

# LOCALLY COMPACT SEMIGROUPS IN WHICH A SUBGROUP WITH COMPACT COMPLEMENT IS DENSE

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The object of this paper is a complete study of the category of all locally compact topological semigroups which contain a dense open subgroup such that the complement of this subgroup is compact. Unlike the theory of compact topological semigroups where considerable progress has been made in the last years, our knowledge of not necessarily compact locally compact topological semigroups is confined to a few very special classes such as totally ordered semigroups (which already belong to the mathematical folklore in this subject), or semigroups on the euclidean plane [8; 12], or semigroups on euclidean spaces or manifolds satisfying certain additional conditions [13; 14]. The investigation of the semigroups described at the beginning is supposed to add a new item to the list of classified locally compact semigroups. The motivation for a study of locally compact groups with compact boundary, as we shall later call those semigroups, has different roots:

(A) If a locally compact semigroup has an identity, then the set of all elements in the semigroup having inverses relative to the identity is a subgroup, called the maximal subgroup. It may or, as examples show, may not be open. If it is open, then it is either compact or open in its closure. If in the latter case the boundary of the maximal subgroup is compact (as e.g. in [14]), then the closure of the maximal subgroup is one of the semigroups which we shall describe completely. In this instance our theory gives all desirable information about the maximal subgroup and the immediately adjacent parts of the semigroup.

(B) A special subclass of the semigroups under consideration plays an important rôle in classical topological algebra. All multiplicative semigroups of locally compact topological division rings are semigroups in which the subgroup of all invertible elements is dense and has certainly a compact complement, namely the set consisting of zero. Apart from the fact that these semigroups have only one nongroup-element they are distinguished by the fact that their topological space is homogeneous, i.e. has a transitive group of homeomorphisms (e.g. the one defined by the additive group of the field). By a celebrated theorem

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Received by the editors September 26, 1961.

<sup>(1)</sup> This work was supported by the National Science Foundation. It has been accepted as part of the author's Habilitationsschrift by the Mathematisch-naturwissenschaftliche Fakultät der Universität Tübingen.

of Pontrjagin a locally compact connected division ring is either isomorphic to the reals, the complexes, or the quaternions. It has been proved that a locally compact semigroup containing a dense subgroup and having one and only one nongroup-element is one of the multiplicative semigroups of these division rings provided it is homogeneous [7]. If a locally compact division ring is not connected, then it is a finite dimensional algebra over a complete  $p$ -adic field or a complete field of Laurent series in one transcendental variable over a prime field of characteristic  $p$ . There is obviously one common feature in the multiplicative groups of all these division rings: They are a direct product of a compact subgroup (the reals or complexes with absolute value 1, the quaternions with norm 1, the elements with value zero in the case of the fields with a valuation) and a central not compact subgroup which is isomorphic to the additive reals in the connected case and is infinite cyclic in the totally disconnected case (see e.g. [16]). It turns out that, from our point of view, this is the most typical property of the field-semigroups.

(C) According to a terminology introduced by Freudenthal, a locally compact group is said to have two ends if it admits a two point compactification, i.e. if it can be compactified by adding two not isolated points. Freudenthal characterized the two ended locally compact connected separable groups as those groups which are isomorphic to a direct product of the reals and a compact connected separable group [5] (see also [9a]). In these groups one can make one of the new points a nonisolated zero and obtain a semigroup of our class. The notion of having two ends is not very manageable in the case of not necessarily connected groups without further assumptions. Suppose now that  $G$  is a locally compact group to which a nonisolated zero 0 can be added so as to make it a topological semigroup. Then  $G$  certainly admits a two point compactification, namely the one point compactification of  $G \cup \{0\}$ . In this sense the (not necessarily connected) locally compact groups which admit a continuous extension of their multiplication to a not isolated zero are a natural generalisation of the two-ended connected locally compact groups.

If  $S$  is a locally compact semigroup in which an open subgroup  $G$  is dense in such a way that  $S \setminus G$  is compact, then  $S \setminus G$  is a compact semigroup ideal as we shall see; this is in fact an almost trivial observation. The quotient semigroup of  $S$  modulo the congruence relation which collapses all of  $S \setminus G$  onto one point and leaves  $G$  unaffected is then the union of a group isomorphic to  $G$  and a not isolated zero, namely the coset of  $S \setminus G$ . These semigroups are called groups with zero. The first section is devoted to the study of locally compact groups with zero; the result is one final theorem. *We prove that the locally compact groups, which can be made into a topological semigroup by adding a nonisolated zero are the product of a characteristic maximal compact subgroup  $C$  and a noncompact subgroup  $M$  which is either isomorphic to the additive reals*

or is an infinite cyclic group. The first case occurs if and only if the identity of the group and the zero are contained in some connected subspace. Furthermore, in the first case this product is direct, i.e.  $M$  is normal; examples show that in the second case  $M$  need not be normal. The reader may observe that no connectivity assumptions are made; this is reflected in the fact that  $C$  can be any compact group. This result is the general setting in which the field-semigroups may be looked at in an intrinsic fashion. In this context a result might be of interest which recently has been proved [7]:

If  $G$  is a locally compact two-ended group which is compact modulo the component of the identity, the  $G$  is the semidirect product of a normal subgroup  $M$  isomorphic to the additive reals and a compact subgroup  $C$ ; moreover  $C$  contains a subgroup  $C'$  of index 2 and  $C$  being normal in  $G$  such that  $MC'$  is a direct product; clearly  $MC'$  is a normal subgroup of  $G$  of index two. Comparing this result with the theorem mentioned above, we observe that in general no nonisolated zero can be added to such a  $G$  to make it a topological semigroup; it can, however, be attached to "one half" of the group, namely to a normal subgroup of index 2. The precise reason for this phenomenon is given in Proposition 1.12.

Chu [3] has proved that a locally compact group without any nontrivial compact subgroup which is the product of an infinite cyclic subgroup and a compact subset is either isomorphic to the reals or is infinite cyclic, a result that should be mentioned at this place and in the context of Proposition 1.20 and 1.21.

The second chapter is concerned with the investigation of the general case, i.e. with the structure of the ideal  $S \setminus G$ , which turns out to be a group, and with the way  $G$  and  $S \setminus G$  are related in  $S$ . The final results are too complicated to be stated here in full generality. We may, however, indicate the most typical source of complication in a characteristic example in which the maximal subgroup  $G$  is very simple, namely isomorphic to the additive reals.

Let  $M_0$  be the semigroup of all non-negative reals and  $C$  the circle group, i.e. the group of all complex numbers  $e^{2\pi ir}$  of absolute value 1. The direct product  $M_0 \times C$  is the semigroup one obtains if one "blows up" the zero in the complex plane to a circle; this is just another way of saying that one gets the complex plane if one collapses the ideal  $\{0\} \times C$  to a point. Let  $G$  be the subgroup of all elements  $(e^{-r}, e^{2\pi ir})$ ,  $-\infty < r < \infty$ ; evidently  $G$  is isomorphic to the additive group of reals. Let the semigroup  $S$  be the closure of  $G$  in  $M_0 \times C$ ; then  $S = G \cup (\{0\} \times C)$ , for as  $r$  tends to  $\infty$ , the point  $(e^{-r}, e^{2\pi ir})$  "spirals down" to the bottom edge of the closed cylinder  $M_0 \times C$  in a fashion which appears to be an analogue of the behaviour of the graph of  $\sin r^{-1}$  for  $r$  approaching 0. Thus  $S$  is a semigroup of the type we consider, for  $G$  is a dense open subgroup and  $\{0\} \times C$  is the compact complement, which in this case is a circle group.  $S$  is connected but not arc-wise connected. It is an example of

a connected semigroup in which no one-parameter semigroup runs straight from the identity to the unit of the ideal, as e.g. in the semigroup  $M_0 \times C$ , where  $(e^{-r}, 1)$ ,  $-\infty < r < \infty$  is the required one-parameter semigroup. This gives rise to a trichotomy into the cases where the units of  $G$  and of  $S \setminus G$  are in no connected subset, where they are in some connected subset but cannot be joined by an arc, and finally where they can be joined by an arc. The last case is the one which one would expect, and in fact the semigroups having this property are easy to describe: They are direct products of  $M_0$  with any compact group  $C$  and homomorphic images of these obtained by collapsing in  $\{0\} \times C$  cosets modulo some normal subgroup  $\{0\} \times N$  of  $\{0\} \times C$  and leaving the maximal subgroup unaffected. The example described above has the property that the "bad" semigroup  $S$  can be embedded in the "good" semigroup  $M_0 \times C$ . This suggests the question whether or not in the general case, given a semigroup  $S$  in the class under consideration, it can always be embedded in some "good" semigroup of our class. This question is answered in the affirmative.

