

ON A THEOREM OF D. RIDOUT IN THE THEORY OF DIOPHANTINE APPROXIMATIONS

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1. **Introduction.** The following extension of the Thue-Siegel-Roth Theorem [5] was proved by D. Ridout [4].

THEOREM I. *Let α be any algebraic number not 0. Let $\{P_1, \dots, P_s\}$ and $\{Q_1, \dots, Q_t\}$ be sets of primes. Let μ, ν, c be real numbers satisfying*

$$(1.1) \quad 0 \leq \mu \leq 1, \quad 0 \leq \nu \leq 1, \quad c > 1.$$

Let p, q be restricted to integers of the form

$$(1.2) \quad p = p^* P_1^{\rho_1} \dots P_s^{\rho_s}, \quad q = q^* Q_1^{\sigma_1} \dots Q_t^{\sigma_t},$$

where $\rho_1, \dots, \rho_s, \sigma_1, \dots, \sigma_t$ are non-negative integers and p^, q^* are any integers satisfying*

$$(1.3) \quad |p^*| < c |p|^\mu, \quad 0 < q^* < c q^\nu.$$

Then the inequality

$$(1.4) \quad 0 < |\alpha - p/q| < q^{-(\mu+\nu+\zeta)}$$

has only a finite number of solutions in p, q for every $\zeta > 0$.

In § 2 we show that the case of Theorem I is the usual case, in that inequality (1.4) has infinitely many solutions for almost no real α . In fact, we shall prove a stronger result. Let C_1 and C_2 be two classes of positive integers p' and q' with the properties

$$\sum_{p' \in C_1} (p')^{-\varepsilon} < \infty, \quad \sum_{q' \in C_2} (q')^{-\varepsilon} < \infty$$

for every $\varepsilon > 0$. Let r be a positive integer, μ, ν, c satisfy (1.1), and let

$$(1.5) \quad p_i = p_i^* p', \quad q = q^* q', \quad p' \in C_1, \quad q' \in C_2,$$

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where p_i^*, q^* are integers satisfying

$$(1.6) \quad |p_i^*| < c |p_i|^\mu, \quad 0 < q^* < cq^v.$$

We prove

THEOREM II. *Let $\zeta > 0$. The set of inequalities*

$$|\alpha_i - p_i/q| < q^{-(\mu+v/r+\zeta)} \quad (i = 1, \dots, r)$$

has infinitely many solutions in vectors $(p_1/q, \dots, p_r/q)$, p_i, q of the form (1.5), for almost no real vectors $(\alpha_1, \dots, \alpha_r)$.

The assertion that (1.4) has only a finite number of solutions for almost all real α , is what Theorem II specializes to for $r = 1$, and p_i, q of the form

$$(1.7) \quad p_i = p_i^* P_1^{\rho_1} \dots P_s^{\rho_s}; \quad q = q^* Q_1^{\sigma_1} \dots Q_t^{\sigma_t}$$

Indeed, in this case C_1 and C_2 consist of integers of the form

$$p' = P_1^{\rho_1} \dots P_s^{\rho_s}; \quad q' = Q_1^{\sigma_1} \dots Q_t^{\sigma_t},$$

and $\sum_{p' \in C_1} (p')^{-\varepsilon} = \prod_1^s (1 - P_i^{-\varepsilon})^{-1} < \infty$, and similarly $\sum_{q' \in C_2} (q')^{-\varepsilon} < \infty$.

On the other hand, there are clearly infinitely many (transcendental numbers for which (1.4) has infinitely many solutions. Thus, numbers of the form

$$\alpha = \sum_{j=0}^{\infty} 10^{-i_j},$$

where $i_j - i_{j-1} \geq 2\zeta i_{j-1}$ for infinitely many j , $i_0 = 1$, $\zeta > 0$, are transcendental, because the rationals

$$p_k/q_k = \sum_{j=0}^k 10^{-i_j}$$

satisfy (1.4) with $\mu = 1$, $v = 0$ for infinitely many k .

In §2 we also prove

THEOREM III. *Let $k, \eta > 0$, and let $0 \leq \mu < 1$, $0 < v \leq 1$, $c > 1$, r a positive integer, v and r not both 1. Let p_i, q be restricted to integers of the form (1.5), where p_i^*, q^* are integers satisfying*

$$|p_i^*| < |p_i|^{\mu-\eta}, \quad 0 < q^* < cq^v \quad (i = 1, \dots, r).$$

Then the set of inequalities

$$|\alpha_i - p_i/q| < 1/kq(p')^{(\mu+v/r-1)/(1-\mu)} \quad (i = 1, \dots, r)$$

has infinitely many solutions in vectors $(p_1/q, \dots, p_r/q)$ for almost no real vectors $(\alpha_1, \dots, \alpha_r)$.

In §3 we prove Theorems IV and V below. Theorem III is used as a lemma in the proof of Theorem V.

THEOREM IV. *Let $\alpha_1, \dots, \alpha_r$ be real numbers, and let*

$$(1.8) \quad S_1 = \{P_1, \dots, P_s\}, \quad S_2 = \{Q_1, \dots, Q_t\}$$

be sets of primes. Let $K > 0$, and let μ, ν satisfy (1.1), and either

$$\mu + \nu/r \geq 1$$

or

$$\nu = 0$$

or

$$\mu = 0.$$

Then there is a constant $c > 1$ depending on $K, \max |\alpha_i|, \mu, \nu, r$ and the primes in S_1 and S_2 , such that the set of inequalities

$$(1.9) \quad |\alpha_i - p_i/q| < Kq^{-(\mu+\nu/r)} \quad (i = 1, \dots, r)$$

has infinitely many solutions in vectors $(p_1/q, \dots, p_r/q)$, p_i, q of the form (1.7) subject to (1.6), with the provision that $K = 1$ if $\mu = \nu = 1$.

Let $r = 1$. If α is irrational, (1.9) can be replaced by the stronger inequality

$$(1.10) \quad 0 < |\alpha - p/q| < Kq^{-(\mu+\nu)}.$$

If however $\alpha = a/b$ is rational, (1.10) clearly has only a finite number of solutions if $\mu + \nu > 1$. Similarly, if $\mu + \nu = 1$, $K \leq 1/|b|$, then $0 < |\alpha - p/q| < Kq^{-1}$ is satisfied only finitely often⁽²⁾. But for $\mu = 0$, $0 \leq \nu < 1$ or $\nu = 0$, $0 \leq \mu < 1$, (1.10) has infinitely many solutions under certain conditions also when α is rational. Thus, Theorem IV with $\alpha = 1$ states for the case $\mu = \nu = 0$, that

$$0 < |P^\rho - Q^\sigma| < Q^\sigma$$

for infinitely many positive integers ρ, σ if $P \neq Q$. Hence by Ridout's Theorem I, if $\varepsilon > 0$, we have actually

$$Q^{\sigma(1-\varepsilon)} < |P^\rho - Q^\sigma| < Q^\sigma$$

infinitely often.

