FAMILIES OF MEASURES AND REPRESENTATIONS OF ALGEBRAS OF OPERATORS

BY

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Introduction. The purpose of the present paper is most easily explained by examining one part of it in detail. Let $\{B_\omega : \omega \in \Omega\}$ be a set of commuting bounded hermitian operators on a Hilbert space H, and let T be the cartesian product $\times_{\omega \in \Omega} [-\|B_\omega\|, \|B_\omega\|]$. Two of the most useful tools in the study of the operators are the simultaneous spectral resolution of the B_ω by means of a projector-valued measure π on a σ-field $\mathfrak{A}$ of subsets of T, and the multiplicative representation. For this last we choose a set $\{x_j : j \in J\}$ of vectors in H such that if $H(x_j)$ is the closed linear extension of the set $\{\pi(A)x_j : A \in \mathfrak{A}\}$, the $H(x_j)$ are mutually orthogonal and their closed linear extension is H. Then H is the direct sum of the $H(x_j)$, and for each j in J there is a non-negative measure m_j on $\mathfrak{A}$ such that $L_2[T, m_j]$ is isomorphic with $H(x_j)$, and for every (complex) bounded Baire function f on T the operator $f(B_w) = \int f(t) d\pi$ corresponds in the direct sum $H_0 = \sum_j \oplus L_2[T, m_j]$ to the multiplication operator that maps $(\phi(t, j) : t \in T, j \in J)$ on $(f(t)\phi(t, j) : t \in T, j \in J)$. If $\mathfrak{B}$ is the smallest weakly (or strongly) closed commutative self-adjoint algebra of operators that contains all the B_ω, the operators thus defined by multiplication form a subalgebra of $\mathfrak{B}$; but this may be a proper subalgebra and may fail to be weakly closed. On the other hand, if for every bounded complex function $(f(t, j) : t \in T, j \in J)$ which is $\mathfrak{A}$-measurable on T for each j in J we construct the operator that maps the vector $(\phi(t, j) : t \in T, j \in J)$ of the direct sum H_0 on

$$(f(t, j)\ \phi(t, j) : t \in T, j \in J).$$

we obtain too many operators; there are such operators which, under the isomorphism of H_0 and H, do not correspond to members of $\mathfrak{B}$.

One of our principal results is that if we suitably restrict the functions f of the preceding sentence we obtain the multiplicative representation of the algebra $\mathfrak{B}$ itself. The functions f allowed are those which depend on j only in the following heavily restricted way. To each countable subset J_c of J there corresponds an $\mathfrak{A}$-measurable function $(f(t) : t \in T)$ such that for each j in J_c, $f(t) = f(t, j)$ except on a set whose m_j-measure is 0. Such functions on $T \times J$ we call "quasi-functions on T." For such functions f we can also define $\int_T f d\pi$, by first projecting each vector x of H into the subspaces $H(x_j)$, applying $\int f(t, j) d\pi$ to the projection on $H(x_j)$, and adding the results. The integrals thus formed constitute an algebra of operators, and this algebra is

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exactly the smallest weakly closed self-adjoint algebra of operators containing all the B_ω. It is worth noticing that if J is countable we can replace $f(t, j)$ by the $f(t)$ of the definition (with $J_c = J$); the "new" integral is then the standard $\int f(t)d\pi$, and we conclude that if the cyclic set $\{x_j : j \in J\}$ is countable the algebra of operators $f(B_w)$ defined by the integral is weakly closed, even though Ω may be uncountable and H nonseparable.

The second part of this paper is devoted to showing that the "quasi-functions on T" have this desirable property, and that they also have somewhat similar uses in extending the Radon-Nikodym theorem and the Riesz representation theorem. [Since this was written, A. C. Zaanen has presented to the American Mathematical Society a paper in which the Radon-Nikodym theorem is extended just as it is here. The "quasi-functions" appropriate to this situation are called "cross-sections" by Zaanen. See abstract, Notices Amer. Math. Soc. **7**, no. 7 (1960), 986; his paper will appear(1) in the Nederl. Akad. Wetensch. Proc. Ser. A.] The first part of the paper consists largely of the extension (to infinite sets of measure-functions) of a few of the results obtained by I. E. Segal [2] for (single) "localizable" measures. My apology for presenting these extensions is that they are needed in the second part.

1. Measures. Here and henceforth T will denote a set and $\mathfrak{A}$ a σ-ring of subsets of T. J will denote a nonempty set, used only for indices to distinguish objects. For each j in J, m_j is a non-negative extended-real-valued function on $\mathfrak{A}$, and is countably additive. If $F \subset T$ and $j \in J$, F is said to be m_j-finite if $F \in \mathfrak{A}$ and $m_j F < \infty$, and F is said to be m_j-null if $F \in \mathfrak{A}$ and $m_j F = 0$. For brevity, we use the expression "F is m_J-finite" to mean that F is m_j-finite for all j in J, while "F is m_J-null" means that F is m_j-null for all j in J. The class of all m_J-finite sets will be denoted by $\mathfrak{R}$.

We use a device of I. E. Segal's [2] to eliminate the nuisance of sets of infinite measure that contain no subset of finite positive measure. We define A' to be the collection of all subsets E of T such that $E \cap F \in \mathfrak{R}$ for all F in $\mathfrak{R}$. This is a σ-ring, and is in fact a σ-algebra since T is in it. Next, for each j in J and each E in $\mathfrak{A}'$ we define $m_j' E$ to be the supremum of $m_j(E \cap F)$ for all F in $\mathfrak{R}$. These new measures are countably additive on $\mathfrak{A}'$, and $m_j' F = m_j F$ for all j in J and all F in $\mathfrak{R}$. If we treat $\mathfrak{A}'$ and $\{m_j' : j \in J\}$ as we treated $\mathfrak{A}$ and $\{m_j : j \in J\}$ we obtain them back again. It is with $\mathfrak{A}'$ and the measures m_j' that we work henceforth. We drop the prime from the notation, and we list the properties henceforth assumed.

(1.1) *$\mathfrak{A}$ is an algebra of subsets of a set T.*

For each j in a set J, m_j is a non-negative extended-real-valued countably additive measure in $\mathfrak{A}$.

$\mathfrak{R}$ is the set of all m_J-finite sets (i.e., all F in $\mathfrak{A}$ with $m_j F < \infty$ for all j in J).

For all subsets S of T, $S \in \mathfrak{A}$ if and only if $S \cap F \in \mathfrak{R}$ for all F in $\mathfrak{R}$, and in that case $m_j S = \sup\{m_j(S \cap F) : F \in \mathfrak{R}\}$ for every j in J.

(1) See reference [6].

A set S is *measurable* if it belongs to $\mathfrak{A}$. An extended-real-valued function f on T is *measurable* if $f^{-1}(I)\in\mathfrak{A}$ for every interval I in $[-\infty, \infty]$.

To aid in remembering the domains of functions, we denote the identity function on T by t and the identity function on J by j. Functions will be assumed to be extended-real-valued unless the contrary is stated. Thus $g(t, j)$ will denote a function $g: T\times J\to[-\infty, \infty]$.

2. **Localizability.** A measure m_j will be called *localizable* (by a small modification of the definition in [2]) if

(2.1) *to each nonempty collection* $\mathcal{K}$ *of measurable sets there corresponds a supremum* B *in the sense that*

(i) *for each* S *in* $\mathcal{K}$, $S-B$ *is* m_j*-null;*

(ii) *if* $B'\in\mathfrak{A}$ *and* $S-B'$ *is* m_j*-null for every* S *in* $\mathcal{K}$, *so is* $B-B'$.