I am indebted to Helmut Wielandt for giving me the hint to apply transfer to the problem of splitting a one-parameter subgroup with finite index; this suggestion initiated a part of the methods of proof in 1.18 and 1.20. Paul S. Mostert has been a patient participant in many conversations about the present subject.

This investigation will be continued by a study of those of the semigroups characterized in this article which are in addition homogeneous.

## I. Groups with zero.

1.1. DEFINITION. Let  $S$  be a topological Hausdorff space which satisfies the following two conditions:

(i)  $S$  is a topological semigroup,

(ii) there is an element  $0$  in  $S$  which is not isolated such that  $S \setminus \{0\}$  is a group.

Then  $S$  is called a *topological group with zero*. The complement  $S \setminus \{0\}$  will always be denoted with  $G$ . Before we discuss the structure of locally compact groups with zero, we compile some known information as necessary background material; in some places proofs will be given for the convenience of the reader even if proofs are available in the literature or could at least be taken over from those by only minor modifications [16; 18]. The term "*locally compact topological group with zero*" will be abbreviated as *l.c.g.z.*

1.2. PROPOSITION. *In a l.c.g.z.  $S$ , the group  $G$  is a locally compact topological group, and  $0$  is a zero for  $S$ , and  $1$ , the unit of  $G$ , is a unit of  $S$ .*

**Proof.** It has been proved by Ellis that every locally compact topological semigroup with algebraic group structure is a topological group [4]. Therefore  $G$  is topological. Since  $g1 = 1g = g$  for all  $g \in G$  and  $0$  is in the closure of  $G$  (1.1, (ii)),  $01 = 10 = 0$ , i.e.  $1$  is a unit of  $S$ . For all group elements  $g$  the map-

pings  $x \rightarrow gx$  and  $x \rightarrow g^{-1}x$  are continuous mappings of  $S$  into itself; because  $g^{-1}gx = gg^{-1}x = 1x = x$  for all  $x \in S$ , they are inverses of each other and therefore homeomorphisms of  $S$ ; since  $G$  is mapped onto itself,  $0$  is left fixed. The same statements clearly can be made about the mappings  $x \rightarrow xg$  and  $x \rightarrow xg^{-1}$ . As  $0$  is fixed under both left and right translations with elements of  $G$ , we have  $S0 \cup 0S \subset G0 \cup 0G \subset \overline{G0} \cup \overline{0G} = \{0\}$ . This proves that  $0$  is in fact a zero for  $S$ .

1.3. PROPOSITION. *Let  $S$  be a l.c.g.z.,  $K$  any compact subspace of  $S$ , and  $U$  a neighborhood of  $0$ . Then there is a neighborhood  $V$  of  $0$  such that  $VK \cup KV \subset U$ .*

**Proof.** For each  $x$  in  $K$  there is, because of the continuity of multiplication, a neighborhood  $V_x$  of  $0$  and a neighborhood  $W_x$  of  $x$  such that  $V_x W_x \cup W_x V_x \subset U$ . Since  $K$  is compact, it is covered by a finite number of neighborhoods  $W_{x_1}, \dots, W_{x_n}$ . The intersection  $V = V_{x_1} \cap \dots \cap V_{x_n}$  is the required neighborhood of  $0$ .

1.4. COROLLARY. *A l.c.g.z.  $S$  has a countable base for the filter of neighborhoods of  $0$  if and only if there is a (countable) sequence of points of  $G$  converging to  $0$ .*

**Proof.** If  $S$  has a countable base at  $0$ , then, since  $0$  is not isolated, there is a sequence of points of  $G$  converging to  $0$ ; it is sufficient to take a point different from  $0$  in each of the neighborhoods of a countable base. Conversely, let  $\{x_n : n = 1, 2, \dots\}$  converge to zero with  $x_n \neq 0$  for all  $n$ . If  $K$  is an arbitrary compact neighborhood of  $0$ , then  $\{Kx_n : n = 1, 2, \dots\}$  is a neighborhood base for  $0$ ; first of all  $Kx_n$  is a neighborhood of  $0$  because right-multiplication with group elements is a homeomorphism (1.2). If  $U$  is a given neighborhood, choose  $V$  as in 1.3; then we find an  $x_n$  in  $V$  since the sequence of the  $x_n$  is converging to  $0$ ; but this implies  $Kx_n \subset KV \subset U$ .

1.5. LEMMA. *Let  $S$  be a l.c.g.z.,  $U$  an open neighborhood of zero with a compact closure which does not contain  $1$ . Then the following statements are equivalent:*

- (i)  $\lim_{n \rightarrow \infty} g^n = 0$ ,
- (ii) *there is a natural number  $k$  such that  $(\bar{U} \cup \{1\})g^k \subset U$ .*

**Proof.** (i) implies (ii): This follows from 1.3, since  $g^k$  can be made to lie in any prescribed neighborhood  $V$  of  $0$  because  $\lim_{n \rightarrow \infty} g^n = 0$ .

(ii) implies (i): Let  $(\bar{U} \cup \{1\})g^k \subset U$ . Then for all natural numbers  $n$  we have  $(\bar{U} \cup \{1\})g^{kn} \subset U$ ; for if this inclusion holds for  $n$  then  $(\bar{U} \cup \{1\})g^{k(n+1)} \subset Ug^k \subset (\bar{U} \cup \{1\})g^k \subset U$ , i.e. it holds also for  $n + 1$ . In particular, we can infer from this that  $\{g^k, g^{2k}, g^{3k}, \dots\} \subset U \subset \bar{U}$ .

The closure  $G(g^k)$  in  $G$  of the cyclic group generated by  $g^k$  is either an infinite cyclic discrete or compact subgroup [11, p. 102, and 17, p. 96]. If  $G(g^k)$  were compact, then any neighborhood of  $1$  would contain some positive power of  $g^k$  [11, p. 102], i.e.  $1 \in \{g^k, g^{2k}, \dots\} \subset \bar{U}$ , and this is a contradiction to the

assumption that 1 be not in  $\bar{U}$ . But in the compact space  $\bar{U}$  a sequence must have a cluster point [10, p. 138]; since the only possible cluster point is 0, we have indeed  $\lim_{n \rightarrow \infty} g^{kn} = 0$ . This implies on the other hand  $\lim_{n \rightarrow \infty} g^{kn+i} = 0$  for all  $i = 0, 1, \dots, n-1$ ; from these  $n$  identities we conclude  $\lim_{n \rightarrow \infty} g^n = 0$ .

**1.6. LEMMA.** *In a l.c.g.z.  $S$ , the set  $S_0 = \{g : \lim_{n \rightarrow \infty} g^n = 0\}$  is an open neighborhood of 0.*

**Proof.** Let  $U$  be defined as in 1.5, and let  $(\bar{U} \cup \{1\})g^k \subset U$ . Then, because of the continuity of multiplication, for any  $x$  in  $\bar{U} \cup \{1\}$  there is a neighborhood  $V_x$  of  $x$  and a neighborhood  $W_x$  of  $g$  such that for all  $w \in W_x$  we have  $V_x w^k \subset U$ . As  $\bar{U} \cup \{1\}$  is compact, it is covered by a finite number of neighborhoods  $V_{x_1}, \dots, V_{x_m}$ . If we put  $W = W_{x_1} \cap \dots \cap W_{x_m}$  then for all  $w \in W$  we get  $\bar{U} \cup \{1\}w^k \subset U$ . This, however, proves that each point in  $S_0$  is an inner point.

