

ON THE ALGEBRAIC CLOSURE OF CERTAIN PARTIALLY ORDERED FIELDS⁽¹⁾

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Introduction. This paper is divided into five parts:

- Part I. Graduated fields.
- Part II. Fields of meromorphic functions.
- Part III. Examples for Part II.
- Part IV. Applications to algebraic differential equations.
- Part V. Appendix.

Part I deals with an abstract algebraic structure called a *graduated field*, which may be considered as a generalization of a field having a non-Archimedean valuation which is trivial over the prime field. (Cf. Appendix, Note 1.) A graduated field is an ordered quadruple $(\mathcal{K}_0, <, \mathcal{U}, \mathcal{C}_0)$ in which $\mathcal{K}_0$ is a commutative field of characteristic zero, $<$ is a partial order relation (cf. Appendix, Note 2) in $\mathcal{K}_0$, $\mathcal{C}_0$ is a subfield of $\mathcal{K}_0$, $\mathcal{U}$ is a subgroup of the multiplicative group $\mathcal{K}$ (cf. the notational conventions in Appendix, Note 3), and various hypotheses are made about the interrelations of $<$ with $\cdot$ and $+$ in $\mathcal{K}_0$, in $\mathcal{C}_0$, and in $\mathcal{U}$ (§§1,2). In terms of $<$ an equivalence relation $\sim$ is introduced into $\mathcal{K}$ by the definition: $f \sim g$ means $f - g < g$. Two subgroups $\mathcal{M}$ and $\mathcal{A}$ of $\mathcal{K}$ are defined by setting $\mathcal{M}$ equal to the product group $\mathcal{C}_0\mathcal{U}$ and by setting $\mathcal{A} = \{f: \exists m \in \mathcal{M} \text{ such that } f \sim m\}$. Polynomials whose coefficients lie in a field $\mathcal{F}_0$ satisfying $\mathcal{M} \subset \mathcal{F}_0 \subset \mathcal{A}_0$ are said to be *asymptotically inscribed*.

For asymptotically inscribed polynomials F the element p of $\mathcal{M}$ is called a *point of stability* if $q \sim p$ implies $F(q) \sim F(p)$, a *point of instability* otherwise. If p is a point of instability for F , its *multiplicity* is defined to be the minimum s such that p is a point of stability for the s th derivative $F^{(s)}$. The principal result in Part I is Theorem I, §8, which states, in effect, that every graduated field having the following two properties (a) and (b):

- (a) $\mathcal{C}_0$ is algebraically closed, and
- (b) whenever an asymptotically inscribed polynomial F has a point of instability p , of multiplicity 1, then F has a root y with $y \sim p$;

must also have the following property (c) as well:

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(c) every asymptotically inscribed polynomial of degree n has n roots in $\mathcal{A}_0$. (Cf. Appendix, Note 4.)

The crux of the proof of Theorem I is Fundamental Lemma A, §38, which, in the case where $\mathcal{C}_0$ is algebraically closed, implies that if p is a point of instability for F , of multiplicity $k \geq 2$, and if, for some s in $\{1, 2, \dots, k-1\}$, q is a root of $F^{(s)}$ such that $q \sim p$, then the polynomial $F(q+z)$ either has the solution $z=0$ or has a point of instability of multiplicity $\leq k-1$. Since $F^{(s)}$ is of lower degree than F , this lemma makes it possible to obtain (§§39–50) the proof of Theorem I by means of inductive arguments on the degree and the multiplicity.

In Part II, Theorem I is applied to prove a theorem (Theorem II, §62) on the algebraic closure of certain fields of meromorphic functions. For this application, $\mathcal{K}_0$ is the field (cf. Appendix, Note 5) of all functions meromorphic over a system $\bar{N}$ of regions in the complex plane, and $\mathcal{C}_0 = \mathbf{C}$ (cf. Appendix, Note 6). $<$ and $\mathcal{U}$ are obtained by means of constructions (§§56–58, and §§59–61, 63–64, respectively) which permit a great deal of choice. For the graduated fields created in this way, hypothesis (a) above is automatically satisfied, and an easy use of the implicit function theorem suffices to verify hypothesis (b). The conclusion (c) in this context (in which $\sim$ is always some concept of *asymptotic* equivalence), then gives, for polynomials whose coefficients have prescribed asymptotic behavior or expansions, very precise information on the asymptotic behavior or expansions of the roots.

In Part III, various explicit choices of $\mathcal{U}$ and $<$ are made, to illustrate Theorem II.

A typical choice of $\mathcal{U}$ (§70) causes the multiplicative group $\mathcal{M}$ to be the set of all “logarithmic monomials” $cx^{r_0}(\log x)^{r_1}(\log \log x)^{r_2} \dots (\log_p x)^{r_p}$ with $c \in \mathbf{C} - \{0\}$, r_j real, and p fixed ($\mathcal{U}$ then being the set of all such functions with $c=1$).

In §71, this choice of $\mathcal{U}$ is fixed, and it is indicated that the aspect of Theorem II changes in a marked, even qualitative, manner, as the choice of $<$ is varied.

In Part IV a brief sketch is given of some applications of these theorems and methods to obtain new results in the theory of algebraic differential equations.

PART I. GRADUATED FIELDS

1. DEFINITION. Let $\mathcal{K}_0$ be a commutative field of characteristic zero. Let $<$ be a partial order of $\mathcal{K}_0$ such that

- (1.1) If $f < g$, then $f \neq g$.
- (1.2) $0 < 1$.
- (1.3) If $f < g$ and $h \neq 0$, then $fh < gh$.
- (1.4) If $f < g$ and $h < g$ then $f-h < g$.
- (1.5) If $f < g$ and $h < g$ then $f < g+h$.

We shall call the ordered pair $(\mathcal{K}_0, <)$ a *field with asymptotic order*.

2. DEFINITION. Let $(\mathcal{K}_0, <)$ be a field with asymptotic order. Let $\mathcal{U}, \mathcal{C}_0$ be subsets of $\mathcal{K}_0$ such that

(2.1) $\mathcal{U}$ is a subgroup of the multiplicative group $\mathcal{K}$.

(2.2) For every $f \in \mathcal{U}$ and every integer $p \geq 1$ there is a unique $g \in \mathcal{U}$ such that $g^p = f$.

(2.3) $\mathcal{U}$ is completely ordered by $<$.

(2.4) $\mathcal{C}_0$ is a subfield of $\mathcal{K}_0$.

(2.5) If $\{f, g\} \subset \mathcal{K}_0$, and $f < g$, and $c \in \mathcal{C}_0$, then $cf < cg$.

We shall call the ordered quadruple $(\mathcal{K}_0, <, \mathcal{U}, \mathcal{C}_0)$ a *graduated field*.

3. DEFINITION. Let $X = (\mathcal{K}_0, <, \mathcal{U}, \mathcal{C}_0)$ be a graduated field. Then $f \sim g$ means $f - g < g$.

4. DEFINITION. Let $X = (\mathcal{K}_0, <, \mathcal{U}, \mathcal{C}_0)$ be a graduated field. Then

(4.1) The product group $\mathcal{C}\mathcal{U}$ will be denoted by $\mathcal{M}(X)$ (or, briefly, by $\mathcal{M}$).

(4.2) The set $\{f: f \in \mathcal{K}; \text{there exists } g \in \mathcal{M} \text{ such that } f \sim g\}$ will be denoted by $\mathcal{A}(x)$ (or, briefly by $\mathcal{A}$).

(4.3) Every field $\mathcal{F}_0$ such that $\mathcal{M} \subset \mathcal{F}_0 \subset \mathcal{A}_0$ will be called *asymptotically constrained over X*. (Cf. Appendix, Note 7.)

(4.4) If $F(y) = \sum_{i=0}^n a_i y^i$ is a polynomial in y , such that $\{a_0, \dots, a_n\}$ is included in a field $\mathcal{F}_0$ asymptotically constrained over X , then F is said to be *asymptotically inscribed over X*.

5. DEFINITION. Let F be asymptotically inscribed over X . Let $p \in \mathcal{M}$. Then F is called *stable* at p if $F(q) \sim F(p)$ for all $q \in \mathcal{K}_0$ such that $q \sim p$; the *instability multiplicity of p for F* (briefly, $\text{inst}(p, F)$) is equal to $\min \{s: F^{(s)} \text{ is stable at } p\}$; if $\text{inst}(p, F) \geq 1$, then p is called a *point of instability* for F ; if $\text{inst}(p, F) = 1$, then p is called a *simple point of instability* for F ; if $\text{inst}(p, F) = 0$, then p is called a *point of stability* for F .

6. DEFINITION. The graduated field X is called *loosely closed* if both the following conditions are satisfied:

(6.1) $\mathcal{C}_0$ is algebraically closed.

(6.2) Whenever F is asymptotically inscribed over X , and p is a *simple point of instability* for F , there exists a $y_0 \sim p$ such that $F(y_0) = 0$.

7. DEFINITION. The graduated field X is called *tightly closed* if the following condition holds:

For every field $\mathcal{F}_0$ which is asymptotically constrained over X there exists an asymptotically constrained field $\mathcal{G}_0$ which is algebraically closed and includes $\mathcal{F}_0$.

8. THEOREM I. Let X be a graduated field. Then

(8.1) If X is loosely closed, X is tightly closed.

(8.2) If X is tightly closed, X is loosely closed.

Proof. The proof is given in §50.

9. LEMMA. Let $(\mathcal{K}_0, <)$ be a field with asymptotic order. Then

- (9.1) If $g \neq 0$, then $0 < g$.
- (9.2) If $f < g$, then $g \neq 0$.
- (9.3) If $f < g$, then $-f < g$.
- (9.4) If $f < g$, and $h < g$, then $f + h < g$.
- (9.5) If $f < g$, and $h < k$, then $fh < gk$.
- (9.6) If $f < g$, and $f \neq 0$, then $g^{-1} < f^{-1}$.
- (9.7) If $f < g$, then $f < f + g$.

Proof. (9.1) follows from (1.2) and (1.3). (9.2) follows from (1.1) if $f = 0$, and follows from (9.1), the transitivity of $<$, and (1.1) if $f \neq 0$. (9.3) follows from (9.2), (9.1), and (1.4). (9.4) follows from (1.4) and (9.3). (9.5) follows from (9.1), (9.2), and (1.3). (9.6) follows from (1.3), with $h = f^{-1}g^{-1}$. (9.7) follows from (1.5) with $h = f$.

10. LEMMA. Let $(\mathcal{K}_0, <)$ be a field with asymptotic order. Then

- (10.1) If $f \sim g$, then $f \neq 0$ and $g \neq 0$.
- (10.2) If $f \neq 0$, then $f \sim f$.
- (10.3) If $f \sim g$, then $g \sim f$.
- (10.4) If $f \sim g$, and $g \sim h$, then $f \sim h$.
- (10.5) If $f \sim g$, and $h \sim k$, then $fh \sim gk$.
- (10.6) If $f \sim g$, then $f^{-1} \sim g^{-1}$.

