

FOURIER TRANSFORMS OF CERTAIN CLASSES OF INTEGRABLE FUNCTIONS⁽¹⁾

BY
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1. Introduction. Let G be an arbitrary locally compact abelian group with character group $\hat{G}$. This paper is devoted to characterizing those functions $\phi \in L_\infty(\hat{G})$ which are equal almost everywhere (a.e.) to the Fourier transform of some function $f \in L_1(G) \cap L_p(G)$ where $1 \leq p \leq \infty$. The characterizations presented originate from the following theorem by I. J. Schoenberg [12]:

THEOREM 1. *Let $V(-\infty, \infty)$ denote the set of all functions of bounded variation on the real line, and for $\mu \in V(-\infty, \infty)$ define $\hat{\mu}$ by $\hat{\mu}(x) = \int_{-\infty}^{\infty} e^{-ixy} d\mu(y)$. If $\phi \in L_\infty(-\infty, \infty)$ then $\phi(x) = \hat{\mu}(x)$ a.e. for some $\mu \in V(-\infty, \infty)$ if and only if there exists a constant $K > 0$ such that*

$$\left| \int_{-\infty}^{\infty} f(x) \phi(x) dx \right| \leq K \sup_{-\infty < x < \infty} \left| \int_{-\infty}^{\infty} e^{-ixy} f(y) dy \right|$$

for all $f \in L_1(-\infty, \infty)$.

This theorem characterizes those functions $\phi \in L_\infty(-\infty, \infty)$ of the form $\phi(x) = \hat{\mu}(x)$ a.e. in terms of a continuity condition on the linear functional defined by $F(f) = \int_{-\infty}^{\infty} f(x) \phi(x) dx$ for $f \in L_1(-\infty, \infty)$. Given a subset $N \subset V(-\infty, \infty)$ it is possible to ask whether additional continuity conditions on F can be found which, combined with the above condition, are necessary and sufficient in order that $\phi(x) = \hat{\mu}(x)$ a.e. for some $\mu \in N$. The following theorem by A. C. Berry [1] illustrates such a condition when N is the class absolutely continuous functions of bounded variation. Here we define

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-ixy} f(y) dy \text{ for } f \in L_1(-\infty, \infty).$$

Presented to the Society, November 19, 1960; received by the editors October 11, 1960, and, in revised form, October 6, 1961.

⁽¹⁾ This paper is taken from my doctoral dissertation written under the direction of Professor W.A.J. Luxemburg at the Claifornia Institute of Technology. The original work was done while I was a National Science Foundation Cooperative Graduate Fellow. The preparation for publication and the revision was partially supported by N. S. F. Grant No. G-14002 and by the U. S. Army's Mathematical Research Center, University of Wisconsin under contract DA-11-022-ORD-2059.

THEOREM 2. Suppose that $\phi \in L_\infty(-\infty, \infty)$. The following conditions are necessary and sufficient in order that $\phi(x) = \hat{g}(x)$ a.e. for some $g \in L_1(-\infty, \infty)$:

(1) There exists a constant $K > 0$ such that $|F(f)| \leq K \|f\|_\infty$ for all $f \in L_1(-\infty, \infty)$.

(2) For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|F(f)| \leq \varepsilon \|f\|_\infty$ whenever $f \in L_1(-\infty, \infty)$, $\hat{f} \in L_1(-\infty, \infty)$ and $\|\hat{f}\|_1 \leq \delta \|f\|_\infty$.

Condition (1) is just a restatement of Theorem 1; it is this part of the hypothesis which insures that ϕ be equal a.e. to the Fourier transform of some $\mu \in V(-\infty, \infty)$. Condition (2) implies that μ is absolutely continuous.

A theorem similar to Berry's theorem was previously proved for the circle group by R. Salem [10; 11].

THEOREM 3. Let (Z) be the class of functions $\omega(x) = \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$ which are continuous and differentiable with $|\omega(x)| < 1$ and such that the Fourier series of ω' is absolutely convergent. The following conditions are necessary and sufficient in order that (a_n, b_n) be the Fourier coefficients of an integrable function:

(A) The formally integrated series $\sum_{n=1}^{\infty} (a_n n^{-1} \sin nx - b_n n^{-1} \cos nx)$ converges to a continuous function.