**1.7. COROLLARY.** *Let  $S$  be a l.c.g.z. and  $S \cup \{\infty\}$  its one point compactification. Then 0 and  $\infty$  have a countable neighborhood base, or as we say shortly,  $S$  is countable at zero and infinity.*

**Proof.** From the fact that 0 is not isolated and from 1.6 we infer the existence of a  $g \in G$  such that  $\lim_{n \rightarrow \infty} g^n = 0$ . Proposition 1.4 establishes then the existence of a countable neighborhood base for 0. We prove that  $S = \bigcup \{Wg^{-n} : n = 1, \dots\}$  which will show that  $\infty$  has a countable base of neighborhoods. Clearly  $0 \in Wg^{-1} \subset \bigcup \{Wg^{-n} : n = 1, \dots\}$ . Let now  $h \in G$ . Since  $\lim_{n \rightarrow \infty} hg^n = 0$ , we find a natural number  $n$  so that  $hg^n \in W$  or, equivalently,  $h \in Wg^{-n}$  which is in the union of all these sets. This finishes the proof.

**1.8. LEMMA.** *Let  $S$  be a l.c.g.z. and  $S \cup \{\infty\}$  its one point compactification. If  $\{g_n : n = 1, \dots\}$  is a sequence in  $G$  then the following relations are equivalent:*

$$\lim_{n \rightarrow \infty} g_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} g_n^{-1} = \infty.$$

**Proof.** (i)  $\lim_{n \rightarrow \infty} g_n = 0$  implies  $\lim_{n \rightarrow \infty} g_n^{-1} = \infty$ : The sequence  $\{g_n^{-1} : n = 1, \dots\}$  must have clusterpoints in the compact space  $S \cup \{\infty\}$ . Assume that  $h$  is such a clusterpoint and assume in addition that  $h \in S$ . Then there is a subnet  $\{g_{n(i)}^{-1} : i \in I\}$  of the sequence converging to  $h$ . But then

$$1 = \lim_I g_{n(i)} g_{n(i)}^{-1} = \lim_I g_{n(i)} \lim_I g_{n(i)}^{-1} = 0h = 0$$

and this is a contradiction. Therefore  $\infty$  is the only cluster point of  $\{g_n^{-1} : n = 1, \dots\}$ .

(ii)  $\lim_{n \rightarrow \infty} g_n = \infty$  implies  $\lim_{n \rightarrow \infty} g_n^{-1} = 0$ : From  $\lim_{n \rightarrow \infty} g_n = \infty$  we know that  $\{g_n : n = 1, \dots\}$  has no clusterpoint in  $G$ . Therefore  $\{g_n^{-1} : n = 1, \dots\}$  has no clusterpoint in  $G$ . We have to prove that  $\infty$  cannot be among its clusterpoints which must exist on the compact space  $S \cup \{\infty\}$ ; this will show that 0 is the only clusterpoint of  $\{g_n^{-1} : n = 1, \dots\}$  which will finish the proof. Assume now that  $\infty$

is a clusterpoint of this sequence. Then, because  $S$  is countable at infinity, there is a subsequence of it converging to  $\infty$ , and by picking this subsequence and changing notation we may as well assume that  $\lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} g_n^{-1} = \infty$ . From 1.6 we deduce the existence of an element  $h \in G$  with  $\lim_{n \rightarrow \infty} h^n = 0$ . Let  $U$  be an open neighborhood of 0 with compact closure. For any natural number  $k$  the spaces  $\bar{U}h^{-k}$  and  $h^{-k}\bar{U}$  are compact; therefore the sequences  $\{g_n : n = 1, \dots\}$  and  $\{g_n^{-1} : n = 1, \dots\}$  finally are outside of these sets respectively; hence it can never happen that a subsequence of  $\{g_n h^k : n = 1, \dots\}$  or of  $\{h^k g_n^{-1} : n = 1, \dots\}$  is completely in  $\bar{U}$ . On the other hand, both of the sequences  $\{g_n h^m : m = 1, \dots\}$  and  $\{h^m g_n^{-1} : m = 1, \dots\}$  converge to zero for all  $n = 1, \dots$ , so that we find minimal natural numbers  $m_n$  and  $m'_n$  such that both of the sets  $\{g_n h^{m_n}, g_n h^{m_n+1}, \dots\}$  and  $\{h^{m'_n} g_n^{-1}, h^{m'_n+1} g_n^{-1}, \dots\}$  are in  $\bar{U}$ . We observe that the sequences  $\{m_n : n = 1, \dots\}$  and  $\{m'_n : n = 1, \dots\}$  of natural numbers increase indefinitely as  $n$  tends to infinity; for if, e.g.  $\{m_n : n = 1, \dots\}$  had a subsequence  $\{m_{n(i)} : i = 1, \dots\}$  bounded by  $k$ , then the sequence  $\{g_{n(i)} h^k : i = 1, \dots\}$  were completely in  $\bar{U}$  which is impossible after the preceding remark. We may now, without losing generality, assume that all natural numbers  $m_n$  and  $m'_n$  are greater than 1; otherwise we drop a finite number of terms in our sequences. Because of the minimality of  $m_n$  and  $m'_n$ , we know that  $g_n h^{m_n-1}$  is not in  $\bar{U}$ , hence in particular not in  $U$ ; likewise  $h^{m'_n-1} g_n^{-1}$  is not in  $U$ . Therefore the sequences  $\{g_n h^{m_n} : n = 1, \dots\}$  and  $\{h^{m'_n} g_n^{-1} : n = 1, \dots\}$  lie in the compact subspaces  $\bar{U} \setminus Uh$  and  $\bar{U} \setminus hU$  respectively, i.e. the sequence  $\{(g_n h^{m_n}, h^{m'_n} g_n^{-1}) : n = 1, \dots\}$  is in the compact product space  $(\bar{U} \setminus Uh) \times (\bar{U} \setminus hU)$  and has, therefore, a convergent subnet  $\{(g_{n(i)} h^{m_{n(i)}}, h^{m'_{n(i)}} g_{n(i)}^{-1}) : i \in I\}$  with some directed set  $I$ . Note that there may be no convergent subsequence since  $G$  does in general not satisfy the first axiom of countability. If, however, the limit of this net is  $(a, b)$  then clearly  $a \neq 0$  and  $b \neq 0$  whence  $ba \neq 0$ . On the other hand we have

$$ba = \lim_I h^{m'_{n(i)}} g_{n(i)}^{-1} \lim_I g_{n(i)} h^{m_{n(i)}} = \lim_I h^{m'_{n(i)} + m_{n(i)}} = 0$$

since the net  $(m'_{n(i)} + m_{n(i)} : i \in I)$  is in definitely increasing as a subnet of the indefinitely increasing sequence  $(m'_n + m_n : n = 1, \dots)$ . This contradiction finally proves the lemma.

**1.9. COROLLARY.** *If  $S$  is a l.c.g.z. and  $S \cup \{\infty\}$  onto itself is defined by  $f(g) = g^{-1}$  for  $g \in G$  and by  $f(0) = \infty$  and  $f(\infty) = 0$ , then  $f$  is an involutive homeomorphism of  $S \cup \{\infty\}$ , i.e.  $f$  is continuous and  $f^2$  is the identity mapping.*

**Proof.** Clearly  $f$  is an involution, i.e.  $f^2$  is the identity mapping of  $S \cup \{\infty\}$ ; it is continuous on  $G$ . If  $\{g_n : n = 1, \dots\}$  is a sequence converging to 0 (or to  $\infty$ ), then, by 1.8,  $\lim_{n \rightarrow \infty} f(g_n)$  is  $\infty$  (or 0 respectively). But since  $S \cup \{\infty\}$  is countable at 0 and at  $\infty$ , this establishes the continuity of  $f$  in 0 and  $\infty$ .

