NONEXISTENCE OF LOW DIMENSION RELATIONS BETWEEN STIEFEL WHITNEY CLASSES(1)

BY

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1. **Introduction.** Let $\mathcal{W}_m$ be the polynomial algebra $Z_2[w_1, w_2, \cdots, w_m]$ where $w_1, w_2, \cdots, w_m$ are indeterminants and Z_2 denotes the integers modulo two. We make $\mathcal{W}_m$ into a graded algebra by the condition: dim $w_i = i$. For each differentiable m-manifold M let

$$\psi_M: \mathcal{W}_m \rightarrow H^*(M; Z_2)$$

be the algebra homomorphism defined by $\psi_M(w_i) = W_i$ where W_i is the ith Stiefel Whitney class of M. If $\mathfrak{M}_m$ is some class of differentiable m-manifolds let

$$I(\mathfrak{M}_m) = \bigcap \ker \psi_M: M \in \mathfrak{M}_m.$$

Thus $I(\mathfrak{M}_m)$ is the ideal of polynomial relations satisfied by the Stiefel Whitney classes of all the manifolds in $\mathfrak{M}_m$. Let $\mathfrak{N}_m$ be the class of C^∞, compact, m-manifolds without boundary. An unsolved problem is the determination of $I(\mathfrak{N}_m)$ for each m. If A is a graded algebra, A_q will denote its elements of dimension q. By dimensional considerations, $I(\mathfrak{M}_m)_q = \mathcal{W}_{m,q}$ for $q > m$. In [2] Dold has shown that all the elements of $I(\mathfrak{N}_m)_m$ can be obtained from the Wu formulas. For low values of m one can also show that all the elements of $I(\mathfrak{N}_m)$ come from the Wu formulas (the writer has checked this for $m \leqq 5$), but it is an unsolved problem whether this is true in general. In this paper we prove a conjecture of Dold, namely:

THEOREM I. *$I(\mathfrak{N}_m)_q = 0$, if $2q \leqq m$.*

In fact we prove the following:

LEMMA 1.1. *For each integer m there is a compact, C^∞ m-manifold M (M may be chosen with or without boundary) such that*

$$\psi_M: \mathcal{W}_{m,q} \rightarrow H^q(M; Z_2)$$

is an injection for $2q \leqq m$.

The construction of M satisfying 1.1 makes use of the following construction which the writer understands is similar to one defined by Barry Mazur. Recall the mapping cylinder C_f of a continuous function $f: X \rightarrow Y$ is the identification space formed from $X \times [0, 1] \cup Y$ by identifying $(x, 1)$ and $f(x)$

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for each $x \in X$. We also identify x and $(x, 0)$ so that $X \subset C_f$. Let K be a CW complex. A C^∞ manifold M with boundary N will be called a *tubular neighborhood* of K if there is a continuous function $F: N \to K$ such that C_F is the underlying topological space of M, the inclusion map of $N \times [0, 1)$ into M is differentiable of class C^∞ and the inclusion of K in M is a C^∞ embedding on each open cell of K.

If X is a topological space, $A \subset X$, $i: A \to X$ is the inclusion map and ξ is a bundle over X, then the induced bundle over A, $i^*\xi$, will be denoted by $\xi | A$. For any differentiable manifold M, $\tau(M)$ will denote its tangent bundle.

THEOREM II. *If K is a finite, n dimensional CW complex, η is a real m-plane bundle over K and $2n \leqq m$, then there is a finite CW complex K' and a homotopy equivalence $g: K' \to K$ such that K' has an m-dimensional tubular neighborhood M for which $g^*\eta$ and $\tau(M) | K'$ are equivalent.*

As will be seen in the proof of Theorem II, roughly speaking, K' is obtained from K by deforming the attaching maps of the cells of K.

In §3 we show that 1.1 follows from II by taking K to be $[m/2]$ skeleton of BO_m, the classifying space of the orthogonal group O_m, and η to be the m-plane bundle over K induced from the canonical m-plane bundle over BO_m.

REMARK. Let $\mathfrak{M}_m^0$ be the subclass of $\mathfrak{M}_m$ consisting of the orientable manifolds. By the same techniques as are used to prove Theorem I one can show that

$$I(\mathfrak{M}_m^0)_q = \{w_1\}_q$$

if $2q \leqq m$ and $\{w_1\}$ is the ideal generated by w_1. One uses the same argument as is used for Theorem I except that one uses BSO_m in place of BO_m.