Proof. (10.1) follows from (9.2), (9.3), and (1.1). (10.2) follows from (9.1). (10.3) follows from (9.7) and (9.3). In (10.4), using (10.3), we have $h \sim g$; hence $f - g < g$, and $h - g < g$; by (1.4), $f - h < g$; by (1.5), $f - h < g + (h - g) = h$. In (10.5), $f = g + e$, with $e < g$, and $h = k + m$, with $m < k$; therefore $fh = gk + t$, with $t = gm + ek + em$, and $t < gk$ by (10.1), (1.3), (9.5), and (9.4). (10.6) follows from (10.1), together with (1.3) applied to $f - g < g$ multiplied by $f^{-1}g^{-1}$, together with (10.3).

11. LEMMA. Let $(\mathcal{K}_0, <)$ be a field with asymptotic order. Then

- (11.1) If $f < g$, and $g \sim h$, then $f < h$.
- (11.2) If $h \sim f$, and $f < g$, then $h < g$.
- (11.3) If $f < g$, and $h \sim g$, then $f + h \sim g$.
- (11.4) If $f \sim g$, then f is not $< g$.

Proof. In (11.1) (using (10.3)), we have $f < g$ and $h - g < g$; by (1.5), $f < g + (h - g)$. In (11.2) we have $f < g$, $h - f < f$; by transitivity of $<$, $h - f < g$; by (9.4), $f + (h - f) < g$. (11.3) follows from (9.4). In (11.4), if we assume the contrary we have $f < g$, and, by (10.3), $g \sim f$; then, by (11.1), $f < f$, in contradiction to (1.1).

12. GENERAL HYPOTHESIS. In §§13–50 the following notations will be fixed:

$X = (\mathcal{K}_0, <, \mathcal{U}, \mathcal{C}_0)$ will be a graduated field, and $\mathcal{M}(X)$, $\mathcal{A}(X)$ will be denoted by $\mathcal{M}$, $\mathcal{A}$, respectively.

13. LEMMA. (13.1) $\mathcal{U} \cap \mathcal{C} = \{1\}$.

(13.2) If $g \in \mathcal{M}$, there is a unique ordered pair (c, u) such that $c \in \mathcal{C}$, $u \in \mathcal{U}$ and $g = cu$.

Proof of (13.1). Obviously $1 \in \mathcal{U} \cap \mathcal{C}$. Suppose $f \in \mathcal{U} \cap \mathcal{C} - \{1\}$. Since $\mathcal{U}$ is completely ordered by $<$, either $1 < f$ or $f < 1$. If $1 < f$, then, by (2.5) (since $f \in \mathcal{C}$), we have $f < f$, in contradiction to (1.1). Suppose $f < 1$. Since $f \in \mathcal{U}$, f^{-1} exists, and is in $\mathcal{U}$. Also $f^{-1} \in \mathcal{C}$. By (9.6), $1 < f^{-1}$. Hence we are led, by the considerations above, to the contradiction $f^{-1} < f^{-1}$.

Proof of (13.2). This is a corollary of (13.1).

14. LEMMA. (14.1) If $\{c, d\} \subset \mathcal{C}_0$, and $c < d$, then $c = 0$.

(14.2) If $\{f, g\} \subset \mathcal{K}_0$, and $f < g$, and $\{c, d\} \subset \mathcal{C}_0$, and $d \neq 0$, then $cf < dg$.

(14.3) If $\{f, g\} \subset \mathcal{M}$, and $f \sim g$, then $f = g$.

(14.4) For each element f of $\mathcal{A}$ there is a unique element g of $\mathcal{M}$ such that $f \sim g$.

Proof. In (14.1), if $c \neq 0$, then $c^{-1}d$ exists and $\in \mathcal{C}_0$; by (2.5) we have $c^{-1}dc < d$, contradicting (1.1). In (14.2) we have $cd^{-1}f < g$, by (2.5), and then $cf < dg$ by (1.3). In (14.3) we have $f = cu$, $g = dv$, with $\{c, d\} \subset \mathcal{C}$, $\{u, v\} \subset \mathcal{M}$; since $\mathcal{U}$ is completely ordered by $<$, either $u < v$, or $v < u$, or $u = v$; if $u < v$, then, by (14.2), $cu < dv$, which, by (11.4), contradicts $f \sim g$; similarly $v < u$ is impossible, and therefore $u = v$; thus $cu \sim du$, which, by (10.2) and (10.5), implies $c \sim d$ or $c - d < d$; by (14.1), $c - d = 0$. In (14.4), the definition of $\mathcal{A}$ implies that there exists a $g \in \mathcal{M}$ such that $f \sim g$; if h is any element of $\mathcal{M}$ such that $f \sim h$, then $g \sim h$ by (10.3) and (10.4); by (14.3), $g = h$.

15. DEFINITION. Let $\{f, g\} \subset \mathcal{K}_0$. We write $f \approx g$ if there exists an element c of $\mathcal{C}$ such that $f \sim cg$, and we write $h \preceq k$ if either $h < k$ or $h \approx k$.

16. LEMMA. (16.1) If $f \sim g$, then $f \approx g$.

(16.2) Statements (10.1)–(10.6), (11.1)–(11.4) remain valid if $\sim$ is replaced, everywhere in them, by $\approx$. (Cf. the notational convention in Appendix, Note 8.)

(16.3) If $\{f, g\} \subset \mathcal{U}$, and $f \approx g$, then $f = g$.

(16.4) If $f \in \mathcal{C}$, then $f \approx 1$.

(16.5) If $f \preceq g$, and $h \preceq g$, then $f + h \preceq g$.

(16.6) If $f \preceq g$, and $g \preceq h$, then $f \preceq h$.

(16.7) If $f \preceq g$, and $h \neq 0$, then $fh \preceq gh$.

Proof. (16.1) and (16.2) are obvious. In (16.3), $f \sim cg$ for some $c \in \mathcal{C}$; by (14.3), $f = cg$; hence $fg^{-1} = c$; thus $fg^{-1} \in \mathcal{C} \cap \mathcal{U}$; by (13.1), $fg^{-1} = 1$. (16.4) is obvious. In (16.5) we have $f = cg + eg$, $h = dg + sg$ with $\{c, d\} \subset \mathcal{C}_0$, $e < 1$,

$s < 1$; hence $f + h = (c + d)g + (s + e)g$, with $c + d \in \mathcal{C}_0$ and $s + e < 1$ (because of (9.4)); if $c + d = 0$, then $f + h < g$, while if $c + d \neq 0$, then $f + h \approx g$. (16.6) is a corollary of (16.2)/[(10.4), (11.1), (11.2)] and the transitivity of $<$. (16.7) follows at once from (1.3) and (16.2)/[(10.2), (10.5)].

17. DEFINITION. (17.1) Let $f \in \mathcal{A}$. Let g be the unique element of $\mathcal{M}$ such that $f \sim g$ (cf. (14.4)). Let (c, u) be the unique element of $\mathcal{C} \times \mathcal{U}$ such that $g = cu$ (cf. (13.2)). Then u is called the *gauge* of f and is denoted by $]f[$. We extend this definition by setting: gauge $0 =]0[= 0$.

(17.2) Let $f \approx 1$. Then there exists an element c of $\mathcal{C}$ such that $f \sim c$. By (14.4), c is unique. We shall call c the *projection* of f , and denote it by the symbol $\text{proj}(f)$. We extend this definition by writing $\text{proj}(h) = 0$ for all $h < 1$. (Thus for all $k \lesssim 1$, $\text{proj}(k)$ is the uniquely determined element of $\mathcal{C}_0$ such that $k - \text{proj}(k) < 1$.)

18. NOTATION. If $f \in \mathcal{U}$, and p, q are integers with $q \neq 0$, the unique $g \in \mathcal{U}$ such that $g^q = f^p$ will be denoted by $f^{p/q}$. (Remark: It is easy to see that $f^{p/q} = f^{r/s}$ if $p/q = r/s$.)

19. LEMMA. If $\{f, g\} \subset \mathcal{U}$ and $f < g$, then there exists $h \in \mathcal{U}$ such that $f < h < g$.

Proof. Let $h = (fg)^{1/2}$. Then $h^2 = fg < g^2$. Hence $h < g$. (For $\mathcal{U}$ is completely ordered by $<$, and if $h = g$ or $g < h$ the inequality $h^2 < g^2$ would be impossible.) Similarly $f < h$.

20. GENERAL HYPOTHESIS (for §§21–37). $\mathcal{F}_0$ is a field which is asymptotically constrained over X , $F(y) = \sum_{i=0}^n a_i y^i$ is a polynomial with $\{a_0, \dots, a_n\} \subset \mathcal{F}_0$, and $na_n \neq 0$.

21. LEMMA. Assume §20. Let p be a point of instability for F , with $\text{inst}(p, F) = s \geq 1$. Then F' is asymptotically inscribed, and $\text{inst}(p, F') = s - 1$.

Proof. Obvious.

22. LEMMA. Assume §20. Let $h \in \mathcal{F}$, $w \in \mathcal{M}$. Let $G(z) = hF(wz)$. Then G is asymptotically inscribed, and $\text{inst}(p, F) = \text{inst}(p/w, G)$ for all $p \in \mathcal{M}$.

Proof. G is asymptotically inscribed, since its coefficients lie in $\mathcal{F}_0$; the remaining statement is a corollary of the obvious relation $G^{(j)}(z) = hw^j F^{(j)}(wz)$.

23. LEMMA. Assume §20. Let $p \in \mathcal{M}$. Let $m = \max \{]a_i p^i[: i \geq 0 \}$. (We remark that because of (10.5), $a_i p^i \in \mathcal{A}_0$.) Then

(23.1) If $q \sim p$, and j is any non-negative integer, $q^j F^{(j)}(q) \lesssim m$.

(23.2) If $q \sim p$, then $\text{inst}(p, F) = \min \{ j : q^j F^{(j)}(q) \approx m \}$.

Proof of (23.1). This follows at once from (10.5), (16.1), (16.6), and (16.5).

Proof of (23.2). Let $c_i = a_i p^i m^{-1}$. Then $c_i \lesssim 1$, and $c_i \in \mathcal{F}_0$.

Let $\sigma < 1$. Let $q = p(1 + \sigma)$. Then $q \sim p$. Also, for every non-negative integer j we have $q^j F^{(j)}(q) = m(1 + \sigma)^j \cdot G^{(j)}(1 + \sigma)$, where $G(z) = \sum_{i=0}^n c_i z^i$.

Let $\gamma_i = \text{proj}(c_i)$, and let $\varepsilon_i = c_i - \gamma_i$. Then $G = H + K$, where $H(z) = \sum_{i=0}^n \gamma_i z^i$ and $K(z) = \sum_{i=0}^n \varepsilon_i z^i$. Let $s = \min\{r : H^{(r)}(1) \neq 0\}$. (Then $0 \leq s \leq s_0$, where $s_0 = \max\{i :]a_i p^i[= m\}$.) Evidently if $j \leq s$, then $H^{(j)}(1 + \sigma) \sim [(s-j)!]^{-1} \sigma^{s-j} H^{(s)}(1)$. In particular, $H^{(s)}(1 + \sigma) \sim H^{(s)}(1)$. Now $\varepsilon_i \in \mathcal{F}_0$ and $\varepsilon_i < 1$. Let $\delta = \min\{] \varepsilon_i[: i \geq 0\}$. Then $\delta < 1$, and $\delta \in \mathcal{U}_0$. If $\delta = 0$, then $K^{(j)}(1 + \sigma) = 0$ for all σ . If $\delta \neq 0$, then $K^{(j)}(1 + \sigma) \lesssim \delta$ for all $\sigma < 1$. In either case, $K^{(j)}(1 + \sigma) < 1$ for all $\sigma < 1$.