1.10. PROPOSITION. *Let  $S$  be a l.c.g.z. and let the subsets  $S_0$ ,  $S_1$ , and  $S_2$  of the one point compactification  $S \cup \{\infty\}$  be defined as follows:*

$$S_0 = \{g : g \in S, \lim_{n \rightarrow \infty} g^n = 0\},$$

$$S_1 = \{g : g \in G, g \text{ is contained in a compact subgroup of } G\},$$

$$S_2 = \{g : g \in G, \lim_{n \rightarrow \infty} g^n = \infty\} \cup \{\infty\}.$$

*Then these sets are pairwise disjoint, their union is  $S \cup \{\infty\}$ , the sets  $S_0$  and  $S_2$  are open and the set  $S_1$  is compact in  $G$ . The involution  $f$  defined in 1.9 maps  $S_1$  onto itself and interchanges  $S_0$  and  $S_2$ .*

**Proof.** By Weil's lemma ([11, p. 102] or [17, p. 96]), the closure  $G(g)$  in  $G$  of the cyclic subgroup generated in  $G$  by an element  $g \in G$  is either infinite cyclic discrete or compact. In the latter case,  $g \in S_1$ . In the former, the sequence  $\{g^n : n = 1, \dots\}$  cannot have a clusterpoint in  $G$  because the space of its points is discrete in  $G$ . It must, however, have a clusterpoint in the compact space  $S \cup \{\infty\}$  [10, p. 138]; its clusterpoints, therefore, are contained in the set  $\{0, \infty\}$ . We prove that it can have only one clusterpoint.

From 1.6 we know that  $S_0$  is an open neighborhood of 0. If  $\{\infty\}$  is not the only clusterpoint of  $\{g^n : n = 1, \dots\}$  and hence the limit of this sequence, there is at least one element of this sequence in each neighborhood of 0, the only other possible clusterpoint of it. Let  $g^m \in S_0$ . Then we know that  $\lim_{n \rightarrow \infty} g^{mn} = 0$  from the definition of  $S_0$ . This, however, implies  $\lim_{n \rightarrow \infty} g^{mn+i} = 0$  for all  $i = 0, 1, \dots, m-1$  from which we conclude  $\lim_{n \rightarrow \infty} g^n = 0$ . We have proved that any  $g \in G$  is either in  $S_1$  or  $S_2$ , or in  $S_0$ . Since  $S_0$  is open in  $S$ , it is open in  $S \cup \{\infty\}$ ; but because  $f$  as defined in 1.9 is a homeomorphism,  $f(S_0)$  is open in  $S \cup \{\infty\}$ , and from 1.8 it is clear that  $f(S_0) = S_2$ . The complement of  $S_0 \cup S_2$  is closed and therefore compact in  $S \cup \{\infty\}$ ; neither 0 nor  $\infty$  is adherent to  $S_1$ , because the neighborhoods  $S_0$  of 0 and  $S_2$  of  $\infty$  fail to meet  $S_1$ . Therefore we have finally established the compactness of  $S_1$  in  $G$  which finishes the proof.

1.11. PROPOSITION. *Let  $S$  be a l.c.g.z. and  $g \in G$  such that  $\lim_{n \rightarrow \infty} g^n = 0$ , and let  $G(g)$  denote the infinite cyclic and discrete subgroup of  $G$  generated by  $g$ . Then the quotient space  $G/G(g)$  of left cosets modulo  $G(g)$  is compact.*

**Proof.** Let  $U$  be an open neighborhood of 0 with compact closure. If  $h$  is an element in  $G$ , then  $\lim_{n \rightarrow \infty} hg^n = 0$  and  $\lim_{n \rightarrow \infty} hg^{-n} = \infty$ . Therefore we find a minimal integer number  $n(h)$  so that  $hg^{n(h)}, hg^{n(h)+1}, \dots$  is in  $\bar{U}$ . Because of minimality,  $hg^{n(h)-1}$  is not in  $\bar{U}$  and hence a fortiori not in  $U$ . Hence  $hg^{n(h)} \in \bar{U} \setminus Ug$ . This proves that every left coset  $hG(g)$  modulo  $G(g)$  intersects the compact subspace  $\bar{U} \setminus Ug$  which is now entirely in  $G$ . Therefore the quotient-space  $G/G(g)$  is a continuous image of the compact space  $\bar{U} \setminus Ug$  and is consequently compact.



**REMARK.** It should be noted that in all of the preceding propositions and lemmas we did not use the full power of the associative law for  $S$ . Whenever a product of group-elements occurred, it was the product of powers of two elements. This observation shows that all results are still valid for locally compact di-associative topological loops with zero whose definition should be obvious after 1.1. The generalisation of 1.11, however, needs the additional remark that in a di-associative loop a cyclic subgroup always defines a quotient set of left cosets modulo this subgroup similarly to the group case.

The following propositions are destined to detect the fine structure of the maximal subgroup  $G$  of a locally compact topological group with zero.

**1.12. PROPOSITION.** *Let  $S$  be a l.c.g.z. and assume that  $G$  contains a subgroup  $M$  which is either isomorphic to the additive group of all integers or is an infinite cyclic discrete group and suppose that  $M$  is normal. Then  $M$  lies in the center of  $G$ .*

**Proof.** Since the only element in  $M$  (in both cases) which lies in a compact subgroup is 1, we know that  $M \cap S_1 = \{1\}$ . Therefore (1.10) there is an element  $g \in M \cap S_0$ ; the space of left cosets modulo  $G(g)$  in  $G$  is compact (1.11); the quotient group  $G/M$  is homeomorphic to the continuous image  $(G/G(g))/(M/G(g))$  of  $G/G(g)$  and is consequently compact. The mapping  $f$  from  $G$  into the group of automorphisms of  $M$  endowed with the compact open topology, i.e. the topology of uniform convergence on compact sets in  $M$ , which assigns to an element  $h \in G$  the inner automorphism  $f(h)$  mapping  $x$  onto  $h^{-1}xh$  is a continuous homomorphism. The kernel of this homomorphism is the subgroup of  $G$  comprising all elements which commute with all elements of  $M$  and this is the centralizer  $Z$  of  $M$ . Certainly  $Z$  contains  $M$ , since  $M$  is abelian, therefore  $G/Z$  being isomorphic to the homomorphic image  $(G/M)/(Z/M)$  of  $G/M$  is a compact group. If  $p$  denotes the coset mapping of  $G$  onto  $G/Z$ , then there is an algebraic homomorphism  $f$  of  $G/Z$  into the automorphism group of  $M$  such that  $f = \tilde{f} \circ p$  and  $\tilde{f}$  is continuous (with respect to the compact open topology on  $M$ ) if and only if  $f$  is continuous [1, Chapter I, Paragraph 9, Theorem 1]. We shall prove that  $\tilde{f}(h)$  is the identity automorphism for all  $h \in G$  and thus  $G = Z$  which will prove the proposition. The continuous automorphisms of an infinite cyclic group are the identity automorphism and the inversion  $x \rightarrow x^{-1}$ , the continuous automorphisms of the additive group of reals are of the form  $r \rightarrow ar$  with some real number  $a \neq 0$ . The automorphism group of  $R$  is therefore isomorphic to the multiplicative group of nonzero real numbers. The maximal compact subgroup of this group is  $\{1, -1\}$  because all other automorphisms generate unbounded cyclic groups. Since  $G/Z$  is compact and  $\tilde{f}$  is continuous, we know that  $\tilde{f}(G/Z)$  is a compact group of automorphisms of  $M$  which now in both cases can contain only the identity automorphism and the inversion

$x \rightarrow x^{-1}$ . We prove that the latter case leads to a contradiction. Let  $x^{f(h)} = h^{-1}xh = x^{-1}$  for all  $x \in M$ . Let  $x \in S_0 \cap M$  and observe that  $0 = h^{-1}0h = h^{-1}(\lim_{n \rightarrow \infty} x^n)h = \lim_{n \rightarrow \infty} h^{-1}x^n h = \lim_{n \rightarrow \infty} (h^{-1}xh)^n = \lim_{n \rightarrow \infty} x^{-n} = \infty$ , because  $x \in S_0$  implies  $x^{-1} \in S_2$  (1.10). This contradiction finishes the proof.

