SUMMABILITY OF FOURIER SERIES IN $L^p(d\mu)$

BY

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1. **Introduction.** Let μ be a non-negative finite Borel measure on the unit circle C such that $\mu(C) > 0$. For each p, $1 \leqq p < \infty$, let $L^p(d\mu)$ be the Banach space of μ-measurable complex-valued functions $f(e^{i\phi})$ such that

$$\|f\|_p = \left[\int |f(e^{i\phi})|^p d\mu(\phi)\right]^{1/p}$$

$< \infty$. σ shall be normalized Lebesgue measure on C. $\mathscr{P}$ and $\mathscr{P}_0$ are the classes of trigonometric polynomials of the form $\sum_n c_n e^{in\phi}$ and $\sum_{n \geqq 0} c_n e^{in\phi}$ respectively. $P_r(e^{i\phi})$ shall be the Poisson kernel and $*$ the symbol of Fourier convolution. Thus if $f(e^{i\phi}) = \sum c_n e^{in\phi} \in \mathscr{P}$, then $(P_r * f)(e^{i\phi}) = \sum c_n r^{|n|} e^{in\phi} \in \mathscr{P}$. $\delta_1, \delta_2, \ldots$ shall be fixed positive numbers and $K_1, K_2, \ldots$ absolute constants. We omit writing subscripts or use the same subscript in different contexts when we believe that no confusion can arise.

Our main concern shall be the following problems:

Problems A, B. Characterize the classes $\mathscr{Q}_p$ and $\mathscr{B}_p$ of measures μ such that

$$\sup\{\|P_r * f\|_p : \delta < r < 1\} \leqq K \|f\|_p \tag{1.1}$$

for all $f \in \mathscr{P}_0$ and $\mathscr{P}$ respectively.

We shall also be concerned with variations of problem B, where Abel summability is replaced by Fejér and several other types of summability. These problems follow a line of investigations in harmonic analysis with non-translation-invariant measures that dates back to Hardy and Littlewood [8]. Subsequent work was done by Babenko [1], Hirschman [10], Gapoškin [6], Edwards [4], Chen [3] and Helson and Szegö [9]. Our work follows up certain consequences of Helson and Szegö's results. These authors classify the class $\mathscr{D}_2$ of measures μ for which

$$\sup_n \|D_n * f\|_2 \leqq \sum K \|f\|_2 \tag{1.2}$$

for all $f \in \mathscr{P}$. D_n is the Dirichlet kernel. They prove that $\mu \in \mathscr{D}_2$ if and only if

(1.3) (i) μ is absolutely continuous, $d\mu = w d\sigma$, and

(1.3) (ii) $w = \exp(u + \tilde{v})$, where u and v are σ-essentially bounded real functions such that σ-ess sup $|v| < \pi/2$. $v \to \tilde{v}$ is the Fourier conjugation operator.

From (1.3) one can deduce that if $f \in L^2(d\mu)$, $\mu \in \mathscr{D}_2$, then the Fourier coefficients of f are well-defined and the Fourier series of f converges in $L^2(d\mu)$ norm to f. Given this, it seems reasonable to ask when the Fourier series of any $f \in L^p(d\mu)$

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is Abel summable to f in $L^p(d\mu)$ norm. This, in turn, leads us to problem B. Suppose $\mu \in \mathscr{B}_p$. Then the densely defined linear functionals $l_n(f) = \int f(e^{i\phi})e^{-in\phi}\,d\sigma(\phi)$, $n=0, \pm 1, \pm 2, \ldots$ can be shown to be bounded on $\mathscr{P}$ and thus have unique continuous extensions to all of $L^p(d\mu)$. Thus the Fourier coefficients of any $f \in L^p(d\mu)$ are well-defined. Similarly the densely defined operators $f \to P_r * f$, r fixed, $\delta < r < 1$, are bounded on $\mathscr{P}$ and thus one can speak meaningfully of the Abel means of the Fourier series of any $f \in L^p(d\mu)$. Finally (still under the assumption that $\mu \in \mathscr{B}_p$) one can deduce from (1.1) that these Abel means converge in $L^p(d\mu)$ norm to f.

Problem A leads to a generalization of Hardy spaces. Let $H^p(d\mu)$ be the class of functions $f(z)$, $z = re^{i\phi}$, holomorphic in $0 \leqq r < 1$ and such that

$$|||f|||_p = \sup\{[\int |f(re^{i\phi})|^p\,d\mu(\phi)]^{1/p} : 0 \leqq r < 1\} \tag{1.4}$$

is finite. Let $L_0^p(d\mu)$ be the closure of $\mathscr{P}_0$ in $L^p(d\mu)$. A classical theorem [11, p. 284] states that $H^p(d\sigma)$ is vector space isomorphic and isometric to $L_0^p(d\sigma)$ under the operator $T: f(z) \to f(e^{i\phi})$. Our generalization is as follows: If $\mu \in \mathscr{Q}_p$ then the operator T is a vector space isomorphism mapping $H^p(d\mu)$ onto $L_0^p(d\mu)$ such that T and T^{-1} are bounded. If T is an isometry then μ is a multiple of Lebesgue measure.