It follows that $G^{(s)}(1 + \sigma) \sim H^{(s)}(1)$ for all $\sigma < 1$, and therefore that $q^s F^{(s)}(q) \sim m H^{(s)}(1)$ for all $q \sim p$. This establishes the stability of $F^{(s)}$ at p , and the relation $q^s F^{(s)}(q) \approx m$. Also, if $j \leq s-1$, then $H^{(j)}(1 + \sigma) < 1$, so $G^{(j)}(1 + \sigma) < 1$, and therefore $q^j F^{(j)}(q) < m$. Thus $s = \min\{j : q^j F^{(j)}(q) \approx m\}$.

If $\delta = 0$, then $K = 0$, so $p^j F^{(j)}(p) = m H^{(j)}(1) = 0$ for $j \leq s-1$, and therefore p is a point of instability for $F^{(j)}$, if $j \leq s-1$. If $\delta \neq 0$, then $\delta \in \mathcal{U}$, and therefore we may (and do) take $\sigma \in \mathcal{U}$ such that $\delta^{1/n} < \sigma < 1$ (cf. §19). Then, if $j \leq s-1$, we have $H^{(j)}(1 + \sigma) \sim \lambda_j$, where $\lambda_j = [(s-j)!]^{-1} \sigma^{s-j} H^{(s)}(1)$. But $K^{(j)}(1 + \sigma) < \sigma^n \lesssim \sigma^{s-j}$. Hence $G^{(j)}(1 + \sigma) \sim \lambda_j$, and therefore $q^j F^{(j)}(q) \sim m \lambda_j$. If $q_0 = p(1 + 2\sigma)$, then by the same argument $(q_0)^j F^{(j)}(q_0) \sim m \lambda_j 2^{s-j}$, which is not $\sim q^j F^{(j)}(q)$, although $q_0 \sim p$ and $q \sim p$. Thus $F^{(j)}$ is not stable at p , if $j \leq s-1$.

Therefore $s = \text{inst}(p, F)$, and from above we have $s = \min\{j : q^j F^{(j)}(q) \approx m\}$. This establishes (23.2).

24. LEMMA. Assume §20. Let $q \sim p \in \mathcal{M}$. Let $G(y) = (y - q)F(y) = \sum_{i=0}^{n+1} b_i y^i$. Let G be asymptotically inscribed. Then

$$(24.1) \quad \text{inst}(p, G) = 1 + \text{inst}(p, F).$$

$$(24.2) \quad \text{If } r \in \mathcal{M} - \{p\}, \text{ then } \text{inst}(r, G) = \text{inst}(r, F).$$

Proof of (24.1). We have $b_i = a_{i-1} - qa_i$. Hence $b_i p^i = a_{i-1} p^i - a_i q p^i$. Let $m = \max\{]a_i p^i[: i \geq 0\}$, $m^* = \max\{]b_i p^i[: i \geq 0\}$. Let $k = \min\{i :]a_i p^i[= m\}$. Then $a_{k-1} p^k < mp$, and $a_k q p^k \approx mp$. Hence $b_k p^k \approx mp$. Also $b_i p^i \lesssim mp$ for all i . Thus $m^* = m[p[$.

Let $\phi_j = p^j F^{(j)}(p)$, $\gamma_j = p^j G^{(j)}(p)$. Then $\gamma_j = (p - q)\phi_j + jp\phi_{j-1}$. Let $t = \text{inst}(p, F)$. Then $\phi_j < m$ for $j \leq t-1$, $\phi_t \approx m$, $\phi_{t+1} \lesssim m$. Also $p - q < p$. Hence $\gamma_j < pm \approx m^*$ if $j \leq t$, and $\gamma_{t+1} \approx pm \approx m^*$. Therefore, by §23, $\text{inst}(p, G) = t + 1$. This proves (24.1).

Proof of (24.2). This is similar to the proof of (24.1).

25. LEMMA. Assume §20. Let k be an integer ≥ 1 . Let $G(y) = y^k F(y)$. Then for every $p \in \mathcal{M}$, we have $\text{inst}(p, G) = \text{inst}(p, F)$.

Proof. It suffices to consider the case $k = 1$. The argument for that special case is similar to that for (24.1).

26. LEMMA. Assume §20. Let $p \in \mathcal{M}$. Let $y_1 \sim p$, $y_2 \sim p$, $y_1 \neq y_2$. Then

$$(26.1) \quad \text{If } F(y_1) = 0, \text{ inst}(p, F) \geq 1.$$

(26.2) If $F(y_1) = F(y_2) = 0$, $\text{inst}(p, F) \geq 2$.

Proof of (26.1). Assume the contrary. Then p is a point of stability for F . Hence $F(y_1) \sim F(p)$, and therefore, by (10.1), we have the contradiction $F(y_1) \neq 0$.

Proof of (26.2). Assume the contrary. Then, using (26.1), we have $\text{inst}(p, F) = 1$. Let $G(z) = F(y_1 + z) = \sum b_i z^i$. Let $m = \max\{]a_i p^i[: i \geq 0\}$. We have $b_0 = F(y_1) = 0$. By §23, $b_1 \approx m p^{-1}$, and $b_j \lesssim m p^{-j}$ for $j \geq 2$. Let $z_1 = y_2 - y_1$. Then $z_1 \neq 0$, and $G(z_1) = 0$. Hence $(*) -b_1 = \sum_{i=2}^n b_i z_1^{i-1}$. But $-b_1 \approx m p^{-1}$, while since $z_1 < p$ we have $b_i z_1^{i-1} < m p^{-i} p^{i-1} = m p^{-1}$ for $i \geq 2$. Thus we have the contradiction that the two members of $(*)$ cannot be equal.

27. DEFINITION. Assume §20. Assume $a_0 \neq 0$. Let $u \in \mathcal{U}$. Let (r, s) be an ordered pair of integers such that $0 \leq r < s \leq n$. Then the ordered triple (r, s, u) is called a *Newton triple* for F , over X (or, briefly, a *Newton triple* for F), if

$$(27.1) \quad a_i u^i \lesssim a_r u^r \approx a_s u^s$$

for all i such that $0 \leq i \leq n$, with the following strict inequality

$$(27.2) \quad a_i u^i < a_r u^r$$

holding for all $i \leq r-1$ and for all $i \geq s+1$.

28. LEMMA. Assume §20. Assume $a_0 \neq 0$. Then there exists a unique finite sequence $(\tau_1, \tau_2, \dots, \tau_k)$ of Newton triples for F , with $\tau_j = (r_j, s_j, u_j)$, such that $u_1 < u_2 < \dots < u_k$, and such that $\sum_{j=1}^k (s_j - r_j) = n$. For every Newton triple σ there exists a unique j such that $\sigma = \tau_j$. The sequence $(\tau_1, \dots, \tau_k)$ can be constructed as follows: Let $b_i =]a_i[$, ($i = 0, 1, \dots, n$). For $b_h \neq 0$, $i \neq h$, let $c_{hi} = (b_i/b_h)^{1/(h-i)}$. Let $r_1 = 0$, $u_1 = \min\{c_{0i} : i \geq 1\}$, $s_1 = \max\{i : c_{0i} = u_1\}$. When $(\tau_1, \dots, \tau_q)$ has been determined: if $s_q \leq n-1$ let $r_{q+1} = s_q$, let $u_{q+1} = \min\{c_{\sigma i} : i > \sigma = r_{q+1}\}$ and let $s_{q+1} = \max\{i : c_{\sigma i} = u_{q+1}\}$; if $s_q = n$, let $k = q$.

Proof. The argument follows closely the classic discussion of Newton polygons. We remark, for future application, that in the construction above (in which u_j is asserted to increase with j), r_j obviously increases with j .

29. NOTATION. The sequence $(\tau_1, \tau_2, \dots, \tau_k)$ described in §28 will be called the *Newton sequence* for F , over X .

30. LEMMA. Assume §20. Let $a_0 \neq 0$. Let $\phi \in \mathcal{F}$, $w \in \mathcal{U}$. Let $G(z) = \phi F(wz) = \sum_{i=0}^n b_i z^i$. Then G is asymptotically inscribed, and if $(\tau_1, \dots, \tau_k)$ is the Newton sequence for F , with $\tau_j = (r_j, s_j, u_j)$, then the Newton sequence for G is $(\sigma_1, \dots, \sigma_k)$, with $\sigma_j = (r_j, s_j, w^{-1}u_j)$.

Proof. Since the coefficients of G belong to $\mathcal{F}_0$, G is asymptotically inscribed. Also $b_0 b_n = a_0 a_n w^n \phi^2 \neq 0$. Hence G has a Newton sequence. The remaining conclusions follow from the definition of Newton triple.

31. LEMMA. Assume §20. Let $a_0 \neq 0$. Let (r, s, u) be a Newton triple for F . Let $G(z) = F(uz)(a, u')^{-1} = \sum_{i=0}^n c_i z^i$. Then

$$(31.1) \quad 1 = c_r.$$

(31.2) $c_i \lesssim c_r$ ($i = 0, 1, \dots, n$), with strict inequality if $i \leq r-1$ and if $i \geq s+1$.

(31.3) $c_r \approx c_s$.

Proof. Obviously, $c_r = 1$. By §30, G has the Newton triple $(r, s, 1)$. That is, (31.2) and (31.3) are valid.

32. NOTATION. Under the hypotheses of §31, let $\gamma_i = \text{proj}(c_i)$. Then the polynomial $H(z) = \sum_{i=0}^n \gamma_i z^i$ is called the *Newton polynomial for F associated with (r, s, u)* , and the ordered quadruple (r, s, u, H) is called a *Newton quadruple for F* .

The remaining definitions in this section are given under the assumption that $\mathcal{C}_0$ is algebraically closed.

For each nonzero root ζ of H (evidently $\zeta \in \mathcal{C}$), the element $m = \zeta u$ of $\mathcal{M}$ is called an *approximate zero of F* , and the multiplicity of ζ as a root of H is called the *approximate-zero multiplicity of m for F* , and is denoted by $\text{azm}(m, F)$.

If $\text{azm}(m, F) = 1$, then m is called a *simple approximate zero of F* .

The assertion $\text{azm}(m, F) = 0$ is understood to mean that $m \in \mathcal{M}$ and that m is *not* an approximate zero of F .

33. LEMMA. Assume §20. Assume $a_0 \neq 0$. Assume $\mathcal{C}_0$ is algebraically closed. Let $\{w_1, w_2, \dots, w_a\}$ be the set of all approximate zeros of F . Let $k_j = \text{azm}(w_j, F)$. Then $\sum_{j=1}^a k_j = n$.

Proof. Let (r, s, u, H) be a Newton quadruple. Let $\{\zeta_1, \dots, \zeta_b\}$ be the set of all nonzero roots of H . Let h_i be the multiplicity of ζ_i as a root of H . Then, by §31, $\sum h_i = s - r$. Let $v_i = \zeta_i u$. Then $\text{azm}(v_i, F) = h_i$. Hence $\sum \text{azm}(v_i, F) = s - r$. That is, $\sum \{\text{azm}(w_j, F) : w_j \in u\} = s - r$. Hence

$$\sum_{j=1}^s \text{azm}(w_j, F) = \sum \{s - r : \text{all Newton triples } (r, s, u)\} = n$$

(by §28).