We shall say that

(2.2) *a set* B *is a simultaneous supremum of the collection* $\mathcal{K}$ *of sets with respect to the measures* m_j $(j\in J)$, *or more briefly that* B *is an* m_J*-supremum of* $\mathcal{K}$, *if*

(i) *for each* S *in* $\mathcal{K}$ *and each* j *in* J, $S-B$ *is* m_j*-null;*

(ii) *for each* j *in* J *and each* B' *in* $\mathfrak{A}$, *if* $m_j(S-B')=0$ *for every* S *in* $\mathcal{K}$ *then* $m_j(B-B')=0$.

Likewise,

(2.3) *if* $\mathfrak{F}$ *is a collection of measurable functions on* T, *a function* $b(t)$ *is a simultaneous supremum of* $\mathfrak{F}$ *with respect to the measures* m_j $(j\in J)$, *or more briefly an* m_J*-supremum of* $\mathfrak{F}$, *if*

(i) *for each* f *in* $\mathfrak{F}$ *and each* j *in* J, $b(t)\geqq f(t)$ *except on an* m_j*-null set;*

(ii) *if* $b'(t)$ *is measurable and* j *is in* J, *and for each* f *in* F *it is true that* $b'(t)\geqq f(t)$ *except on an* m_j*-null set, then* $b'(t)\geqq b(t)$ *except on an* m_j*-null set.*

The measures m_j $(j\in J)$ are *simultaneously localizable* if every nonempty collection $\mathcal{K}$ of measurable sets has a simultaneous supremum with respect to the measures m_j $(j\in J)$. This immediately implies an analogous property for functions, as follows.

(2.4) COROLLARY. *Let the measures* m_j $(j\in J)$ *be simultaneously localizable. Let* $\mathfrak{F}$ *be a nonempty collection of measurable functions. Then* $\mathfrak{F}$ *has an* m_J*-supremum, and it is measurable.*

For each rational number r let $\mathcal{K}_r$ consist of the measurable sets $\{t: t\in T, f(t)\geqq r\}$ with f in $\mathfrak{F}$, and let B_r be the m_J-supremum of the class $\mathcal{K}_r$. If $r'>r$, $B_{r'}-B_r$ is m_J-null, so the set $S_r=\bigcup\{B_{r'}: r'$ rational, $r'\geqq r\}$ differs from B_r by an m_J-null set. If $r>r'$, $S_{r'}\supset S_r$. Define $b_r(t)$ by setting $b_r(t)=r$ if $t\in S_r$, $b_r(t)=-\infty$ otherwise; and define $b(t)=\sup\{b_r(t): r$ rational$\}$. This is measurable. Let f belong to $\mathfrak{F}$, and for rational r define $E_r=\{t: f(t)>r>b(t)\}$. Then $E_r\subset\{t: t\in T, f(t)\geqq r\}\in\mathcal{K}_r$, so E_r-B_r is m_J-null, and therefore so is E_r-S_r. But $E_r\cap S_r$ is empty by definition of $b(t)$, so E_r is m_J-null. The set on which $f(t)>b(t)$ is contained in the m_J-null set $\bigcup\{E_r: r$ rational$\}$, and (2.3i) holds.

Now let j be in J and let b' be as in (2.3ii), and for rational r define $B'_r = \{t: t \in T, b'(t) \geqq r\}$. For each f in F, B'_r contains all of $\{t: t \in T, f(t) \geqq r\}$ except an m_j-null subset. By (2.2ii), $m_j(B_r - B'_r) = 0$, so $m_j(S_r - B'_r) = 0$. Hence $N = \bigcup\{S_r - B'_r : r \text{ rational}\}$ is m_j-null. If $t \in T - N$,

$$\begin{aligned} b(t) &= \sup\{r: r \text{ rational}, t \in S_r\} \\ &\leqq \sup\{r: r \text{ rational}, t \in B'_r\} \\ &= b'(t), \end{aligned}$$

and (2.3ii) holds.

3. **Simultaneous localizability.** We now seek conditions that will ensure that the measures in a set $\{m_j: j \in J\}$ are simultaneously localizable.

(3.1) THEOREM. *If J is countable and T is m_J-finite, the measures m_j ($j \in J$) are simultaneously localizable.*

Let $\mathcal{K}$ be a nonempty collection of measurable sets, $\mathcal{K}'$ the collection of all finite unions of sets in $\mathcal{K}$. We may suppose that J is a subset of the positive integers. For each j in J let $b_j = \sup\{m_jK: K \in \mathcal{K}'\}$. For each positive integer n there is a K_n in K' such that

$$b_j - 1/n \leqq m_jK_n \leqq b_j \qquad (j \in J, j \leqq n);$$

and we may suppose that K_n increases with n. Define $B = \bigcup_n K_n$. If $K \in \mathcal{K}$, $K_n \cup K \in \mathcal{K}'$ for all n, so for each j in J $m_j(K_n \cup K) \leqq b_j$, whence $m_j(B \cup K) \leqq b_j = m_jB$. Thus $m_j(K - B) = 0$, and (2.2i) holds. If $j \in J$ and B' is measurable and $m_j(K - B') = 0$ for all K in $\mathcal{K}$, then $m_j(K_n - B') = 0$ for all n, and $m_j(B - B') = 0$, so (2.2ii) holds.

If m is a measure on the σ-ring $\mathcal{A}$, and $E \in \mathcal{A}$, we define the "restriction of m to E" to be the measure defined by

(3.2) $m_ES = m(E \cap S)$ *for all S in $\mathcal{A}$.*

If m has the properties (1.1) so has m_E.

(3.3) THEOREM. *Let* (1.1) *hold. Assume that there is a disjoint collection $\mathcal{E}$ of measurable subsets of T whose union is T and which satisfy*

(i) *for each E in $\mathcal{E}$, the restrictions $m_{j,E}$ ($j \in J$) are simultaneously localizable;*

(ii) *for each subset S of T, $S \in \mathcal{A}$ if and only if $S \cap E \in \mathcal{A}$ for every E in $\mathcal{E}$;*

(iii) *if $j \in J$ and $S \in \mathcal{A}$ and $m_jS > 0$, there exists an E in $\mathcal{E}$ for which $m_j(E \cap S) > 0$.*

Then the measures m_j ($j \in J$) are simultaneously localizable.

Let $\mathcal{K}$ be a nonempty collection of measurable sets. For each E in $\mathcal{E}$, let B_E be the $m_{J,E}$-supremum of $\mathcal{K}$; we may assume $B_E \subset E$. Let B be the union of the B_E; by (ii) this is measurable. If $K \in \mathcal{K}$, $m_{j,E}(K - B) = 0$ for all j in J and all E in $\mathcal{E}$, so by (iii) $K - B$ is m_J-null. Thus (2.2i) holds, and (2.2ii) is proved in practically the same way.

(3.4) REMARK. The hypothesis in Theorem 3.3 that $\mathcal{E}$ is a disjoint collection can be weakened to:

Each set E in $\mathcal{E}$ has points in common with at most countably many other sets in $\mathcal{E}$.

We well-order $\mathcal{E}$ by a relation $\prec$, and for each E in $\mathcal{E}$ we define $E' = E - \bigcup\{E'' : E'' \in \mathcal{E}, E'' \prec E\}$. These sets E' are measurable and disjoint, and each E_0 in $\mathcal{E}$ is the union of countably many sets $E_0 \cap E'$, so the collection of E' satisfies the requirements of Theorem 3.3.