2. **Proof of Theorem II.** We first show that if M_1 is a compact tubular neighborhood of K_1 and M_2 is obtained by attaching a handle to M_1, then M_2 is a tubular neighborhood of a complex K_2 obtained from K_1 by attaching a cell to K_1.

Let R be the real numbers and let

$$R^n = R \times R \times \cdots \times R \quad (R \text{ taken } n \text{ times}),$$
$$|(x_1, x_2, \cdots, x_n)| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2},$$
$$E^n = \{x \in R^n \mid |x| \leqq 1\},$$
$$S^{n-1} = \{x \in R^n \mid |x| = 1\},$$
$$\tilde{E}^n = E^n - S^n.$$

Suppose M_1 is a compact tubular neighborhood of a CW complex K_1, dim $M_1 = m$ and $M_1 = C_{F_1}$ where $N_1 = \partial M_1$ and $F_1: N_1 \to K_1$. Choose a Riemannian metric on N_1. Suppose $f: S^p \to N_1$ is a C^∞ embedding such that the normal bundle ν of $f(S^p)$ in N_1 is trivial. Choose linearly independent cross-sections v_i, $1 \leqq i \leqq m-p-1$, of ν so that the map

$$\bar{h}: S^p \times E^{m-p-1} \to N_1$$

given by

$$\bar{h}(x, y_1, y_2, \cdots, y_{m-p-1}) = \exp_{f(x)} \sum y_i v_i(f(x))$$

is a homeomorphism. Let $I = [0, 1]$ and let

$$h: S^p \times E^{m-p-1} \times I \to M_1$$

be given by

$$h(x, y, t) = (\bar{h}(x, y), t).$$

Roughly speaking, attaching $E^{p+1} \times E^{m-p-1}$ to M_1 as a handle consists in pasting $S^p \times E^{m-p-1} \subset E^{p+1} \times E^{m-p-1}$ to N_1 via $\bar{h}$. This is more or less what we do except that we enlarge $S^p \times E^{m-p-1}$ a little in order to smooth out corners and keep track of the mapping cylinder structure. To do this we first paste $\tilde{E}^{p+1} \times E^{m-p-1}$ to $S^p \times E^{m-p-1} \times I$ and then paste the result to M_1 via h. This first step we do for $p=1$ and $m=2$ and then obtain the construction for arbitrary p and m by rotating everything in sight. The whole construction can be understood if one draws a careful picture for $p=1$ and $m=2$ (see Figure 1).

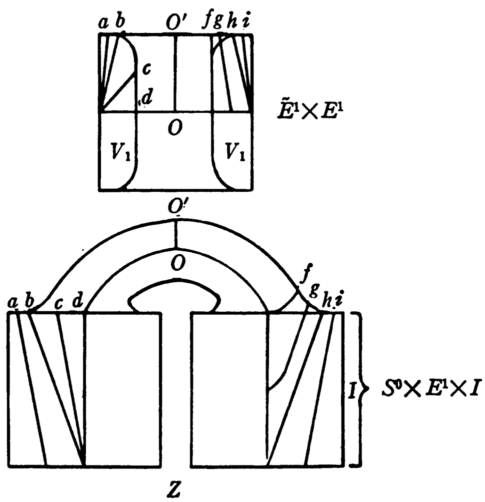

FIGURE 1

Let $\alpha, \beta, \gamma: R \to R$ be C^∞ functions with the following properties: α is even, increasing on $t \geqq 0$, $\alpha(0) = 1/2$ and $\alpha(t) = t$ for $t \geqq 3/4$. β is odd, increasing on $|t| \leqq 3/4$, $\beta'(0) = 1$ and $\beta(t) = 1$ for $t \geqq 3/4$. γ is odd, increasing on $t \geqq 3/4$, $\gamma(1) = 1$ and $\gamma(t) = 0$ for $|t| \leqq 3/4$. Consider the curve $c: R \to R \times R$ defined by $c(t) = (\alpha(t), \beta(t))$. c is a "$\subset$" shaped curve with arms $x \geqq 3/4$, $y = \pm 1$ and passing through $(1/2; 0)$. Let $V_1 \subset \tilde{E}^1 \times E^1$ be all points p of the form:

$$p = s((-1)^i, 0) + (1 - s)((-1)^i\alpha(t), \beta(t))$$

where $i=0, 1, 0 \leqq s < 1$ and $|t| < 1$. V_1 is the intersection of $\tilde{E}^1 \times E^1$ and the "inside" of c together with reflection of this set through the y axis. Let μ_1: $V_1 \rightarrow S^0 \times E^1 \times I$ be defined as follows:

$$\mu_1(p) = ((-1)^i, s\gamma(t) + (1 - s)t, s)$$

where p is given as above.