34. NOTATION. Assume §20. Assume $a_0 \neq 0$. Assume $\mathcal{C}_0$ is algebraically closed. Let $S = (p_1, p_2, \dots, p_n)$ be an ordered n -tuple of elements of $\mathcal{M}$. We shall call S a *sequence of approximate solutions, multiplicities counted, for F* , if for every $q \in \mathcal{M}$ the number of integers i in the set $\{1, 2, \dots, n\}$ such that $p_i = q$ is equal to $\text{azm}(q, F)$.

35. LEMMA. Assume §20. Assume $a_0 \neq 0$. Assume $\mathcal{C}_0$ is algebraically closed. Let (r, s, u, H) be a Newton quadruple for F . Let $\phi \in \mathcal{F}$; $w \in \mathcal{U}$. Let $G(z) = \phi F(wz)$. Then $(r, s, u/w, H)$ is a Newton quadruple for G , and for every $\zeta \in \mathcal{C}$ we have $\text{azm}(\zeta u/w, G) = \text{azm}(\zeta u, F)$.

Proof. This follows immediately from §30 and the definitions of Newton quadruple and azm .

36. LEMMA. Assume §20. Assume $a_0 \neq 0$. Assume $\mathcal{C}_0$ is algebraically closed. Let $p \in \mathcal{M}$. Then $\text{inst}(p, F) = \text{azm}(p, F)$.

Proof. Let $p \in \mathcal{M}$. Let $q_i =]a_i p^i[$. Let $m = \max\{q_i : i \geq 0\}$. Let $J = \{i : q_i = m\}$. Let $r = \min\{i : i \in J\}$, $s = \max\{i : i \in J\}$.

CASE I: $r = s$. Then J has only one element, r . Also, $a_r p^r \approx m$, while $a_i p^i < m$ for all $i \neq r$. Hence $F(p) \approx m$, and therefore, by §23, $\text{inst}(p, F) = 0$. Suppose $\text{azm}(p, F) \geq 1$. Let $u =]p[$. Then there exists a Newton triple (ρ, σ, u) for F . It follows that $a_r u^r \approx a_\rho u^\rho \approx a_\sigma u^\sigma$, and therefore that $m \approx q_r \approx q_\rho \approx q_\sigma \approx m$, which implies that $\{\rho, \sigma\} \subset J$, although J has only one element. This contradiction shows that $\text{azm}(p, F) = 0$. Thus, in Case I, $\text{inst}(p, F) = \text{azm}(p, F)$.

CASE II: $r \leq s - 1$. Let $u =]p[$, $\zeta = p/u$. Then $a_r u^r \approx a_s u^s$, and $a_i u^i \approx a_r u^r$ for all i , with strict inequality if $i \leq r - 1$ and if $i \geq s + 1$. Thus (r, s, u) is a Newton triple for F . Let (r, s, u, H) be a Newton quadruple for F . Let $G(z) = F(uz)(a_r u^r)^{-1} = \sum_{i=0}^n c_i z^i$, let $\gamma_i = \text{proj}(c_i)$, $\varepsilon_i = c_i - \gamma_i$, $E(z) = \sum_{i=0}^n \varepsilon_i z^i$. Then $G = H + E$. Let $\text{azm}(p, F) = k \geq 0$. Then $H^{(j)}(\zeta) = 0$ if $j \leq k - 1$ and $H^{(k)}(\zeta) \neq 0$. But $E^{(j)}(\zeta) < 1$ for all j , since $\varepsilon_i < 1$. Hence $G^{(j)}(\zeta) < 1$ if $j \leq k - 1$, while $G^{(k)}(\zeta) \approx 1$. But, by §31, $1 = \max\{]c_i \zeta^i[: i \geq 0\}$. Hence, by §23, $\text{inst}(\zeta, G) = k$. But then, by §22, $\text{inst}(p, F) = k$. Therefore, in Case II, $\text{inst}(p, F) = \text{azm}(p, F)$.

37. **LEMMA.** Assume §20. Assume $\mathcal{C}_0$ is algebraically closed. Let $\{p_1, \dots, p_s\}$ be the set of all points of instability for F , and let $k_i = \text{inst}(p_i, F)$. Let $\delta = \min\{j : F^{(j)}(0) \neq 0\}$. Then $\delta + \sum_{i=1}^s k_i = n$.

Proof. This is a corollary of §§25, 33, and 36.

38. **FUNDAMENTAL LEMMA A.** Let $F(y) = \sum_{i=0}^n a_i y^i$ be asymptotically inscribed, with $a_n \neq 0$. Let p be a point of instability for F , with instability multiplicity $k \geq 1$. Let q be such that $q \sim p$, and such that the polynomial $G(z) = F(q + z)$ is asymptotically inscribed. Then

(38.1) For every element v of $\mathcal{M}$ such that $v < p$, the instability multiplicity h of v for G satisfies $h \leq k$, with $h \leq k - 1$ holding in the special case where for some σ in $\{0, 1, \dots, k - 1\}$ we have $F^{(\sigma)}(q) = 0$.

(38.2) If $\mathcal{C}_0$ is algebraically closed, and $w_1, w_2, \dots, w_T$ are the points of instability of G such that $w_i < p$, and if $\lambda_i = \text{inst}(w_i, G)$, and $\delta = \min\{i : G^{(i)}(0) \neq 0\}$, then $\delta + \sum_{i=1}^T \lambda_i = k$.

Proof of (38.1) Let $G(z) = \sum b_j z^j$. Let $m = \max\{]a_i p^i[: i \geq 0\}$. We have $b_j = (j!)^{-1} F^{(j)}(q)$. Hence, by §23, we have $p^j b_j \approx m$, with $<$ for $j \leq k - 1$, and with $\approx$ for $j = k$. Let $m^* = \max\{]b_j v^j[: j \geq 0\}$. Let $c_j = b_j v^j / m^*$. Then $c_j \approx 1$. Let $\gamma_j = \text{proj}(c_j)$.

Now $b_k v^k \approx m(v/p)^k$, so $m(v/p)^k \approx m^*$. If $j \geq k + 1$, then $b_j v^j \approx m(v/p)^j < m(v/p)^k \approx m^*$, so $c_j < 1$ if $j \geq k + 1$.

Let $d_i = \sum_{j=i}^n j(j-1) \dots (j-i+1) c_j$. By (23.2), $v^h G^{(h)}(v) \approx m^*$. That is, $\sum_{j=h}^n j(j-1) \dots (j-h+1) b_j v^j \approx m^*$, whence $d_h \approx 1$. Since $c_j < 1$ for $j \geq k + 1$, it follows at once that $h \leq k$.

Suppose $h = k$. Then $d_k \approx 1$. Also, by (23.2), $v^i G^{(i)}(v) < m^*$ if $i \leq k - 1$. Therefore $d_i < 1$, for $i \leq k - 1$.

Let $\tau = \text{proj}(d_k)$. Let $\lambda_j = \gamma_j/\tau$. Then $\lambda_j = 0$ if $j \geq k + 1$. Hence we have $\sum_{j=i}^k j(j-1)\dots(j-i+1)\lambda_j = \delta_{ik}$ ($j = 0, 1, \dots, k$). This is a linear system in $\lambda_0, \dots, \lambda_k$ with nonvanishing determinant $0!1!\dots k!$. Hence its solution is unique. But obviously the following is a solution $(**) \lambda_j = (-1)^{k-j} [j!(k-j)!]^{-1}$ (as we see by defining $\phi(x) = (k!)^{-1}(x-1)^k$ and expanding the equations $\phi^{(i)}(1) = \delta_{ik}$).

If now for some σ in $\{0, 1, \dots, k-1\}$ we have $F^{(\sigma)}(q) = 0$, then $0 = b_\sigma = c_\sigma = \gamma_\sigma = \lambda_\sigma$. But the relation $\lambda_\sigma = 0$ contradicts $(**)$. This contradiction shows that in this special case we must have $h \leq k - 1$.

Proof of (38.2). Let $G(z) = z^\delta J(z)$. Then $J(0) \neq 0$. Obviously $\delta \leq k$, since $b_k \neq 0$. If $\delta = n$, then $\delta = k = n$, and (38.2) is a trivial consequence of §25. Assume $\delta \leq n - 1$. Let $J(z) = \sum h_i z^i$. Then $h_i = b_{i+\delta}$. Let (r, s, u) be a Newton triple for J such that $r \leq k - \delta < s$. Then $h_{k-\delta} u^{k-\delta} \lesssim h_r u^r \approx h_s u^s$, or $b_k u^{k-\delta} \lesssim b_{r+\delta} u^r \approx b_{s+\delta} u^s$. Now $b_k \approx mp^{-k}$ and $b_{s+\delta} \lesssim mp^{-s-\delta}$. Hence $mp^{-k} u^{k-\delta} \lesssim mp^{-s-\delta} u^s$, and therefore $(u/p)^{k-\delta-s} \lesssim 1$. Since $k-\delta-s \leq -1$, this implies that $p \lesssim u$. Now $b_{r+\delta} \lesssim mp^{-r-\delta}$ with strict inequality if $r + \delta \leq k - 1$. Hence $mp^{-k} u^{k-\delta} \lesssim mp^{-r-\delta} u^r$ with strict inequality if $r + \delta \leq k - 1$. Therefore $(u/p)^{k-\delta-r} \lesssim 1$, with strict inequality if $r + \delta \leq k - 1$. But since $p/u \lesssim 1$, this means that $r + \delta = k$.

Thus there is a Newton triple $(k - \delta, s, u)$ for J , and $p \lesssim u$.

Now if t is such that $(t, k - \delta, v)$ is a Newton triple for J , we have $h_t v^t \approx h_{k-\delta} v^{k-\delta}$, or $b_{t+\delta} v^t \approx b_k v^{k-\delta}$, and since $b_k \approx mp^{-k}$ while $b_{t+\delta} < mp^{-t-\delta}$, we have $mp^{-k} v^{k-\delta} < mp^{-t-\delta} v^t$, or $(v/p)^{k-\delta-t} < 1$, whence $v < p$. Hence all Newton triples for J of the form (α, β, w) with $\beta \leq k - \delta$ have $w < p$. It follows that if $(r_1, r_2, \dots, r_{n-\delta})$ is a sequence of approximate solutions, multiplicities counted, for J , and $r_1 \lesssim r_2 \lesssim \dots \lesssim r_{n-\delta}$ we have $r_{k-\delta} < p$ while $p \lesssim r_{k-\delta+1}$. Hence (by §§36 and 25) we have $\sum_{i=1}^T \lambda_i = k - \delta$. This establishes (38.2).

39. DEFINITION. Let F be asymptotically inscribed and of degree ≥ 1 . We shall say that F is *normal* if every asymptotically constrained field $\mathcal{F}_0$ containing the coefficients of F has an asymptotically constrained extension containing at least one root of F .

If F is asymptotically inscribed and of degree ≥ 1 , we shall say that F is *abnormal* if F is not normal.

40. LEMMA. If F is asymptotically inscribed and F is of the first degree, then F is normal.

Proof. Obvious.

41. DEFINITION. Let X be a graduated field. Let n be an integer ≥ 1 . We shall say that n is *regular* for X if every polynomial asymptotically inscribed over X whose degree is ≥ 1 but $\leq n - 1$ is normal.