(3.5) COROLLARY. *If T is the union of a countable collection $\mathcal{L}$ of measurable sets, and for each L in $\mathcal{L}$ the restrictions $m_{j,L}$ ($j \in J$) are simultaneously localizable, then the m_j ($j \in J$) are simultaneously localizable.*

As a special case, if J is countable and T is the union of countably many m_J-finite sets, the m_j are simultaneously localizable. Still more particularly, every σ-finite measure is localizable.

(3.6) COROLLARY. *If J is countable, and there exists a collection $\mathcal{E}$ of m_J-finite sets whose union is T and no one of which has points in common with more than countably many other members of $\mathcal{E}$, and for each j in J and each S in $\mathcal{A}$ such that $m_j S > 0$ there is an E in $\mathcal{E}$ such that $m_j(S \cap E) > 0$, then the measures m_j ($j \in J$) are simultaneously localizable.*

By the proof of (3.4) we may assume that $\mathcal{E}$ is disjoint, and by Theorem 3.1, (3.3i) holds. For each m_J-finite set F let $\mathcal{E}[F]$ be the (countable) set of E in $\mathcal{E}$ such that $m_j(E \cap F) > 0$ for at least one j in J, and let D_F be the (measurable) set $F - \bigcup\{F \cap E : E \in \mathcal{E}[F]\}$. Clearly $D_F \cap E$ is m_J-null for all E in $\mathcal{E}$, so D_F is m_J-null. If $S \subset T$ and $S \cap E \in \mathcal{A}$ for all E in $\mathcal{E}$, then whenever F is m_J-finite $S \cap F$ is the union of the countably many measurable sets $S \cap E \cap F$ ($E \in \mathcal{E}[F]$) and the m_J-null set $S \cap D_F$, so by (1.1) $S \in \mathcal{A}$. In particular, $T - \bigcup\mathcal{E}$ is m_J-null, and we may adjoin it to any one set of $\mathcal{E}$ without disturbing the requirements. Now by Theorem 3.3 the m_j ($j \in J$) are simultaneously localizable.

4. **A decomposition theorem.** The results of §3 suggest the following question. If a measure m is localizable, can T be subdivided into a disjoint collection of sets of finite measure in such a way that (3.3ii, iii) are satisfied? We partially answer this in Theorem 4.2. We consider countably many measures m_j simultaneously, since this creates no added difficulty.

(4.1) LEMMA. *Let J be countable. Let $\mathcal{E}^*$ be a maximal family of m_J-finite subsets of T, such that no set in $\mathcal{E}^*$ is m_J-null and each pair of distinct members of $\mathcal{E}^*$ has m_J-null intersection. Then*

(i) *$T - \bigcup\mathcal{E}^*$ is m_J-null;*

(ii) *if $S \subset T$, $S \in \mathcal{A}$ if and only if $S \cap E \in \mathcal{A}$ for every E in $\mathcal{E}^*$;*

(iii) *for each j in J and each S in $\mathcal{A}$ such that $m_j S > 0$, there exists an E in $\mathcal{E}^*$ such that $m_j(S \cap E) > 0$.*

For each m_J-finite set F define $\mathcal{E}^*[F]$ to be the (countable) set of E in $\mathcal{E}^*$ with $F \cap E$ not m_J-null; then $D_F = F - \bigcup \{E \cap F: E \in \mathcal{E}^*[F]\}$ is measurable. Then for E in $\mathcal{E}^*$, $D_F \cap E$ is m_J-null; and by the maximality of $\mathcal{E}^*$, D_F must be m_J-null. Now let S be a subset of T such that $S \cap E \in \mathfrak{A}$ for all E in $\mathcal{E}^*$. For every m_J-finite set F, $S \cap F$ is the union of countably many measurable sets $S \cap F \cap E$ $(E \in \mathcal{E}^*[F])$ and the m_J-null set $S \cap D_F$, so $S \cap F \in \mathfrak{A}$. By (1.1), $S \in \mathfrak{A}$. In particular, if $j \in J$ and $m_j S > 0$, there is an m_J-finite set F for which $m_j(S \cap F) > 0$, and one of the countably many sets $S \cap F \cap E$ $(E \in \mathcal{E}^*[F])$ must have positive m_j-measure; then $m_j(S \cap E) > 0$. This implies that $T - \bigcup \mathcal{E}^*$ is m_J-null, completing the proof.

The Hausdorff maximality principle allows us to prove easily that sets $\mathcal{E}^*$ satisfying the hypotheses of (4.1) exist. Segal uses this [2, p. 287] to show that there exists a measure-preserving map of the measurable sets in T onto measurable sets in another measure-space under which images of sets in $\mathcal{E}^*$ are disjoint. However, this does not provide a decomposition of T itself. If $\mathcal{E}^*$ is countable it is trivial to construct another family whose members are disjoint. We now show that such a construction is possible whenever there are at most continuum-many sets in $\mathcal{E}^*$.

(4.2) THEOREM. *Let the measures m_j $(j \in J)$ be simultaneously localizable. If J is countable, and there exists a family $\mathcal{E}^*$ satisfying the hypotheses of Lemma* 4.1 *and having cardinal number not greater than that of the continuum, there exists a decomposition $\mathcal{E}$ of T with the properties* (ii) *and* (iii) *of* (3.3), *the sets in $\mathcal{E}$ being m_J-finite and mutually disjoint.*

There is a mapping r of $\mathcal{E}^*$ onto a subset R_0 of the open interval (0, 1). Let $\mathfrak{F}$ be the set of functions f_{E^*} $(E^* \in \mathcal{E}^*)$, where f_{E^*} has value $r(E^*)$ on E^* and value 0 on $T - E^*$. By Corollary 2.4, $\mathfrak{F}$ has a supremum $b(t)$. Then $b(t) \geqq 0$ except on an m_J-null set, and we may suppose it non-negative on T. Let E' be in $\mathcal{E}^*$; define $h(t)$ by

$$h(t) = r(E')(t \in E'); \qquad h(t) = b(t)(t \in T - E').$$

For each E^* in $\mathcal{E}^*$ let $D_{E^*} = \{t: t \in T,\ f_{E^*}(t) > h(t)\}$. If $E^* = E'$ this is empty; if $E^* \neq E'$, $D_{E^*} \cap [T - E']$ is contained in the m_J-null set $\{t: t \in T, f_{E^*}(t) > b(t)\}$ and $D_{E^*} \cap E'$ is contained in the m_J-null set $E^* \cap E'$, so in all cases D_{E^*} is m_J-null. By definition of m_J-supremum, $h(t) \geqq b(t)$ except on an m_J-null set. In particular, the subset of E' on which $h(t) \geqq b(t)$ fails to hold is m_J-null. The reversed inequality is obvious, so $b(t) = r(E')$ on all of E' except an m_J-null subset.

Now for each r in R_0 we define $E(r) = b^{-1}(r)$, and we define $\mathcal{E}$ to be $\{E(r): r \in R_0\}$. The mapping $E^* \to E(r(E^*))$ is a one-to-one correspondence between $\mathcal{E}^*$ and $\mathcal{E}$, and corresponding sets differ by m_J-null sets. Thus $\mathcal{E}$ is a disjoint collection of sets none of which is m_J-null. The set $\mathcal{E}$ satisfies the hypotheses placed upon $\mathcal{E}^*$ in Lemma 4.1. Thus $T - \bigcup \mathcal{E}$ is m_J-null; we adjoin it to any one set of $\mathcal{E}$, and the amended $\mathcal{E}$ has union T. The other conclusions hold by Lemma 4.1.

5. **Definition and lattice properties.** We continue to assume that (1.1) is satisfied.