2. **Solution of problem** B. Let $d\mu(\phi) = w(e^{i\phi})d\sigma(\phi) + d\mu_s(\phi)$ be the Lebesgue decomposition of μ. We shall first show that if $\mu \in \mathscr{Q}_p$, then $\mu_s = 0$ and log $w \in L^1(d\sigma)$. These properties are incidentally shared by any $\mu \in \mathscr{D}_2$.

LEMMA 1. (i) $\mathscr{Q}_p = \mathscr{Q}_2$ *for all* p, $1 \leqq p < \infty$.
(ii) If $\mu \in \mathscr{Q}_p$, *then* log $w \in L^1(d\sigma)$.

Proof. (i) is an easy consequence of the Blaschke factorization of any $f \in \mathscr{P}_0$. We shall prove (ii) by contradicting the assumption that log $w \notin L^1(d\sigma)$, while assuming that $\mu \in \mathscr{Q}_2$. By [7, p. 50], $\{e^{in\phi}\}_0^\infty$ is total in $L^2(d\mu)$, so for each positive integer n there is a sequence $\{h_{n,j}\}_{j=0}^\infty \subset \mathscr{P}_0$ such that $(*)$ $\|h_{n,j}(e^{i\phi}) - e^{-in\phi}\|_2 \to 0$. Now fix r, $\delta < r < 1$. From $(*)$ and (1.1) it follows that there exists $k_n \in L^2(d\mu)$ such that $\| h_{n,j}(re^{i\phi}) - k_n(e^{i\phi}) \| \to 0$ *as* $j \to \infty$. But

$$\lim_{j\to\infty} \| r^n e^{i\phi n} h_{n,j}(re^{i\phi}) - 1 \| \leqq \lim_{j\to\infty} K \| e^{i\phi n} h_{n,j}(e^{i\phi}) - 1 \| = 0,$$

so $k_n(e^{i\phi}) = r^{-n} e^{-i\phi n}$ in $L^2(d\mu)$ norm. Thus

$$r^{-n} \| 1 \| = \| k_n \| = \lim_{j\to\infty} \| h_{n,j}(re^{i\phi}) \| \leqq \lim_{j\to\infty} K \| h_{n,j} \| = K \| 1 \|.$$

Take $n \to \infty$ to obtain a contradiction of $\| 1 \| > 0$. Thus the proof of (ii) is complete.

LEMMA 2. *Suppose* $\mu \in \mathscr{Q}_p$. *Then* $d\mu(\phi) = |g(e^{i\phi})|\,d\sigma(\phi)$, *where* $g \in H^1(d\sigma)$ *is an outer function.*

Proof. Any non-negative function w such that w and $\log w \in L^1(d\sigma)$ has a representation of the form $w(e^{i\phi}) = |g(e^{i\phi})|$, where g is as above (see [2]). Hence we have only to prove that $\mu_s = 0$. Let E be σ-null set such that the mass of μ_s is concentrated on E. Now, by [11, p. 276], there exists a bounded outer function b such that $b(0) > 0$ and $\lim_{s \to 1} b(se^{i\phi}) = 0$ for all $\phi \in E$. By Lemma 1 we may assume $\mu \in \mathscr{Q}_2$. Let $0 < r,\ s < 1$. Then for all $f \in \mathscr{P}_0$

$$\|f(re^{i\phi})\, b(sre^{i\phi})\|_2^2 \leqq K^2 \|f(e^{i\phi})\, b(se^{i\phi})\|_2^2.$$

Take s to 1 and obtain

$$\|f(re^{i\phi})\, b(re^{i\phi})\|^2 \leqq K^2 \|f \cdot b\|^2 = K^2 \int |f \cdot h|^2 d\sigma,$$

where $h = b \cdot g^{1/2}$ is an outer function. Next let $z = te^{i\psi}$ be any complex number with $|z| < 1$. Then, since h is outer, there exists $\{f_n\} \subset \mathscr{P}_0$ with $f_n(e^{i\phi})\, h(e^{i\phi}) \to (1 - te^{i(\phi-\psi)})^{-1} g^{1/2}(e^{i\phi})$ in $L^2(d\sigma)$ norm. Thus

$$\int |1 - tre^{i(\phi-\psi)}|^{-2} d\mu(\phi) \leqq K^2 \int |1 - te^{i(\phi-\psi)}|^{-2} w(e^{i\phi})\, d\sigma(\phi).$$