(B) The expression $\sum_{n=1}^{\infty} (a_n \alpha_n + b_n \beta_n)$ tends to zero when ω varies in (Z) in such a way that $\sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2)$ tends to zero.

Here condition (A) implies that $(-b_n n^{-1}, a_n n^{-1})$ is the set of Fourier coefficients of some continuous function. Condition (B) implies that this function is absolutely continuous and hence an integral of some function $f \in L_1(0, 2\pi)$. It then follows that (a_n, b_n) is the set of Fourier coefficients for f .

In this paper we prove for an arbitrary locally compact abelian group a theorem which gives as a special case the second part of Berry's theorem. This, combined with the generalization of Schoenberg's theorem, constitutes a characterization of the Fourier transforms of $L_1(G)$. We also include a slightly different statement and proof of Salem's theorem. Using the same ideas we characterize the Fourier transforms of $L_1(G) \cap L_p(G)$ for $1 < p \leq \infty$. The statement and proof of Theorem 1 generalize directly for a locally compact abelian group (see Eberlein [3]). Thus, in the statements of our theorems we will generally assume that ϕ , the function under investigation, is equal almost everywhere to the Fourier transform of some bounded Radon measure μ . Here we are concerned with the additional conditions which ϕ must satisfy in order that $d\mu(x) = f(x)dx$ for some $f \in L_1(G) \cap L_p(G)$.

§ 2 contains notation and definitions; § 3 treats $L_1(G)$; § 4 is devoted to $L_1(G) \cap L_p(G)$ for $1 < p \leq \infty$.

The author wishes to thank the referee for the very simple proof of Lemma 1 and the useful comments on the proof of Theorem 5.

2. **Preliminaries.** Throughout this paper G denotes an arbitrary locally compact abelian group, and the group operation is denoted by $+$. The character group of G is denoted by $\hat{G}$. The complex number $(x, \hat{x})$ is the value of the character $\hat{x} \in \hat{G}$ at the point $x \in G$.

The space of all bounded, continuous, complex valued functions defined on G is denoted by $C(G)$. We give $C(G)$ the usual norm:

$$\|f\|_{\infty} = \sup_{x \in G} |f(x)|, \quad f \in C(G).$$

$C_{\infty}(G)$ denotes the subspace of functions $f \in C(G)$ which vanish at infinity, i.e., for each $f \in C_{\infty}(G)$ and $\varepsilon > 0$ there exists a compact set $A \subset G$ such that $|f(x)| < \varepsilon$ for $x \in A'$ (= the complement of A). $C_{\infty\infty}(G)$ denotes the set of functions in $C_{\infty}(G)$ which have compact support.

Let $\mathcal{B}$ denote the smallest σ -algebra of subsets of G containing the compact subsets⁽²⁾, and let $M(G)$ denote the space of all complex valued, bounded, regular and countably additive set functions defined on $\mathcal{B}$. $M(G)$ is identified with the space of all bounded linear functionals on $C_{\infty}(G)$, and an element $\mu \in M(G)$ is called a bounded Radon measure. For detailed discussions of $M(G)$ and references we refer the reader to the survey articles by Hewitt [7] and Rudin [9].

Let m be a nontrivial Haar measure defined on $\mathcal{B}$. $L_p(G)$ for $1 \leq p < \infty$ denotes the space of all $\mathcal{B}$ -measurable, complex valued functions defined on G for which

$$\|f\|_p = \left(\int_G |f(x)|^p dm(x) \right)^{1/p} < +\infty.$$

$L_{\infty}(G)$ denotes the space of $\mathcal{B}$ -measurable functions such that

$$\|f\|_{\infty} = \inf \left\{ \alpha \mid m \{x \mid x \in G, |f(x)| > \alpha\} = 0 \right\} < +\infty.$$

(If $f \in C(G)$ then the two definitions of $\|f\|_{\infty}$ agree.)

There corresponds to each $f \in L_1(G)$ a unique measure $\mu_f \in M(G)$ defined by $\mu_f(E) = \int_E f(x) dx$ for all $E \in \mathcal{B}$ ⁽³⁾. By the Radon-Nikodym theorem $\mu = \mu_f$ for some $f \in L_1(G)$ if and only if μ is absolutely continuous with respect to m .