**EXAMPLE.** Let  $R$  be the additive group of reals and  $\alpha$  the automorphism  $r \rightarrow -r$ . Let  $D$  be the group  $\{\alpha^2, \alpha\}$  of the two elements and define on the cartesian product  $G = R \times D$  the following multiplication:  $(r, \beta)(s, -\gamma) = (r^\gamma s, \beta\gamma)$ . Then  $G$  is a topological group in which  $M = R \times \{\alpha^2\}$  is a normal but not central subgroup of  $G$  isomorphic to the reals. Because of this property  $G$  cannot be given a nonisolated zero without destroying the continuity of multiplication, and this happens in spite of the simplicity of the topological structure of  $G$  which is nothing but the product space of  $G$  and  $D$ , i.e. the disjoint union of two intervals.

It may be observed that this example presents, in a sense, already all the possible complications which arise in the study of two-ended groups which are compact modulo their component (see [7] and the references to this study made in the introduction).

**1.13. PROPOSITION.** *If  $S$  is a l.c.g.z. and  $H$  a compact normal subgroup of  $G$ , then the collection of left cosets  $gH$  modulo  $H$  in  $G$  and the set  $\{0\}$  forms a locally compact Hausdorff space denoted by  $S/H$  which under the obvious multiplication on  $G/H$  and with  $\{0\}gH = gH\{0\} = \{0\}$  is a locally compact topological group with zero and  $G/H$  as maximal subgroup.*

**Proof.** The subspace  $G/H$  of the quotient-space is certainly a locally compact Hausdorff group in the usual sense of the quotient group. We have to prove that  $S/H$  is Hausdorff, locally compact in 0, that the multiplication on  $S/H$  is continuous and that  $\{0\}$  is not isolated in  $S/H$ . Let  $gH \in G/H$  and  $U$  a compact neighborhood of 1 in  $G$ . Then  $\{ugH : u \in U\}$  is a neighborhood of  $gH$  in  $G/H$ ; since  $UgH$  as the product of two compact subspaces in  $G$  is compact and 0 is not adherent to any compact set in  $G$ , there is an open neighborhood  $V$  of 0 which does not meet  $UgH$ . Clearly the set  $VH$  does not intersect  $UgH$  either, because the relations  $vh = ugh'$  and  $v = ug(h'h^{-1})$  with  $v \in V$ ,  $u \in U$ , and  $h, h'$ , and therefore  $h'h^{-1}$ , in  $H$  are equivalent. But  $\{vH : v \in V\}$  is a neighborhood of  $\{0\}$  in  $S/H$ , because  $VH$  is open in  $S$  since the sets  $V$  and  $Vh$ ,  $h \in H$  are open neighborhoods of 0 and  $h$  respectively. We have proved that  $S/H$  is Hausdorff. If  $W$  is a compact neighborhood of 0, then  $WH$  is compact as a continuous image of the compact product space  $W \times H$ ; the set  $\{wH : w \in W\}$  in  $S/H$  is compact as a continuous image of  $WH$  and it is a neighborhood of 0. Since  $G/H$  is locally compact anyway, this proves the local compactness. If  $p$  is the quotient mapping of  $S$  onto  $S/H$ , then  $p$  is continuous and open, for it is clearly open on the maximal subgroup; but if  $V$  is an open set containing 0, then  $VH$  is the union of the sets  $V$  and  $Vh$ ,  $h \in H$  all of which are open, therefore

$\{vH : v \in V\} = p(V)$  is open. The multiplication  $(x, y) \rightarrow xy = m(x, y)$  mapping  $S/H \times S/H$  onto  $S/H$  can then be written in the form  $m(x, y) = p(p^{-1}(x)p^{-1}(y))$  where  $p^{-1}(x)p^{-1}(y)$  is the ordinary set product in  $S$ ; if  $g \in p^{-1}(x)$ ,  $h \in p^{-1}(y)$  and  $U$  is a neighborhood of  $gh$  in  $S$ , then  $p(U)$  is, because of openness of  $p$ , a neighborhood of  $m(x, y)$ , but since multiplication is continuous in  $S$ , we find two neighborhoods  $V$  and  $W$  of  $g$  and  $h$  respectively, such that  $VW \subset U$ . The images  $p(V)$ ,  $p(W)$  are neighborhoods of  $x$  and  $y$  respectively and we have  $p(U) \supset p(VW) = p(V)p(W)$  which proves the continuity of multiplication.

If  $U$  is a neighborhood of  $\{0\}$  in  $S/H$ , then  $p^{-1}(U)$  is a neighborhood of  $0$  which contains some  $g \in G$  since  $0$  is not isolated in  $G$ . Therefore  $\{0\}$  is not isolated in  $S/H$ .

**1.14. PROPOSITION.** *Let  $S$  be a l.c.g.z.; then the maximal group  $G$  contains a unique maximal compact normal and characteristic subgroup  $C$ , and  $S/C$  is a l.c.g.z. whose maximal subgroup  $G/H$  is a locally compact group without nontrivial compact normal subgroups.*

**Proof.** Let  $(\mathfrak{C}, \subset)$  be the collection of all compact normal subgroups of  $G$  partially ordered under inclusion  $\subset$ . This partially ordered set is inductive: Every member  $K \in \mathfrak{C}$  is contained in  $S_1$  since all elements  $g \in K$  lie in a compact subgroup of  $G$ . If  $\mathfrak{R}$  is a totally ordered subset of  $\mathfrak{C}$ , then its union is a normal subgroup (as the union of an ascending chain of normal divisors) and it is still contained in  $S_1$  since every one of its elements is contained in some compact subgroup of the chain. But  $S_1$  is compact; hence the closure of the union is a closed normal subgroup of  $G$  contained in  $S_1$  and is therefore compact and gives us an upper bound in  $\mathfrak{C}$  of the given totally ordered subset  $\mathfrak{R}$ . By Zorn's lemma we find indeed a compact normal subgroup  $C$  maximal in  $\mathfrak{C}$ . If  $C$  and  $C'$  are two maximal compact normal subgroups, then  $CC'$  is compact and again normal. Hence, because of maximality  $C = C' = CC'$ ; therefore  $C$  is uniquely determined and characteristic.  $S/C$  is a l.c.g.z. from 1.13. We prove that the maximal subgroup  $G/C$  does not contain a nontrivial compact subgroup. Let  $C'$  be a subgroup of  $G$  containing  $C$  such that  $C'/C$  is normal in  $G/C$  and  $C'/C$  is compact. Since  $C'$  is normal in  $G$  modulo  $C$  and  $C$  is contained in  $C'$ , it is normal in  $G$ , and because it is an extension of a compact group  $C$  by a compact group  $C'/C$ , it is compact. Then, because of the maximality of  $C$  we get  $C' = C$  which proves the assertion.

**1.15. LEMMA.** *Let  $S$  be a l.c.g.z. and  $G$  its maximal subgroup. Then  $G$  is a projective limit of Lie groups or it contains an open and compact subgroup in  $S_1$  (which then is a projective limit of Lie groups).*

**Proof.** Every locally compact topological group contains an open subgroup  $G'$  which is the projective limit of Lie groups [11, Chapter IV, in particular p. 175], i.e. in every neighborhood  $U$  of  $1$  in  $G$  there is a compact subgroup

