ON THE ORTHOGONALITY OF MEASURES INDUCED BY L-PROCESSES(1)

BY

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1. **Introduction and summary.** Let $\{X(t),\ 0 \leqq t \leqq 1\}$ be a real, centered stochastic process with independent increments with no fixed points of discontinuity and with $X(0) = 0$. The random variable $X(t)$ has then, for any $0 \leqq t \leqq 1$, an infinitely divisible distribution function $F(t,x)$ with characteristic function $\phi(t,v)$ satisfying Lévy's [9] formula

$$\text{(1)} \qquad \log \phi(t,v) = i\gamma(t)v - \frac{1}{2}\sigma^2(t)v^2 + \left[\int_{-\infty}^{0-} + \int_{0+}^{\infty} A(u,v) d_u H(t,u) \right],$$

where $\gamma(t)$ and $\sigma(t) \geqq 0$ are continuous functions and $\sigma(t)$ is nondecreasing, $A(u,v) = e^{iuv} - 1 - ivu/(1+u^2)$, $H(t,u)$ is, for any $t \in [0,1]$, defined and nondecreasing for $u < 0$ and $u > 0$, $H(t, -\infty) = H(t, +\infty) = 0$ and, for any finite $\varepsilon > 0$,

$$\left[\int_{-\varepsilon}^{0-} + \int_{0+}^{\varepsilon} u^2 d_u H(t,u) \right] < \infty .$$

For any $t \in [0,1]$ and $u < 0 (u > 0)$, the function $H(t,u)\,(-H(t,u))$ is equal to (see Doob [3, VIII, §7]) the expected number of jumps of the process $X(t)$ before time t of size less than u (larger than u).

We remind the reader that an infinitely divisible distribution function $F(x)$ is said to belong to the class L $(F \in L)$ if it is a limit, in the sense of weak convergence, of a sequence of distribution functions $F_n(x)$ of the form

$$F_n(x) = P\left(\frac{X_1 + \dots + X_n}{B_n} - A_n < x \right),$$

where $\{X_j\}$ $(j = 1, 2, 3, \dots)$ is a sequence of independent random variables, $B_n > 0$ and A_n are some sequences of constants, and X_j/B_n is asymptotically constant.

If $F \in L$, the function $H(t,u)$ assigned to F by formula (1), has for every $t \in [0,1]$, at any point $u < 0$ and $u > 0$ right and left derivatives in u and $uH'(t,u)$ is nonincreasing for $u < 0$ and $u > 0$, where $H'(t,u)$ denotes either the right or the left

Presented to the Society, April 13, 1962; received by the editors December 4, 1961.

(1) Research supported in part by National Science Foundation Grant NSF-G14146 at Columbia University.

derivative in u. The function $H(t,u)$ satisfies for arbitrary $u_1 < u_2 < 0$ and for arbitrary $0 < u_1 < u_2$ the inequality

$$(2) \qquad H(t,u_2) - H(t,u_1) \geqq H\left(t, \frac{u_2}{\alpha}\right) - H\left(t, \frac{u_1}{\alpha}\right)$$

for any $0 < \alpha < 1$. (See Gnedenko and Kolmogorov [5, §30].)

We introduce the following

DEFINITION. The stochastic process $\{X(t),\ 0 \leqq t \leqq 1\}$ with independent increments will be called a *L-process* if, for any $0 \leqq t \leqq 1$, the distribution function $F(t,x)$ of $X(t)$ belongs to L.

By Kolmogorov's [7] theorem, any real stochastic process $\{X(t),\ 0 \leqq t \leqq 1\}$ induces in the space $\mathfrak{A}$ of real functions a probability measure P_X, defined on the minimal Borel field $\mathscr{F}$ of subsets of $\mathfrak{A}$, generated by the cylindric sets, i.e., by the sets of all real functions $f(t)$ such that, for $n = 1,2,3,\cdots$, and any $t_1,\cdots,t_n$ from the interval $[0,1]$, the vector $\{f(t_1),\ldots,f(t_n)\}$ takes on values from Borel sets in the n-dimensional Euclidean space.