(5.1) *A function $f(t, j)$ will be called a "quasi-function on T" with respect to the measures m_j $(j\in J)$ if to each countable subset J_c of J there corresponds a function $f(t)$ such that $m_j\{t: t\in T, f(t, j)\neq f(t)\}=0$ for every j in J_c.*

Occasionally we shall use the symbol $f(t, *)$ for a quasi-function; the use of the asterisk in place of j shall connote the relation between the various $f(t, j)$ specified in the definition of quasi-function. A quasi-function $f(t, *)$ is called *measurable* if all the functions $f(t, j)$ $(j\in J)$ are measurable.

Each function $f(t)$ defines a quasi-function $f(t, *)$ such that $f(t, j)=f(t)$ for all t in T and all j in J. Conversely, if $f(t, j)$ is independent of j it defines a function on T. By a familiar abuse of language we then say that the quasi-function $f(t, *)$ *is* the function on T that it defines.

Algebraic and lattice operations on quasi-functions need no discussion, since each quasi-function is an extended-real-valued function on $T\times J$. Two quasi-functions $f(t, *)$, $g(t, *)$ are *equivalent* if for each j in J we have $m_j\{t: t\in T, f(t, j)\neq g(t, j)\}=0$. Clearly, if J is countable every quasi-function is equivalent to a function on T.

Let $\mathfrak{F}$ be a nonempty set of quasi-functions on T, with respect to the m_j $(j\in J)$. Then:

(5.2) *A quasi-function $b(t, *)$ is called an m_J-supremum of $\mathfrak{F}$ if*

(i) *for each j in J and each f in $\mathfrak{F}$, $b(t, j)\geqq f(t, j)$ except on an m_j-null set;*

(ii) *for each j in J, if $b'(t)$ is a function on T such that for every f in $\mathfrak{F}$ the inequality $b'(t)\geqq f(t, j)$ holds except on an m_j-null set, then $b'(t)\geqq b(t, j)$ except on an m_j-null set.*

Evidently any two m_J-suprema of a set $\mathfrak{F}$ are equivalent.

Quasi-functions have lattice properties stronger than those of ordinary functions, as the following theorem shows.

(5.3) THEOREM. *Let (1.1) hold, and for each countable subset J_c of J let the measures m_j $(j\in J_c)$ be simultaneously localizable. Then each nonempty collection $\mathfrak{F}$ of measurable quasi-functions on T (with respect to the measures m_j, $j\in J$) has an m_J-supremum, and this is a measurable quasi-function.*

By Corollary 2.4, for each j in J there is a function $b(t, j)$ satisfying (5.2i, ii). It remains to show that $b(t, j)$ is a quasi-function. Let J_c be a countable subset of J. For each $f(t, j)$ in $\mathfrak{F}$ we can and do choose an $f(t)$ such that for all j in J_c, $f(t)=f(t, j)$ except on an m_j-null set. Denote by $\mathfrak{F}_c$ the set of $f(t)$ thus chosen. By Corollary 2.4, $\mathfrak{F}_c$ has a supremum $b(t)$ with respect to the measures m_j $(j\in J_c)$. For each j in J_c we have $b(t, j)\geqq f(t, j)$ except on an m_j-null set whenever $f(t, j)\in\mathfrak{F}$, so $b(t, j)\geqq f(t)$ except on an m_j-null set whenever $f(t)\in\mathfrak{F}_c$, and so $b(t, j)\geqq b(t)$ except on an m_j-null set. On the other hand, if $f(t, j) \in \mathfrak{F}$, we have $b(t) \geqq f(t) = f(t, j)$ except on an m_j-null set, whence by (5.2ii) $b(t) \geqq b(t, j)$ except on an m_j-null set. Thus for all j in J_c we have $b(t, j) = b(t)$ except on an m_j-null set, and the proof is complete.

6. **Summable quasi-functions.** Suppose that we have a system satisfying (1.1) with J a singleton; we then drop the subscript j from m_j. Let $\mathfrak{R}$ be the collection of m-finite sets, and $\mathfrak{L}$ the collection of all measurable subsets L of T such that the restriction of m to L is localizable (cf. (3.2)). Then the system $\{T, \mathfrak{A}, \{m_R: R \in \mathfrak{R}\}\}$ satisfies (1.1), and so does $\{T, \mathfrak{A}, \{m_L: L \in \mathfrak{L}\}\}$. If we define $\boldsymbol{L}$ and $\boldsymbol{R}$ to be the identity functions on $\mathfrak{L}$ and $\mathfrak{R}$ respectively, we can construct two classes of quasi-functions, $f(t, \boldsymbol{L})$ and $f(t, \boldsymbol{R})$. Since $\mathfrak{R} \subset \mathfrak{L}$, if $f(t, \boldsymbol{L})$ is a quasi-function with respect to the measures m_L $(L \in \mathfrak{L})$, its restriction $f(t, \boldsymbol{R})$ to $T \times \mathfrak{R}$ is a quasi-function with respect to the measures m_R $(R \in \mathfrak{R})$. Conversely, every quasi-function with respect to the m_R can be extended to a quasi-function with respect to the m_L, as we now show.

(6.1) Lemma. *If $f(t, \boldsymbol{R})$ is a quasi-function with respect to the measures m_R $(R \in \mathfrak{R})$, it has an extension $f(t, \boldsymbol{L})$ to $T \times \mathfrak{L}$ which is a quasi-function with respect to the measures m_L $(L \in \mathfrak{L})$. In particular, if m is localizable $f(t, \boldsymbol{R})$ is equivalent to a measurable function $f(t)$ on T.*

For each R in $\mathfrak{R}$ let g_R be the function coinciding with $f(t, R_R)$ on R and identically $-\infty$ on $T - R$. By Corollary 3.5 the hypotheses of Theorem 5.3 are satisfied with $J = \mathfrak{L}$, so the set $\{g_R: R \in \mathfrak{R}\}$ has a supremum $b(t, \boldsymbol{L})$ among the measurable quasi-functions with respect to the m_L $(L \in \mathfrak{L})$. Let R_1 and R_2 be in $\mathfrak{R}$. Then there is a measurable function $f(t)$ such that $f(t) = f(t, R_i)$ except on an m_{R_i}-null set $(i = 1, 2)$. Hence $g_{R_1}(t) = g_{R_2}(t)$ at all points of $R_1 \cap R_2$ except those of an m-null subset. This implies $g_{R_1}(t) \geqq g_{R_2}(t)$ at all points of T except an m_{R_1}-null set, and since R_2 is any member of $\mathfrak{R}$, $g_{R_1}(t) \geqq b(t, R_1)$ except on an m_{R_1}-null set. The reversed inequality is an immediate consequence of the definition of b, so $f(t, R_1) = g_{R_1}(t) = b(t, R_1)$ except on an m_{R_1}-null set. Without loss of generality we may assume that $b(t, R_1)$ was already defined so as to be identical with $f(t, R_1)$ for all R_1 in $\mathfrak{R}$. Then $b(t, \boldsymbol{L})$ is an extension of $f(t, \boldsymbol{R})$ to $T \times L$, and is a quasi-function with respect to the measures m_L $(L \in \mathfrak{L})$. In particular, if m is localizable, T is in $\mathfrak{L}$ and $m = m_T$; and $b(t, T)$ is a function on T equivalent to $b(t, \boldsymbol{L})$. This completes the proof.