Let $r \to 1$, so

$$\int P_t(e^{i(\phi-\psi)})\, d\mu(\phi) \leqq K^2 \int P_t(e^{i(\phi-\psi)})\, w(e^{i\phi})\, d\sigma(\phi)$$

for all t, $0 \leqq t < 1$ and all real ψ. Thus $P_t * (K^2 w d\sigma - d\mu)$ is a positive harmonic function in $|z| < 1$ and consequently $K^2 w d\sigma - d\mu$ is a positive measure. Thus μ is absolutely continuous. Our solution of problem A is contained in

THEOREM 1. *$\mu \in \mathscr{Q}_p$ if and only if*

(i) *μ is absolutely continuous, $d\mu = w d\sigma$, where $w = |g|$, $g \in H^1(d\sigma)$ is an outer function and*

(2.1) (ii) $\displaystyle\int P_r(e^{i(\phi-\psi)})\, |g(e^{i\phi})/g(re^{i\phi})|\, d\sigma(\phi) \leqq K_2$ *for all r, $0 \leqq r < 1$ and real ψ.*

Proof. Suppose $\mu \in \mathscr{Q}_p$. Then Lemmas 1 and 2 prove that (i) is true. We shall show that necessarily (ii) holds. First of all we note that for each r, $\delta < r < 1$, and real ψ, $(1 - re^{i(\phi-\psi)})^{-1} g^{-1/2}(e^{i\phi})$ is in the closed linear span of $\{e^{in\phi}\}_0^\infty$ in $L^2(d\mu)$. This is a simple consequence of the fact that $\{e^{in\phi} g^{1/2}(e^{i\phi})\}_0^\infty$ is total in $H^2(d\sigma)$. Hence by (1.1)

$$\begin{aligned} \int |(1 - r^2 e^{i(\phi-\psi)})^{-1} g^{-1/2}(re^{i\phi})|^2 w(e^{i\phi})\, d\sigma(\phi) &\leqq K \int |1 - re^{i(\phi-\psi)}|^{-2} d\sigma(\phi) \\ &= K(1 - r^2)^{-1}. \end{aligned}$$

From this and the elementary inequality $4(1 - r^2)|1 - r^2 e^{i\phi}|^{-2} \geqq Pr(e^{i\phi})$ we deduce that (ii) holds for $\delta < r < 1$. Then it clearly holds for all r, $0 \leqq r < 1$.

Conversely suppose that (i) and (ii) hold. As indicated in Lemma 1 it is sufficient to derive (1.1) for those $f \in \mathscr{P}_0$ that are the boundary functions of functions zerofree and holomorphic in $|z| < 1$, and thus we may restrict our proof to the case $p = 1$.

$$\int |f(re^{i\phi})| \, w(e^{i\phi}) \, d\sigma(\phi)$$

$$= \int |f(re^{i\phi})g(re^{i\phi})| \, |g(e^{i\phi})/g(re^{i\phi})| \, d\sigma(\phi)$$

$$\leqq \int \left[\int |f(e^{i\psi})g(e^{i\psi})| \, P_r(e^{i(\phi-\psi)}) d\sigma(\psi) \right] |g(e^{i\phi})/g(re^{i\phi})| \, d\sigma(\phi),$$

which by the Fubini theorem and (ii) is

$$\leqq K_2 \int |f(e^{i\psi})| \, w(e^{i\psi}) \, d\sigma(\psi).$$

Thus (1.1) is true for $0 \leqq r < 1$, and the proof is complete.

3. **Approximate identities.** We shall set about stating some problems equivalent to problem B.

DEFINITION 1. By an *approx id* (approximate identity)$\{k_\lambda\} \lambda \in A$ we mean a sequence or generalized sequence of non-negative functions k_λ such that

(i) $\int k_\lambda(e^{i\phi}) \, d\sigma(\phi) = 1$ for all $\lambda \in A$, and

(ii) $\lim_\lambda k_\lambda * f = f$ for all $f \in \mathscr{P}$.

DEFINITION 2. Let $k = \{k_\lambda\}_{\lambda \in \mathscr{A}}$ and $k' = \{k'_\lambda\}_{\lambda \in \mathscr{B}}$ be approx ids. k is *weaker* than k' (with respect to $L^p(d\mu)$) if whenever

$$\sup_\lambda \| k_\lambda * f \|_p \leqq K_1 \|f\|_p \text{ for all } f \in \mathscr{P},$$

then there exists K_2 such that

$$\sup_\lambda \| k'_\lambda * f \|_p \leqq K_2 \|f\|_p \text{ for all } f \in \mathscr{P}.$$

If k is weaker than k' and k' is weaker than k, then k and k' are said to be *equisummable* (with respect to $L^p(d\mu)$).