Let now P_1 and P_2 be two measures defined on a Borel field $\mathscr{F}$ from some space $\mathfrak{A}$. The measure P_2 is said to be absolutely continuous with regard to P_1 ($P_2 \ll P_1$) if, for any set $A \in \mathscr{F}$, the equation $P_1(A) = 0$ implies $P_2(A) = 0$. If both $P_1 \ll P_2$ and $P_2 \ll P_1$, the measures P_1 and P_2 are called equivalent ($P_1 \sim P_2$). The measures P_1 and P_2 are said to be orthogonal or mutually singular ($P_1 \perp P_2$) if for some $A \in \mathscr{F}$ both of the equations

$$P_1(A) = 0, \qquad P_2(\mathfrak{A} - A) = 0$$

hold.

The question of equivalence and orthogonality of measures in function spaces has attracted much attention. The pioneering work is due to Kryloff and Bogoliuboff [8]. An important result is due to Kakutani [6] who has shown that if P_i ($i = 1,2$) is a probability measure induced by a sequence of independent random variables X_{ij} ($j = 1,2,\cdots$) and for every j the probability measures of X_{1j} and X_{2j} are equivalent, then either $P_1 \sim P_2$ or $P_1 \perp P_2$. The problem of equivalence and orthogonality of measures induced by Gaussian processes has been discussed by Cameron and Martin [2], Prohorov [10], Baxter [1] and Feldman [4]. Necessary and sufficient conditions for the relation $P_2 \ll P_1$ when P_1 and P_2 are probability measures induced by processes with independent increments whose parameter range is finite have been given by Skorohod [11]. It is the purpose of this note to give conditions for $P_{X_1} \perp P_{X_2}$ when P_{X_1} and P_{X_2} are induced by centered L-processes with finite parameter range. It is shown, in particular, that if the L-processes are stable processes with unequal $H(t,u)$ functions, then $P_{X_1} \perp P_{X_2}$.

2. **Theorems and proofs.** Let $\{X_i(t),\ 0 \leqq t \leqq 1\}$ ($i = 1,2$) be L-processes with $H_i(t,u)$ in formula (1). If, for some t and $-\infty < u < 0$ or $0 < u < \infty$, both

$H_1(t,u)$ and $H_2(t,u)$ are identically 0, we shall agree to say that, on the considered half-line in the plane (t,u), $H_2'(t,u)/H_1'(t,u) = 1$. We shall prove the following theorems.

THEOREM 1. *Let P_{X_1} and P_{X_2} be probability measures induced in the space of real functions by the centered L-processes $\{X_1(t),\ 0 \leqq t \leqq 1\}$ and $\{X_2(t),\ 0 \leqq t \leqq 1\}$ with no fixed points of discontinuity, with $X_1(0) = X_2(0) = 0$ and with $\gamma_i(t)$, $\sigma_i(t)$ and $H_i(t,u)$ $(i = 1,2)$ in formula* (1). *Let $H_i'(t,u)$ $(i = 1,2,)$ denote the left-hand and right-hand derivatives in u of $H(t,u)$ for $u < 0$ and $u > 0$, respectively. If, for some $t_0 \in (0,1]$, $H_1(t_0,u)$ and $H_2(t_0,u)$ are not identically* 0, *the limits in* (3) *and* (4), *finite or infinite, exist, and at least one of the relations*

$$\rho_-(t_0) = \lim_{u \uparrow 0-} \frac{H_2'(t_0,u)}{H_1'(t_0,u)} = 1, \tag{3}$$

$$\rho_+(t_0) = \lim_{u \downarrow 0+} \frac{H_2'(t_0,u)}{H_1'(t_0,u)} = 1 \tag{4}$$

does not hold, then $P_{X_1} \perp P_{X_2}$.