7. **An extension of the Radon-Nikodym theorem.** Let $f(t, \boldsymbol{R})$ be a quasi-function with respect to the m_R $(R \in \mathfrak{R})$; and suppose that for each R in $\mathfrak{R}$, $f(t, R)$ is m_R-summable over T (or, equivalently, m-summable over R). Then $\int f(t, R') dm_R(t)$ has the same value for all R' in $\mathfrak{R}$ that contain R. This integral is said to have a limit k (in $[-\infty, \infty]$) *as R expands* if to each neighborhood U of k there corresponds an R_U in $\mathfrak{R}$ such that whenever $R \in \mathfrak{R}$ and $R \supset R_U$, $\int f(t, R) dm_R(t) \in U$. In this case we write

$$\int f(t, *) dm(t) = k.$$

(All integrals are over T.) If f is non-negative the limit will surely exist; and in fact we can relax the requirement that $f(t, R)$ be m_R-summable and require that it have an integral, finite or infinite. It is easy to see that on the class of summable quasi-functions (those having finite integrals) the integral is linear, and the class is a vector lattice.

Now consider a system satisfying (1.1) with $J=\{1, 2\}$. To save subscripts we write m for m_1 and n for m_2. (The assumption that n is non-negative is not vital, but saves some bother.) $\mathfrak{L}$ will be the class of sets L in $\mathfrak{A}$ such that m_L is localizable.

(7.1) THEOREM. *Let m and n be as just described. Assume that for each R in $\mathfrak{R}$, if $mR=0$ then $nR=0$. Then there exists a non-negative finite valued measurable quasi-function $f(t, L)$ such that for all S in $\mathfrak{A}$,*

$$nS = \int f(t, *)K_S(t)dm(t).$$

If m is localizable $f(t, *)$ can be taken to be a measurable function $f(t)$ on T.

If S is σ-finite, by the standard form of the Radon-Nikodym theorem ([1], p. 128) there is a measurable real-valued function f on S such that for every measurable subset E of S,

$$n(E) = \int_E f(t)dm(t).$$

For each σ-finite set S in $\mathfrak{A}$ we choose such an integrand, and we define $f_S(t)$ to be $f(t)$ for t in S and to be 0 for t in $T-S$. These f_S may be regarded as quasi-functions on T with respect to the m_L ($L\in\mathfrak{L}$), and as such they have an $m_{\mathfrak{L}}$-supremum $f(t, L)$, which we may suppose to be everywhere non-negative.

If R is in $\mathfrak{R}$ it is in $\mathfrak{L}$, so the inequality $f(t, R)\geqq f_R(t)$ holds except on an m_R-null set. If S is σ-finite, f_S and f_R have the same integral $n(E)$ over every measurable subset E of $R\cap S$, so they coincide on all of $R\cap S$ except an m-null set. But f_S vanishes on $R-S$, so $f_R\geqq f_S$ except on an m_R-null set. By definition of supremum, $f_R\geqq f(t, R)$ except on an m_R-null set. With the previous inequality, this proves that $f_R(t)=f(t, R)$ except on an m_R-null set. So for all S in $\mathfrak{A}$

$$n(R\cap S) = \int f(t, R)K_S(t)dm_R(t).$$

The integral increases as R expands, so for fixed S in $\mathfrak{A}$ its limit as R expands is the same as its supremum for all R in $\mathfrak{R}$, whence

$$nS = \int f(t, *)K_S(t)dm(t).$$

The last conclusion follows readily. (Cf. [2].)

8. **The Riesz representation theorem.** A function $f(t, j)$ on $T\times J$ (not necessarily a quasi-function with respect to the measures m_j, $j\in J$) will be called *measurable* if $f(t, j)$ is measurable on T for each j in J; it is *essentially bounded* if there is a real number r such that for each j in J, $|f(t, j)| \leqq r$ except on an m_j-null set. The infimum of r is the *essential supremum* of $|f(t, j)|$. If $f(t, j)$ and $f'(t, j)$ are equivalent quasi-functions and one is essentially bounded, so is the other, and the essential suprema of their absolute values are equal. The set of all equivalence classes of essentially bounded quasi-functions will be denoted by $L_{\infty *}[T, \{m_j: j\in J\}]$, or by $L_{\infty *}$ when this will not cause confusion. If $\phi\in L_{\infty *}$ and $f(t, j)\in\phi$, we define $\|\phi\|_\infty$ to be the essential supremum of $|f(t, j)|$. As usual, if $f(t, j)$ is essentially bounded we shall (incorrectly, but conveniently) say that f belongs to $L_{\infty *}$ and has norm $\|f(t, j)\| = \text{ess. sup. } |f(t, j)|$.

If $b(t, *)$ is an essentially bounded quasi-function with respect to the measures m_R ($R\in\mathfrak{R}$) or the measures m_L ($L\in\mathfrak{L}$), it is easy to see that for every m-summable function $f(t)$ the integral

$$l_b(f) = \int f(t)b(t, *)dm(t)$$

exists and is finite. This defines a continuous linear functional l_b over $L_1[T, m]$, and $\|l_b\| = \|b(t, *)\|$. We now prove the converse of this, which is a sort of extension of the Riesz representation theorem. (For a different kind of extension see [3].)

(8.1) THEOREM. *Let l be a continuous linear real-valued functional over $L_1[T, m]$. Then there exists an essentially bounded measurable quasi-function $b(t, L)$ with respect to $\{m_L: L\in\mathfrak{L}\}$ such that $\|b(t, *)\|_\infty = \|l\|$ and for every function f in $L_1[T, m]$*

$$l(f) = \int f(t)b(t, *)dm(t).$$

The quasi-function $b(t, *)$ is unique up to equivalence.

By a well-known device we can represent l as the difference of two nonnegative functionals, so without loss of generality we may assume l nonnegative. For each S in $\mathfrak{A}$ we define nS to be the supremum of $l(K_{R\cap S})$ for all R in $\mathfrak{R}$; this is a countably additive measure on $\mathfrak{A}$. By Theorem 7.1 there is a quasi-function $b(t, L)$ with respect to $\{m_L: L\in\mathfrak{L}\}$ such that for each S in $\mathfrak{A}$

$$nS = \int b(t, *)K_S(t)dm(t).$$

It is easy to see that the essential supremum of $|b(t, L)|$ is $\|l\|$. If f is a "simple function," meaning that f is a finite linear combination of characteristic functions of sets of finite measure, this implies

$$l(f) = \int b(t, *)f(t)dm(t).$$

Both members of this equation define continuous linear functions on $L_1[T, m]$, and the simple functions are dense in that space, so the equation holds for all f in $L_1[T, m]$. The uniqueness of b, up to equivalence, is easy to establish.

9. **Decomposition of a Hilbert space by means of a cyclic set.** Let T be a set and $\mathfrak{A}$ a σ-algebra of subsets of T. Let H be a complex Hilbert space (not necessarily separable) and π a resolution of the identity assigning to each set S in $\mathfrak{A}$ a projector $\pi(S)$ in H such that $\pi(T)$ is the identity **1**, $\pi(\emptyset)$ is the projector **0** on the origin, $\pi(S_1)\pi(S_2)=\pi(S_1\cap S_2)$ and π is countably additive. For each x in H let $H(x)$ be the closure of the set of all linear combinations of projectors $\pi(S)x$ $(S\in\mathfrak{A})$. If y is orthogonal to $H(x)$, for all S and S' in $\mathfrak{A}$ we have $(\pi(S)y, \pi(S')x)=(y, \pi(S\cap S')x)=0$, so $H(y)$ is orthogonal to $H(x)$. From the Hausdorff maximality principle we deduce that there is a set $\{x_j: j\in J\}$ of unit vectors in H such that $H(x_j)$ and $H(x_k)$ are orthogonal whenever j and k are distinct members of J, and such also that the closed linear span of the $H(x_j)$ $(j\in J)$ is H.