The following observation may be helpful in understanding the formulas to follow. Let $Z = (S^0 \times E^1 \times I) \cup (\tilde{E}^1 \times E^1)$ with $\mu_1(x, y)$ and (x, y) identified for each $(x, y) \in V_1$. Then the following line segments define a mapping cylinder structure on Z.

$$(1 - t)(x, \pm 1) + t(\beta(x), 0) \in \tilde{E}^1 \times E^1, \qquad 0 \leqq t \leqq 1, x \in \tilde{E}^1,$$
$$(\pm 1, \pm 1, t) \in S^0 \times E^1 \times I, \qquad 0 \leqq t \leqq 1.$$

In Figure 1 we indicate how $\tilde{E}^1 \times E^1$ is pasted to $S^0 \times E^1 \times I$ to form Z. The lines on the left indicate how μ_1 maps V_1 into $S^0 \times E^1 \times I$, namely the lines above go into the lines below linearly. The curves on the right indicate the mapping cylinder structure.

Let

$$V = \{(x, y) \in \tilde{E}^{p+1} \times E^{m-p-1} \mid (|x|, |y|) \in V_1\}.$$

Let μ: $V \rightarrow S^p \times E^{m-p-1} \times I$ be defined as follows:

$$\mu(x, y) = \left(\frac{x}{|x|}, u\frac{y}{|y|}, t\right)$$

where $\mu_1(|x|, |y|) = (1, u, t)$. If $y=0$, take $uy/|y|$ to be 0.

Let $M_2 = \mathfrak{M}(M_1, f, v_1, v_2, \cdots, v_{m-p-1})$ be the identification space formed from $M_1 \cup \tilde{E}^{p+1} \times E^{m-p-1}$ by identifying (x, y) and $h\mu(x, y)$ for each $(x, y) \in V$. M_2 is easily seen to be a C^∞ manifold with boundary N_2 where

$$N_2 = (N_1 - \bar{h}(S^p \times \tilde{E}^{m-p-1})) \cup \tilde{E}^{p+1} \times S^{m-p-2}.$$

The only questionable behavior goes on around K_1 and here we already know it is locally Euclidean. Let $K_2 = K_1 \cup E^{p+1}$ with x and $Ff(x)$ identified for each $x \in S^p$. Embed E^{p+1} in $E^{p+1} \times E^{m-p-1}$ by identifying x and $(x, 0)$. This embedding together with the embedding of K_1 in M_1 embeds K_2 in M_2. Let F_2: $N_2 \rightarrow K_2$ be defined as follows:

$$F_2(u) = F_1(u), \qquad u \in N_1 - \bar{h}(S^p \times \tilde{E}^{m-p-1}),$$

$$F_2(x, y) = \begin{cases} \beta(|x|)\dfrac{x}{|x|}, & |x| \leqq 3/4, \\ h\left(\dfrac{x}{|x|}, \gamma(|x|)y, 1\right) & 3/4 \leqq |x| < 1; \end{cases} \quad (x, y) \in \tilde{E}^{p+1} \times S^{m-p-2}.$$

Finally let $g\colon C_{F_2}\to M_2$ be defined as follows:

$$g(u,t)=(u,t)\quad 0\leqq t\leqq 1\quad u\in N_1-\bar{h}(S^p\times \tilde{E}^{m-p-1})$$

$$g(x,y,t)=\begin{cases}(1-t)(x,y)+t\left(\dfrac{x}{|x|},\gamma(|x|)y\right), & t<1,\\ F_2(x,y),\quad t=1; & (x,y)\in \tilde{E}^{p+1}\times S^{m-p-2}.\end{cases}$$

Although somewhat tedious, it is fairly easy to check that F_2 and g are well defined, g is a homeomorphism, $g|N_2\times[0,1)$ is C^∞ and the inclusion map of $\tilde{E}^{p+1}$ in M_2 is a C^∞ embedding. Hence we have:

LEMMA 2.1. *$M_2=\mathfrak{M}(M_1,f,v_1,v_2,\cdots,v_{m-p-1})$ is a tubular neighborhood of K_2.*

We next show how the tangent bundle of M_2 can be controlled by the choice of the v_i. Suppose K_i, M_i, f and ν are as above.

