 that $E^3-S$ is locally simply connected at each point of $S$. Since $S$ is compact, $E^3-S$ is 1-ULC. Theorem 2 implies that $S$ is tame. Since $S$ is tame, $M_2$ is locally tame at $p$.

Since a closed set in a 3-manifold is tame if it is locally tame [2; 10] we have the following result.

THEOREM 7. *A 2-manifold $M_2$ in a triangulated 3-manifold $M_3$ is tame if and only if $M_3-M_2$ is locally simply connected at each point of $M_2$.*

COROLLARY. *A 2-manifold $M_2$ in a 3-manifold $M_3$ is tame if $M_3-M_2$ is* 1-ULC.

## 7. Tame 2-manifolds-with-boundaries.

In this section we extend our results about 2-manifolds to 2-manifolds-with-boundaries.

THEOREM 8. *Suppose $M_2$ is a 2-manifold-with-boundary embedded in a 3-manifold $M_3$ and $U$ is an open subset of $M_2$ such that $M_3-M_2$ is locally simply connected at each point of $U$. Then $M_2$ is locally tame at each point of $U$.*

**Proof.** Since we only look at $M_3$ locally, we suppose that $M_3$ is $E^3$ and $U$ is $M_2$. Since Theorem 6 takes care of points of Int $M_2$, we only show that $M_2$ is locally tame at a point $p$ of Bd $M_2$.

Let $D$ be a disk in $M_2$ such that Bd $D\cdot$Bd $M_2$ is an arc containing $p$ as a non end point. An argument like that used in the proof of Theorem 3 shows that $E^3 - D$ is locally simply connected at each point of $D$. We finish the proof of Theorem 8 by showing that $D$ is tame.

Since Theorem 6 shows that $D$ is locally tame at each point of Int $D$, it follows from [4; 9] that there is a homeomorphism $h$ of $E^3$ onto itself such that $h(D)$ is locally polyhedral at each point of $h(\text{Int } D)$. Hence, we suppose with no loss of generality that $D$ is locally polyhedral at each point of Int $D$.

By pushing $D$ to one side at points of Int $D$, one obtains a disk $D'$ such that Bd $D$ = Bd $D'$, $D'$ is locally polyhedral at points of Int $D'$, and $D+D'$ bounds a topological cube $C$. Since $E^3 -$ Bd $C$ is locally simply connected at each point of Bd $C$, it follows from either Theorem 2 or 6 that Bd $C$ is tame. Hence $D$ is tame and $M_2$ is locally tame at $p$.

8. **Extensions of Theorem** 0. As pointed out in [5], we can use Theorem 6 to extend Theorem 0 as follows.

THEOREM 9. *A 2-manifold $M_2$ in a 3-manifold $M_3$ is locally tame at a point $p$ of $M_2$ if there is a disk $D$ with $p \in \text{Int } D \subset M_2$ such that for each positive number $\epsilon$, there are disks $D'$, $D''$ on opposite sides of $M_2$ such that*

$$H(D, D') < \epsilon, \qquad H(D, D'') < \epsilon.$$

**Proof.** Theorem 9 follows from Theorem 6 when we show that $M_3 - M_2$ is locally simply connected at each point of Int $D$.

Suppose $E$ is a disk, $q$ is a point of Int $D$, and $f$ is a map of Bd $E$ into a small subset of $M_3 - M_2$ near $q$. Suppose $f$ is extended to map $E$ into a small subset of $M_3$. Let $D'$, $D''$ be disks on opposite sides of $M_2$ which are homeomorphically close to $D$ and whose sum separates $f(\text{Bd } E)$ from $M_2 \cdot f(E)$ in $f(E)$. If $E'$ is the component of $E - f^{-1}(D' + D'')$ containing Bd $E$, $f$ on $E'$ can be extended to map $E$ into a small subset of $f(E') + D' + D''$. Hence, $M_3 - M_2$ is locally simply connected at $q$.

Since locally tame closed subsets are tame in triangulated 3-manifolds [2; 10], we have the following as a corollary of Theorem 9.

COROLLARY. *A 2-manifold $M_2$ in a triangulated 3-manifold $M_3$ is tame if and only if for each positive number $\epsilon$ the appropriate one of the following conditions is satisfied:*

CASE 1. *If $M_2$ is two sided in some neighborhood $N$ of $M_2$ there are 2-manifolds $M'$, $M''$ on opposite sides of $M_2$ in $N$ such that*

$$H(M_2, M') < \epsilon, \qquad H(M_2, M'') < \epsilon.$$

CASE 2. *If $M_2$ is one sided in each neighborhood of $M_2$ there is a connected double covering $M'$ of $M_2$ with projection map $\pi$ and a homeomorphism $h'$ of $M'$ into $M_3 - M_2$ such that*

$$p(h(x), \pi(x)) < \epsilon, \qquad x \in M'.$$

Neither Theorem 9 nor its corollary can be extended to 2-manifolds-with-boundaries. Besides having to speak with care about the two sides of a 2-manifold-with-boundary in a 3-manifold, one would have to contend with the example of Stallings [12] in which he describes an uncountable family of mutually exclusive wild disks in $E^3$. It would follow from an application of Theorem 9 that most of these disks are locally tame except on their boundaries.

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