THEOREM V. *Let $\alpha_1, \dots, \alpha_r$ be real numbers and let $\delta > 0$. Let μ, ν satisfy*

$$(1.11) \quad 0 < \mu < 1, \quad 0 < \nu < 1,$$

$$(1.12) \quad \nu \leq \mu + \nu/r < 1.$$

(2) For the cases $\mu + \nu > 1$ and $\mu + \nu = 1$, $K \leq 1/|b|$, the theorem thus states in particular, that given any rational a/b , there are infinitely many rationals $p/q = a/b$, where p, q are of the form (1.2) subject to (1.3).

Then if $S_1 \cap S_2 \neq 0$, the result of Theorem IV holds. If μ, ν satisfy

$$0 < \mu < 1, \quad 0 < \nu \leq 1$$

$$\nu > \mu + \nu/r,$$

or if $S_1 \cap S_2 = 0$, $\mu + \nu/r < 1$, then the set of inequalities

$$|\alpha_i - p_i/q| < q^{-(\mu + \nu/r - \delta)} \quad (i = 1, \dots, r)$$

has infinitely many solutions in vectors $(p_1/q, \dots, p_r/q)$, p_i, q of the form (1.7) subject to (1.6), for almost all real vectors $(\alpha_1, \dots, \alpha_r)$.

The specializations $r = 1$, α algebraic not 0 of Theorems IV and V show that Ridout's Theorem I is best possible for the case $S_1 \cap S_2 \neq 0$. We do not know at present whether the set of excluded vectors $(\alpha_1, \dots, \alpha_r)$ in the case $S_1 \cap S_2 = 0$ or $\nu > \mu + \nu/r$ is empty or not.

In the final §4 we give an upper bound to the number of solutions of (1.4). Using a method of Davenport and Roth [2], we prove

THEOREM VI. *Let*

$$(1.13) \quad f(x) = x^n + a_1 x^{n-1} + \dots + a_n,$$

a_i (rational) integers, be the defining polynomial of α , and let $0 < \zeta \leq 1/3$. Then (1.4) has less than

$$D(D+2) +$$

$$(2\zeta^{-1} \log C + \exp(280(n+1)^2 \zeta^{-2})) \sum_{i=1}^s \binom{u+s+i-1}{s-1} \sum_{t=1}^t \binom{u+t+i-1}{t-1}$$

solutions in integers p, q of the form (1.2) with $(p, q) = 1$, where ⁽³⁾

$$(1.14) \quad D = 2^{90/\zeta} + (2/|\alpha|)^{90/\zeta} + c^{8100/\zeta(90-\zeta)},$$

$$(1.15) \quad C = 3 + \log(1 + |\alpha|) + 2 \log(1 + A),$$

$$A = \max(|a_1|, \dots, |a_n|),$$

$$u = [90 \max(s, t)/\zeta] + 1.$$

A corresponding result for algebraic numbers which are not integers, is contained in the following

THEOREM VII. *Let β satisfy the irreducible polynomial*

$$g(x) = b_0 x^n + b_1 x^{n-1} + \dots + b_n, \quad b_i \text{ (rational) integers, } b_0 \geq 2.$$

Then if

$$0 < \xi \leq 2/3,$$

the inequality

⁽³⁾ $[x]$ stands for the integral part of x , and $\{x\}$ for the fractional part.

$$(1.16) \quad 0 < |\beta - p/q| < q^{-(\mu+\nu+\zeta)}$$

has less than

$$F(F+2) + (4\xi^{-1} \log \log(1+2B^n) + \exp(1122(n+1)^2 \xi^{-2})) \sum_{i=1}^s \binom{u+s+i-1}{s-1} \\ \cdot \sum_{i=1}^t \binom{u+t+i-1}{t-1}$$

solutions in integers p, q of the form (1.2) with $(p, q) = 1$, where

$$B = \max(|b_0|, \dots, |b_n|), \\ F = 2^{180/\xi} + (2/b_0 |\beta|)^{180/\xi} + 2(cb_0)^{1802/\xi(180-\xi)}.$$

2. **Proof of Theorems II and III.** The proof is preceded by two lemmas.

LEMMA 1. Let $N(x)$ be the number of $p' \leq x$, $p' \in C_1$. Then $N(x) = O(x^\varepsilon)$ for every $\varepsilon > 0$.

Proof. If $\varepsilon_0 > 0$ has the property that $N(x) > x^{\varepsilon_0}$ for arbitrarily large x , then clearly $N(p_k') > (p_k')^{\varepsilon_0}$ for arbitrarily large $p_k' \in C_1$. Hence,

$$\sum_{p' \in C_1, p' \leq p_k'} (p')^{-\varepsilon_0/2} \geq (p_k')^{-\varepsilon_0/2} N(p_k') > (p_k')^{\varepsilon_0/2} \rightarrow \infty,$$

contradicting the convergence of the series $\sum (p')^{-\varepsilon}$.

LEMMA 2. Let $\{(x_{1j}, \dots, x_{rj})\}$ ($j = 1, 2, \dots$) be a sequence of real vectors, and let ϕ be a positive function defined on the sequence satisfying

$$\sum (\phi(x_{1j}, \dots, x_{rj}))^r < \infty.$$

Then the set of inequalities

$$|\alpha_i - x_{ij}| < \phi(x_{1j}, \dots, x_{rj}) \quad (i = 1, \dots, r; j = 1, 2, \dots)$$

has infinitely many solutions (x_{1j}, \dots, x_{rj}) for almost no real vectors $(\alpha_1, \dots, \alpha_r)$.

This is an immediate generalization of Lemma 1, Chapter VII of [1].