The approx ids we shall consider are the

(i) *Abel* $\{P_r : \delta < r < 1\}$;

(ii) *generalized Abel* $\{P_{\alpha,r} : \delta < r < 1\}$, where

$$P_{\alpha,r}(e^{i\phi}) = k_{\alpha,r}(1-r)^{2\alpha-1} |1 - re^{i\phi}|^{-2\alpha}; \tag{3.1}$$

$\alpha > 1/2$ and $k_{\alpha,r}$ is chosen so $\int P_{\alpha,r} \, d\sigma = 1$;

(iii) *moving average* $\{Q_h : 0 < h < \delta \leqq \pi\}$, where $Q_h(e^{i\phi}) = \pi/h$ if $|\phi| \leqq h$ and 0 if $\pi \geqq |\phi| > h$; and

(iv) *Fejér* $\{F_n : n = N, N+1, N+2, \ldots\}$,

where

$$F_n(e^{i\phi}) = \frac{1}{n+1}\frac{\sin^2[1/2(n+1)\phi]}{\sin^2(\phi/2)}, \quad \text{and}$$

N is a fixed non-negative integer.

We shall list some pertinent properties of the less familiar $P_{\alpha,r}$ later. Now it will be expedient to introduce the approx id $\{Q_{h_j}\}$, where $h_j = \pi(j+1)^{-1}$ with j ranging over all sufficiently large positive integers.

LEMMA 3. *The first four approx ids listed above are all weaker than* $\{Q_{h_j}\}$.

Proof. This is a sequence of the inequalities

$$P_{\alpha,r} \geqq K_3 Q_{1-r} \text{ for fixed } \alpha, \tag{3.3}$$

and $F_j \geqq K_4 Q_{h_j}$. This second inequality is easily deduced from the elementary inequalities $|\sin\psi| \leqq |\psi|$ and $\sin\phi \geqq 2/\pi\,\phi$ for $\phi \in [0, \pi/2]$. (3.3) will be proved later.

LEMMA 4. *Suppose h is a fixed number in $(0,\pi]$, and $\|Q_h * f\|_p \leqq K_1\|f\|_p$ for all $f \in \mathscr{P}$. Then*

(i) $\|1 * f\|_p \leqq K_2\|f\|_p$ *for all* $f\in\mathscr{P}$ *and*

(ii) *the linear functionals*

$$l_n(f) = \int f(e^{i\phi})e^{-in\phi}\,d\sigma(\phi),\ f\in\mathscr{P}, \text{ are } L^p(d\mu) \text{ continuous.}$$

Proof. Let A be the operator on $\mathscr{P} \subset L^p(du)$ that maps any f into $Q_h * f$. For some sufficiently large n there exists $\varepsilon > 0$ such that $A^n f \geqq \varepsilon \int f\,d\sigma$ for all non-negative $f \in \mathscr{P}$. Thus under the assumptions of the lemma

$$\varepsilon\int |f|\,d\sigma\left(\int du\right)^{1/p} \leqq \|A^n f\|_p \leqq K_1^n\|f\|_p\ ,$$

so (i) is true. (ii) is an immediate consequence of (i).

LEMMA 5. $\{Q_{h_j} : j \geqq M\}$ *is a weaker approx id than* $\{Q_h\}_{0<h\leqq\pi}$.

Proof. Assume $\sup_j \|Q_{h_j} * f\|_p \leqq K\|f\|_p$ for all $f\in\mathscr{P}$. First suppose that $0 < h \leqq h_M$, so there exists an integer $n \geqq M$ such that $h_{n+1} < h \leqq h_n$. Clearly $h_n \leqq 2h$, so $Q_h \leqq Q_{h_{n+1}} + 2Q_{h_n}$ and $\|f * Q_h\| \leqq 3K\|f\|$ for all $f\in\mathscr{P}$.