LEMMA 2.2. *If η is a real m-plane bundle over K_2, $\eta|K_1$ and $\tau(M_1)|K_1$ are equivalent and* 2 dim *$K_2\leqq m$, then ν is trivial and $v_1, v_2, \cdots, v_{m-p-1}$ can be chosen so that η and $\tau(M_2)|K_2$ are equivalent.*

Proof. Let $GL(n)$ be the group of $n\times n$ nonsingular, real matrices. Let $\lambda\colon GL(n)\to GL(n+k)$ be given by

$$\lambda(A)=\begin{pmatrix}I_k & 0\\ 0 & A\end{pmatrix}$$

where I_k is the $k\times k$ identity matrix and 0 is the appropriate zero matrix. It is well known [4] that

$$\lambda_*\colon \pi_i(GL(n))\to\pi_i GL(n+k))$$

is 1-1 for $i<n-1$ and onto for $i\leqq n-1$.

Let

$$\bar{E}^p=\{x\in E^p\mid |x|\leqq 1/2\},$$
$$\bar{S}^{p-1}=\{x\in E^p\mid |x|=1/2\}.$$

Let $K'=K_2-\text{Int }\bar{E}^p$. We embed K' in M_1 by identifying x and $(f(x/|x|),2|x|-1)\in C_{F_1}$ for each $x\in E^{p+1}-\text{Int }\bar{E}^{p+1}$. Note that under this identification $\bar{S}^p=f(S^p)$ and hence ν is the normal bundle of $\bar{S}^p$ in N_1. Since K_1 is a deformation retract of K' and $\tau(M_1)|K_1$ and $\eta|K_1$ are equivalent by assumption, there is a bundle equivalence $\psi\colon\tau(M)|K'\to\eta|K'$. Therefore $\tau(M_1)|\bar{S}^p$ and $\eta|\bar{S}^p$ are equivalent. But $\eta|\bar{S}^p$ is trivial since η lies over K_2 and hence $\tau(M_1)|\bar{S}^p$ is trivial.

$$\tau(M_1)\mid\bar{S}^p=\tau(\bar{S}^p)+\xi+\nu$$

where ξ is the trivial line bundle consisting of the vectors in $\tau(M_1)|\bar{S}^p$ normal to N_1. As is well known, $\tau(S^p)$ plus a trivial line bundle is trivial. Let $a\in\pi_{p-1}(GL(m))$ and $b\in\pi_{p-1}(GL(m-p-1))$ represent the characteristic functions of $\tau(M_1)|\bar{S}^p$ and ν, respectively [4]. Then $\lambda_*b=a$. Since $p-1<m-p-2$, λ_* is $1-1$. $a=0$ and hence $b=0$. Therefore ν is trivial.

Recall $V\subset\tilde{E}^{p+1}\times E^{m-p-1}$ and $h\mu\colon V\to M_1$ is the map used to paste $\tilde{E}^{p+1}\times E^{m-p-1}$ to M_1 to form M_2. Let $d(h\mu)\colon\tau(V)\to\tau(M_1)$ denote the differential of $h\mu$. Identify $x\in E^{p+1}$ and $(x, 0)\in E^{p+1}\times E^{m-p-1}$ so that $E^{p+1}\subset E^{m-p-1}$. Note that $\tau(M_2)|K_2$ may be viewed as the union of $\tau(M_1)|K'$ and $\tau(E^{p+1}\times E^{m-p-1})|\bar{E}^{p+1}$ with $v\in\tau(E^{p+1}\times E^{m-p-1})|\bar{S}^p$ and $d(h\mu)(v)\in\tau(M_1)|\bar{S}^p$ identified. Let X_i and Y_i be the vector fields on $\tilde{E}^{p+1}\times E^{m-p-1}$ given as follows:

$$X_i = \frac{\partial}{\partial x_i}, \qquad 1 \leqq i \leqq p+1,$$

$$Y_i = \frac{\partial}{\partial y_i}, \qquad 1 \leqq i \leqq m-p-1,$$

where $(x_1, x_2, \cdots, x_{p+1}, y_1, y_2, \cdots, y_{m-p-1})$ are the cartesian coordinates. Under the embedding of K' in M_1, $h\mu|(\tilde{E}^{p+1}-\operatorname{Int}\bar{E}^{p+1})$ is simply the inclusion map. Hence $\{d(h\mu)X_i|\bar{S}^p\}$ is independent of the choice of the v_i's and also provides linearly independent cross-sections of $\nu+\xi$. Let $Z_i=d(h\mu)X_i|\bar{S}^p$. $\{Z_i, v_j\}$ are then linearly independent cross-sections of $\tau(M_1)|\bar{S}^p$. One may also check that