42. LEMMA. Let n be regular for X . Let G be asymptotically inscribed,

of degree n , and normal. Let $\mathcal{F}_0$ be any asymptotically constrained field containing the coefficients of G . Then there exists an extension $\mathcal{G}_0$ of $\mathcal{F}_0$ such that $G(y) = c(y - y_1) \dots (y - y_n)$, with $\{y_1, \dots, y_n\} \subset \mathcal{G}_0$.

Proof. Since G is normal, there exists an asymptotically constrained extension $\mathcal{G}_{10}$ of $\mathcal{F}_0$ such that G has a root y_1 in $\mathcal{G}_{10}$. Therefore $G(y) = (y - y_1)H(y)$, where H has coefficients in $\mathcal{G}_{10}$. Since H is asymptotically inscribed, and $(\text{degree } H) = n - 1$, H is normal, and the degree of H is regular for X , unless $(\text{degree } H) = 0$. If $(\text{degree } H) = 0$, we have the required factorization with $c = H(y)$. If $(\text{degree } H) \geq 1$, H may be treated in the same way as G . Repetition of this procedure produces the required factorization.

43. **LEMMA.** Let n be regular for X . Let F be abnormal and of degree n . Let $s \in \{1, 2, \dots, n - 1\}$. Let y_1 be an element of $\mathcal{K}_0$ such that $F^{(s)}(y_1) = 0$, and let $G(z) = F(y_1 + z)$. Then G is abnormal.

Proof. Assume the contrary. Let $\mathcal{F}_0$ be any asymptotically constrained field containing the coefficients of F . Then $\mathcal{F}_0$ contains the coefficients of $F^{(s)}$. Since $1 \leq (\text{degree } F^{(s)}) \leq n - 1$, $F^{(s)}$ is normal. Of course, the degree of $F^{(s)}$ is regular. Hence, by §42, there exists an asymptotically constrained extension $\mathcal{G}_0$ of $\mathcal{F}_0$ in which $F^{(s)}$ has a factorization into linear factors. Then $y_1 \in \mathcal{G}_0$. Hence $\mathcal{G}_0$ contains the coefficients of G . Thus G is asymptotically inscribed. Therefore, since G is not abnormal, G is normal. Since G is normal, there exists an asymptotically constrained extension $\mathcal{H}_0$ of $\mathcal{G}_0$ in which G has a root z_1 . Then $y_1 + z_1$ is a root of F . But $y_1 + z_1 \in \mathcal{H}_0$. Thus $\mathcal{H}_0$ is an asymptotically constrained extension of $\mathcal{F}_0$ containing a root of F . Therefore F is normal. This contradiction establishes the lemma.

44. **DEFINITION.** Let X be loosely closed. Let n be regular for X . Let F be an abnormal polynomial of degree n . (Cf. Appendix, Note 9.) Let p be a point of instability for F . Let $k = \text{inst}(p, F)$. Let η be the unique root of $F^{(k-1)}$ such that $\eta \sim p$. (We note that $\text{inst}(p, F^{(k-1)}) = 1$, by §21. Hence, by (6.2), the loose closure of X implies the existence of η . The uniqueness of η is implied by (26.2).) Then

(44.1) the ordered quadruple (F, p, k, η) will be called an *instability quadruple*.

(44.2) If (F, p, k, η) and (G, q, h, ζ) are instability quadruples, then (G, q, h, ζ) will be called a *descendant* of (F, p, k, η) if $G(z) = F(\eta + z)$, and $q \prec p$.

(44.3) If $Q = (Q_1, Q_2, \dots, Q_s)$ is a finite sequence of instability quadruples, with $Q_j = (F_j, p_j, k_j, y_j)$, we shall say that Q is a *reduction sequence* for Q_1 if Q_{j+1} is a descendant of Q_j ($j = 1, 2, \dots, s - 1$), and $k_s = 1$.

45. **LEMMA.** Let (F, p, k, η) be an instability quadruple. Let $k = 1$. Then (F, p, k, η) has no descendants.

Proof. Since $k = 1$, $F(\eta) = 0$. Hence if $G(z) = F(\eta + z)$, then $G(0) = 0$. Hence G is not abnormal. Hence F has no descendants.

46. LEMMA. Let $Q_1 = (F_1, p_1, k_1, \eta_1)$ be an instability quadruple. Then the number of distinct reduction sequences for Q_1 is equal to k_1 .

Proof. If $k_1 = 1$, then, by §45, Q_1 has no descendants. Hence the only reduction sequence for Q_1 is the one-term sequence (Q_1) . This establishes the result for the case $k_1 = 1$.

Suppose that r is an integer ≥ 2 that the conclusion holds whenever $k_1 \leq r - 1$. Let $Q_1 = (F_1, p_1, k_1, \eta_1)$ be an instability quadruple with $k_1 = r$. Let $F_2(z) = F_1(\eta_1 + z)$. Then, by §43, F_2 is abnormal. This implies that $F_2(0) \neq 0$. Let $\{q_1, q_2, \dots, q_m\}$ be the set of those points of instability for F_2 which satisfy $q_i < p$. Let $h_i = \text{inst}(q_i, F_2)$. For each i let ζ_i be the root of $F_2^{(h_i-1)}$ such that $\zeta_i \sim q_i$, and let $Q_{2i} = (F_2, q_i, h_i, \zeta_i)$. Then $S = \{Q_{21}, Q_{22}, \dots, Q_{2m}\}$ is the set of all descendants of Q_1 . Hence $(Q_1, Q_2, \dots, Q_s)$ is a reduction sequence for Q_1 if and only if $Q_2 \in S$ and $(Q_2, \dots, Q_s)$ is a reduction sequence for Q_2 . By (38.1), $h_i \leq r - 1$, and therefore it follows from the original assumption about r that there are h_i distinct reduction sequences for Q_{2i} . Since each reduction sequence for Q_{2i} is distinct from each reduction sequence for Q_{2j} if $i \neq j$, it follows that the number of distinct reduction sequences for Q_1 is $\sum_{i=1}^m h_i$. By (38.2), this sum has the value k_1 . This shows that the conclusion holds for $k_1 = r$, and establishes the lemma by induction with respect to k_1 .

47. LEMMA. Let $Q_1 = (F_1, p_1, k_1, \eta_1)$ be an instability quadruple. For each reduction sequence $Q = (Q_1, \dots, Q_s)$ for Q_1 , with $Q_i = (F_i, p_i, k_i, \eta_i)$, let $y(Q) = \eta_1 + \eta_2 + \dots + \eta_s$. Then

(47.1) $y(Q)$ is a root of F_1 .

(47.2) $y(Q) \sim p_1$.

(47.3) If Q and Q^* are reduction sequences for Q_1 , and $Q \neq Q^*$, then $y(Q) \neq y(Q^*)$.

Proof of (47.1). $F_1(\eta_1 + \eta_2 + \dots + \eta_s) = F_2(\eta_2 + \dots + \eta_s) = \dots = F_s(\eta_s) = 0$, since $k_s = 1$.

Proof of (47.2). $\eta_1 \sim p_1$, while $\eta_2 \sim p_2 < p_1$, and likewise $\eta_i < p_1$ for all $i \geq 2$. Hence $y(Q) \sim p_1$.

Proof of (47.3). Let $Q^* = (Q_1^*, \dots, Q_t^*)$, with $Q_i^* = (F_i^*, p_i^*, k_i^*, \eta_i^*)$. We may assume that $s \leq t$. There must exist $i \leq s$ such that $Q_i \neq Q_i^*$. (For if $Q_i = Q_i^*$ for all $i \leq s$, then $k_s^* = k_s = 1$, so Q_s^* has no descendants, and therefore $t = s$, whence $Q^* = Q$.) Let $j = \min\{i: Q_i \neq Q_i^*\}$. Then $j \geq 2$. Now $Q_{j-1}^* = Q_{j-1} = (F_{j-1}, p_{j-1}, k_{j-1}, \eta_{j-1})$, and therefore $F_j^*(z) = F_{j-1}(\eta_{j-1} + z) = F_j(z)$. If $p_j^* = p_j$, then by definition of instability quadruple we have $k_j^* = k_j$, $\eta_j^* = \eta_j$, and therefore $Q_j = Q_j^*$, which is a contradiction. Hence $p_j^* \neq p_j$.

Let $\sigma = \eta_1 + \eta_2 + \dots + \eta_{j-1}$. Then $y(Q) - \sigma = \eta_j + \dots + \eta_s \sim p_j$, while

$y(Q^*) - \sigma = \eta_j^* + \dots + \eta_t^* \sim p_j^*$ which is not $\sim p_j$ (cf. (14.3)). Hence $y(Q) - \sigma \neq y(Q^*) - \sigma$, and therefore $y(Q) \neq y(Q^*)$.

48. LEMMA. *Let X be loosely closed. Let n be regular for X . Let F be abnormal, of degree n . Then*

$$(48.1) \quad F(y) = c(y - y_1)(y - y_2) \dots (y - y_n), \text{ with } \{y_1, \dots, y_n\} \subset \mathcal{A}.$$

Proof. According to §§47, 46, and 37, F has at least n (hence exactly n) distinct roots $y_1, \dots, y_n$ in $\mathcal{A}$. (We note that F 's abnormality implies that $F(0) \neq 0$.) Therefore we have a factorization (48.1) for some c in $\mathcal{K}_0$.

49. LEMMA. *Let X be loosely closed. Then every asymptotically inscribed polynomial of degree ≥ 1 is normal.*

Proof. Assume the contrary. Let

$$n = \min \{s: \exists \text{ abnormal polynomial of degree } s\}.$$

By §40, $n \geq 2$. Obviously, n is regular. Let F be an abnormal polynomial of degree n . Let $\mathcal{F}_0$ be an asymptotically constrained field containing the coefficients of F . Let y_1 be a root of F in $\mathcal{A}$ (cf. §48). Let $\mathcal{G}_0 = \mathcal{F}_0(y_1)$. Now if h is any element of $\mathcal{G}_0$, h is a root of a polynomial G , with coefficients in $\mathcal{F}_0$, such that the degree d of G satisfies $1 \leq d \leq n$. Then G is asymptotically inscribed. By §§42 and 48, we see that $h \in \mathcal{A}_0$. Hence $\mathcal{G}_0 \subset \mathcal{A}_0$. Thus $\mathcal{G}_0$ is an asymptotically constrained extension of $\mathcal{F}_0$ containing a root y_1 of F . Hence F is normal. This contradiction proves the Lemma.

50. PROOF OF THEOREM I, §8.

Proof of (8.1). Let X be loosely closed. Let $\mathcal{F}_0$ be a field asymptotically constrained over X . By Zorn's Lemma, there exists a maximal (in the sense of inclusion) asymptotically constrained field $\mathcal{J}_0$ containing $\mathcal{F}_0$. By §49, $\mathcal{J}_0$ is algebraically closed.