$H_U$  normal in  $G'$  such that  $G'/H_U$  is a Lie group. If  $G' = G$ , then  $G$  itself is a projective limit of Lie groups. Assume now that  $G$  is not a projective limit of Lie groups. We shall prove that every open subgroup  $G'$  which is projective limit of Lie groups is compact and therefore contained in  $S_1$ ; this will prove the proposition. Let  $G'$  be open in  $G$  and be not compact; then it contains an element  $g$  of  $S_0$  because otherwise it would be contained in  $S_1$  and hence would be compact (see 1.10). The quotient space  $G/G(g)$  of left cosets modulo  $G(g)$  is compact; hence a fortiori the quotient space  $G/G'$  of left cosets modulo  $G'$  is compact since  $G'$  contains  $G(g)$  and  $G/G'$  is a continuous image of the compact quotient  $G/G(g)$  under the mapping assigning to a coset  $hG(g)$  modulo  $G(g)$  the coset  $hG'$  modulo  $G'$ . But since  $G'$  is open, the quotient  $G/G'$  is discrete and must therefore be finite. Now  $G$  is a finite union of right cosets  $G', G'g_1, \dots, G'g_{n-1}$  modulo  $G'$ . Let  $H$  be a compact normal subgroup of  $G'$  with the property that  $G'/H$  is a Lie group; we may choose  $H$  in any given neighborhood of 1, a fact which we shall use presently. Every element  $k$  of  $G$  is of the form  $g'g_i$  with  $g' \in G'$  and  $g_i = g_0 = 1$  or  $i = 1, \dots, n-1$ ; all the conjugates  $k^{-1}G'k$  of  $G'$  can therefore be expressed in the form  $k^{-1}G'k = g_i^{-1}g'^{-1}G'g'g_i = g_i^{-1}G'g_i$  and all the conjugates of  $H$  can be written  $k^{-1}Hk = g_i^{-1}g'^{-1}Hg'g_i = g_i^{-1}Hg_i$  because  $H$  is normal in  $G'$ . Therefore  $G'$  and  $H$  have at most  $n$  different conjugates in  $G$ . The intersection  $G'' = \bigcap \{k^{-1}G'k : k \in G\}$  is normal in  $G$  and as it is in fact only a finite intersection of open subgroups, it is open. Because of the continuity of inner automorphisms we find a neighborhood  $U$  of 1 in  $G$  such that  $U, g_1^{-1}Ug_1, \dots, g_{n-1}^{-1}Ug_{n-1}$  are in  $G''$ . Now we may assume that  $H$  has been chosen in  $U$  which implies that all its conjugates are in  $G''$ . The intersection  $D$  of the conjugates of  $H$  is normal in  $G$  and it is compact.

We shall prove that  $G/D$  is a Lie group thus showing that  $G$  is a projective limit of Lie groups which will finish the proof because we explicitly assumed that  $G$  is not a projective limit of Lie groups. Since  $G''$  is open in  $G$ , it is sufficient to show that  $G''/D$  is a Lie group.  $G''/H$  is a Lie group as a closed subgroup of the Lie group  $G'/H$ . All groups  $D_m = H g_1^{-1}H g_1 \dots g_m^{-1}H g_m$ ,  $1 \leq m \leq n-1$  are normal in  $G''$  since  $g_i^{-1}Hg_i$  is normal in  $g_i^{-1}Hg_i$  and is contained in  $G''$  which is a subgroup of  $g_i^{-1}Hg_i$ . Assume now that for  $D_m$   $1 \leq m < n-1$  it has been proved that  $G''/D_m$  is a Lie group. We show that  $G''/D_m$  is a Lie group thereby completing the argument by induction.  $G''/g_{m+1}^{-1}Hg_{m+1}$  is isomorphic to  $G''/H$  and is therefore a Lie group.  $D_m g_{m+1}^{-1}Hg_{m+1}/D_m$  is a closed subgroup of  $G''/D_m$  and is hence a Lie group; on the other hand this group is isomorphic to  $g_{m+1}^{-1}Hg_{m+1}/D_{m+1}$  [17, p. 17], because  $D_{m+1} = D_m g_{m+1}^{-1}Hg_{m+1}$ . But as  $G''/g_{m+1}^{-1}Hg_{m+1}$  and  $g_{m+1}^{-1}Hg_{m+1}/D_{m+1}$  both are Lie groups  $G''/D_{m+1}$  certainly is a Lie group. (Compare also [11, p. 175].)

The following is an essentially group theoretic lemma.

1.16. LEMMA. *Let  $G$  be a locally compact topological group with a compact*

open subgroup  $H$ . Let  $K$  be the set of all elements which are contained in some compact subgroup of  $G$ ; assume that  $K$  is compact. Then  $G$  contains a finite number of maximal compact open subgroups whose intersection is a unique greatest maximal compact open normal divisor.

REMARK.  $K$  plays the role of  $S_1$  in the former propositions.

**Proof.** (a) The set of all compact subsets of the compact space  $K$  can be made into a compact topological space  $\mathfrak{K}$  in a well known manner [1], which we can describe in our case as follows: For each neighborhood  $U$  of 1 in  $K$  (which is also a neighborhood in  $G$  since  $H$  is in  $K$  and  $H$  is a neighborhood of 1 in  $G$ ) and each compact set  $C \subset K$  we consider the set  $\mathfrak{U}(U, C)$  of all compact subsets  $C'$  of  $K$  such that  $C' \subset CU$  and  $C \subset C'U$ . For fixed  $C$  the collection of all  $\mathfrak{U}(U, C)$  will be a base for the neighborhoods of  $C$  in  $\mathfrak{K}$  when  $U$  varies over all neighborhoods of 1 in  $K$ . The details do not belong to the present discussion and we refer the reader e.g. to [1, Chapter II, Paragraph 2, ex. 7 and Paragraph 4, ex. 5, 6].

It is almost trivial that the collection of all compact open subgroups of  $G$  is inductive with respect to inclusion; for the union of an ascending chain of compact open subgroups is open on one hand and contained in  $K$  on the other hand; as an open subgroup it is closed in  $G$  and therefore compact as a subset of  $K$ . More specifically, we can prove by the same argument that the collection of all compact open subgroups containing a given compact open subgroup is inductive. Hence, by Zorn's lemma, each compact open subgroup is contained in a maximal compact open subgroup.

We shall denote with  $\mathfrak{C}$  the subset of  $\mathfrak{K}$  comprising all maximal compact open subgroups of  $G$ . Let  $C \in \mathfrak{C}$ , then  $\mathfrak{U}(C, C)$  is a neighborhood of  $C$  in  $\mathfrak{K}$  since  $C$  is open in  $G$ . Therefore  $C' \in \mathfrak{U}(C, C)$  implies in particular  $C' \subset CC = C$ . Let  $C''$  be any maximal compact open subgroup which is contained in  $\mathfrak{U}(C, C)$ ; then  $C'' \subset C$  which because of maximality of  $C''$  implies  $C'' = C$ . This means that for any  $C$  in  $\mathfrak{C}$  there is a neighborhood of  $C$  in  $\mathfrak{K}$  containing no  $C'' \in \mathfrak{C}$  other than  $C$ . Consequently  $\mathfrak{C}$  is a discrete subspace of the compact space  $\mathfrak{K}$  and is therefore finite. We have proved that  $G$  contains at most a finite number of maximal compact open subgroups which we shall call  $C_1, \dots, C_n$ . Observe that  $n$  might be 0 in which case no maximal open subgroups exist.

(b) We shall now show that actually  $K = C_1 \cup \dots \cup C_n$ . For that purpose it is sufficient to prove that any compact subgroup is contained in an open compact subgroup which then, in turn, is in some maximal compact open subgroup after (a). But every element in  $K$  is in some compact subgroup and therefore in the union of the maximal open compact groups. Let now  $C$  be a compact group and consider  $D = \bigcap \{c^{-1}Hc : c \in C\}$ ; then  $D$  is invariant under inner automorphisms with elements from  $C$ , i.e.  $C$  is in the normaliser of  $D$ . Therefore  $CD$  is a group which is compact since  $C$  and  $D$  are compact; if we show that

$D$  is open then this will be the required compact open group containing  $C$ . But it is well known that  $D$  is open [11, p. 55]; this can in this special case be inferred from the fact that  $C$  is as a compact space covered by a finite number of open right cosets  $Hc_1, \dots, Hc_m$ ; if now  $c'$  is an element of  $C$ , it is of the form  $c' = hc_i$ ,  $h \in H$  and  $c^{-1}Hc = c_i^{-1}h^{-1}Hhc_i = c_i^{-1}Hc_i$  so that  $D$  in fact is a finite intersection of open conjugates of  $H$ .

(c) Any automorphism of  $G$  (which is by definition also a homeomorphism) maps a compact open subgroup on a compact open subgroup; the image of a maximal compact open subgroup  $C$  is contained in a maximal compact open subgroup whose image under the inverse of the automorphism under consideration is a compact open subgroup containing  $C_i$  and has to coincide with  $C_i$  because of the maximality of  $C_i$ . This proves that the automorphic image of a maximal compact open subgroup is maximal compact open. In other words: Every automorphism permutes  $C_1, \dots, C_n$ . Therefore the intersection of these groups is characteristic.