$$d(h\mu)Y_i|\bar{S}^p = v_i. \tag{2.3}$$

Thus varying the choice of the v_i's varies the way in which $\tau(\tilde{E}^{p+1}\times E^{m-p-1})|\bar{E}^p$ is pasted to $\tau(M_1)|K'$. On the other hand, up to equivalence, any extension of $\tau(M_1)|K'$ to a bundle over K_2 may be obtained by pasting $\tau(\tilde{E}^{p+1}\times E^{m-p-1})|\bar{E}^{p+1}$ to $\tau(M_1)|K'$ via a bundle map of $\tau(\tilde{E}^{p+1}\times E^{m-p-1})|\bar{S}^p$ onto $\tau(M_1)|\bar{S}^p$. Hence we must show that up to bundle equivalence, these two ways of pasting are equivalent. Let $\bar{v}_i$, $1\leqq i\leqq m-p-1$ be linearly independent cross-sections of ν. Let $\bar{w}_i$, $1\leqq i\leqq m$ be linearly independent cross-sections of $\eta|\bar{E}^{p+1}$ which exist since $\bar{E}^{p+1}$ is contractible. $\bar{w}_i|\bar{S}^p$ and $\{\psi Z_i, \psi\bar{v}_i\}$ both give linearly independent cross-sections of $\eta|\bar{S}^p$. Therefore there is a map $B\colon\bar{S}^p\to GL(m)$ such that

$$\bar{w}_i(x) = \sum_{j=1}^{p+1} B_{ij}(x)\psi Z_i(x) + \sum_{j=p+2}^{m} B_{ij}(x)\psi\bar{v}_{j-p+1}(x)$$

for $x\in\bar{S}^p$. $\lambda_*\colon\pi_p(GL(m-p-1))\to\pi_p(GL(m))$ is onto since $p<m-p-1$. Therefore there is a map $C\colon\bar{S}^p\to GL(m-p-1)$ and a map $D\colon\bar{E}^{p+1}\to GL(m)$ such that

$$D(x) = (\lambda C(x))B(x)^{-1}, \qquad x\in\bar{S}^p.$$

Let

$$w_i(x) = \textstyle\sum_{j\,\epsilon} D_{ij}(x)\bar{w}_j(x),$$
$$v_i(x) = \textstyle\sum_{j\,\epsilon} C_{ij}(x)\bar{v}_j(x),$$

where $\epsilon > 0$ is chosen so that $h \mid S^p \times E^{m-p-1}$ is a homeomorphism. Then

$$(2.4)\qquad \begin{aligned} w_i \mid \bar{S}^p &= \psi Z_i, & 1 \leqq i \leqq p+1,\\ &= \psi v_{i-p-1}, & p+2 \leqq i \leqq m. \end{aligned}$$

Let $\phi: \tau(M_2) \mid K_2 \to \eta$ be the bundle map defined as follows:

$$\begin{aligned} \phi(v) &= \psi(v), & v \in \text{total space of } \tau(M_2) \mid K',\\ \phi(X_i(x)) &= w_i(x), & 1 \leqq i \leqq p+1,\ x \in \bar{E}^p,\\ \phi(Y_i(x)) &= w_{i+p+1}(x), & 1 \leqq i \leqq m-p-1,\ x \in \bar{E}^p. \end{aligned}$$

By (2.3) and (2.4) ϕ is well defined. Therefore $\tau(M_2) \mid K_2$ and η are equivalent.

Proof of Theorem II. Let K be a finite, n-dimensional CW complex and let $r(K)$ be the number of cells in K of positive dimension. We prove by induction on r, if η is real m-plane bundle over K and $2n \leqq m$, then there is a finite, n-dimensional CW complex K' and a homotopy equivalence $g: K' \to K$ such that $r(K') = r(K)$ and K' has a tubular neighborhood M for which $\tau(M) \mid K'$ and $g^*\eta$ are equivalent.

If $r = 0$, let $K' = K$, $g =$ identity and $M = C_F$ where $F: S^{m-1} \times K \to K$ is the projection. M is then a disjoint union of closed m-cells and η and $\tau(M) \mid K$ are both trivial.