THEOREM 2. *Let P_{X_1} and P_{X_2} have the same meaning as heretofore. If $X_1(t)$ and $X_2(t)$ are centered, stable processes and, for some $t_0 \in (0,1]$, $H_1(t_0,u) \not\equiv H_2(t_0,u)$, then $P_{X_1} \perp P_{X_2}$.*

The proof of Theorems 1 and 2 will be preceded by the proof of three lemmas concerning the H function of $F \in L$. For the sake of brevity, we shall write in the formulation of the lemmas and their proofs $H(u)$, without referring to the argument t.

LEMMA 1. *Let the distribution function $F \in L$ and let $H(u)$ correspond to F by formula* (1). *Then for $u < 0$ $(u > 0)$ the relation*

$$\lim_{u \uparrow 0-} H(u) = \infty \left(\lim_{u \downarrow 0+} H(u) = -\infty \right) \tag{5}$$

holds, unless $H(u) \equiv 0$ for $u < 0 (u > 0)$.

Proof of Lemma 1. Suppose that $H(u) \not\equiv 0$ for $u < 0$ and that relation (5) does not hold. Since $H(u)$ is nondecreasing,

$$\lim_{u \uparrow 0-} H(u) = a < \infty, \tag{6}$$

and, by the continuity of $H(u)$, it would be possible to find for arbitrary $\varepsilon > 0$ and $\eta > 0$ two numbers $u_1 < u_2 < 0$ such that $|u_1| < \eta$ and $H(u_2) - H(u_1) < \varepsilon$. Since η is arbitrary, it would then follow from formula (2) that the increment of H on an arbitrary large interval $[u_1/\alpha,\ u_2/\alpha]$ is less than ε. Taking into account that

$\varepsilon > 0$ may be arbitrarily small, we would get $H(u) \equiv 0$ for $u < 0$, contrary to the assumption; relation (5), therefore, holds.

The case of $u > 0$ may be proved in the same way.

LEMMA 2. *Let $F \in L$ and let $H(u)$ be the function assigned to F by formula* (1). *If, at some point $u_0 < 0$ $(u_0 > 0)$, $H(u_0) > 0$ $(H(u_0) < 0)$, the function $H(u)$ is for all $u_0 \leqq u < 0$ $(0 < u \leqq u_0)$ strictly increasing.*

Proof of Lemma 2. Let $H(u_0) > 0$ at $u_0 < 0$ and suppose that at two points u' and u'' $(u_0 \leqq u' < u'' < 0)$ the equality $H(u') = H(u'')$ holds. Since $H(u)$ is nondecreasing this would imply $H(u) = \text{const.}$ for $u' \leqq u \leqq u''$.

Now relation (2) implies that for any u_1, u_2 such that $u_1 < u_2 < u''$ and $u_2 - u_1 = u'' - u'$ the inequality

$$H(u'') - H(u') \geqq H(u_2) - H(u_1) \tag{7}$$

holds. Indeed, suppose for the moment that $u' < u_2$ and take $\alpha = u''/u_2$. We have then by (2)

$$H(u'') - H(u') \geqq H(u_2) - H\left(\frac{u'u_2}{u''}\right) \geqq H(u_2) - H(u_1).$$

If we drop the assumption $u' < u_2$, we arrive at (7) by repeating the argument a finite number of times.

Take now points $u_1 < u_0 < u_2 < \cdots < u_k < u' < u_{k+1} < u'' < 0$ such that $u_j - u_{j-1} = u'' - u'$ $(j = 2, \cdots, k+1)$. Since $H(u)$ is constant for $u' \leqq u \leqq u''$, the same will, by relation (7), be true for the interval $[u_1, u'']$. Since we can extend this procedure to any interval $[a, u'']$ with $a < u_0$, the increase of $H(u)$ on an arbitrary large interval (a, u'') would be equal 0, contrary to the assumption that $H(u_0) - H(-\infty) > 0$.