The convolution of two functions $f \in L_1(G)$ and $g \in L_p(G)$ with $1 \leq p \leq \infty$ is defined in the usual way:

$$f * g(x) = \int_G f(x - y) g(y) dy.$$

This integral exists for almost every $x \in G$ and defines a function $f * g \in L_p(G)$ with $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. If $\{u_{\alpha} \mid \alpha \in \mathcal{A}\}$ is an approximate identity for the

(2) See Halmos [5] for measure theoretic terminology not explained here.

(3) The differential of Haar measure will be written dx , dy etc.

algebra $L_1(G)$ then it is well known that $\lim_{\alpha} \|u_{\alpha} * f - f\|_p = 0$ for all $f \in L_p(G)$ with $1 \leq p < \infty$. (See Loomis [8].) Since G is a locally compact Hausdorff space we may assume that $u_{\alpha} \in C_{\infty}(G)$ for all $\alpha \in \mathcal{A}$. If $f \in C(G)$ is uniformly continuous it is easy to show that $\lim_{\alpha} u_{\alpha} * f(x) = f(x)$ uniformly for $x \in G$.

The following notation will be used for the various Fourier transforms:

$$\hat{\mu}(\hat{x}) = \int_G (-x, \hat{x}) d\mu(x), \quad \mu \in M(G).$$

$$\hat{f}(\hat{x}) = \int_G (-x, \hat{x}) f(x) dx, \quad f \in L_1(G).$$

If $g \in L_1(\hat{G})$ we will write

$$\hat{g}(x) = \int_{\hat{G}} (x, \hat{x}) g(\hat{x}) d\hat{x}.$$

The Haar measure on G is normalized so that $\int_G |f(x)|^2 dx = \int_{\hat{G}} |\hat{f}(\hat{x})|^2 d\hat{x}$ for $f \in L_1(G) \cap L_2(G)$. If $N \subset M(G)$ then $N^{\wedge}$ denotes the set of functions $\hat{\mu}$ where $\mu \in N$.

$P(G)$ denotes the set of continuous, positive definite functions defined on G , and $[L_1(G) \cap P(G)]$ denotes the linear space spanned by $L_1(G) \cap P(G)$. If $f \in [L_1(G) \cap P(G)]$ then $\hat{f} = g \in L_1(\hat{G})$ and $f(x) = \hat{g}(x)$ for all $x \in G$ [8]. From this it is seen that $[L_1(G) \cap P(G)]^{\wedge} = [L_1(\hat{G}) \cap P(\hat{G})]$. A simple argument using an approximate identity shows that $[L_1(G) \cap P(G)]$ is dense in $L_p(G)$ for $1 \leq p < \infty$. Since $(L_1(\hat{G}))^{\wedge}$ is dense in $C_{\infty}(G)$ it follows that $[L_1(G) \cap P(G)]^{\wedge} = [L_1(\hat{G}) \cap P(\hat{G})]^{\wedge}$ is dense in $C_{\infty}(G)$.

3. $L_1(G)$. Throughout this section we assume that $\phi \in L_{\infty}(\hat{G})$ is of the form $\phi(\hat{x}) = \hat{\mu}(\hat{x})$ a.e. for some $\mu \in M(G)$. Theorems 4 and 5 present respectively necessary and sufficient conditions on ϕ in order that $\phi(\hat{x}) = \hat{f}(\hat{x})$ a.e. for some $f \in L_1(G)$. A special case of these theorems, combined with the generalization of Theorem 1, gives Berry's theorem.

LEMMA 1. If $f \in L_1(G)$ then the linear functional defined by

$$F(g) = \int_G g(x) f(x) dx$$

for $g \in L_{\infty}(G)$ satisfies the following condition;

For every p with $1 \leq p < \infty$ and every $\varepsilon > 0$ there exists a $\delta > 0$ depending only upon p, ε and f and such that

$$|F(g)| \leq \varepsilon \|g\|_{\infty}$$

whenever $g \in L_p(G) \cap L_{\infty}(G)$ and

$$\|g\|_p \leq \delta \|g\|_{\infty}.$$

Proof. Let p and ε be given and fixed. Assume that $\|g\|_{\infty} = 1$. Then we must find a $\delta > 0$ such that $\|g\|_p \leq \delta$ implies $|F(g)| \leq \varepsilon$. If this were not possible

there would exist a sequence of functions $\{g_n\}$ with $\|g_n\|_\infty = 1$, $\|g_n\|_p < 1/n$ and $|F(g_n)| > \varepsilon > 0$. Then there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \rightarrow 0$ a.e. on the set where $f(x) \neq 0$. Hence by the theorem on dominated convergence $F(g_{n_k}) = \int_G g_{n_k}(x)f(x)dx$ tends to zero. This contradiction proves the lemma.