If $\pi \geqq h > h_M$, then Lemma 4(i) guarantees that

$$\|f * Q_h\| \leqq K_2\|f\| \text{ for all } f\in\mathscr{P}.$$

LEMMA 6. *Suppose $k(e^{i\phi})$ is an even σ-absolutely continuous function and put $k_1(e^{i\phi}) = (d/d\phi)k(e^{i\phi})$. Suppose further that*

(i) $|k(e^{i\pi})| \leqq K_1$, $\int|\phi k_1(e^{i\phi})|\,d\sigma(\phi) \leqq K_2$, *and*

(ii) $\sup\{\|Q_h * f\|_p : 0 < h \leqq \pi\} \leqq K_3\|f\|_p$ for *all* $f\in\mathscr{P}$. *Then*

$$\|k * f\|_p \leqq (K_1K_3 + 2K_2K_3)\|f\|_p \text{ for all } f\in\mathscr{P}.$$

Proof. Assume (i), (ii) and let $0 < \psi \leqq \pi$. Then

$$k(e^{i\psi}) - k(e^{i\pi}) = -2\pi \int_{\psi}^{\pi} k_1(e^{i\phi})\, d\sigma(\phi) = -2\int_{0}^{\pi} Q_\phi(e^{i\psi})\, \phi\, k_1(e^{i\phi})\, d\sigma(\phi),$$

so

$$\| (k - k(e^{i\pi})) * f \|_p \leqq 2 \int_0^{\pi} \| Q_\phi * f \|_p \,|\phi\, k_1(e^{i\phi})|\, d\sigma(\phi) \leqq 2K_3K_2 \|f\|_p.$$

Finally

$$\begin{aligned} \| k * f \|_p &\leqq \| (k - k(e^{i\pi})) * f \|_p + |k(e^{i\pi})|\ \| 1 * f \|_p \\ &\leqq (2K_2K_3 + K_1K_3)\, \|f\|_p \text{ for all } f \in \mathscr{P}. \end{aligned}$$

THEOREM 2. *The Abel, generalized Abel, Fejér and moving average approximate identities are equisummable with respect to $L^p(d\mu)$.*

Proof. In view of Lemmas 3 and 5 we have only to show that $\{Q_h\}_{0 \leqq h < \pi}$ is weaker than the generalized Abel and Fejér approx ids. The argument in [11, p. 155] shows $P_{\alpha,r}$ satisfies the hypotheses of Lemma 6, as does a dominant of the Fejér kernel.

4. **Generalized Abel approximate identities.** We list here some properties of the functions $P_{\alpha,r}$ defined in (3.1). From [5, p. 81] we have

$$F(\alpha,\alpha,1,r^2) = \int |1 - re^{i\phi}|^{-2\alpha} d\sigma(\phi),\ \alpha > \frac{1}{2},$$

where F is the hypergeometric function. Since

$$\lim_{r \to 1}\ (1-r)^{2\alpha-1} F(\alpha,\alpha,1,r^2) = \frac{\Gamma(2\alpha - 1)}{(\Gamma(\alpha))^2} \tag{4.1}$$

[12, p. 299] it follows that for fixed α, $\infty > \alpha > 1/2$, $\{k_{\alpha,r} : 0 \leqq r < 1\}$ is bounded away from 0 and ∞.

Now we prove (3.3). If $|\phi| \leqq 1 - r \leqq 1$, then

$$\begin{aligned} P_{\alpha,r}(e^{i\phi}) &= k_{\alpha,r}(1-r)^{2\alpha-1}[(1-r)^2 + 4r\sin^2\phi/2]^{-\alpha} \\ &\geqq k_{\alpha,r}(1-r)^{2\alpha-1}[(1-r)^2 + \phi^2]^{-\alpha} \\ &\geqq K Q_{1-r}(e^{i\phi}). \end{aligned}$$

The inequalities

$$K|1 - re^{i\phi}|^{-1} \geqq |1 - r^2e^{i\phi}|^{-1} \tag{4.2}$$

and

$$K|1 - r^3e^{i\phi}|^{-1} \geqq |1 - r^2e^{i\phi}|^{-1} \geqq K_2|1 - re^{i\phi}|^{-1} \tag{4.3}$$

are also easily demonstrated.

Of course $P_r = P_{1,r}$. A paraphrase of the proof of (2.1) of Theorem 1 shows that the condition

$$\int P_{\alpha,r}(e^{i(\phi-\psi)})\, |g(e^{i\phi})/g(re^{i\phi})|\ d\sigma(\phi) \leqq K \tag{4.4}$$

is necessary for $\mu \in \mathscr{Q}_2$ if α (fixed) is $> 1/2$. We shall use (4.4) later.

5. **Solution of problem B.** We introduce the notation f_h for $Q_h * f$, so $f_h(e^{i\phi}) = (\pi/h) \int_{-h}^{h} w(e^{i(\phi+\psi)})\, d\sigma(\psi)$. We first treat problem B for the easy case $p = 1$.