For $u > 0$ the proof runs along the same lines.

LEMMA 3. *Let $H(u)$ be the function assigned to $F \in L$ by formula* (1). *Then $H(u)$ is an absolutely continuous function on $(-\infty, 0-)$ and on $(0+, \infty)$.*

Proof. For the proof it is sufficient to show that $H(u)$ is absolutely continuous on any interval $[a, b]$ with $a<b<0$ or $0<a<b$. Let $a < b < 0$ and $H(b) \neq 0$. For an arbitrary $\varepsilon > 0$ take $c < b$ such that $H(b) - H(c) < \varepsilon$. Put $\delta = b - c$ and consider the disjoint intervals (a_i, b_i) $(i = 1, 2, \cdots, n)$ with $a \leqq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leqq b$ such that $\sum_{i=1}^n (b_i - a_i) < \delta$ and otherwise arbitrary. By (7) we get

$$\sum_{i=1}^{n} [H(b_i) - H(a_i)] < \varepsilon.$$

Since n is arbitrary, the absolute continuity of $H(u)$ on $[a, b]$ has been proved. For intervals on the half-line $u > 0$, the proof is analogous.

Proof of Theorem 1. We remark first of all that, without restricting the generality of our considerations, we may assume that $X_1(t)$ and $X_2(t)$ are separable. Indeed, if they were not separable, we would consider the separable processes $X_1^*(t)$ and $X_2^*(t)$ that are stochastically equivalent to $X_1(t)$ and $X_2(t)$, repectively. (See Doob [3, p. 57].) Since, for $i = 1,2$, the finite dimensional distributions of $X_i(t)$ and $X_i^*(t)$ are identical, the probability measure $P_{X_i^*}$ induced in $\mathfrak{A}$ by $X_i^*(t)$ is equal to P_{X_i} for any set A from $\mathscr{F}$. Therefore, to show that $P_{X_1} \perp P_{X_2}$ it would be enough to show that for some set A from $\mathscr{F}$ both of the equations $P_{X_1^*}(A) = 0$, $P_{X_2^*}(A) = 1$ hold. We therefore assume, in the proofs of Theorems 1 and 2, that $X_1(t)$ and $X_2(t)$ are separable and, in proving orthogonality, we shall use sets $A \in \mathscr{F}$ only.

We remark now that the sample functions of a centered, separable L-process $\{X(t), 0 \leqq t \leqq 1\}$ with no fixed points of discontinuity and with $H(t_0, u) \not\equiv 0$ for some $t_0 \in (0,1]$ are discontinuous, with probability 1. Indeed, let N_n denote the number of jumps before t_0 either of size $\in (-2^{-n}, -2^{-n-1}]$ or of size $\in (2^{-n-1}, 2^{-n}]$. Then the N_n form a sequence of independent Poisson variables with

$$\lambda_n = E(N_n) = H(-2^{-n-1}) - H(-2^{-n}) + H(2^{-n}) - H(2^{-n-1}).$$

By Lemma 1, we have $\sum_{n=1}^{\infty} \lambda_n = \infty$. This implies (see [3, p. 115, Theorem 2.7 (ii)]) that $\sum_{n=1}^{\infty} N_n = \infty$, with probability 1.