Proof of (8.2). Let X be tightly closed. Let $F(y) = \sum a_i y^i$ be any polynomial of degree ≥ 1 with coefficients in $\mathcal{C}_0$. We may and do assume that $F(0) \neq 0$. Let y_0 be a root of F , with y_0 in $\mathcal{A}_0$. Then $y_0 \in \mathcal{A}$. Let $y_0 \sim p \in \mathcal{M}$. Then, by §26, p is a point of instability for F . Hence, by §36, p is an approximate zero for F . Let $u =]p[$, and let $\zeta = p/u$. Let (r, s, u) be a Newton triple for F . Then $a_r u^r \approx a_s u^s$, whence $u^{r-s} \approx 1$, which by (16.3) and (2.2) implies $u = 1$. Hence $p = \zeta \in \mathcal{C}$. Let $y_0 = p + e$. Then $e < 1$. We have $0 = F(y_0) = F(p) + \varepsilon$, where $\varepsilon = \sum_{j=1}^n e^j F^{(j)}(p) (j!)^{-1}$. Evidently $\varepsilon < 1$. Hence $F(p) < 1$. But $F(p) \in \mathcal{C}_0$. Hence, by (14.1), $F(p) = 0$. Thus F has a root p in $\mathcal{C}$. Hence $\mathcal{C}_0$ is algebraically closed. This gives (6.1).

Now let F be asymptotically inscribed over X , and let w be a point of instability for F . Let $\mathcal{F}_0$ be an asymptotically constrained field containing the coefficients of F . Let $\mathcal{G}_0$ be an asymptotically constrained field which is algebraically

closed and includes $\mathcal{F}_0$. Then $F = c(y - y_1) \dots (y - y_n)$ with $\{c, y_1, \dots, y_n\} \subset \mathcal{G}_0$. If no y_i is $\sim w$, it follows from repeated applications of §§24 and 25 that $\text{inst}(w, F) = 0$. This contradiction gives the result that for every point of instability w there is a root y_0 of F such that $y_0 \sim w$. In particular, it gives (6.2).

PART II. FIELDS OF MEROMORPHIC FUNCTIONS

51. DEFINITION. By a *domain system* will be meant a filter base $\bar{N}$ each element of which is a region in the finite complex plane. (Cf. Appendix, Note 10.)

52. DEFINITION. Let $\bar{N}$ be a domain system. Then the function ϕ will be called *analytic over $\bar{N}$* if the domain of ϕ includes at least one element of $\bar{N}$, and ϕ is analytic at every point of the domain of ϕ . (Cf. Appendix, Note 11.) The concept *meromorphic over $\bar{N}$* is defined in a similar manner.

53. NOTATION. Let $\bar{N}$ be a domain system. Then $\tilde{K}_0(\bar{N}) = \{\phi : \phi \text{ meromorphic over } \bar{N}\}$.

54. DEFINITION. If $\bar{N}$ is a domain system, and $\{\phi, \gamma\} \subset \tilde{K}_0(\bar{N})$, we shall say that $\phi = \gamma$ (over $\bar{N}$) if there exists an element W of $\bar{N}$ such that $\phi(x) = \gamma(x)$ for all x in W .

55. DEFINITION. Let $\bar{N}$ be a domain system. Let $\sigma \in \mathbf{C}$. Let $\phi \in \tilde{K}_0(\bar{N})$. Then we write $\phi \rightarrow \sigma$ (over $\bar{N}$) if for every $\varepsilon > 0$ there exists $W \in \bar{N}$ such that $|\phi(x) - \sigma| < \varepsilon$ for all $x \in W$. (Cf. Appendix, Note 12.)

56. DEFINITION. Let $\bar{N}$ be a domain system. Let $\tilde{E}_0$ be a subset of $\tilde{K}_0(\bar{N})$. Then $\tilde{E}_0$ will be called a *subconstant class over $\bar{N}$* , and the ordered pair $(\bar{N}, \tilde{E}_0)$ will be called a *subconstant pair*, if all the following conditions are satisfied:

(56.1) If $\phi \in \tilde{E}_0$, and $\gamma = \phi$ (over $\bar{N}$), then $\gamma \in \tilde{E}_0$.

(56.2) If $\phi \in \tilde{E}_0$, then $\phi \rightarrow 0$ (over $\bar{N}$).

(56.3) If $\phi \in \tilde{E}_0$, and $\gamma \in \tilde{E}_0$, then $\phi\gamma \in \tilde{E}_0$, and $\phi - \gamma \in \tilde{E}_0$ and $(1 + \gamma)^{-1}\phi \in \tilde{E}_0$.

(56.4) If $\phi \in \tilde{E}_0$, and $\sigma \in \mathbf{C}$, then $\sigma\phi \in \tilde{E}_0$.

57. DEFINITION. Let $(\bar{N}, \tilde{E}_0)$ be a subconstant pair. Then $(\bar{N}, \tilde{E}_0)$ is called *quasi-linearly closed* if for every non-negative integer n and every ordered $(n+1)$ -tuple $(\varepsilon_0, \dots, \varepsilon_n)$ of elements of $\tilde{E}_0$ there exists an element η of $\tilde{E}_0$ such that $\eta = \sum_{j=0}^n \varepsilon_j \eta^j$.

58. DEFINITION. Let $(\bar{N}, \tilde{E}_0)$ be a subconstant pair. Let $\{\phi, \gamma, \zeta\} \subset \tilde{K}_0(\bar{N})$. We write $\phi < \gamma$ (over $(\bar{N}, \tilde{E}_0)$) if $\gamma \neq 0$ and $\phi\gamma^{-1} \in \tilde{E}_0$, and we write $\zeta \sim \gamma$ (over $(\bar{N}, \tilde{E}_0)$) if $\zeta - \gamma < \gamma$ (over $(\bar{N}, \tilde{E}_0)$).

59. NOTATION. Let $\tilde{V}$ be a subset of the real numbers, such that $\tilde{V}$ is a vector space over the field of rational numbers. We shall call $\tilde{V}$ a *real // rational vector space*.

60. DEFINITION. Let $(\bar{N}, \tilde{E}_0)$ be a subconstant pair. Let $\tilde{V}$ be a real//rational vector space. Let $\tilde{L}$ be a nonempty set of functions analytic over $\bar{N}$. We shall say that $(\bar{N}, \tilde{E}_0, \tilde{V}, \tilde{L})$ is a *logarithmic quadruple* if

(60.1) Whenever $\phi \in \tilde{L}$ and $\gamma \in \tilde{L}$, then either $\phi = \gamma$, or $\exp \phi < \exp \gamma$, or $\exp \gamma < \exp \phi$.

(60.2) Whenever $\phi \in \tilde{L}$ and $\gamma \in \tilde{L}$, and $\exp \phi < \exp \gamma$, then $\exp(\sigma\phi) < \exp(\tau\gamma)$ for all (strictly) positive σ and τ in $\tilde{V}$.

61. DEFINITION. Let $Y = (\tilde{N}, \tilde{E}_0, \tilde{V}, \tilde{L})$ be a logarithmic quadruple. Then

(61.1) Every function meromorphic over $\tilde{N}$ which has, in some element of $\tilde{N}$, a representation of the form $\exp(v_1\lambda_1 + v_2\lambda_2 + \dots + v_s\lambda_s)$ with $v_i \in \tilde{V}$ and $\lambda_i \in \tilde{L}$ will be called a *unit Y-monomial*. The set of all unit Y -monomials will be denoted by $\tilde{U}(Y)$.

(61.2) Every function of the form cu , where $c \in \mathbf{C} - \{0\}$ and $u \in \tilde{U}(Y)$ will be called a *Y-monomial*. The set of all Y -monomials will be denoted by $\tilde{M}(Y)$.

(61.3) Every function h for which there exists an element g of $\tilde{M}(Y)$ such that $h \sim g$ will be called an *asymptotic Y-monomial*, and the set of all asymptotic Y -monomials will be denoted by $\tilde{A}(Y)$.

(61.4) Every field (cf. Appendix, Note 5) $\tilde{F}_0$ such that $\tilde{M}(Y) \subset \tilde{F}_0 \subset \tilde{A}(Y) \cup \{0\}$ will be called a *Y-field*.

62. THEOREM II. Let $Y = (\tilde{N}, \tilde{E}_0, \tilde{V}, \tilde{L})$ be a logarithmic quadruple. Let $(\tilde{N}, \tilde{E}_0)$ be quasi-linearly closed. Then if $\tilde{F}_0$ is any Y -field, there exists an extension $\tilde{G}_0$ of $\tilde{F}_0$ such that $\tilde{G}_0$ is a Y -field and is algebraically closed.

Proof. The proof is given in §65.

63. NOTATION. If $\tilde{N}$ is a domain system, and $\phi \in \tilde{K}_0(\tilde{N})$, the symbol $\phi \#$ will represent the set $\{\gamma: \gamma \in \tilde{K}_0(\tilde{N}); \gamma = \phi \text{ (over } \tilde{N})\}$. If $\tilde{H} \subset \tilde{K}_0(\tilde{N})$, $\tilde{H} \#$ will represent the set $\{\zeta \#: \zeta \in \tilde{H}\}$.

64. FUNDAMENTAL LEMMA B. Let $Y = (\tilde{N}, \tilde{E}_0, \tilde{V}, \tilde{L})$ be a logarithmic quadruple. Let $\mathcal{X}_0 = (\tilde{K}_0) \#$, and for $\{\phi \#, \gamma \#\} \subset \mathcal{X}_0$, let the following definitions be made:

$$\phi \# + \gamma \# = (\phi + \gamma) \#,$$

$$\phi \# \gamma \# = (\phi\gamma) \#,$$

$$\phi \# < \gamma \# \text{ iff } \phi < \gamma.$$

Let $\mathcal{U} = (U(Y)) \#$. Let $\mathcal{C}_0 = \mathbf{C} \#$. Then

(64.1) $(\mathcal{X}_0, <, \mathcal{U}, \mathcal{C}_0)$ is a graduated field.

(64.2) If $(\tilde{N}, \tilde{E}_0)$ is quasi-linearly closed, then $(\mathcal{X}_0, <, \mathcal{U}, \mathcal{C}_0)$ is loosely closed.

Proof of (64.1). This is verified in a completely straightforward fashion.

Proof of (64.2). It suffices to verify (6.2). We assert first that if $\mathcal{E}_0 = (\tilde{E}_0) \#$, then for every non-negative integer n and every ordered $(n+1)$ -tuple $(e_0, \dots, e_n)$ of elements of $\mathcal{E}_0$ there exists an element z of $\mathcal{E}_0$ such that $z = \sum_{i=0}^n e_i z^i$. To establish this assertion we note that by definition of $\mathcal{E}_0$ there exists an ordered $(n+1)$ -tuple $(\varepsilon_0, \dots, \varepsilon_n)$ of elements of $\tilde{E}_0$ such that $e_i = \varepsilon_i \#$ ($i = 0, \dots, n$). Then, since $(\tilde{N}, \tilde{E}_0)$ is quasi-linearly closed, there exists ζ in $\tilde{E}_0$ such that

$\zeta = \sum_{i=0}^n e_i \zeta^i$. Then if $z = \zeta \neq \#$, we have $z = \sum_{i=0}^n e_i z^i$, and $z \in \mathcal{E}_0$, which establishes the assertion.

Now let $\mathcal{F}_0$ be an asymptotically constrained subfield of $\mathcal{K}_0$. Let $F(y) = \sum_{i=0}^n f_i y^i$, with $f_i \in \mathcal{F}_0$, and let us assume, as we may without essential loss of generality, that $f_0 f_n \neq 0$. Let $\text{inst}(w, F) = 1$.