THEOREM 4. *If $\phi = f^\wedge$ a.e. for some $f \in L_1(G)$ then the linear functional defined for $g \in L_1(\hat{G})$ by*

$$F(g) = \int_{\hat{G}} g(\hat{x}) \phi(\hat{x}) d\hat{x}$$

satisfies the following condition:

For every p with $1 \leq p < \infty$ and every $\varepsilon > 0$ there exists a $\delta > 0$ depending only upon ε , p and ϕ and such that

$$|F(g)| \leq \varepsilon \|\hat{g}\|_\infty$$

whenever $g \in [L_1(\hat{G}) \cap P(\hat{G})]$ and

$$\|\hat{g}\|_p \leq \delta \|\hat{g}\|_\infty.$$

Proof. If $g \in [L_1(\hat{G}) \cap P(\hat{G})]$ then $\hat{g} \in [L_1(G) \cap P(G)]$ and hence $g \in L_p(G) \cap L_\infty(G)$. By Fubini's theorem

$$F(g) = \int_{\hat{G}} g(\hat{x}) \phi(\hat{x}) d\hat{x} = \int_G \hat{g}(x) f(-x) dx.$$

The result now follows from the lemma.

THEOREM 5. *Suppose $\phi = \hat{\mu}$ a.e. for some $\mu \in M(G)$. Then $\phi = f^\wedge$ a.e. for some $f \in L_1(G)$ if the functional defined by*

$$F(g) = \int_{\hat{G}} g(\hat{x}) \phi(\hat{x}) d\hat{x}$$

for $g \in L_1(\hat{G})$ satisfies the following condition:

There exists a p with $1 \leq p < \infty$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ depending only upon ε , p and ϕ and such that

$$|F(g)| \leq \varepsilon \|\hat{g}\|_\infty$$

whenever $g \in [L_1(\hat{G}) \cap P(\hat{G})]$ and

$$\|\hat{g}\|_p \leq \delta \|\hat{g}\|_\infty.$$

Proof. If $g \in [L_1(\hat{G}) \cap P(\hat{G})]$ then by Fubini's theorem

$$F(g) = \int_{\hat{G}} (\hat{x}) \phi(\hat{x}) d\hat{x} = \int_G \hat{g}(-x) d\mu(x).$$

Hence it is sufficient to show that if μ is not absolutely continuous then there exists a sequence of functions $\{g_n\}$ with $g_n \in [L_1(G) \cap P(G)] = [L_1(\hat{G}) \cap P(\hat{G})]^\wedge$

and such that $\|g_n\|_\infty \rightarrow 1$, $\|g_n\|_p \rightarrow 0$ but such that $\int_G g_n(x) d\mu(x)$ does not tend to zero as $n \rightarrow \infty$. Here p is an arbitrary number with $1 \leq p < \infty$. It is also sufficient to assume that μ is a real valued measure since we will make the functions g_n real valued.

Now assume that μ has a nontrivial singular part λ . In general $\|\lambda\| = \sup \{ |\int_G h(x) d\lambda(x)| \mid h \in C_{\infty\infty}(G), h \text{ is a real valued and } \|h\|_\infty = 1 \}$. However since λ is singular with respect to the Haar measure m the supremum may be taken over just those functions for which $m(\text{support of } h) \leq \eta$ where η is an arbitrary positive number. In order to see this assume that λ is positive. The general case involves considering the positive and negative variations of λ . Then since λ is regular $\|\lambda\| = \sup \{ \lambda(K) \mid K \subset G \text{ is compact, } m(K) = 0 \}$ where $\lambda(K) = \inf \{ \int_G g(x) d\lambda(x) \mid g \in C_{\infty\infty}(G), g(x) = 1 \text{ for } x \in K \text{ and } 0 \leq g(x) \leq 1 \}$. Since $m(K) = 0$ it is clear that the functions in this last expression may be taken with $m(\text{support of } g) \leq \eta$, $\eta > 0$ and η independent of K .