Let now $H_1(t_0, u)$ and $H_2(t_0, u)$ not be identically 0. Denote by $N_i(t_0, 0-)$ and $N_i(t_0, 0+)$ $(i = 1,2)$ the number of jumps before t_0 of the process $X_i(t)$, of negative and positive size, respectively. If $H_1(t_0, u) \equiv 0$ for $u < 0$ and $H_2(t_0, u) \equiv 0$ for $u > 0$, we have $EN_1(t_0, 0-) = EN_2(t_0, 0+) = 0$. Since the sample functions of $X_1(t)$ and $X_2(t)$ are discontinuous, the $X_1(t)$ and $X_2(t)$ processes are entirely concentrated on functions with negative and positive jumps, respectively. Hence $P_{X_1} \perp P_{X_2}$. Denote for $i = 1,2$

$$a_i = a_i(t_0) = \inf\{u : u < 0, H_i(t_0, u) > 0\}$$

and suppose now that both $a_1 < 0$ and $a_2 < 0$. Make the unrestrictive assumption that $a_2 \leqq a_1$. We shall show that for any u_1, u_2 with $a_1 \leqq u_1 < u_2 \leqq 0-$

$$\int_{u_1}^{u_2} d_u H_2(t_0, u) = \int_{u_1}^{u_2} \frac{H_2'(t_0, u)}{H_1'(t_0, u)} d_u H_1(t_{,0} u). \tag{8}$$

To see this, let us notice first that, by Lemma 2, $H_1'(t_0, u) > 0$ for $a_1 < u < 0$. Next, the derivatives of $H_1(t_0, u)$ in u exist everywhere, except possibly at points u belonging to a set of Lebesgue measure 0. By Lemma 3, this exceptional set has H_1-measure equal to 0_j thus relation (8) holds.

Let us now assume that relation (3) does not hold; hence $\rho_- = \rho_-(t_0) \neq 1$. Suppose $\rho_- < \infty$. For $\varepsilon > 0$ arbitrary, we could then find a $c < 0$ such that for all u from $[c, 0)$

$$\left| \frac{H_2'(t_0, u)}{H_1'(t_0, u)} - \rho_- \right| \leqq \varepsilon. \tag{9}$$

Take $\varepsilon = (1/2)|1 - \rho_-|$ and a number c such that (9) holds. Let us choose a sequence of points u_n from $[c, 0)$ such that $H_1(t_0, u_n) - H_1(t_0, c) = n$ $(n = 1, 2, 3, \cdots)$. Denote by $M_i(t_0, c, u_n)$ $(i = 1, 2)$ the number of jumps of size $\in [c, u_n]$ of the process $X_i(t)$ before t_0. Then $M_1(t_0, c, u_n)$ is a Poisson variable with parameter equal to n, while $M_2(t_0, c, u_n)$ is a Poisson variable with parameter equal to $H_2(t_0, u_n) - H_2(t_0, c)$.

By the Chebyshev Inequality, we have for any $\delta > 0$

$$P_{X_1}\left(\left| \frac{M_1(t_0, c, u_n)}{n} - 1 \right| \geqq \delta \right) \leqq \frac{1}{\delta^2 n^2}. \tag{10}$$

By the Borel-Cantelli Lemma, since $\sum_{n=1}^{\infty} 1/n^2 < \infty$, relation (10) implies

$$P_{X_1}\left(\lim_{n \to \infty} \frac{M_1(t_0, c, u_n)}{n} = 1 \right) = 1. \tag{11}$$

Write $H_2(t_0, u_n) - H_2(t_0, c) = G(t_0, c, u_n)$. We have, as before, for any $\delta > 0$

$$P_{X_2}\left(\left| \frac{M_2(t_0, c, u_n)}{G(t_0, c, u_n)} - 1 \right| \geqq \delta \right) \leqq \frac{1}{\delta^2 G^2(t_0, c, u_n)}. \tag{12}$$

By relations (8) and (9), we have

$$(\rho_- - \varepsilon)n \leqq G(t_0, c, u_n) \leqq (\rho_- + \varepsilon)n; \tag{13}$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{G^2(t_0, c, u_n)} < \infty. \tag{14}$$

Again, by the Borel-Cantelli Lemma

$$P_{X_2}\left(\lim_{n \to \infty} \frac{M_2(t_0, c, u_n)}{G(t_0, c, u_n)} = 1 \right) = 1. \tag{15}$$