Proof of Theorem II. If we let $\{(x_{1j}, \dots, x_{rj})\} = \{(p_1/q, \dots, p_r/q)\}$, p_i, q of the form (1.5), then by Lemma 2 it is sufficient to show that

$$\sum_q \sum_{P_i} q^{-(\mu+\nu/r+\zeta)r} < \infty,$$

where the summation extends over all vectors $(p_1/q, \dots, p_r/q)$, p_i, q of the form (1.5). Let M be a positive number, and consider the expression

$$S = \sum_q \sum_{0 < p_i < Mq} q^{-(\mu+\nu/r+\zeta)r},$$

i.e., the summation extends only over those vectors which approximate numbers in the range $0 \leq \alpha_i \leq M$ ($i = 1, \dots, r$). It is sufficient to show that $S + S' < \infty$ for arbitrarily large M , where

$$S' = \sum_q \sum_{-Mq < p_i < 0} q^{-(\mu + \nu/r + \zeta)r}.$$

Let $M \geq 1$ be fixed for the present, and introduce the notation

$$S_1 = \sum_{p_i, i} q^{-r(\mu + \zeta) - \nu}, \quad S_2 = \sum_{p'} S_1, \quad S_3 = \sum_{q^*} S_2, \\ S \leq S_4 = \sum_{q'} S_3,$$

where p' , q are fixed in the first summation, q is fixed in the second and q' is fixed in the third summation.

We first consider the case

$$(2.1) \quad 0 \leq \mu < 1, \quad 0 \leq \nu < 1.$$

The first inequality of (1.6) can then be written in the form

$$(2.2) \quad |p_i^*| < c^{1/(1-\mu)} (p')^{\mu/(1-\mu)}.$$

Similarly, the second inequality of (1.6) is equivalent to

$$(2.3) \quad 0 < q^* < c^{1/(1-\nu)} (q')^{\nu/(1-\nu)}.$$

We may also assume, without loss of generality, that ζ is sufficiently small to satisfy

$$(2.4) \quad 0 < \delta = \zeta(r - \frac{1}{2}) < 1 - \nu.$$

Since $p_i < Mq$, we have

$$S_1 = \sum q^{-r(\mu + \zeta) - \nu},$$

summed over all p_i^* satisfying $0 < p_i^* < \min(c^{1/(1-\mu)} (p')^{\mu/(1-\mu)}, Mq/p')$, $1 \leq i \leq r$, or,

$$S_1 \leq q^{-r(\mu + \zeta) - \nu} (\min(c^{1/(1-\mu)} (p')^{\mu/(1-\mu)}, Mq/p'))^r.$$

We thus have two cases, depending upon the value of p' . Either

$$c^{1/(1-\mu)} (p')^{\mu/(1-\mu)} < Mq/p',$$

i.e.,

$$(2.5) \quad p' < c^{-1} (Mq)^{1-\mu},$$

or $p' \geq c^{-1}(Mq)^{1-\mu}$. Since $p' \leq p_i < Mq$ we have

$$(2.6) \quad c^{-1}(Mq)^{1-\mu} \leq p' < Mq$$

for the second case. Hence,

$$S_2 \leq q^{-r(\mu+\zeta)-\nu} \left(\sum c^{r/(1-\mu)} (p')^{r\mu/(1-\mu)} + \sum (Mq/p')^r \right),$$

where the first summation is over all p' satisfying (2.5), and the second over all p' satisfying (2.6). From this we obtain easily (for all sufficiently large q),

$$S_2 \leq 2c^r q^{-r(\mu+\zeta)-\nu} \sum (Mq)^{r\mu} \quad (p' < Mq).$$

By Lemma 1,

$$S_2 < q^{-r(\mu+\zeta)-\nu} (Mq)^{r\mu} O((Mq)^{\zeta/2}).$$

Thus,

$$(2.7) \quad S_2 < c_1 M^{r\mu+\delta} q^{-\nu-\delta},$$

where c_1 is a constant independent of q and M , and δ is given by (2.4). Hence by (2.3),

$$S_3 < c_1 M^{r\mu+\delta} (q')^{-\nu-\delta} \sum (q^*)^{-\nu-\delta} \quad (q^* < R),$$

where $R = c^{1/(1-\nu)} (q')^{\nu/(1-\nu)}$. Now

$$\sum (q^*)^{-\nu-\delta} < 1 + \int_1^R x^{-\nu-\delta} dx < (1-\nu-\delta)^{-1} c^{(1-\nu-\delta)/(1-\nu)} (q')^{\nu-\nu\delta/(1-\nu)}$$

(by (2.4)). Hence

$$S_3 < c_2 M^{r\mu+\delta} (q')^{-\delta},$$

where c_2 is a constant independent of M and q' . Hence

$$S < c_2 M^{r\mu+\delta} \sum (q')^{-\delta} \quad (q' \in C_2).$$

The series $\sum (q')^{-\delta}$ converges by hypothesis, and thus $S < \infty$ for arbitrarily large M . these arguments can be repeated for α in the range $-M \leq \alpha < 0$. Hence the theorem follows subject to (2.1).

We now turn to the case

$$0 \leq \mu < 1, \quad \nu = 1.$$

The above proof up to (2.7) is still valid, except that (2.3) and (2.4) do not apply and $\zeta > 0$ is unrestricted. It follows from (2.7) that

$$S < c_1 M^{r\mu+\delta} \sum_{q=1}^{\infty} q^{-1-\delta} < \infty,$$

which proves the result for this case.

If

$$\mu = 1, \quad 0 \leq \nu < 1,$$

then

$$S_2 = \sum_{p_i < Mq, 1 \leq i \leq r} q^{-r(1+\zeta)-\nu} \leq M^r q^{-(r\zeta+\nu)} \leq M^{r+\delta} q^{-\nu-\delta}.$$

The rest of the proof is the same as the above from (2.7) on.

We finally have to consider the case

$$\mu = \nu = 1,$$

which, for $r = 1$, is the case arising from the Thue-Siegel-Roth Theorem. Here

$$S = \sum_{q=1}^{\infty} \sum_{p_i < Mq, 1 \leq i \leq r} q^{-r(1+\zeta)-1} \leq M^r \sum_{q=1}^{\infty} q^{-1-r\zeta} < \infty.$$

The proof is complete by Lemma 2.