Thus for every $n = 1, 2, 3, \dots$ there exists a real valued function $h_n \in C_{\infty\infty}(G)$ such that $\|h_n\|_\infty = 1$, $\|\lambda\| - |\int_G h_n(x) d\lambda(x)| \leq n^{-1}$ and $m(\text{support of } h_n) \leq n^{-1}$. For each h_n we can choose a real valued function $u_n \in C_{\infty\infty}(G)$ from an approximate identity such that

$$|h_n(x) - h_n * u_n(x)| \leq n^{-1}.$$

The functions $g_n = h_n * u_n$ then have the required properties. In particular being a convolution of functions in $C_{\infty\infty}(G)$, $g_n \in [L_1(G) \cap P(G)]$ and

$$\|g_n\|_p \leq \|h_n\|_p \|u_n\|_1 = \|h_n\|_p \leq [m(\text{support of } h_n)]^{1/p} \leq n^{-1/p}.$$

From $|h_n(x) - g_n(x)| \leq n^{-1}$ it follows that

$$\|g_n\|_\infty \rightarrow 1 \text{ and } \int_G g_n(x) d\lambda(x) \rightarrow \|\lambda\| \text{ as } n \rightarrow \infty.$$

Writing

$$\int_G g_n(x) d\mu(x) = \int_G g_n(x) d\lambda(x) + \int_G g_n(x) d(\mu - \lambda)(x)$$

we see that $\int_G g_n(x) d\mu(x) \rightarrow \|\lambda\| \neq 0$ since by Theorem 4 the second integral tends to zero. This proves the theorem.

Theorems 4 and 5 combined with Theorem 1 reduce to Berry's theorem (Theorem 2) by letting G be the real line and $p = 1$.

We now apply Theorem 5 to prove Salem's theorem with slightly weaker hypothesis. The notation is the same as in the statement of Theorem 3.

THEOREM 6. *Suppose that*

$$-b_n n^{-1} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos nx \, dx, \quad a_n n^{-1} = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \dots,$$

for some continuous function f defined on the unit circle. Then f is an absolutely continuous function of bounded variation and

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f'(x) \cos nx \, dx, \quad b_n = \frac{1}{2\pi} \int_0^{2\pi} f'(x) \sin nx \, dx$$

if and only if condition (B) of Theorem 3 holds.

Proof. We will prove only the sufficiency of the condition.

Let Z be the class of all functions $g(x) = \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$ for which g' exists and has an absolutely convergent Fourier series. Then $g \in (Z)$ if and only if $g \in Z$ and $\|g\|_{\infty} \leq 1$.

For $g \in Z$ define the linear functional F by

$$F(g) = -\frac{1}{2\pi} \int_0^{2\pi} f(x) g'(x) dx = \sum_{n=1}^{\infty} (a_n \alpha_n + b_n \beta_n).$$

The continuity condition (B) and the fact that

$$\sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2) = \frac{1}{2\pi} \int_0^{2\pi} |g(x)|^2 dx \leq \|g\|_{\infty}^2$$

imply immediately that F is a bounded linear functional on Z . Since Z is dense in $C(0, 2\pi)$ F can be uniquely extended to a bounded linear functional F defined on $C(0, 2\pi)$. By the Riesz representation theorem for linear functional of $C(0, 2\pi)$ there exists a function of bounded variation α such that

$$F(g) = \frac{1}{2\pi} \int_0^{2\pi} g(x) d\alpha(x), \quad g \in C(0, 2\pi).$$

Letting $g(x)$ be $\cos nx$ and $\sin nx$ gives

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \cos nx d\alpha(x) \text{ and } b_n = \frac{1}{2\pi} \int_0^{2\pi} \sin nx d\alpha(x).$$

This proves that (a_n, b_n) is the set of Fourier coefficients of a function of bounded variation. The theorem now follows from Theorem 5 by taking G to be the circle group and $p = 2$ and observing that if condition (B) holds for $g \in Z$ then it must also hold for $g \in [L_1(0, 2\pi) \cap P(0, 2\pi)]$.