Now, if $\rho_- < 1$, $\rho_- + \varepsilon < 1$. Since, by (13),

$$\frac{M_2(t_0, c, u_n)}{n} \leqq (\rho_- + \varepsilon) \frac{M_2(t_0, c, u_n)}{G(t_0, c, u_n)}, \tag{16}$$

we have by (15)

$$P_{X_2}\left(\limsup_{n \to \infty} \frac{M_2(t_0, c, u_n)}{n} < 1 \right) = 1. \tag{17}$$

Similarly, if $\rho_- > 1$, $\rho_- - \varepsilon > 1$. Since, again by (13),

$$\frac{M_2(t_0, c, u_n)}{n} \geqq (\rho_- - \varepsilon) \frac{M_2(t_0, c, u_n)}{G(t_0, c, u_n)}, \tag{18}$$

we have by (15)

$$P_{X_2}\left(\liminf_{n\to\infty} \frac{M_2(t_0, c_0, u_n)}{n} > 1 \right) = 1. \tag{19}$$

It follows from (11), (17) and (19) that $P_{X_1} \perp P_{X_2}$.

If $\rho_- = \infty$, one gets the same result by interchanging the role of H_2 and H_1. The proof is analogous if $H_1(t_0, u)$ and $H_2(t_0, u)$ are not identically 0 for $u > 0$. This remark completes the proof of Theorem 1.

Proof of Theorem 2. Let $X_1(t)$ and $X_2(t)$ be separable, centered stable processes, without fixed discontinuity points, and let, for some $t_0 \in (0,1]$, $H_1(t_0,u) \not\equiv H_2(t_0,u)$. If the exponent, say, α_1 of the process $X_1(t)$ equals 2, $H_1(t_0,u) \equiv 0$, while $H_2(t_0,u) \not\equiv 0$. We have thus $P_{X_1} \perp P_{X_2}$, since the sample functions of $X_1(t)$ are continuous, with probability 1, while those of $X_2(t)$ are almost all (P_{X_2}) discontinuous. Let now $0 < \alpha_i < 2 (i = 1,2)$. If $H_1(t_0,u) = 0$ for all $u < 0$ while $H_2(t_0,u) = 0$ for all $u > 0$, then evidently $P_{X_1} \perp P_{X_2}$. If both $H_1(t_0,u)$ and $H_2(t_0,u)$ are not identically equal to 0 for, say, all $u < 0$, we have

$$\frac{H_2'(t_0,u)}{H_1'(t_0,u)} = k|u|^{\alpha_1-\alpha_2}, \tag{20}$$

where k is some constant. Since $H_1(t_0,u) \not\equiv H_2(t_0,u)$, we have either $\alpha_1 \neq \alpha_2$, or $k \neq 1$, or both. Consequently, ρ_- equals either 0, or ∞, or $k \neq 1$. By Theorem 1, it follows $P_{X_1} \perp P_{X_2}$.

Remark. The following example shows that there exist L-processes $X_1(t)$ and $X_2(t)$ such that P_{X_2} is absolutely continuous with regard to P_{X_1}. Relations (3) and (4) are, of course, then satisfied.

Example. Consider stationary, centered L-processes with $\gamma_1 = \gamma_2 = \sigma_1 = \sigma_2 = 0$ and with

$$H_1(u) = \begin{cases} 0 & (u < 0), \\ 2\log u & (0 < u \leqq 1/2), \\ 2\log u/2 & (1/2 \leqq u \leqq 2), \\ 0 & (u \geqq 2), \end{cases}$$

and

$$H_2(u) = \begin{cases} 0 & (u < 0), \\ 2\log u & (0 < u \leqq 1), \\ 0 & (u \geqq 1). \end{cases}$$

By using Skorohod's [11] results, it is easy to check that P_{X_2} is absolutely continuous with regard to P_{X_1}.

Acknowledgement. The author is most grateful to the referee for his stimulating comments.

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