Proof of Theorem III. The proof is very similar to the proof of Theorem II. We estimate

$$S = \sum_q \sum_{0 < p_i < Mq} 1/k^r q^r (p')^{r(\mu+\nu/r-1)/(1-\mu)}, \quad M \geq 1,$$

and show that $S < \infty$ for arbitrarily large M . Using the notation of the proof of Theorem II, we write

$$S_1 \leq \sum 1/k^r q^r (p')^{r(\mu+\nu/r-1)/(1-\mu)},$$

summed over $p_i^* < \min(c^{1/(1-\mu+\eta)}(p')^{(\mu-\eta)/(1-\mu+\eta)}, Mq/p')$, and $1 \leq i \leq r$. Thus,

$$S_1 \leq \frac{1}{k^r q^r (p')^{r(\mu+\nu/r-1)/(1-\mu)}} (\min(c^{1/(1-\mu+\eta)}(p')^{(\mu-\eta)/(1-\mu+\eta)}, Mq/p'))^r.$$

We thus have again the cases (2.5) and (2.6) of Theorem II, with μ replaced by $\mu - \eta$, and hence

$$S_2 \leq \frac{1}{k^r q^r} \left(\sum c^{r/(1-\mu+\eta)} (p')^{(r(1-\mu)-\nu(1-\mu+\eta))/(1-\mu)(1-\mu+\eta)} + \sum (Mq)^r (p')^{-\nu/(1-\mu)} \right),$$

where the first summation is over all $p' < c^{-1}(Mq)^{1-\mu+\eta}$, and the second is over all p' satisfying $c^{-1}(Mq)^{1-\mu+\eta} \leq p' < Mq$. For η sufficiently small, as we may assume without loss of generality, the exponent of p' in the first summand is positive unless $r = \nu = 1$. Hence if r and ν are not simultaneously 1, we obtain

$$S_2 \leq \frac{2c^{\nu/(1-\mu)}}{k^r q^r} \sum (Mq)^{r-\nu(1-\mu+\eta)/(1-\mu)} \quad (p' < (Mq)^{1+\eta}).$$

Let $0 < \varepsilon < v\eta$. By Lemma 1,

$$S_2 < \frac{c_3 M^{\varepsilon + r - v(1-\mu+\eta)/(1-\mu)}}{q^{v(1-\mu+\eta)/(1-\mu) - \varepsilon}} = \frac{c_3 M^{\varepsilon + r - v(1-\mu+\eta)/(1-\mu)}}{q^{v+\delta}},$$

where c_3 is a constant independent of M and q , and $\delta = v\eta/(1-\mu) - \varepsilon > 0$. It is thus seen that $S < \infty$ if $v = 1$, and the same holds for $0 < v < 1$ by the proof of Theorem II following (2.7). The result now follows by Lemma 2.

Proof of Theorems IV and V.

LEMMA 3. Let μ, v satisfy (1.1). Then there are positive constants k_1, k_2 and infinitely many pairs (p', q') of the form

$$(3.1) \quad p' = P_1^{\rho_1} \dots P_s^{\rho_s}, \quad q' = Q_1^{\sigma_1} \dots Q_t^{\sigma_t}$$

which satisfy

$$(3.2) \quad k_1 < (q')^{1-\mu}/(p')^{1-v} < k_2.$$

Proof. By choosing k_1, k_2 to satisfy $0 < k_1 < 1 < k_2$, we see that the lemma is true if either $\mu = 1$ or $v = 1$. Hence we may assume $0 \leq \mu < 1$, $0 \leq v < 1$. Pick two (not necessarily distinct) primes $P = P_i \in S_1$, $Q = Q_j \in S_2$ from the sets (1.8). We now restrict our attention to $p' = P^\rho$, $q' = Q^\sigma$. It suffices to show that

$$\log k_1 < \sigma(1-\mu) \log Q - \rho(1-v) \log P < \log k_2,$$

or equivalently, that

$$G = \frac{\log k_1}{(1-v) \log P} < \sigma \frac{(1-\mu) \log Q}{(1-v) \log P} - \rho < \frac{\log k_2}{(1-v) \log P} = H$$

has infinitely many solutions in positive integers ρ, σ for suitable positive k_1, k_2 .

If $(1-\mu) \log Q / (1-v) \log P = \theta$ is irrational, then by Kronecker's Theorem (see e.g. [3]), there are infinitely many numbers of the form $\sigma\theta - \rho$ in the interval (G, H) for every $0 < k_1 < k_2$.

If $(1-\mu) \log Q / (1-v) \log P = l/m$ is rational, then the Diophantine equation $\sigma l/m - \rho = a$ has infinitely many solutions in positive integers ρ, σ for any integer a . Thus, if $k_2/k_1 > P^{1-v}$ then the interval (G, H) contains an integer a , and hence this interval contains

$$\sigma \frac{(1-\mu) \log Q}{(1-v) \log P} - \rho$$

for infinitely many positive integers ρ, σ .

LEMMA 4. Let $\alpha_1, \dots, \alpha_r$ be real, $k \geq 1$ real, and let μ, v satisfy

$$(3.3) \quad 0 \leq \mu \leq 1, \quad 0 \leq v < 1.$$

Then for c sufficiently large, the set of inequalities

$$(3.4) \quad \left| \alpha_i - p_i/q \right| < p'/kq^*(q')^{(v+r(1-v))/r(1-v)} \quad (i = 1, \dots, r)$$

has infinitely many solutions in vectors $(p_1/q, \dots, p_r/q)$, p_i, q of the form (1.7) subject to (1.6), where p', q' are integers of the form (3.1) satisfying (3.2).

Proof. We proceed in analogy to proofs of this type where p and q are unrestricted. Pick a pair (p', q') satisfying (3.2). Let

$$A = [2k(q')^{v/r(1-v)}].$$

Divide the r -dimensional unit cube into A^r boxes, by dividing each of its edges into A parts by means of the points $0, 1/A, 2/A, \dots, (A-1)/A, 1$. Consider the $A^r + 1$ vectors

$$(\{x\alpha_1 q'/p'\}, \{x\alpha_2 q'/p'\}, \dots, \{x\alpha_r q'/p'\})$$

for $x = 0, 1, \dots, A^r$. By Dirichlet's box principle, there exist x_1, x_2 with $0 \leq x_1 < x_2 \leq A^r$, such that

$$|\{x_2 \alpha_i q'/p'\} - \{x_1 \alpha_i q'/p'\}| < A^{-1} \quad (i = 1, \dots, r).$$

Since $k \geq 1$, this implies

$$(3.5) \quad |(x_2 - x_1) \alpha_i q'/p' - ([x_2 \alpha_i q'/p'] - [x_1 \alpha_i q'/p'])| < 1/k(q')^{v/r(1-v)}.$$