For all j in J and S in $\mathfrak{A}$ we define

$$\mathrm{m}_jS = \pi(S)x_j, \qquad m_jS = \|\pi(S)x_j\|^2.$$

If z is a simple function, with values $z_1, \cdots, z_n$ on disjoint measurable sets $S_1, \cdots, S_n$, we define

$$\int z(t)d\,\mathrm{m}_j(t) = \sum_{i=1}^{n} z_i\mathrm{m}_jS_i,$$

$$\int z(t)dm_j(t) = \sum_{i=1}^{n} z_im_jS_i.$$

Then

$$\left\|\int z(t)d\,\mathrm{m}_j(t)\right\|^2 = \sum_i \|z_i\mathrm{m}_jS_i\|^2$$
$$= \sum_i |z_i|^2m_jS_i$$
$$= \int |z|^2dm_j(t).$$

This isometry can be extended to all of (complex) $L_2[T, m_j]$; to each $z(t)$ in

$L_2[T, m_j]$ there corresponds a unique vector-integral $\int z(t)d\mathrm{m}_j(t)$, and its length is the norm of $z(t)$ in $L_2[T, m_j]$.

(9.1) LEMMA. *Let H_0 be the direct sum of the spaces $L_2[T, m_j]$ $(j\in J)$. For each function $z(t, j)$ in H_0 define*

$$\psi(z) = \sum_{j\in J}\int z(t, j)d\mathrm{m}_j(t).$$

Then ψ is an isometric linear mapping of H_0 onto H.

The linearity of ψ is obvious. If $z(t, j)\in H_0$, for each j in J, $z(t, j)$ is in $L_2[T, m_j]$ and therefore has an integral y_j with respect to m_j, and

$$\|y_j\|^2 = \int |z(t, j)|^2 dm_j(t).$$

The y_j are mutually orthogonal, and the sum of the right members is $\|z(t, j)\|^2$, so $\sum_j y_j$ exists and has length $\|z(t, j)\|$. Thus the mapping ψ is defined on H_0 and is an isometry. The range of ψ includes all vectors $\pi(S)x_j$ $(S\in \mathfrak{A}, j\in J)$, and the linear combinations of these are dense in H, so the range of ψ is H.

For each bounded linear operator B_0 on H_0 we define

$$\psi_0(B_0) = \psi B_0\psi^{-1}.$$

Then ψ_0 is an isometric isomorphism of the algebra of all bounded linear operators on H_0 onto the algebra of all bounded linear operators on H, and is weakly continuous.

10. **Multiplication operators.** Each essentially bounded measurable function $f(t, j)$ on $T\times J$ (not necessarily a quasi-function on T) defines a bounded linear operator $f^\times$ on H_0 by the relation

$$f^\times(z) = f(t, j)z(t, j) \qquad (z(t, j) \in H_0).$$

When $f(t, j)$ is a quasi-function on T with respect to the measures m_j $(j\in J)$ the operator $f^\times$ will be called a *multiplication operator*; the set of all multiplication operators will be denoted by $\mathfrak{M}$. It is obvious that $\mathfrak{M}$ is a commutative $*$-algebra over the complex field.

The algebra $\mathfrak{M}$ is mapped isometrically and isomorphically onto an algebra $\psi_0(\mathfrak{M})$ of operators in H. One special case is worth noting. Let S be measurable, and let q be the characteristic function of $S\times J$. Then $\psi_0(q^\times) = \pi(S)$. For if $j\in J$ and z is the characteristic function of a set $E\times\{j\}$ with E in $\mathfrak{A}$, we have

$$\psi(z) = \int z(t, j)d\mathrm{m}_j(t) = \mathrm{m}_jE = \pi(E)x_j,$$

so

$$\pi(S)\psi(z) = \pi(E\cap S)x_j = \mathrm{m}_j(E\cap S) = \psi(q(t, j)z(t, j)) = \psi(q^\times(z)).$$

This extends to linear combinations of such z, and by continuity to the closure of the set of linear combinations of such z, which is H_0. If $y \in H$ then there exists a z in H_0 such that $y = \psi(z)$, so

$$\pi(S)y = \psi(q^{\times}(\psi^{-1}(y))) = \psi_0(q^{\times})y.$$

11. **The multiplicative representation.** We can now state our principal theorem on algebras of operators.

(11.1) THEOREM. *With the notation of the two preceding sections,*

(i) *every bounded linear operator C that commutes with every projector $\pi(S)$ $(S \in \mathfrak{A})$ also commutes with all operators in $\psi_0(\mathfrak{M})$;*

(ii) *the algebra $\psi_0(\mathfrak{M})$ is weakly closed, and is the smallest weakly closed algebra of linear operators on H that contains all the projectors $\pi(S)$ $(S \in \mathfrak{A})$.*

For any set $\mathfrak{N}$ of bounded linear operators on H (or on H_0) let $\mathfrak{N}'$ denote the set of all bounded linear operators on H (or on H_0, respectively) that commute with every operator in $\mathfrak{N}$. Let $\Pi = \{\pi(S) : S \in \mathfrak{A}\}$. Since $\Pi \subset \psi_0(\mathfrak{M})$, it is clear that $\Pi' \supset [\psi_0(\mathfrak{M})]'$. Conclusion (i) is the assertion $\Pi' \subset [\psi_0(\mathfrak{M})]'$, and implies the equality of the two sets.

We shall use certain kinds of characteristic functions often enough to warrant abbreviation. If $j \in J$ and $S \subset T$, we shall use the symbols $K_{S,j}$, K_j, K_S to denote the characteristic functions of the respective sets $S \times \{j\}$, $T \times \{j\}$, $S \times J$.

By §10, conclusion (i) is equivalent to $\{K_S^{\times} : S \in \mathfrak{A}\}' \subset \mathfrak{M}'$. Let then C_0 be an operator on H_0 that commutes with $K_S^{\times}$ for all S in $\mathfrak{A}$; we must prove that C_0 commutes with all $q^{\times}$ in $\mathfrak{M}$. So consider any $q(t, j)$ in $L_{\infty}^{\bullet}$. Let $z(t, j)$ belong to H_0, and let $y = C_0 z$. Then $\int |z(t, j)|^2 dm_j$ and $\int |y(t, j)|^2 dm_j$ vanish except for a countable subset J_c of J. There is a bounded measurable function $q_0(t)$ such that for every j in J_c, $q_0(t) = q(t, j)$ except on an m_j-null set. This $q_0(t)$ is the uniform limit of simple functions $q_n(t)$ $(n = 1, 2, \cdots)$. Each q_n is a finite linear combination

$$q_n = \sum_{k=1}^{m_n} c_{n,k} K_{S_{n,k}} \qquad (S_{n,k} \in \mathfrak{A}),$$

so $q_n^{\times}$ is a sum of operators each of which commutes with C_0. Hence

$$C_0[q_n(t)z(t, j)] = q_n(t)C_0[z(t, j)] = q_n(t)y(t, j),$$

so that

$$C_0[q_0(t)z(t, j)] = q_0(t)y(t, j).$$

If $j \in J_c$, the equation

$$q_0(t)z(t, j) = q(t, j)z(t, j)$$

holds except on an m_j-null set, by definition of q_0; if $j \in J - J_c$, it holds except on the m_j-null set on which $z(t, j) \neq 0$. So the two members of the equation represent the same point of H_0. Similarly, $q_0(t)y(t, j)$ and $q(t, j)y(t, j)$ represent the same point of H_0, and the last equation of the preceding paragraph implies

$$C_0 q^{\times} z = q^{\times} C_0 z.$$

This holds for all z in H_0, and (i) is established.