**1.17. COROLLARY.** *Let  $S$  be a l.c.g.z. such that its maximal subgroup  $G$  does not contain nontrivial compact normal subgroups. Then  $G$  is a Lie group.*

**Proof.** From 1.14,  $G$  is a projective limit of Lie groups and hence, since it has only the trivial compact normal subgroup, is a Lie group; or else it contains a compact open subgroup. Then  $G$  satisfies the hypotheses of 1.16, with  $K = S_1$  and contains therefore a compact open normal subgroup. But the only compact normal subgroup is 1 which consequently is open. Hence  $G$  is discrete and thus is a Lie group.

**1.18. PROPOSITION.** *Let  $S$  be a l.c.g.z. such that its subgroup  $G$  is connected and does not contain nontrivial compact normal subgroups. Then  $G$  is isomorphic to the multiplicative group of positive real numbers and  $S$  is isomorphic to the semigroup of all non-negative real numbers.*

**Proof.** 1.17 implies that  $G$  is a connected Lie group.  $G$  is not compact but if  $g \in G \cap S_0$  then  $G(g)$  is infinite cyclic discrete and  $G/G(g)$  is compact (1.11).  $G$  is a Lie group with two ends in the sense of Freudenthal, namely 0 and  $\infty$ , when  $S \cup \{\infty\}$  is the one point compactification of  $S$ , see [5, p. 289]. Therefore, using a result of Freudenthal's,  $G$  is isomorphic to the direct product of a one dimensional real vector-group and a compact connected group which in our case must be trivial since  $G$  does not contain nontrivial compact normal subgroups. The multiplicative group of positive reals is isomorphic to the additive group of all reals; this proves the first part of the claim. Let  $f_0$  be an isomorphism of the multiplicative group of real numbers on  $G$  such that the reals of absolute value less than 1 are mapped on  $G \cap S_0$ . This can, of course, be done in many ways. If we set  $f(r) = f_0(r)$  for all positive real numbers  $r$  and  $f(0) = 0$ , the former zero denoting the real number zero, then the extended

mapping  $f$  is an isomorphism: If  $\{r_n : n = 1, \dots\}$  is a sequence of positive real numbers converging to 0, then  $\{f(r_n) : n = 1, \dots\}$  is a sequence of elements in  $S_0$  which is unbounded in  $C$ ; but in the compact space  $S_0 \cup S_1$  (it is compact as a closed subspace of the compact space  $S_0 \cup S_1 \cup S_2$ ) it must have a clusterpoint which only can be zero. Therefore the extended map  $f$  is continuous in 0. The real interval  $[0, 1]$  is mapped continuously in a one-to-one fashion hence homeomorphically since  $[0, 1]$  is compact and  $S$  is Hausdorff. Hence, the extended map  $f$  is not only continuous one to one but also open on the set of non-negative reals.

1.19. PROPOSITION. *A l.c.g.z.  $S$  having a maximal subgroup  $G$  without non-trivial compact normal subgroups has precisely one of the two following properties:*

(i) *0 and 1 lie in a connected l.c.g.z. isomorphic to the multiplicative semigroup of non-negative reals and the component of 1 in  $G$  is the isomorphic to the positive reals under multiplication;*

(ii)  *$G$  is discrete and 0 is the only accumulation point of  $S$ .*

**Proof.** Let  $G'$  be the component of 1 in  $G$ ; then  $G'$  is a normal subgroup closed and open in  $G$ , since  $G$  is a Lie group (1.17). If  $G'$  is not compact, the closure  $\overline{G'}$  of  $G'$  in  $S$  is  $G' \cup \{0\}$ , for  $G'$  is not entirely contained in  $S_1$  and consequently contains a  $g \in S_0$ ; then  $\lim_{n \rightarrow \infty} g^n = 0 \in \overline{G'}$  (see 1.10); and since  $G'$  is closed in  $G$ , the only point not contained in the closure in  $S$  is 0. Now  $G' \cup \{0\}$  is a l.c.g.z. because 0 is not isolated, and its maximal subgroup  $G'$  has no non-trivial compact normal subgroups. Then 1.16 describes the structure of  $G' \cup \{0\}$  which gives the necessary information for case (i).

If, however,  $G'$  is compact, then  $G' = \{1\}$ , hence  $\{1\}$  is open in  $G$  and  $G$  is discrete. By the definition of a l.c.g.z. 0 is not isolated. This is case (ii).

1.20. PROPOSITION. *Let  $S$  be a l.c.g.z. such that its maximal subgroup  $G$  has no nontrivial compact normal subgroups and let 0 and 1 be connected (i.e. lie in some connected subspace). Then  $S$  is isomorphic to the multiplicative semigroups of all non-negative real numbers.*

**Proof.** From the previous numbers we know that the component  $G'$  of 1 is isomorphic to the positive reals under multiplication or, what amounts to the same, to the additive group of all reals and  $G'$  is open in  $G$ . If  $g \in G \cap S_0$ , then  $G/G(g)$  is compact; then a fortiori,  $G/G'$  is compact since  $G(g)$  is a subgroup of  $G'$ . On the other hand  $G/G'$  is discrete. Hence  $G'$  has finite index in  $G$ . Let now  $s : G/G' \rightarrow G$  be a cross section, i.e. if  $x \in G/G'$  is a left coset modulo  $G'$  then  $s(x)$  is an element in  $x$ . If  $g$  is in  $G$  and  $x = hG'$  in  $G/G'$ , we let  $gx$  be the  $ghG' \in G/G'$ . With this convention we define a function  $f$  from  $G \times G/G'$  into  $G$  by setting  $f(g, x) = gs(x)s(gx)^{-1}$ . Since  $gs(x)$  is an element of the left coset  $gx$ , there exists a  $g' \in G'$  such that  $gs(x) = s(gx)g'$  whence  $s(gx)g's(gx)^{-1} \in G'$  since

$G'$  is normal. Thus  $f$  is a mapping into  $G'$ . From the definition of  $f$  we derive the following identities:  $f(gh, x)s(ghx) = ghs(x) = g(f(h, x)s(hx)) = f(h, x)gs(hx)$  because  $f(h, x) \in G'$  and  $G'$  is not only normal but even central in  $G$  after 1.12. But  $gs(hx) = f(g, hx)s(ghx)$ , which finally implies the functional equation  $f(gh, x) = f(g, hx)f(h, x)$ , where we have used that  $G'$  is abelian. Since  $G'$  is isomorphic to the additive reals, every element in  $G'$  has unique roots of all possible orders; if  $n$  is the index of  $G'$  in  $G$ , then  $G'$  has, in particular,  $n$ th roots. Therefore the function  $t$  from  $G$  to  $G'$  is well defined and continuous if we put  $t(g) = \prod \{f(g, x)^{1/n} : x \in G/G'\}$ . Because of the commutativity of  $G'$  we have  $\prod \{f(g, hx)^{1/n} : x \in G/G'\} = \prod \{f(g, x)^{1/n} : x \in G/G'\}$  since  $x \rightarrow hx$  is a permutation of the  $n$  cosets of  $G/G'$ . Again using the commutativity of  $G'$ , once more we get the identity

$$t(gh) = t(g)t(h)$$

showing that  $t$  is an endomorphism of  $G$  into  $G'$ . If  $g'$  is in  $G'$  and  $x = hG'$  is in  $G/G'$ , then  $g'x = g'hG' = hG' = x$  since  $G'$  is in the center. This implies that  $f(g', x) = g's(x)s(g'x)^{-1} = g'$  independently of  $x$ ; therefore  $t(g') = (g')^{1/n} = g'$ . Let the kernel of  $t$  be  $N$ . If  $g' \in N \cap G'$ , then  $g' = t(g') = 1$ . On the other hand  $t(t(g)^{-1}g) = t(tg)^{-1}t(g) = 1$  since  $t^2 = t$  because  $t(g) \in G'$ . Hence  $G = NG'$ ,  $N \cap G' = 1$ , and this product is direct. But since  $G/G'$  is finite discrete,  $N$  is finite discrete because it intersects each (open) coset modulo  $G'$  in one and only one point. As  $G$  does not have nontrivial compact normal subgroups,  $N = \{1\}$  and  $G = G'$ , which finishes the proof.