Now $1 \leq x_2 - x_1 \leq A^r \leq (2k)^r(q')^{v/(1-v)}$, and, for $\mu < 1$,

$$\begin{aligned} |[x_2 \alpha_i q'/p'] - [x_1 \alpha_i q'/p']| &\leq |[x_2 \alpha_i q'/p']| \\ &\leq (2k)^r |\alpha_i| (q')^{1/(1-v)} (p')^{-1} \\ &< (2k)^r |\alpha_i| k_2^{1/(1-\mu)(1-v)} (p')^{\mu/(1-\mu)} \end{aligned}$$

by the right side of (3.2). Thus letting

$$c > \max((2k)^{r(1-v)}, ((2k)^r |\alpha|)^{1-\mu} k_2^{1/(1-v)}), \quad |\alpha| = \max_i |\alpha_i|,$$

we see from (2.2) and (2.3) that $[x_2 \alpha_i q'/p'] - [x_1 \alpha_i q'/p']$ and $x_2 - x_1$ have the form p_i^* and q^* respectively of (1.6). However, any c satisfying the weaker inequality

$$(3.6) \quad c > (2k)^{r(1-v)}$$

already satisfies (2.3). If $\mu = 1$ we choose c to satisfy (3.6), and define $p_i = p_i^* = [x_2 \alpha_i q'/p'] - [x_1 \alpha_i q'/p']$. Since $c > 1$, (1.6) is satisfied. In either case (3.5) can be rewritten in the form

$$|\alpha_i q/p' - p_i^*| < 1/k(q')^{v/r(1-v)},$$

leading to (3.4).

Now repeat the application of the box principle with another pair (p', q') satisfying (3.2), where q' is larger than the common denominator q of the first solution vector. This gives rise to a second solution vector which is distinct from the first. Repeating this process, which is possible since (3.2) has infinitely many solutions, an infinite sequence of solution vectors is obtained.

LEMMA 5. *Let $\alpha_1, \dots, \alpha_r$ be real nonzero, $k \geq 1$ real, and let*

$$0 \leq \mu < 1, \quad 0 \leq \nu \leq 1.$$

Then for c sufficiently large, the set of inequalities

$$(3.7) \quad |\alpha_i - p_i/q| < 1/kq(p')^{(\mu+\nu/r-1)/(1-\mu)} \quad (i = 1, \dots, r)$$

has infinitely many solution vectors $(p_1/q, \dots, p_r/q)$, p_i, q of the form (1.7) subject to (1.6), where p', q' are integers of the form (3.1) satisfying (3.2).

Proof. The proof of Lemma 4 with

$$A = [2k(p')^{\nu/r(1-\mu)}]$$

applies. If $\nu < 1$ we let

$$c > \max((2k)^{r(1-\nu)} k_1^{-\nu/(1-\mu)}, ((2k)^r |\alpha|)^{1-\mu} k_2), \quad |\alpha| = \max_i |\alpha_i|.$$

Then $[x_2 \alpha_i q'/p'] - [x_1 \alpha_i q'/p']$ and $x_2 - x_1$ have the required form p_i^* and q^* . Any c satisfying

$$(3.8) \quad c \geq ((2k)^r (1 + |\alpha|))^{1-\mu} k_2$$

already satisfies (2.2). If $\nu = 1$ we choose c to satisfy (3.8) and let $q = q^* = x_2 - x_1$. Thus

$$|\alpha_i q/p' - p_i^*| < 1/k(p')^{\nu/r(1-\mu)},$$

leading to the desired result.

Proof of Theorem IV. We consider several cases:

(i) $0 \leq \mu \leq 1, \quad 0 \leq \nu < 1, \quad \mu + \nu/r \geq 1.$

By Lemma 4 and by the left side of (3.2), the set of inequalities

$$(3.9) \quad |\alpha_i - p_i/q| < c_1/q^*(q')^{(\nu+r(\mu-\nu))/r(1-\nu)} \quad (i = 1, \dots, r)$$

has infinitely many solution vectors $(p_1/q, \dots, p_r/q)$, p_i, q of the form (1.7) subject to (1.6), where $c_1 = (kk_1^{1/(1-\nu)})^{-1}$. This is equivalent to

$$|\alpha_i - p_i/q| < c_1/q(q')^{(\mu+\nu/r-1)/(1-\nu)} \quad (i = 1, \dots, r).$$

From (2.3),

$$(q')^{1/(1-\nu)} > c^{-1/(1-\nu)} q,$$

where c is sufficiently large to satisfy (3.6), say $c = (3k)^{r(1-\nu)}$. Thus, since $\mu + \nu/r \geq 1$, the set of inequalities

$$|\alpha_i - p_i/q| < c_2/q^{\mu+\nu/r} \quad (i = 1, \dots, r)$$

has infinitely many solutions of the required form, where

$$\begin{aligned} c_2 &= c_1 c^{(\mu+\nu/r-1)/(1-\nu)} = (kk_1^{1/(1-\nu)})^{-1} (3k)^{r(\mu+\nu/r-1)} \\ &\leq 3^r k_1^{-1/(1-\nu)} k^{-(r(1-\mu)+1-\nu)} \leq K \end{aligned}$$

for all sufficiently large k . This completes the proof for this case.

(ii) $0 \leq \mu < 1$, $\nu = 1$, $\mu + \nu/r \geq 1$.

By renaming the α_i if necessary, we may assume that $\alpha_1, \dots, \alpha_j$ are not zero, $\alpha_{j+1} = \dots = \alpha_r = 0$. From (2.2)

$$(p')^{1/(1-\mu)} > c^{-1/(1-\mu)} |p_i|,$$

where c is sufficiently large to satisfy (3.8). For $1 \leq i \leq j$, (3.7) has infinitely many solutions with $p_i \neq 0$.

Hence by Lemma 5 and since $\mu + 1/r \geq 1$, the set of inequalities

$$|\alpha_i - p_i/q| < c^{(\mu+1/r-1)/(1-\mu)} / kq |p_i|^{\mu+1/r-1} \quad (i = 1, \dots, j)$$

has infinitely many solutions of the required form. There is a constant $c' > 0$ depending only on $\alpha_1, \dots, \alpha_j$, such that $|p_i| > c'q$ for all p_i, q satisfying this set of inequalities. Hence

$$|\alpha_i - p_i/q| < c_3/q^{\mu+1/r} \quad (i = 1, \dots, j)$$

has infinitely many solutions of the required form, where (with equality in (3.8)),

$$\begin{aligned} c_3 &= c^{(\mu+1/r-1)/(1-\mu)} (c')^{1-\mu-1/r} k^{-1} \\ &= (2k)^{r(\mu+1/r-1)} (1 + |\alpha|)^{\mu+1/r-1} k_2^{(\mu+1/r-1)/(1-\mu)} (c')^{1-\mu-1/r} k^{-1} \leq K \end{aligned}$$

for all sufficiently large k . For $i = j+1, \dots, r$, we let $p_i = 0$, and thus the set of inequalities (1.9) has infinitely many solutions of the required form.