If $\mathfrak{N}$ is any set of bounded linear operators on H such that for all A in $\mathfrak{N}$, A^* is also in $\mathfrak{N}$, the smallest weakly closed algebra containing $\mathfrak{N}$ is $\mathfrak{N}''$ [5, p. 44]. Since (i) holds, $\Pi'' = [\psi_0(\mathfrak{M})]''$, and (ii) will be established when we prove $[\psi_0(\mathfrak{M})]'' = \psi_0(\mathfrak{M})$, or (more conveniently) $\mathfrak{M}'' = \mathfrak{M}$. Let A be any operator in $\mathfrak{M}''$; we shall prove $A \in \mathfrak{M}$.

For $j \in J$, the characteristic function K_j of $T \times \{j\}$ is in H_0, so AK_j is an element $r_j(t, j)$ of H_0. Since $K_j^{\times}$ is in $\mathfrak{M}'$,

$$K_j^{\times} r_j = A K_j^{\times}(K_j) = AK_j = r_j.$$

We may therefore suppose that the chosen representation r_j of AK_j already has the property of vanishing except on $T \times \{j\}$, and we define

$$r(t, j) = \sum_{j \in J} r_j(t, j);$$

at each point in $T \times J$ at most one term in the sum is different from 0.

If $j \in J$ and $S \in \mathfrak{A}$ we have $K_{S,j} = K_{S,j} K_j$, so

$$A(K_{S,j}) = A K_{S,j}^{\times}(K_j) = K_{S,j}^{\times} A(K_j) = K_{S,j}(t, j) r_j(t, j) = K_{S,j}(t, j) r(t, j).$$

Hence if $z(t, j)$ is a function assuming finitely many values, each on a set of the form $S \times \{j\}$ with $j \in J$ and $S \in \mathfrak{A}$, we obtain

$$A(z) = z(t, j) r(t, j) = r^{\times}(z).$$

By the usual device we deduce that the essential supremum of $|r(t, j)|$ is $\|A\|$. Furthermore, the simple functions of the type just described are dense in H_0, and A and $r^{\times}$ are continuous, so the last equation implies $A = r^{\times}$. It remains to prove that r is a quasi-function on T with respect to the m_j $(j \in J)$, so that $r^{\times}$ will belong to $\mathfrak{M}$.

Let J_c be a countable subset of J; it can be mapped one-to-one on a set of positive integers by a function $(n(j) : j \in J)$. For each S in $\mathfrak{A}$ define

$$mS = \sum_{j \in J_c} 2^{-n(j)} m_j S.$$

This is a countably additive non-negative measure on $\mathfrak{A}$, with $m(T) \leqq 1$; and for each j in J and each S in $\mathfrak{A}$, $m_j S \leqq 2^{n(j)} mS$. By the Radon-Nikodym theorem, for each j in J there is a measurable function $f_j(t)$ on T, with $0 \leqq f_j \leqq 2^{n(j)}$, such that for all S in $\mathfrak{A}$

$$\int_S f_j(t)dm(t) = m_j S.$$

With the standard definition of the signum function (so that $\operatorname{sgn} f_j(t)$ is 1 if $f_j(t)>0$ and is 0 if $f_j(t)=0$), for all t in T we define

$$r(t) = \sup\{r_j(t, j) \operatorname{sgn} f_j(t) : j \in J_c\}.$$

This is measurable and essentially bounded. We shall now prove

(*) *for each j in J_c, $r(t)=r(t, j)$ except on an m_j-null set.*

Suppose this false; for some k in J_c, $r(t)\neq r(t, k)$ on a set S_1 with $m_k S_1>0$. Since $m_k S_1$ is the limit as n increases of $m_k\{t: t\in S_1, f_k(t)\geqq 1/n\}$, there is a subset S_2 of S_1 with $m_k S_2>0$ on which f_k has a positive lower bound. By definition of r, this implies $r(t)\geqq r_k(t, k)$ on S_2, and they are unequal on S_1, so for each t in S_2 there is a j in J_c for which $r_j(t, j)>r_k(t, k)$ and $f_j(t)>0$. Since J_c is countable and $mS_2 > 0$, there is an h in J_c for which the set $S_3 = \{t: t\in S_2, f_h(t)>0, r_h(t, h)>r_k(t, k)\}$ has positive m-measure. Since $f_k>0$ on S_3, $m_k S_3>0$. As in defining S_2, we find that there is a subset S_4 of S_3 with $m_h S_4>0$ on which $f_h(t)$ has a positive lower bound. Summarizing, $m_h S_4>0$ and $m_k S_4>0$; on S_4, both f_h and f_k have positive lower bounds and finite upper bounds; and on S_4, $r_h(t, h)>r_k(t, k)$.

Define C on H_0 by setting $Cz(t, j)=y(t, j)$, where

$$y(t, j) = z(t, j) \text{ unless } t \in S_4 \text{ and } j \in \{h, k\},$$
$$y(t, h) = z(t, k) \quad\text{and}\quad y(t, k) = z(t, h) \ (t \in S_4).$$

This is a bounded linear operator which is easily seen to commute with $K_S^\times$ for every S in $\mathfrak{A}$, so by conclusion (i) C is in $\mathfrak{M}'$. Now let z be the characteristic function of $S_4\times\{h\}$. Then at each point (t, k) of $S_4\times\{k\}$ we have

$$Cr^\times(z)(t, k) = r_h(t, h),$$
$$r^\times Cz(t, k) = r_k(t, k),$$

and the right members differ on all of S_4. So $r^\times$ does not commute with the element C of $\mathfrak{M}'$, and is therefore not in $\mathfrak{M}''$. But $r^\times=A$, and by hypothesis A is in $\mathfrak{M}''$. This contradiction establishes (*), so $r(t, j)$ is a quasi-function and A is in $\mathfrak{M}$. Thus $\mathfrak{M}''\subset\mathfrak{M}$, and the proof is complete.

12. **The algebra of quasi-functions of operators.** Consider now a weakly closed commutative *-algebra $\mathfrak{B}$ of bounded operators on a Hilbert space H (a von Neumann algebra, in Dixmier's terminology). Let $(B_\omega: \omega\in\Omega)$ be a set of hermitian operators in $\mathfrak{B}$ such that the smallest weakly closed algebra containing the set is $\mathfrak{B}$ itself. Let T_ω be the interval $[-\|B_\omega\|, \|B_\omega\|]$, and let T be the cartesian product $\times_\omega T_\omega$. By a proof in [4, pp. 125 et seq.] we know that there exists a σ-field $\mathfrak{A}$ of subsets of T including all Borel subsets of T, and on $\mathfrak{A}$ a projector-valued measure $(\pi(S): S\in\mathfrak{A})$ which is a resolution of the identity, such that for each ω in Ω the integral of the coordinate-function

t_ω with respect to π is B_ω. Whenever $q(t)$ is bounded and measurable the integral $\int q(t)d\pi(t)$ exists and is a bounded operator on H, which we denote by $q(B_w)$ to indicate its dependence on all the B_ω; these operators obey the usual rules of an operational calculus. As we shall shortly prove, the set of all such $q(B_w)$ is contained in $\mathcal{B}$. However, it may be a proper subset of $\mathcal{B}$ and may fail to be weakly closed, as we now show.