By §36, $\text{azm}(w, F) = 1$. Let $m =]w[$. Let $t = w/m$. Let (r, s, m, D) be a Newton quadruple for F . Let $G(z) = (f, m')^{-1} F(mz) = \sum_{i=0}^n g_i z^i$. Then $g_i \in \mathcal{F}_0$, and $g_i \lesssim 1$, and if $d_i = \text{proj}(g_i)$, we have $D(z) = \sum_{i=0}^n d_i z^i$. Now $D(t) = 0 \neq D'(t)$, so $G(t) < 1$ and $G'(t) \approx 1$.

Let $Q(u) = G(t+u) = \sum_{i=0}^n q_i u^i$. Then $q_i \in \mathcal{F}_0$. Also, since $q_i = (i!)^{-1} G^{(i)}(t)$, we have $q_0 < 1$, $q_1 \approx 1$, $q_j \lesssim 1$.

If $q_0 = 0$, then $Q(0) = 0$, so $G(t) = 0$, and therefore $F(w) = 0$. In this case we may take $y_0 = w$.

If $q_0 \neq 0$, let $b =]q_0[$. Then $b \in \mathcal{U}$ and $b < 1$. Let $h = b^{1/2}$. Then $q_0 < h < 1$. The equation $Q(u) = 0$ is transformed by the substitution $u = hv$, followed by division by $q_1 h$, into an equation $R(v) = 0$, where $R(v) = v - \sum_{i=0}^n e_i v^i$, with $e_0 = -q_0 q_1^{-1} h^{-1} < 1$, $e_1 = 0$, $e_i = -q_i q_1^{-1} h^{i-1} < 1$ ($i \geq 2$). By the assertion in the first paragraph of the proof of (64.2) there exists $v_0 < 1$ such that $R(v_0) = 0$. Then if $y_0 = w(t + hv_0)$, we have $y_0 \sim w$, and $F(y_0) = 0$. This completes the verification of (6.2).

65. Proof of Theorem II. This is a corollary of Fundamental Lemma B (in §64) and Theorem I (in §8).

PART III. EXAMPLES FOR PART II

66. PRELIMINARY NOTATIONS.

(66.1) Let γ satisfy $0 < \gamma \leq \pi$. Let r be a non-negative real number. Then by $S(r, \gamma)$ will be meant the sector $\{x : x \in \mathbf{C}; |x| \geq r; |\arg x| < \gamma\}$.

(66.2) The functions $\log x, \log \log x, \dots, \log_p x, \dots$ are (for the purposes of this paper) determined by the stipulations that domain $(\log_p x) = S(e_{p-1}(0), \pi)$ (where $e_0(x) = x$ and $e_{j+1} = \exp(e_j)$), and that $\log_p x > 0$ for all x such that $x > e_p(0)$.

(66.3) Let $\tilde{V}$ be any real//rational vector space. The function f will be called a $(\tilde{V}, p)$ -logarithmic monomial (or briefly, a logarithmic monomial), if there exists a nonzero complex number c and an ordered $(p+2)$ -tuple $(r_0, r_1, \dots, r_{p+1})$ in $\tilde{V}$ such that $f(x) = c \exp(r_0 \cdot 0 + r_1 \log x + r_2 \log \log x + \dots + r_{p+1} \log_{p+1} x)$ (which is also written as $c x^{r_1} (\log x)^{r_2} \dots (\log_p x)^{r_{p+1}}$.) The set of all $(\tilde{V}, p)$ -logarithmic monomials will be denoted by $L \cdot M(\tilde{V}, p)$.

67. EXAMPLES OF DOMAIN SYSTEMS. For each γ such that $0 < \gamma \leq \pi$, let $\bar{N}_1(\gamma) = \{S(r, \gamma) : r \geq 0\}$. For each $r \geq 0$, let $\bar{N}_2(r) = \{S(r, \gamma) : 0 < \gamma \leq \pi\}$. Then each $\bar{N}_1(\gamma)$ is a domain system, and each $\bar{N}_2(r)$ is a domain system.

68. DEFINITION. Let $\bar{N}$ be any domain system which has the following two properties:

(68.1) No element of $\bar{N}$ meets the nonpositive real axis.

(68.2) $\lim \bar{N} = \infty$, i.e., for every $G > 0$ there exists an element A of $\bar{N}$ such that $|x| > G$ whenever $x \in A$.

As in [A] §3, (cf. Appendix, Note 13), we shall call such an $\bar{N}$ a *complex neighborhood system of $+\infty$* , or, briefly, a $\bar{CS}$. We note that in §67 each $\bar{N}_1(\gamma)$ is a $\bar{CS}$, while no $\bar{N}_2(r)$ is a $\bar{CS}$.

It is obvious that if $\bar{N}$ is any $\bar{CS}$, then all the functions $\log x$, $\log \log x$, ... are analytic over $\bar{N}$.

69. EXAMPLES OF SUBCONSTANT PAIRS.

(69.1) Let $\bar{N}$ be any domain system. Let q be a non-negative integer. Let $\bar{E}_0(q, \bar{N}) = \{f: f \text{ analytic over } \bar{N}; f^{(j)} \rightarrow 0 \text{ over } \bar{N} (j = 0, 1, \dots, q)\}$. Let $\bar{E}_0(\infty, \bar{N}) = \{f: f \text{ analytic over } \bar{N}; f^{(j)} \rightarrow 0 \text{ over } \bar{N} \text{ (all } j)\}$. Then $(\bar{N}, \bar{E}_0(q, \bar{N}))$ is a subconstant pair, and so is $(\bar{N}, \bar{E}_0(\infty, \bar{N}))$. It follows readily from the implicit function theorem that all these subconstant pairs are quasi-linearly closed.

(69.2) Let $\bar{N}$ be a $\bar{CS}$. A subconstant class $\bar{E}_0 * (\bar{N})$ over $\bar{N}$ can be defined in the following way: $f \in \bar{E}_0 * (\bar{N})$ iff for every ordered pair of non-negative integers (j, p) we have $(x \log x \log \log x \dots \log_p x d/dx)^j f \rightarrow 0$ over $\bar{N}$. With this definition of $\bar{E}_0$ the order relation $<$ becomes the one used in [A] and [P] for the study of differential equations, where it possesses certain crucial properties in its behavior under differentiation (e.g. if $f < g \sim h$, where h is a nonconstant logarithmic monomial, then $f' < g'$). It is easy to see that $(\bar{N}, \bar{E}_0 * (\bar{N}))$ is quasi-linearly closed. (Cf. [A], §36.)

(69.3) Let $\bar{N}$ be a $\bar{CS}$. Let $\bar{V}$ be any real//rational vector space. Let p be an integer ≥ 0 . Let $\bar{E}_{p0}(\bar{N}, \bar{V})$ be the set of functions f defined as follows: $f \in \bar{E}_{p0}(\bar{N}, \bar{V})$ if there exists an integer $s \geq 1$ and a function $G(x_1, x_2, \dots, x_s)$ of s complex variables, analytic and vanishing at the origin, and an ordered s -tuple $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s)$ of elements of $L \cdot M(\bar{V}, p)$ with $\varepsilon_i \rightarrow 0$ over $\bar{N}$, such that, in some element of $\bar{N}$, f has a representation $f = G(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s)$. It is easy to see, by the implicit function theorem, that $(\bar{N}, \bar{E}_{p0}(\bar{N}, \bar{V}))$ is a quasi-linearly closed subconstant pair.

70. EXAMPLES OF LOGARITHMIC QUADRUPLES. Let $\bar{N}$ be any $\bar{CS}$. Let $\bar{V}$ be any real//rational vector space. Let p be any integer such that $p \geq 0$. Let $\bar{S}_p = \{0, \log x, \log \log x, \dots, \log_{p+1} x\}$. Then if $\bar{E}_0$ is chosen as any one of the following subconstant classes over $\bar{N}$: $\bar{E}_0(q, \bar{N}), \bar{E}_0(\infty, \bar{N}), \bar{E}_0 * (\bar{N}), \bar{E}_{p0}(\bar{N}, \bar{V})$, the ordered quadruple $Y = (\bar{N}, \bar{E}_0, \bar{V}, \bar{S}_p)$ will be a logarithmic quadruple.

We note that in all these cases the set $\bar{M}(Y)$ (in the notation of (61.2)), that is, the set such that $\bar{M}(Y) \neq$ is the group $\mathcal{M}$ of the corresponding graduated field (cf. §§63, 64) is a set of logarithmic monomials. (Cf. Appendix, Note 14.)

71. EXAMPLES OF THE APPLICATION OF THEOREM II.

(71.1) If we apply Theorem II to the logarithmic quadruple $(\bar{N}, \bar{E}_0(q, \bar{N}), \mathbf{R}, \bar{S}_p)$ we get this result:

Let $\bar{F}_0$ be a field of functions which includes $L \cdot M(\mathbf{R}, p)$ and which has the

following property: For every nonzero element g of $\tilde{F}_0$ there exists an element h of $L \cdot M(\mathbf{R}, p)$ such that $g \sim h$ in the sense that $D^j(g^{-1}h - 1) \rightarrow 0$ over $\tilde{N}$ ($j = 0, 1, \dots, q$).

Then every polynomial P of degree $n \geq 1$ with coefficients in $\tilde{F}_0$ will have n roots $y_1, \dots, y_n$ each of which is $\sim$ (in the same sense) to an element of $L \cdot M(\mathbf{R}, p)$.

(71.2) If we replace, in (71.1), $\tilde{E}_0(q, \tilde{N})$ by $\tilde{E}_0 * (\tilde{N})$, we get a corresponding result in which the coefficients of the polynomial P , and the roots $y_1, \dots, y_n$, are asymptotically equivalent to logarithmic monomials in a much more stringent sense (cf. Appendix, Note 15). This result is very useful in the study of differential equations. (Cf. Part IV below.)

(71.3) If we choose the logarithmic quadruple $Y = (\tilde{N}, \tilde{E}_{p_0}(\tilde{V}, \tilde{N}), \tilde{V}, \tilde{S}_p)$, with $\tilde{N}$ any $\tilde{C}\tilde{S}$, and $\tilde{V}$ any real//rational vector space, we find that (in the terminology of (61.3)), $\tilde{A}(Y) \cup \{0\}$ is identical with the totality $\tilde{B}_{p_0}(\tilde{V}, \tilde{N})$ of all functions having, in an element of $\tilde{N}$, a representation $mG(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s)$, where G is analytic at $(0, 0, \dots, 0)$ and $m, \varepsilon_1, \dots, \varepsilon_s$ are elements of $L \cdot M(\tilde{V}, p)$ such that $\varepsilon_j \rightarrow 0$ over $\tilde{N}$. (Cf. Appendix, Note 16.) From this it is easy to see that $\tilde{A}_0(Y)$ is a field. Then it follows at once from Theorem II that $\tilde{A}_0(Y)$ is algebraically closed.

(71.4) If in the result of (71.3) we specialize p to 0 and $\tilde{V}$ to $\mathbf{Q}$ we get the result that the field of all functions f having, in an element of $\tilde{N}$, a representation of the form $(*) f(x) = x^{p/q} \sum_{i=0}^{\infty} c_i x^{i/q}$ (where p and q are integers, with $q \leq -1$) is algebraically closed. This gives in particular the classic theorem on the expansion of an algebraic function of one variable in the neighborhood of a branch point. (The branch point here is taken at ∞ ; if we take $\tilde{N} = \tilde{N}_1(\gamma)$ (for any γ such that $0 < \gamma \leq \pi$) the sectorial region of validity of the expansions $(*)$ can obviously, in the case of algebraic functions, be extended by analytic continuation to a neighborhood of ∞ on an appropriate Riemann surface.)