4. $L_1(G) \cap L_p(G)$ for $1 < p \leq \infty$. Presented in this section are two necessary and sufficient conditions on $\phi \in L_{\infty}(\hat{G})$ in order that $\phi = \hat{f}$ a.e. for some $f \in L_1(G) \cap L_p(G)$: Theorem 7 gives a continuity condition on the functional $F(g) = \int_{\hat{G}} g(\hat{x}) \phi(\hat{x}) d\hat{x}$ for $g \in L_1(\hat{G})$; Theorem 9 presents a multiplier condition on ϕ .

THEOREM 7. Suppose that $\phi = \hat{\mu}$ a.e. for some $\mu \in M(G)$ and define

$$F(g) = \int_{\hat{G}} g(\hat{x}) \phi(\hat{x}) d\hat{x}$$

for $g \in L_1(\hat{G})$. In order that $\phi = \hat{f}$ a.e. for some $f \in L_1(G) \cap L_p(G)$ with $1 < p \leq \infty$ it is necessary and sufficient that there exist a constant $K > 0$ such that

$$|F(g)| \leq K \|\hat{g}\|_q$$

for all $g \in [L_1(\hat{G}) \cap P(\hat{G})]$ where $1/p + 1/q = 1$.

Proof. A proof of the necessity of the condition is readily constructed by using the techniques of the sufficiency proof in reverse. We therefore proceed directly to a proof of the sufficiency.

Assume that there exists a $K > 0$ such that

$$|F(g)| \leq K \|\hat{g}\|_q \text{ for all } g \in [L_1(\hat{G}) \cap P(\hat{G})].$$

Define the linear functional H on $[L_1(\hat{G}) \cap P(\hat{G})]^\wedge = [L_1(G) \cap P(G)]$ by

$$H(\hat{g}) = F(g).$$

Since Fourier transforms are unique H is well defined. Furthermore

$$|H(g)| \leq K \|g\|_q$$

for all $g \in [L_1(G) \cap P(G)]$. This shows that H is a bounded linear functional defined on a dense subset of $L_q(G)$, and hence H can be extended uniquely to all of $L_q(G)$ without changing its norm. Let $\bar{H}$ be the extension of H . Then there exists a unique function $f \in L_p(G)$ such that

$$\bar{H}(g) = \int_G g(x)f(-x)dx$$

for all $g \in L_q(G)$. If $g \in [L_1(\hat{G}) \cap P(\hat{G})]$

$$F(g) = \int_{\hat{G}} g(\hat{x}) \phi(\hat{x}) d\hat{x} = \int_{\hat{G}} g(\hat{x}) \left[\int_G (-x, \hat{x}) d\mu(x) \right] d\hat{x} = \int_G \hat{g}(-x) d\mu(x).$$

Since $F(g) = \bar{H}(\hat{g})$ we have

$$\int_G \hat{g}(-x) d\mu(x) = \int_G \hat{g}(x)f(-x)dx = \int_G \hat{g}(-x)f(x)dx$$

for all $\hat{g} \in [L_1(\hat{G}) \cap P(\hat{G})]^\wedge = [L_1(G) \cap P(G)]$. The fact that $[L_1(G) \cap P(G)]$ is dense in $L_q(G)$ and $C_\infty(G)$ implies that

$$\int_G g(x) d\mu(x) = \int_G g(x)f(x)dx$$

for all $g \in C_\infty(G)$. From this it follows that

$$\|f\|_1 = \sup \left\{ \left| \int_G g(x)f(x)dx \right| \mid g \in C_\infty(G), \|g\|_\infty \leq 1 \right\} < +\infty.$$

Thus $f \in L_1(G)$ as well as $L_p(G)$, and $\hat{\mu}(\hat{x}) = \hat{f}(\hat{x})$ for all $\hat{x} \in \hat{G}$. This proves the theorem.

If G is compact it is not necessary to assume that $\phi = \hat{\mu}$ a.e. for some $\mu \in M(G)$. In this case $\|\hat{g}\|_q \leq \|\hat{g}\|_\infty$ for $g \in L_1(\hat{G})$, and the condition $|F(g)| \leq K \|\hat{g}\|_q$

implies that $|F(g)| \leq K \|\hat{g}\|_\infty$. Since for G compact $L_1(\hat{G}) = [L_1(\hat{G}) \cap P(\hat{G})]$ this holds for all $g \in L_1(\hat{G})$. The generalization of Theorem 1 then insures that $\phi = \hat{\mu}$ for some $\mu \in M(G)$.