Suppose $r(K) > 0$. Then $K = L \cup \tilde{E}^{p+1}$ where $r(L) = r(K) - 1$, $2 \dim L \leqq m$ and $2(p+1) \leqq m$. By inductive hypothesis there is a finite CW complex L', a homotopy equivalence $k: L' \to L$ and a tubular neighborhood N of L' such that $\dim L' = \dim L$, $r(L') = r(L)$ and $\tau(N) \mid L'$ and $k^*(\eta \mid L)$ are equivalent. Let $f': S^p \to L$ be the map by which E^{p+1} is attached to L to form K. Let $\bar{k}$ be a homotopy inverse of k. By a simple variation of 11A [5] one may show that because $\dim L + p < n$, $\bar{k}f'$ is homotopic in N to a map $f'': S^p \to N$ such that $f''(S^p) \cap L' = 0$. Since $N = C_F$ where $F: \partial N \to L'$, f'' is homotopic in N to a map $f''': S^p \to \partial N$. Since $2p+1 \leqq \dim \partial N$, f''' is homotopic to a C^∞ embedding $f: S^p \to \partial N$ [6]. Let $K' = L' \cup \tilde{E}^{p+1}$ with $\tilde{E}^{p+1}$ attached by $Ff: S^p \to L'$. f and $\bar{k}f'$ are homotopic in N and F can be extended to a deformation retraction of N into L'. Therefore Ff and $\bar{k}f'$ are homotopic in L' and hence kFf and f' are homotopic in L. Let $H: S^p \times I \to L$ be a homotopy between kFf and f'. Let $g: K' \to K$ be defined as follows:

$$\begin{aligned} g(t) &= k(t), & t \in L',\\ g(x) &= H\left(\frac{x}{|x|},\ 2|x| - 1\right), & 1/2 \leqq |x|,\ x \in E^{p+1},\\ &= 2x, & |x| \leqq 1/2,\ x \in E^{p+1}. \end{aligned}$$

One may easily check that g is a homotopy equivalence. $(g^*\eta)|L' = k^*(\eta|L)$ which in turn is equivalent to $\tau(N)|L'$. Therefore, by Lemma 2.2, $M = \mathfrak{M}(N, f, v_1, v_2, \cdots, v_{m-p-1})$, for some v_i, is a tubular neighborhood of K' such that $\tau(M)|K'$ and $g^*\eta$ are equivalent.

3. **Proof of Theorem I.** Let BO_m be the classifying space of the orthogonal group O_m and let ξ be the canonical m-plane bundle over BO_m. Let K be the $[m/2]$ skeleton of BO_m with respect to the usual cell decomposition of BO_m [3] and let $\eta = \xi|K$. According to Theorem II there is a CW complex K', a homotopy equivalence $g: K' \to K$ and a tubular neighborhood M of K' such that $\tau(M)|K'$ and $g^*\eta$ are equivalent. Let $i: K \to BO_m$ and $j: K' \to M$ be the inclusion maps and let $f: M \to BO_m$ be the classifying map of $\tau(M)$. Since $(ig)^*\xi = g^*\eta$ and $(fj)^*\xi = \tau(M)|K'$ are equivalent, ig and fj are homotopic. Consider the commutative diagram:

$$\begin{array}{ccc} H^q(BO_m; Z_2) & \xrightarrow{i^*} & H^q(K; Z_2) \\ \downarrow f^* & & \downarrow g^* \\ H^q(M; Z_2) & \xrightarrow{j^*} & H^q(K'; Z_2). \end{array}$$

g^* and j^* are isomorphisms because g and j are homotopy equivalences. i^* is an injection for $2q \leqq m$ by the way K was constructed. Therefore f^* is an injection for $2q \leqq m$. But $H^*(BO_m, Z_2)$ is a polynomial algebra generated by $w_1, w_2, \cdots, w_m$, the Whitney classes of ξ [1]. Furthermore, $f^*(w_i) = W_i$, the ith Stiefel Whitney class of M. Hence M has the desired properties.

To obtain a manifold without boundary let V be M doubled, that is, two copies of M pasted together along their boundaries. Let $k: M \to V$ be the homeomorphism taking M onto one of its copies. Then $k^*\tau(V) = \tau(M)$. Consider the diagram:

$$\begin{array}{ccc} W_{m,q} & \xrightarrow{\psi_V} & H^q(V, Z_2) \\ & \psi_M \searrow & \downarrow k^* \\ & & H^q(M, Z_2). \end{array}$$

This diagram is commutative because $k^*W_i(V) = W_i(M)$. Therefore ψ_V is an injection for $2q \leqq m$ since ψ_M is an injection in this range and $k^*\psi_V = \psi_M$.

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