It might be remarked that the problems solved in (71.1) and (71.2) are fundamentally more difficult than those solved in (71.3) and (71.4), precisely because in (71.1) and (71.2) $\tilde{A}_0(Y)$ is not a field: asymptotic equivalence (to logarithmic monomials) as defined by means of $\tilde{E}_0(q, \tilde{N})$ or $\tilde{E}_0 * (\tilde{N})$ is not so stringently specified that it survives, in general, under the operation of addition.

PART IV. APPLICATIONS TO ALGEBRAIC DIFFERENTIAL EQUATIONS

In the terminology of §67, let $\tilde{N} = \tilde{N}_1(\gamma)$ for some γ such that $0 < \gamma \leq \pi$.

We consider first-order algebraic differential equations $F(x, y, y') = \sum_{i,j=0}^n f_{ij}(x) y^i (y')^j = 0$, with $f_{00} \neq 0$, and with the f_{ij} belonging to a set of functions $\tilde{G}_0$ which is a logarithmic differential field of logarithmic rank p over $\tilde{N}$, (briefly, an $\text{LDF}_p(\tilde{N})$). (This means that $\tilde{G}_0$ is a Y -field, in the terminology of (61.4), with $Y = (\tilde{N}, \tilde{E}_0 * (\tilde{N}), \mathbf{R}, \tilde{S}_p)$, and has the additional property that if $f \in \tilde{G}_0$, then $f' \in \tilde{G}_0$.) We seek a solution which is $\sim$ to a logarithmic monomial.

We define a *principal monomial* to be a logarithmic monomial m such that F is not stable at m , but F is stable at every logarithmic monomial h such that $h \prec m$ (where stable is defined as in §5). The principal monomials constitute a finite set which can be determined by an algorithm which is an elaboration of the Newton polygon algorithm described in §28 (cf. [A], §66). It simplifies the discussion to suppose, as we shall, that we are treating the case of a principal monomial equal to 1. (This case can always be realized by a transformation $y = mz$.) Analogous to the instability multiplicity defined in §5 are two indices (corresponding to the two arguments y', y in F). One index is the *singularity multiplicity* $s^* = \min \{k : \partial^k F / (\partial y')^k \text{ is stable at } 1\}$. The other index is the *algebraic multiplicity* $a^* = \min \{h : \partial^h F / (\partial y)^h \text{ is stable at } 1\}$.

In the case $a^* = 1$ and $s^* = 1$, the equation $F = 0$ is reduced to a quasi-linear form by the substitution $y = 1 + z$. Then if subsidiary conditions are fulfilled (e.g., if $\tilde{G}_0$ is Schwarzian-symmetric (by which is meant that for every f in $\tilde{G}_0$ the Schwarzian image g of f , defined by $g(x) = (f(\bar{x}))^-$, is in $\tilde{G}_0$)) a solution y_0 can be found, with $y_0 \sim 1$, over $(\bar{N}_1(\delta), \bar{E}_0^*(\bar{N}_1(\delta)))$, for some δ satisfying $0 < \delta \leq \gamma$.

In the case $s^* = 1$, and a^* arbitrary, we can, using both Theorem II, §62, as applied in (71.2), and the method of proof of Fundamental Lemma A, §38, show that by means of substitutions of the form $y = \eta + z$, with η a solution of an algebraic equation with coefficients in $\tilde{G}_0$, a^* can be reduced to 1, with s^* kept equal to 1, so that the result of the previous paragraph may be applied to show the existence of a solution ~ 1 .

In some cases where both s^* and a^* are arbitrary, similar methods suffice to reduce s^* to 1, after which the above results will apply. This is the case, for instance, with the equation $F(x, y, y') = \sum_{i=0}^n a_i(x)y^i + \sum_{i=0}^n b_i(x)(y')^i$, with the set $\{a_0, \dots, a_n, b_0, \dots, b_n\}$ included in an LDF $_p$.

These questions, and other problems in the theory of differential equations to which the methods and results of the present paper apply, will be treated in forthcoming papers by the author.

PART V. APPENDIX

NOTE 1. Let $\mathcal{K}_0$ be a commutative field having a non-Archimedean valuation ϕ trivial over the prime field π , (cf. van der Waerden, *Modern algebra*, Vol. I, pp. 235, 238, Ungar, New York, 1949). If $f \prec g$ is defined to mean $\phi(f) < \phi(g)$, and $\mathcal{C}_0$ is set equal to π , and $\mathcal{U}$ is set equal to $\{1\}$, then $(\mathcal{K}_0, \prec, \mathcal{U}, \mathcal{C}_0)$ will be a graduated field. As a step in the opposite direction we note that if we define, for elements of $\mathcal{A}$, $\psi(f) =]f[$ (cf. §17), then $f \prec g$ is equivalent, for $\{f, g\} \subset \mathcal{A}$, to $\psi(f) < \psi(g)$, and ψ has all the properties of a non-Archimedean valuation (with values in the ordered Abelian group $\mathcal{U}$), except that its domain is restricted to $\mathcal{A}$, instead of being equal to the field $\mathcal{K}_0$.

NOTE 2. We use the older term "partial order" in preference to the term "order" since we wish to emphasize the fact that we are not dealing here with an "ordered field" (e.g., $a < b$, $c < d$ does not imply $a + c < b + d$).

NOTE 3. Subsets of $\mathcal{K}_0$ which contain 0 will always be denoted by a letter with a zero subscript, such as $\mathcal{E}_0$, and $\mathcal{E}_0 - \{0\}$ will be denoted by the corresponding symbol $\mathcal{E}$ without the zero subscript. Conversely, subsets of $\mathcal{K}_0$ which do not contain 0 will always be denoted by a letter without zero subscript, such as $\mathcal{F}$, and the symbol $\mathcal{F}_0$ will be used for $\mathcal{F} \cup \{0\}$.

NOTE 4. Theorem I asserts the converse as well: (c) implies (a) and (b). This converse statement is by far the easier part of the theorem.

NOTE 5. Since there is, in general, no common domain for the class $\tilde{K}_0(\bar{N})$ of functions meromorphic over $\bar{N}$, there is a technical difficulty in the interpretation of an individual element f of $\tilde{K}_0(\bar{N})$ as an element of a field. What more conveniently plays this role is the class of functions equivalent to f under an appropriate type of analytic continuation. This distinction is observed in the formulation of Fundamental Lemma B, §64, but, in the interests of notational brevity, is otherwise ignored in this paper.

NOTE 6. We use the customary notations $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$ for the fields of complex numbers, real numbers, and rational numbers, respectively. For each c in $\mathbf{C}$ the function having the constant value c at all points of the finite complex plane will also be denoted by c .

NOTE 7. For every graduated field there exist asymptotically constrained fields. The smallest example of such is the field generated by $\mathcal{M}$.

NOTE 8. In the sequel, the reference $(x)/(y)$ means the statement obtained by applying Statement (x) to Statement (y) . Thus, e.g., $(16.2)/(10.1)$ is the statement "If $f \approx g$ then $f \neq 0$ and $g \neq 0$."

NOTE 9. In §§44–49 our objective, which is achieved in §49, will be to show that in the case where X is loosely closed there are no abnormal polynomials. Thus, in fact, the lemmas of §§45–48 are all vacuous.

NOTE 10. That is, $\bar{N}$ is a nonempty set of nonempty connected open sets, such that if Ω and Λ are elements of $\bar{N}$, then there exists an element Δ of $\bar{N}$ such that $\Delta \subset \Omega \cap \Lambda$. (Cf. N. Bourbaki, *Topologie générale*, Chapter I, §5, no. 4.)

NOTE 11. We are not dealing here with the "complete analytic function" in the sense of Weierstrass; it is possible, for example, for a function which is analytic over $\bar{N}$ to have extensions and restrictions which are not analytic over $\bar{N}$.

NOTE 12. This, of course, is the standard definition of convergence over a filter (N. Bourbaki, *Topologie générale*, Chapter I, §6, no. 4), as it would carry over to the case of a filter base.

NOTE 13. References [A] and [P] are to the following papers by the author:

[A] *Contributions to the asymptotic theory of ordinary differential equations in the complex domain*, Mem. Amer. Math. Soc. No. 13 (1954), 81 pp.

[P] *Principal solutions of ordinary differential equations in the complex domain*, Mem. Amer. Math. Soc. No. 26 (1957), 107 pp.

NOTE 14. The emphasis in §§70–71 upon the very special choice of $\tilde{S}_p$ for $\tilde{L}$ is dictated by the importance of logarithmic monomials in the theory of algebraic differential equations with rational coefficients.

NOTE 15. In [P] §44 there is asserted obliquely (without proof) a stronger result to the effect that if the coefficients of a polynomial $F(y) = \sum_{i=0}^n a_i y^i$ belong to a “logarithmic domain of logarithmic rank $\leq p$ ” (cf. [A], §49), and $a_0 a_n$ is “non-trivial,” then the roots of F will be $\sim$ elements of $L \cdot M(\mathbf{R}, p)$ in the strong sense determined by $\tilde{E}_0^*(\tilde{N})$. This stronger assertion is false. The underlying considerations justified only the conclusion that the roots are $\sim$ elements of $L \cdot M(\mathbf{R}, p)$ in the weak sense determined by $\tilde{E}_0(0, \tilde{N})$. (Fortunately, the weaker conclusion sufficed for the purpose then at hand, namely the *selection* of certain logarithmic monomials.) I am indebted to E. W. Chamberlain for drawing my attention to this error in [P].

NOTE 16. This is easy but not completely trivial. Let $\tilde{A}_0(Y) = \tilde{A}(Y) \cup \{0\}$. Since $\tilde{A}(Y)$, by definition, is equal to the set

$$\{m(1+e): m \in L \cdot M(\tilde{V}, p); e \in \tilde{E}_{p0}(\tilde{N}, \tilde{V})\}$$

it is almost obvious that $\tilde{A}_0(Y) \subset \tilde{B}_{p0}(\tilde{V}, \tilde{N})$. Suppose that $f \in \tilde{B}_{p0}(\tilde{V}, \tilde{N})$. Then (**) $f = mG(\delta_1, \dots, \delta_s)$ with $\{m, \delta_1, \dots, \delta_s\} \subset L \cdot M(\tilde{V}, p)$, $\delta_j \rightarrow 0$ over $\tilde{N}$, and $G(x_1, \dots, x_s)$ analytic and vanishing at $x_1 = x_2 = \dots = x_s = 0$. By assembling like terms in the representation (**) we can obtain either $f = 0$ or $f = m_0(1 + e_0)$ with $m_0 \in L \cdot M(\tilde{V}, p)$, and $e_0 \in \tilde{E}_{p0}(\tilde{N}, \tilde{V})$, whence $f \in \tilde{A}_0(Y)$. Thus $\tilde{A}_0(Y) \subset \tilde{B}_{p0}(\tilde{V}, \tilde{N})$. Then since $\tilde{A}_0(Y)$ is obviously closed under multiplication and division, and $\tilde{B}_{p0}(\tilde{V}, \tilde{N})$ is obviously closed under addition and subtraction, $\tilde{A}_0(Y)$ is a field.

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