(iii) $\mu = 0$, $0 < \nu \leq 1$.

By Lemma 5, the set of inequalities

$$|\alpha_i - p_i/q| < (p')^{1-\nu/r}/kq \leq |p_i|^{1-\nu/r}/kq \leq c''q^{1-\nu/r}/kq < Kq^{-\nu/r} \quad (i = 1, \dots, j)$$

has infinitely many solutions, where c'' is a positive constant depending only on $\alpha_1, \dots, \alpha_j$ and k is sufficiently large so that $c''k^{-1} < K$. For $i > j$, take $p_i = 0$ as before.

(iv) $\nu = 0$, $0 \leq \mu < 1$.

By (3.9), the set of inequalities

$$|\alpha_i - p_i/q| < c_1/q^*(q')^\mu \leq c_1/(q^*)^\mu (q')^\mu = c_1/q^\mu \quad (i = 1, \dots, r)$$

has infinitely many solution vectors of the required form, and $c_1 \leq K$ for all sufficiently large k .

(v) $\mu = v = 1$.

The set of inequalities (1.9) reduces to

$$|\alpha_i - p_i/q| < q^{-(1+1/r)} \quad (i = 1, \dots, r),$$

where p_i, q are arbitrary integers with $q > 0$, and the result specializes to a well known theorem (see e.g. Theorem 200 of [3]).

Proof of Theorem V. Let P be a prime belonging to both S_1 and S_2 . For every pair (μ, v) satisfying (1.11) and (1.12) we choose a pair μ_1, v_1 satisfying

$$0 < \mu_1 < 1, \quad 0 < v_1 \leq 1, \quad \mu_1 + v_1/r \geq 1$$

and

$$(3.10) \quad \mu_1/\mu = v_1/v.$$

Such a pair can always be found, by choosing h to satisfy $\mu + v/r \geq h^{-1} \geq v$, $h^{-1} > \mu$, and letting $\mu_1 = h\mu$, $v_1 = hv$. By Theorem IV, the set of inequalities

$$(3.11) \quad |\alpha_i - p_{i1}/q_1| < KP^{-(\mu+v/r)} q_1^{-(\mu_1+v_1/r)} \quad (i = 1, \dots, r)$$

has infinitely many solution vectors $(p_{11}/q_1, \dots, p_{r1}/q_1)$, where

$$p_{i1} = p_i^* p', \quad q_1 = q^* q', \quad |p_i^*| < c |p_{i1}|^{\mu_1}, \quad 0 < q^* < c q_1^{v_1},$$

c a sufficiently large constant. For each q_1 satisfying (3.11) define

$$\rho = \left[\frac{v_1 - v}{v} \frac{\log q_1}{\log P} \right] + 1.$$

Then

$$P^{(v_1-v) \log q_1 / v \log P} < P^\rho \leq P P^{(v_1-v) \log q_1 / v \log P},$$

so that

$$(3.12) \quad q_1^{(v_1-v)/v} < P^\rho \leq P q_1^{(v_1-v)/v}.$$

Letting

$$q = q_1 P^\rho, \quad p_i = p_{i1} P^\rho,$$

we obtain, on multiplying (3.12) by q_1 and $|p_{i1}|$,

$$(3.13) \quad q_1^{v_1/v} < q \leq P q_1^{v_1/v}$$

and

$$|p_{i1}| q_1^{(v_1-v)/v} < |p_i|.$$

There is a constant $c'_i > 0$, depending only on α_i , such that $q_1 > c'_i |p_{i1}|$ for all p_{i1}, q_1 satisfying (3.11). Thus, by (3.10) the last inequality becomes

$$|p_{i1}|^{\mu_1/\mu} < (c')^{-(\mu_1-\mu)/\mu} |p_i|, \quad c' = \min_i c'_i.$$

Hence

$$(3.14) \quad |p_i^*| < c |p_i|^\mu < c(c')^{-(\mu_1-\mu)} |p_i|^\mu.$$

By the left side of (3.13),

$$(3.15) \quad 0 < q^* < cq_1^{v_1} < cq^v.$$

By the right side of (3.13) and by (3.10),

$$q_1^{\mu_1+v_1/r} \geq (P^{-1}q)^{v(\mu_1+v_1/r)v_1} = (P^{-1}q)^{\mu+v/r}.$$

Thus letting $c_4 = \max(c, c(c')^{-(\mu_1-\mu)})$, it follows from this and from (3.14) and (3.15) that for each solution vector $(p_1/q_1, \dots, p_r/q_r)$ of (3.11) there is a solution vector $(p_1/q, \dots, p_r/q)$ to the set of inequalities.

$$|\alpha_i - p_i/q| < Kq^{-(\mu+v/r)},$$

where

$$|p_i^*| < c_4 |p_i|^\mu, \quad 0 < q^* < c_4 q^v.$$

Now suppose that either $v > \mu + v/r$, or that $S_1 \cap S_2 = 0$, $\mu + v/r < 1$. Then not both v and r are 1. Let $\eta > 0$ satisfy $2\eta(1 - \mu - v/r)/(1 - \mu) < \delta$. Since (3.7) has infinitely many solutions, it follows from Theorem III that all but a finite number of them satisfy $|p_i^*| \geq |p_i|^{\mu-\eta}$ for some i , say $i = 1$, for almost all $(\alpha_1, \dots, \alpha_r)$. But then also $|p_i^*| \geq |p_i|^{\mu-\eta-\varepsilon}$ ($i = 2, \dots, r$) for any $\varepsilon > 0$ and for all sufficiently large p_i , since $p_i/q \rightarrow \alpha_i$. Choosing $\varepsilon = \eta$, we have

$$p' \leq |p_i|^{1-\mu+2\eta}.$$

Since $\mu + v/r < 1$, (3.7) now becomes, for the nonzero α_i ,

$$\begin{aligned} |\alpha_i - p_i/q| &< (kq)^{-1} |p_i|^{(1-\mu-v/r)(1-\mu+2\eta)/(1-\mu)} \\ &< (kq)^{-1} |p_i|^{1-\mu-v/r+\delta} \\ &< (c')^{1-\mu-v/r+\delta} / kq^{\mu+v/r-\delta} \\ &< 1/q^{\mu+v/r-\delta} \end{aligned}$$

for all sufficiently large k . If $\alpha_i = 0$, we let $p_i = 0$. This completes the proof of Theorem V.