In the case that G is compact Theorem 7 implies the Riesz-Fisher theorem. For let ϕ be an element of $L_2(\hat{G})$. Then if $g \in [L_1(\hat{G}) \cap P(\hat{G})]$ we have $|F(g)| \leq \|\phi\|_2 \|g\|_2 = \|\phi\|_2 \|\hat{g}\|_2$, and from Theorem 7, $\phi = \hat{f}$ for some $f \in L_1(G) \cap L_2(G) = L_2(G)$.

The next theorem is related to certain older results by H. Cramér [2]. We omit the proof since it is almost a direct consequence of Theorem 7.

THEOREM 8. *Let $\{f_\alpha\}$ be a net of Fourier transforms with $f_\alpha \in L_1(G) \cap L_p(G)$ where $1 < p \leq \infty$, and such that $\|f_\alpha\|_p \leq K < +\infty$. If $\phi = \hat{\mu}$ a.e. for some $\mu \in M(G)$ and if*

$$\lim_\alpha \int_{\hat{G}} g(\hat{x}) f_\alpha(\hat{x}) d\hat{x} = \int_{\hat{G}} g(\hat{x}) \phi(\hat{x}) d\hat{x}$$

for all $g \in [L_1(\hat{G}) \cap P(\hat{G})]$ then $\phi = \hat{f}$ a.e. for some $f \in L_1(G) \cap L_p(G)$.

The final theorem is an addition to the extensive literature on multipliers or factor functions. The proof is modeled after the proof of a similar theorem by Helson [6].

THEOREM 9. *Suppose that $\phi \in L_\infty(\hat{G})$. Then $\phi = \hat{f}$ a.e. for some $f \in L_1(G) \cap L_p(G)$ with $1 < p \leq \infty$ if and only if $\phi \cdot \hat{g} \in (L_1(G) \cap L_p(G))^\wedge$ for all $g \in L_1(G)$.*

Proof. The necessity of the condition is just the fact that $f * g \in L_1(G) \cap L_p(G)$ and $(f * g)^\wedge = \hat{f} \cdot \hat{g}$. To show the sufficiency we first observe that the condition implies that $\phi \cdot \hat{g} \in (L_1(G))^\wedge$ for all $g \in L_1(G)$. Thus by Helson's theorem [6] $\phi = \hat{\mu}$ a.e. for some $\mu \in M(G)$. This μ defines a bounded linear transformation $g \rightarrow \mu * g$ of $L_1(G)$ into $L_1(G)$ with

$$\|\mu * g\|_1 \leq \|\mu\| \|g\|_1, \quad g \in L_1(G).$$

The condition of the theorem implies by means of the closed graph theorem that this transformation is also bounded from $L_1(G)$ into $L_p(G)$. Thus there exists a constant $K > 0$ such that

$$\|\mu * g\|_p \leq K \|g\|_1, \quad g \in L_1(G).$$

Let $\{u_\alpha\}$ be an approximate identity for $L_1(G)$. Then for $g \in [L_1(\hat{G}) \cap P(\hat{G})]$

$$F(g) = \int_{\hat{G}} g(\hat{x}) \phi(\hat{x}) d\hat{x} = \lim_\alpha \int_{\hat{G}} g(\hat{x}) \hat{\mu}(\hat{x}) u_\alpha(\hat{x}) d\hat{x} = \lim_\alpha \int_G \hat{g}(-x) \mu * u_\alpha(x) dx.$$

Combining the above inequalities give

$$|F(g)| \leq \lim_\alpha \|\mu * u_\alpha\|_p \|\hat{g}\|_q \leq \lim_\alpha K \|u_\alpha\|_1 \|\hat{g}\|_q, \text{ and since } \|u_\alpha\|_1 = 1$$

we get

$$|F(g)| \leq K \|\hat{g}\|_q$$

for all $g \in [L_1(\hat{G}) \cap P(\hat{G})]$. The result now follows directly from Theorem 7.

Theorem 9 can also be proved by using a result of Edwards [4] concerning the form of bounded linear transformations from $L_1(G)$ into $L_p(G)$ which commute with translations. From Edwards' theorem and $\|\mu * g\|_p \leq K \|g\|_1$ it follows that $\mu * g = f * g$ for all $g \in L_1(G)$ where $f \in L_p(G)$. It is a simple consequence that $\hat{\mu} = \hat{f}$.

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