LEMMA 7. *$\mu \in \mathscr{B}_1$ if and only if*

(i) *$d\mu = w\, d\sigma$ and*

(5.1) (ii) *$w_h \leqq K w$ a.e. for each h, $0 < h \leqq \pi$ In fact, for fixed h, $0 < h \leqq \pi$*

$$\sup\{\| Q_h * f\|_1 : \|f\|_1 = 1,\ f \in \mathscr{P}\} = \sigma\text{-ess sup } (w_h/w).$$

Proof. $\mathscr{B}_1 \subset \mathscr{Q}_1$ so (i) is certainly necessary. In fact $\log w \in L^1(d\sigma)$ so w vanishes on no set of positive measure. The following statements are equivalent to the statement $\mu \in \mathscr{B}_1$:

(a) $L_1(f) = \int f w_h\, d\sigma$ is a bounded linear functional on $L^1(d\mu)$;

(b) $L_2(f) = \int f(w_h/w)\, d\sigma$ is a bounded linear functional on $L^1(d\sigma)$;

(c) $\|L_2\| = \sigma$-ess sup (w_h/w).

This set of equivalences proves the lemma.

For any $p > 1$ we define q by $p^{-1} + q^{-1} = 1$. By considering the adjoint operator of $A : f \to Q_h * f$ we will prove the following duality result.

THEOREM 3. *Let $p > 1$. $\mu \in \mathscr{B}_p$ if and only if*

(i) *$d\mu = w\, d\sigma$, with*

(ii) *$w^{1-q} \in L^1(d\sigma)$, and*

(iii) *$\sup\{ \int |Q_h * f|^q w^{1-q} d\sigma : 0 < h \leqq \pi\} \leqq K \int |f|^q w^{1-q} d\sigma$ for all $f \in \mathscr{P}$.*

Proof. Suppose $\mu \in \mathscr{B}_p$. Then $\mu \in \mathscr{Q}_p$, so (i) holds and $\log w \in L^1(d\sigma)$. We see from Lemma 4 that $l : f \to \int f d\sigma$ is an element of the adjoint space $L^{p*}(d\mu)$ of $L^p(d\mu)$. Thus there exists $h \in L^q(d\mu)$ with $\int f d\sigma = \int f h w\, d\sigma$ for all $f \in \mathscr{P}$. Necessarily $hw = 1$, so $\infty > \int |h|^q w\, d\sigma = \int w^{1-q} d\sigma$, which proves (ii). To show that (iii) holds we consider the adjoint operator A^* of A. Since $\|A^*\| = \|A\|$ we have

$$\int |w^{-1}[Q_h * (hw)]|^q w d\sigma \leqq \|A\|^q \int |h|^q w d\sigma$$

for all $h \in \mathscr{P}$. Put $f = hw$ to obtain (iii).

Conversely suppose (i), (ii), (iii) hold. Then one can interchange the roles of p and q to derive (1.1).

It should be noted that the moving average approx identity in (iii) may be replaced by any of the approx ids of Theorem 2 without affecting the validity of the proof.

Our solution of problem B is stated in

THEOREM 4. *$\mu \in \mathscr{B}_p$ if and only if*

(i) *μ is absolutely continuous, $d\mu = w d\sigma$,*

(ii) *$w^{1-q} \in L^1(d\sigma)$ if $p > 1$, and*

(iii)

(5.2) *$Q_h * (w_h/w)^{q-1} \leqq K$ for all h, $0 < h \leqq \pi$, if $p > 1$, or (5.1) holds if $p = 1$.*

Proof. Lemma 7 takes care of the case when $p = 1$, so assume that $p > 1$. Then (i), (ii) follow from Theorem 3. We shall defer the rather involved proof of the necessity of (iii) until later.

Conversely, suppose (i), (ii), (iii) hold and let $f \in \mathscr{P}$, $0 < h \leqq \pi$. Then

$$\int |Q_h * f|^q w^{1-q} d\sigma = \int |Q_h * (f w^{-1/p} w^{1/p})|^q w^{1-q} d\sigma,$$

which by the Hölder inequality is $\leqq \int [Q_h * (|f|^q \cdot w^{1-q})] \cdot [w_h/w]^{q-1} d\sigma$. By the Fubini theorem and (iii) this is $\leqq K \int |f|^q w^{1-q} d\sigma$. Thus Theorem 3 guarantees that $\mu \in \mathscr{B}_p$.

We note in passing that due to the duality Theorem 3 we can replace (iii) by the condition

(iii′) $Q_h * [(w^{1-q})_h / w^{1-q}]^{p-1} \leqq K$.

Our task now is to prove that (iii) is a necessary condition for $\mu \in \mathscr{B}_p$. We thus assume that $\mu \in \mathscr{B}_p$, $p > 1$, so $d\mu = |g| d\sigma$ as in Theorem 1.