4. Proof of Theorems VI and VII. Following Roth [5], Ridout's proof [4] consists of showing that if m is a sufficiently large positive integer and $\delta > 0$ a sufficiently small integer, then (1.4) has less than m solutions in relatively prime integers p, q of the form (1.2) arising from the same vectors $\mathcal{B} = (b_1, \dots, b_s)$ $\mathcal{C} = (c_1, \dots, c_t)$ defined in §4 of [4], such that

$$(4.1) \quad |p_j^*| < |p_j|^{\mu+\delta}, \quad 0 < q_j^* < q_j^{v+\delta}, \quad |p_j| > q_j^{1-\delta} \quad (j = 1, \dots, m),$$

$$(4.2) \quad \log q_1 > Cm\delta^{-2}$$

$$(4.3) \quad \log q_j / \log q_{j-1} > 2\delta^{-1} \quad (j = 2, \dots, m),$$

where C is given by (1.15), the same as (5) of [2], (4.1) is (26) of [4], (4.2) is (14) of [2] and (4.3) is (15) of [2]. As in [2], we restrict ζ to satisfy $0 < \zeta \leq 1/3$, but we do not restrict n . Let $\varepsilon > 0$ be a fixed fraction of ζ . Replace the δ appearing in (26) and in §4 of [4] by ε throughout⁽⁴⁾. Then (4.1) reads

$$(4.4) \quad |p_j^*| < |p_j|^{\mu+\varepsilon}, \quad 0 < q_j^* < q_j^{v+\varepsilon}, \quad |p_j| > q_j^{1-\varepsilon} \quad (j = 1, \dots, m),$$

and the next to last inequality of [4] becomes

$$(4.5) \quad (1 + 4\delta)\delta - (\gamma - \eta)\kappa + m(1 + \delta) \\ > (1 - \mu - \varepsilon)(1 - \varepsilon)^2(\gamma - \eta) + (1 - v - \varepsilon)(1 - \varepsilon)\gamma,$$

where

$$\kappa = \mu + v + \zeta, \quad \gamma = (m - \lambda)/2, \quad \lambda = 8(1 + 3\delta)(n + 1)m^{1/2}, \quad \eta = 10^m \delta^{(1/2)m}.$$

The parameters m, δ have to be chosen so that

$$(4.6) \quad 0 < \delta < m^{-1}$$

which is (13) of [2], and so that

$$(4.7) \quad 0 < \frac{m - (2 - \mu - v)\gamma}{\gamma} < \kappa,$$

which is (25) of [4], and so that (4.7) contradicts (4.5), which is the basic contradiction materializing Ridout's proof. It is sufficient to choose them so as to satisfy (4.6) and

$$(4.8) \quad 0 < \frac{m(1 + 2\delta)}{\gamma - \eta} < 2 + \zeta - 6\varepsilon,$$

since it is easily verified that (4.8) implies (4.7) and denies (4.5) if $m \geq 3$. We choose

$$(4.9) \quad \varepsilon = \zeta/90.$$

As in [2], with n replaced by $n + 1$, we let $m = [400(n + 1)^2 \zeta^{-2}] + 1$ and $\eta = 1$, i.e.,

$$\delta = 10^{-m2^m}.$$

With these choices it is evident that (4.6) is satisfied and $\gamma - \eta > 0$. Since the fraction in (4.8) decreases as m increases, we have

$$\begin{aligned} \frac{(1 + 2\delta)}{\gamma - \eta} - 2 &= \frac{8(1 + 3\delta)(n + 1)m^{1/2} + 2\eta + 2\delta m}{m/2 - 4(1 + 3\delta)(n + 1)m^{1/2} - \eta} < \frac{8(n + 1)m^{1/2} + 3}{m/2 - 4(n + 1)m^{1/2} - 2} \\ &< \frac{160(n + 1)^2 \zeta^{-1} + \zeta^{-1}}{200(n + 1)^2 \zeta^{-2} - 80(n + 1)^2 \zeta^{-1} - 2} < \frac{161}{173} \zeta < \frac{14}{15} \zeta = \zeta - 6\varepsilon \end{aligned}$$

⁽⁴⁾ The parameter ε is introduced here merely for the purpose of improving the estimates.

by (4.9), and so (4.8) is verified. For proving the theorems we require the following two lemmas.

LEMMA 6. Let $M_1, M_2, \eta \geq 0$. The number of solutions of

$$(4.10) \quad |\alpha - p/q| < M_1/q^\eta$$

in integers p, q with

$$0 < q < M_2$$

is at most

$$M_2(1 + M_1 + M_1 M_2).$$

Proof. For every solution p/q of (4.10) we have

$$\alpha q - M_1 q^{1-\eta} < p < \alpha q + M_1 q^{1-\eta}.$$

This interval contains less than $2M_1 q^{1-\eta} + 1 \leq 2M_1 q + 1$ integers p . Hence the number of solutions of (4.10) of the required form is less than

$$\sum_{0 < q < M_2} (2M_1 q + 1) < M_2(1 + M_1 + M_1 M_2).$$

LEMMA 7. Let $p_1/q_1, p_2/q_2$ be two distinct solutions of (1.4) in integers of the form (1.2) with $(p_1, q_1) = (p_2, q_2) = 1$, $D \leq q_1 \leq q_2$, which arise from the same \mathcal{B} and \mathcal{C} vectors, where D is given by (1.14). Then

$$\log q_2 / \log q_1 > 1 + 14\zeta/15.$$

Proof. We write

$$(4.11) \quad \left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right| \leq \left| \alpha - \frac{p_1}{q_1} \right| + \left| \alpha - \frac{p_2}{q_2} \right| < \frac{2}{q_1^{\mu+\nu+\zeta}} \leq q_1^{-(\mu+\nu+89\epsilon)}$$

by (1.14) and (4.9). For any solution p, q of (1.4) of the form (1.2) with $q \geq D$ we have,

$$\frac{|p|}{q} > |\alpha| - \frac{1}{q^{\mu+\nu+\zeta}} \geq \frac{2}{q^\epsilon} - \frac{1}{q^{\mu+\nu+\zeta}} > \frac{1}{q^\epsilon},$$

$$c \leq q^{\epsilon(1-\epsilon)} < |p|^\epsilon, \quad c < q^\epsilon,$$

i.e., p_k, q_k ($k = 1, 2$) satisfy (4.4). Hence using the method of §4 of [4], we have by his (20),

$$\rho_i(k) > \frac{b_i - 1}{u \log P_i} (1 - \mu - \epsilon) \log p_k > \frac{b_i - 1}{u \log P_i} (1 - \mu - \epsilon)(1 - \epsilon) \log q_k$$