Let $U=[-1, 1]$, $U^2=U\times U$. Let m_1 be one-dimensional Lebesgue measure, and let $\mathcal{A}$ be the class of all subsets of U^2 that meet each ordinate and each abscissa in an m_1-measurable set. For each S in $\mathcal{A}$ we define

$$mS = \sum_{u\in U} m_1(S\cap[\{u\}\times U]) + \sum_{u\in U} m_1(S\cap[U\times\{u\}]).$$

This system satisfies (1.1). Let H be $L_2[U^2, m]$. The operator B that transforms z into $(u_1z(u_1, u_2):(u_1, u_2)\in U^2)$ is hermitian and has bound 1, so its spectrum is contained in $T=[-1, 1]$. Let π be the resolution of the identity for B, and let $\mathcal{B}$ be the smallest weakly closed algebra of operators on H that contains B. For each τ in T define

$$f_{n,\tau}(t) = [1-(t-\tau)^2/5]^n \qquad (t\in T).$$

As n increases this converges monotonically to the characteristic function of $\{\tau\}$. Hence $\pi(\{\tau\})$ is the weak limit of the $f_{n,\tau}(B)$ and is therefore in $\mathcal{B}$. So too is $\pi(F)$ for every finite subset F of T, and by [4, p. 108] so is

$$C = \sup\{\pi(F): F \text{ finite}, F\subset T\}.$$

It is easy to show that $\pi(\{\tau\})$ is the operation of multiplying by the characteristic function of $\{\tau\}\times U$, so for each z in H the transform $w=Cz$ is defined by

$$\begin{aligned} w(u_1, u_2) &= z(u_1, u_2) \textit{ for all } u_1 \textit{ such that } \int_U |z(u_1, v)|^2dv > 0,\\ &= 0 \textit{ for all other } u_1. \end{aligned}$$

This operator cannot be represented in the form $\int f(t)d\pi(t)$ with any function f. For suppose C thus represented. For each τ in T, let z be the characteristic function of $\{\tau\}\times U$. This is in H, and $\pi(\{\tau\})z=z$. Hence $\pi(T')z=0$ whenever $T'\subset T-\{\tau\}$, and

$$z = Cz = \left[\int_T f(t)d\pi(t)\right]z = f(\tau)z,$$

whence $f(\tau)=1$. This holds for all τ in T, so C is the identity operator. But if z is the characteristic function of $U\times\{0\}$, $Cz=0\neq z$. So C is not represented as an integral, and the class of functions $q(B)$ is not weakly closed.

We correct this inadequacy by enlarging the class of integrands. Let the cyclic set $\{x_j: j\in J\}$, the subspaces $H(x_j)$ and the measures m_j be as in §9 and let P_j be the projector on $H(x_j)$. If $q(t, j)$ is measurable and essentially bounded (cf. §8) the expression

$$q(B_w) = \sum_{j\in J}\int q(t, j)d\pi(t)P_j$$

defines a bounded linear operator on H; for if $y\in H$, $P_jy=0$ except for countably many j, and the sum

$$\sum_{j\in J}\int q(t, j)d\pi(t)P_jy$$

converges in norm to a limit whose norm is at most ess. sup. $|q(t, j)|\cdot\|y\|$, and the linearity is evident. However, we now have too many operators. Easy examples, even with two-dimensional H, show that $q(B_w)$ may fail to belong to $\mathfrak{B}$. This excess is remedied by restricting the $q(t, j)$ to be essentially bounded quasi-functions on T, that is to belong to $L_{\infty*}=L_{\infty*}[T, \{m_j: j\in J\}]$. It is trivially easy to prove all but the last conclusion of the following theorem.

(12.1) THEOREM. *For each quasi-function $q(t, j)$ in $L_{\infty*}$, the operator $q(B_w)$ just defined is linear and has bound* ess. sup. $|q(t, j)|$; *and $\bar{q}(B_w) = [q(B_w)]^*$. If $q_1(_w, j)$ and $q_2(t, j)$ are in $L_{\infty*}$ and c_1 and c_2 are complex numbers, and $q_3(t, j)=c_1q_1(t, j)+c_2q_2(t, j)$ and $q_4(t, j)=q_1(t, j)q_2(t, j)$, then*

$$q_3(B_w) = c_1q_1(B_w) + c_2q_2(B_w) \quad \textit{and} \quad q_4(B_w) = q_1(B_w)q_2(B_w).$$

If $q(t, j)$ is equivalent to $q_0(t)$, then $q(B_w)$ is the same as $q_0(B_w)$ as ordinarily defined. The set of all operators $q(B_w)$ $(q(t, j)\in L_{\infty*})$ is the algebra $\mathfrak{B}$.

Let $q(t, j)$ belong to $L_{\infty*}$. For each C in $\mathfrak{B}'$, to each y in H there corresponds a countable subset J_c of J such that $P_jy=P_jCy=0$ for all j in $J-J_c$. There is a measurable $f(t)$ such that $f(t)=q(t, j)$ except on an m_j-null set for all j in J_c, and by classical theory $f(B_w)$ commutes with C. Thus $Cq(B_w)y=Cf(B_w)y=f(B_w)Cy=q(B_w)Cy$. Since y is arbitrary in H, $q(B_w)$ commutes with the arbitrary member C of $\mathfrak{B}'$, and is therefore in $\mathfrak{B}''$. It remains to prove that every member of $\mathfrak{B}$ can be written in the form $q(B_w)$ for some q in $L_{\infty*}$.

Let K_S be the characteristic function of a measurable subset S of T. By §10, $\psi_0(K_S^\times)=\pi(S)$, so when $f=K_S$ the equation

$$(**) \qquad \int f(t)d\pi(t) = \psi_0(f^\times)$$

is valid. By linearity it holds for all simple functions. If $f(t)$ is bounded and measurable it is the uniform limit of a sequence $s_1, s_2, \cdots$ of simple functions. Then $s_n^\times$ tends to $f^\times$ in the uniform topology of operators, so $\psi_0(s_S^\times)$ tends in the same manner to $\psi_0(f^\times)$. Also, $\int s_n(t)d\pi(t)$ tends in the same manner to $\int f(t)d\pi(t)$, so (**) holds for all bounded measurable functions $f(t)$.

Let $q(t, j)$ belong to $L_{\infty*}$. For each $z(t, j)$ in H_0, let z_j be the function on $T\times J$ defined by $z_j(t, k)=z(t, j)$ if $k=j$, $z_j(t, k)=0$ otherwise. By Lemma 9.1, the vector $y=\psi(z)$ of H is expressed as the sum of integrals, each of which is a vector in the correspondingly labeled subspace $H(x_j)$. Hence

$$P_j y = \int z(t, j)d\,\mathbf{m}_j(t) = \psi(z_j) \qquad (j \in J).$$

From this and (**), with the definition of ψ_0,

$$\left[\int q(t, j)d\pi\right] P_j y = \psi_0(q(t, j)^\times)\psi(z_j)$$

$$= \psi(q(t, j)^\times z_j) \qquad (j \in J).$$

Summing over all j in J yields

$$q(B_w)y = \psi(q^\times z),$$

or

$$\psi^{-1}(q(B_w)\psi(z)) = q^\times z;$$

that is, $q(B_w)=\psi_0(q^\times)$. This last is an arbitrary member of the algebra $\psi_0(\mathfrak{M})$. By Theorem 11.1 this algebra is weakly closed, and it clearly contains all the B_w, so it contains all of $\mathfrak{B}$. Thus each operator in B can be represented in the form $q(B_w)$ for some q in $L_{\infty*}$, and the proof is complete.

As a corollary, if the spectrum has countable multiplicity (that is, if J is countable) the set of operators $f(B_w)$ with f essentially bounded, measurable and complex-valued on T is identical with the algebra $\mathfrak{B}$.

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Added in proof:

6. A. C. Zaanen, *The Radon-Nikodym theorem*, Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 156–187.

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