LEMMA 8. *If* $0 \leqq r < 1$,

$$\int P_r(e^{i(\phi-\psi)})(1 - r^2 e^{i(\xi-\psi)})^{-1} g^{-1/p}(e^{i\psi}) d\sigma(\psi) = J_1(e^{i\phi}) + J_2(e^{i\phi}),$$

where

$$J_1(e^{i\phi}) = (1 - re^{i(\xi-\phi)})^{-1} g^{-1/p}(re^{i\phi})$$

and

$$J_2(e^{i\phi}) = -r(1-r^2)e^{i(\xi-\phi)}(1 - re^{i(\xi-\phi)})^{-1}(1 - r^3 e^{i(\xi-\phi)})^{-1} g^{-1/p}(r^2 e^{i\xi}).$$

Proof. Use the partial fraction expansion of P_r. If $z = re^{i\phi}$, $z^* = re^{-i\phi}$, $\zeta = r^2 e^{i\xi}$, then

$$P_r(e^{i(\phi-\psi)})(1 - \zeta e^{-i\psi})^{-1} = [(1 - ze^{-i\psi})^{-1} - (1 - \zeta e^{-i\psi})^{-1}] e^{i\psi} \cdot (z - \zeta)^{-1}$$
$$+ z^* e^{i\psi}(1 - z^* e^{i\psi})^{-1}(1 - \zeta e^{-i\psi})^{-1}.$$

Since (1.1) holds, necessarily

$$(1-r^2)^{p-1} \int |J_1 + J_2|^p w d\sigma \leqq K_1 (1-r^2)^{p-1} \int |1 - r^2 e^{i(\xi-\psi)}|^{-p} d\sigma(\psi),$$

which is $\leqq K_2$ by (4.1). In addition, (4.4) guarantees that $(1-r^2)^{p-1} \int |J_1|^p w d\sigma \leqq K_3$, so by the Minkowski inequality $(1-r^2)^{p-1} \int |J_2|^p w d\sigma \leqq K_4$. But this implies that

$$r^p (1-r^2)^{2p-1} \int |1 - re^{i(\xi-\phi)})(1 - r^3 e^{i(\xi-\phi)}|^{-p} w(\phi) d\sigma(\phi) \leqq K_4 |g(r^2 e^{i\xi})|.$$

By invoking (4.2), (4.3) and replacing r^2 by r we obtain

LEMMA 9. *Suppose* $\mu \in \mathscr{B}_p$, $p > 1$. *Then* $d\mu = w d\sigma$ *and*

$$\int P_{p,r}(e^{i(\xi-\phi)}) w(\phi) d\sigma(\phi) \leqq K_5 |g(re^{i\xi})| \tag{5.3}$$

for all $r, \frac{1}{2} \leqq r < 1$.

It is an open question whether Lemma 9 is valid if "$P_{p,r}$" is replaced by "P_r". Our preoccupation with the generalized Abel approx ids is, of course, in anticipation of inequality (5.3). With (5.3) we can wrap up the proof of Theorem 4. For, if $r \geqq \frac{1}{2}$

$$(*) \qquad P_{p,r} * [(P_{p,r} * w)/w]^{q-1} \leqq K_5 P_{p,r} * |(P_r * g)/w|^{q-1}$$

$$(**) \qquad = K_5 P_{p,r} * |g^{1-q}/(P_r * g^{1-q})|.$$

By Theorem 3 we know that

$$\sup_r \int |P_r * f|^q w^{1-q} d\sigma \leqq K \int |f|^q w^{1-q} d\sigma$$

for all $f \in \mathscr{P}$, thus this relation is true for all $f \in \mathscr{P}_0$. Thus from (4.4) with g replaced by g^{1-q} we deduce that (**) is $\leqq K_6$. An application of (3.3) to (*) derives (5.2) for all h with $|h| \leqq 1/2$. The inequality (5.2) is obviously true for $h (\leqq \pi)$ bounded away from 0, so the proof of Theorem 4 is complete.

6. **Examples and concluding remarks.** It is an easy matter to show that $\mathscr{D}_p \subset \mathscr{B}_p$. Let $f \in \mathscr{P}$. Then

$$P_r * f = (1 - r) \sum_0^\infty (D_n * f) r^n,$$

so, if $\mu \in \mathscr{D}_p$

$$\| P_r * f \| \leqq (1 - r) \sum_0^\infty \| D_n * f \| r^n$$

$$\leqq K(1 - r) \sum_0^\infty \| f \| r^n = K \| f \|,$$

and thus $\mu \in \mathscr{B}_p$.

Babenko [1] has shown that the measures $w_\alpha(e^{i\phi}) d\sigma(\phi) = |\phi|^\alpha d\sigma$, $-\pi < \phi \leqq \pi$, $-1 < \alpha < p - 1 > 0$ are in $\mathscr{D}_p$. Thus they are also in $\mathscr{B}_p$.