($i = 1, \dots, s$),

and similarly,

$$\sigma_i(k) > \frac{c_i - 1}{u \log Q_i} (1 - v - \varepsilon) \log q_k \quad (i = 1, \dots, t).$$

It follows that a common factor

$$\begin{aligned} &> \prod_{i=1}^s P_i^{((b_i-1)/(u \log P_i)) (1-\mu-\varepsilon) (1-\varepsilon) \log q_1} = \prod_{i=1}^s q_1^{((b_i-1)/u) (1-\mu-\varepsilon) (1-\varepsilon)} \\ &> q_1^{(1-\mu-\varepsilon) (1-\varepsilon)^2} \end{aligned}$$

can be extracted from p_1 and p_2 , and similarly, a common factor

$$> q_1^{(1-v-\varepsilon) (1-\varepsilon)}$$

can be extracted from q_1 and q_2 . Also $p_1/q_1 \neq p_2/q_2$, since $(p_1, q_1) = (p_2, q_2) = 1$. Hence

$$\left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right| = \left| \frac{p_1 q_2 - p_2 q_1}{q_1 q_2} \right| > q_1^{(1-\mu-\varepsilon) (1-\varepsilon)^2 + (1-v-\varepsilon) (1-\varepsilon) - 1} q_2^{-1}.$$

Comparing this with (4.11), we obtain

$$q_1^{(1-\mu-\varepsilon) (1-\varepsilon)^2 + (1-v-\varepsilon) (1-\varepsilon) - 1} q_2^{-1} < q_1^{-(\mu+v+89\varepsilon)},$$

or

$$q_2 > q_1^{\mu+v+89\varepsilon + (1-\mu-\varepsilon) (1-\varepsilon)^2 + (1-v-\varepsilon) (1-\varepsilon) - 1}.$$

But

$$\begin{aligned} &\mu + v + 89\varepsilon + (1 - \mu - \varepsilon)(1 - \varepsilon)^2 + (1 - v - \varepsilon)(1 - \varepsilon) - 1 \\ &= 1 + 87\varepsilon - \varepsilon((2 - \varepsilon)(1 - \mu - \varepsilon) + 1 - v - \varepsilon) > 1 + 84\varepsilon = 1 + 14\zeta/15, \end{aligned}$$

completing the proof.

Proof of Theorem VI. Choose any vectors \mathcal{B} , \mathcal{C} subject to the conditions imposed on them in §4 of [4]. Let $g_1/h_1, g_2/h_2, \dots$ be an enumeration of all solutions of (1.4) of the form (1.2) arising from \mathcal{B} and \mathcal{C} with the properties $(g_j, h_j) = 1$ ($j = 1, 2, \dots$), and $D \leq h_1 \leq h_2 \leq \dots$. By Lemma 7,

$$\log h_{j+1} / \log h_j > 1 + 14\zeta/15 \quad (j = 1, 2, \dots).$$

We now define integers k and l exactly as in [2], but with ζ of [2] replaced by $14\zeta/15$. The number of solutions arising from \mathcal{B} and \mathcal{C} must be less than $k + (m - 1)l$, where

$$k - 3 \leq \frac{\log (Cm\delta^{-2} (\log 2)^{-1})}{\log (1 + 14\zeta/15)} < \frac{2}{\zeta} \log (2Cm\delta^{-2}),$$

$$l - 1 \leq \frac{\log (2\delta^{-1})}{\log (1 + 14\zeta/15)} < \frac{2}{\zeta} \log (2\delta^{-1}).$$

As in [2] we then obtain

$$\begin{aligned} k + (m-1)^l &< 2\zeta^{-1} \log C + 24 m^2 \zeta^{-12^m} \\ &< 2\zeta^{-1} \log C + \zeta^{-1} 2^{1.0023m} \\ &< 2\zeta^{-1} \log C + \exp(280(n+1)^2 \zeta^{-2}). \end{aligned}$$

Letting $u = [\max(s, t)/\varepsilon] + 1 = [90 \max(s, t)/\zeta] + 1$ (see §4 of [4]), it is plain that the total number of \mathcal{B} and \mathcal{C} vectors is

$$\sum_{i=1}^s \binom{u+s+i-1}{s-1} \quad \text{and} \quad \sum_{i=1}^t \binom{u+t+i-1}{t-1}$$

respectively.

By Lemma 6 with $M_1 = 1$, $M_2 = D$, the number of solutions with $q < D$ is at most $D(D+2)$. This gives the estimate of Theorem VI.

Proof of Theorem VII. As in [2], this case is reduced to the case of the algebraic integer $\alpha = b_0\beta$, and the same estimate $C < 7 \log(1 + 2B^n)$ is obtained. Every solution p/q with $(p, q) = 1$ of (1.16) gives rise to exactly one solution $b_0 p/q = p_1/q_1$ with $(p_1, q_1) = 1$ of

$$(4.12) \quad \left| \alpha - p_1/q_1 \right| < b_0 q_1^{-(\mu+\nu+2\zeta)},$$

where $\xi = 2\zeta$. Obviously, p_1, q_1 have the form (1.2) with c replaced by cb_0 . The constant D of (1.14) now becomes

$$(4.13) \quad 2^{180/\xi} + (2/b_0 |\beta|)^{180/\xi} + (cb_0)^{1802/\xi(180-\xi)}.$$

If $q_1^\xi \geq b_0$, then (4.12) implies (1.4). By Lemma 6 with $M_1 = b_0$, $M_2 = b_0^{1/\xi}$, the number of solutions of (4.12) with $q_1^\xi < b_0$ is at most $b_0^{1/\xi}(1 + b_0 + b_0^{1+1/\xi}) < 3b_0^{1+2/\xi}$. Absorbing the square root of this in the expression (4.13) gives the new constant F , on using $b_0 \geq 2$, and establishes the estimate of Theorem VII.

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