The following theorem indicates how certain types of local conditions on w are sufficient to guarantee that $\mu \in \mathscr{B}_p$.

Theorem 5. *Let $w \in L^1(d\sigma)$ and suppose that for each point $\phi_0 \in C$ there exists some measure $v(e^{i\phi}) d\sigma(\phi) \in \mathscr{B}_p$ and constants δ_1, δ_2 such that*

$$0 < \delta_1 v(e^{i\phi}) \leqq w(e^{i\phi}) \leqq \delta_2 v(e^{i\phi})$$

for all ϕ in a neighborhood of ϕ_0. Then $w d\sigma \in \mathscr{B}_p$.

Proof. This follows from Theorem 4 and the Borel-Lebesgue theorem. It is clear that if (5.2) holds for all sufficiently small $h > 0$, and since $w^{1-q} \in L^1(d\sigma)$, necessarily (5.2) holds for all h, $0 < h \leqq \pi$.

When applying Theorem 5 the functions w_α of above or any positive constant function are eligible v's.

It should be noted that if $f \in L^2(d\mu)$ where $\mu \in \mathscr{B}_p$, then

$$\int |f|\, d\sigma = \int |f|\, w^{1/p} w^{-1/p} d\sigma$$
$$\leqq \|f\|_p \cdot \left(\int w^{1-q} d\sigma\right)^{1/q} < \infty,$$

so $f \in L^1(d\sigma)$. Thus the Fourier coefficients and Abel means which are obtained by completion as described in the introduction coincide with the Fourier coefficients and Abel means for $L^1(d\sigma)$ functions. The story is different for $\mu \in \mathscr{Q}_p$. Any $f \in H^p(d\mu)$, $\mu \in \mathscr{Q}_p$, is holomorphic in $|z| < 1$, in fact $f \cdot g^{1/p} \in H^p(d\sigma)$. Thus f is of Nevanlinna class [11, p. 271]. Thus the Fourier coefficients and Abel means are those of Fourier power series. Furthermore f need not be in $L^1(d\sigma)$.

THEOREM 6. *The functions* $w_\alpha(e^{i\phi}) = |\phi|^\alpha$, $-\pi < \phi \leqq \pi$, $\alpha > -1$ *are such that* $w_\alpha\, d\sigma \in \mathscr{Q}_p$.

Proof. If $-1 < \alpha < n - 1 > 0$, $w_\alpha \in \mathscr{B}_n \subset \mathscr{Q}_n = \mathscr{Q}_p$.

Local conditions similar to those of Theorem 5 can be imposed on w that are sufficient to guarantee that $\mu \in \mathscr{Q}_p$.

As observed in the introduction T is bounded and has a bounded inverse. This is so because $|||f|||_p$ and $\|f\|_p$ can be viewed as being equivalent norms for $L^p_0(d\mu)$. We next show that if T is an isometry then necessarily μ is a multiple of Lebesgue measure. This is a consequence of the following

THEOREM 7. *Let* $p > 0$. *Suppose that for some* r, $0 \leqq r < 1$

$$\|P_r * f\|_p \leqq \|f\|_p \tag{6.1}$$

for all $f \in \mathscr{P}_0$. *Then* μ *is a multiple of Lebesgue measure.*

Proof. Normalize μ so $\int d\mu = 1$. Assume (6.1) holds for some $p > 0$. Then it holds for all $p > 0$ and upon letting $p \to 0$ we obtain

$$\exp \int \log |f(re^{i\phi})|\, d\mu(\phi) \leqq \exp \int \log |f|\, d\mu(\phi)$$

so

$$\int (\log |f|) \cdot (P_r * d\mu) d\sigma \leqq \int \log |f|\, d\mu(\phi)$$

for all $f \in \mathscr{P}_0$. By putting $f(z) = \exp \pm (1 + se^{-i\alpha}z)/(1 - se^{-i\alpha}z)$, $0 \leqq s < 1$ we see that

$$P_s * (P_r * d\mu) = P_s * d\mu,$$

so

$$P_{rs} * d\mu = P_s * d\mu,$$

and by comparing Fourier expansions we see that $\mu = \sigma$.

In closing we suggest that the following problems merit further study:

(i) Find representation theorems for the elements of $\mathscr{Q}_p$ and $\mathscr{B}_p$;

(ii) Solve problems A and B for wider classes of approximate identities;

(iii) Characterize the measures μ for which $f \to \sup_r |P_r * f|$ is a bounded operation.

(iv) Extend the results to several variables and more general groups.

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