1.21. PROPOSITION. *Let  $S$  be a l.c.g.z. such that its maximal subgroup  $G$  is discrete and does not contain nontrivial finite (i.e. compact) normal subgroups. Then  $G$  is infinite cyclic and  $S$  is isomorphic to the set of real numbers  $\{0\} \cup \{2^n : n = 0, \pm 1, \dots\}$  under multiplication.*

**Proof.** (a) Let  $g \in G \cap S_0$  so that  $G/G(g)$  is compact and discrete, hence finite. Let  $n$  be the index of  $G(g)$  in  $G$ . Let  $P_n$  be the full permutation group of the set  $g_1G(g), \dots, g_nG(g)$  of left cosets modulo  $G(g)$ ; its order is  $n!$ . Let  $p$  be a mapping from  $G(g)$  into  $P_n$  defined by the convention that  $p(h)$  is the permutation  $g_iG(g) \rightarrow hg_iG(g)$ . Clearly  $p(hh') = p(h)p(h')$  so that  $p$  is a homomorphism. Since  $G(p)$  is infinite and  $P_n$  is finite,  $p$  cannot be one-to-one, i.e. it has a nontrivial kernel. Every subgroup of  $G(g)$  is cyclic. Let  $G(g^m)$  be the kernel of  $p$ . By the definition of  $p$  this implies that  $g^m g_i \in g_i G(p)$  for all  $i = 1, \dots, n$ . If  $i$  is a fixed number among these, then there is an integer  $r$  such that  $g^m g_i = g_i g^r$  or  $g^m = g_i g^r g_i^{-1}$ . Let us denote the inner automorphism  $x \rightarrow g_i x g_i^{-1}$  of  $G$  with  $f$ . Then we have  $f(g^r) = g^m$ . For all integers  $k$  the following identity holds:

$$f^k(g^{r^k}) = g^{m^k}.$$

Since all powers of  $f$  are automorphisms, we have namely  $f^k(g^{r^k}) = (f^{k-1}f(g^r))^{r^{k-1}} = (f^{k-1}(g^m))^{r^{k-1}} = (f^{k-1}(g^{r^{k-1}}))^m$ . If we now assume that the identity holds for



$k-1$  instead of  $k$ , then this is equal to  $g^{m^{k-1} \cdot m}$  which concludes the proof of the identity in question by induction.

Let now  $h$  be any element of  $G$ . Then  $h \in g_j G(g)$  for some  $j \in \{1, \dots, n\}$ ; i.e. there is a  $g' \in G(g)$  such that  $h = g_j g'$ . If  $g''$  is an element of  $G(g)$ , then  $hg''h^{-1} = g_j g' g'' g'^{-1} g_j^{-1} = g_j g'' g_j^{-1}$  since  $G(g)$  is commutative. That means that the restriction of any inner automorphism to  $G(g)$  coincides with the restriction of one out of a finite number of inner automorphisms, namely those defined by  $g_1^{-1}, \dots, g_n^{-1}$ . Among the restrictions of the inner automorphisms  $f, f^2, f^3, \dots$  to  $G(g)$  there are, therefore, at least two,  $f^s$  and  $f^{s'}$ , say, such that  $s < s'$  and  $f^s(g') = f^{s'}(g')$  for all  $g' \in G(g)$ ; because  $f^s$  is an automorphism this implies, however, that  $g' = f^{-s} f^{s'}(g') = f^{s'-s}(g')$ . We put  $s' - s = k$  and get

$$g^{rk} = f^k(g^{rk}) = g^{m^k}.$$

Since  $G(g)$  is infinite cyclic and  $k > 0$ , this implies  $m = \pm r$ . We have now proved that  $g_i g^m g_i^{-1} = g^{\pm m}$ . This means that the normaliser of  $G(g^m)$  contains  $g_1, \dots, g_n$ , but trivially it contains  $G(g)$  and therefore  $g_1 G(g) \cup \dots \cup g_n G(g) = G$ . We have proved that  $G(g^m)$  is normal and then, by (1.12) central in  $G$ .

(b) Let  $s: G/G(g^m) \rightarrow G$  be a cross section, i.e. if  $x \in G/G(g^m)$  is a left coset modulo  $G(g^m)$  then  $s(x)$  is an element in  $x$ . If  $g'$  is in  $G$  and  $x = hG(g^m)$  in  $G/G(g^m)$ , we let  $g'x$  be  $g'hG(g^m) \in G/G(g^m)$ . With this convention we define a function  $f$  from  $G \times G/G$  into  $G$  by setting  $f(g', x) = g's(x)s(g'x)^{-1}$ . Since  $g's(x)$  is an element of the left coset  $g'x$ , there exists a  $g'' \in G(g^m)$  such that  $g's(x) = s(g'x)g''$  whence  $f(g', x) = x s(g'x) g'' s(g'x)^{-1} = g''$  since  $g''$  is in the center of  $G$ . Thus  $f$  is actually a mapping into  $G(g^m)$ . From the definition of  $f$  we derive the following identities:  $f(g'h', x)s(g'h'x) = g'(f(h', x)s(h'x)) = f(h', x)g's(h'x)$  because  $f(h', x)$  is central. Now  $g's(h'x) = f(g', h'x)s(g'h'x)$  which, again under observations of centrality, implies the functional equation

$$f(g'h', x) = f(g', h'x)f(h', x).$$

We define a function  $t$  from  $G$  to  $G(g^m)$  by setting  $t(g') = \Pi\{f(g', x) : x \in G/G(g^m)\}$ . Since  $\Pi\{f(g', h'x) : x \in G/G(g^m)\} = \Pi\{f(g', x) : x \in G/G(g^m)\}$  we have

$$t(g'h') = t(g')t(h').$$

Hence  $t$  is an endomorphism of  $G$  into  $G(g^m)$ . If  $g \in G(g^m)$  then  $f(g', x) = g'$ , because for an element  $g'$  in  $G(g^m)$  we have  $g'x = g'hG(g^m) = hg'G(g^m) = hG(g^m) = x$ . This implies  $t(g') = g'^{m^m}$  because the index of  $G(g^m)$  in  $G$  is the product of the indices of  $G(g^m)$  in  $G(g)$  and of  $G(g)$  in  $G$ . Now let  $N$  be the kernel of  $t$ . If  $g' \in N \cap G(g^m)$  then  $g'^{m^m} = t(g') = 1$ ; but all elements in  $G(g^m)$  have infinite order except 1; hence  $g' = 1$ . From  $N \cap G(g^m) = \{1\}$  it follows that  $N$  is mapped in a one-to-one fashion unto the factor group  $G/G(g^m)$  which is finite (1.11). Hence  $N$  is finite. But  $G$  does not contain nontrivial normal finite subgroups,

hence  $N = \{1\}$  and  $t$  is an isomorphism of  $G$  into  $G(g^m)$ . The image of  $G$  in  $G(g^m)$  is, as a subgroup of a cyclic group, cyclic. Therefore  $G$  is cyclic.

There exists an isomorphism  $f_0$  from the semigroup  $\{2^n : n = 1, \dots\}$  under multiplication onto  $G$  such that  $\{2^n : n = 0, -1, -2, \dots\}$  is mapped onto  $G \cap (S_0 \cup S_1)$ . We extend this isomorphism by mapping the real number 0 on the zero 0 of  $S$ . Both  $\lim_{n \rightarrow \infty} 2^{-n} = 0$  in the reals and  $\lim_{n \rightarrow \infty} g^n = 0$  in  $S$  where  $g$  is the uniquely determined generator out of two generators of  $G$  whose powers converge to 0 (1.10); therefore the extended isomorphism is continuous and continuously invertible because every sequence converging to 0 consisting of powers of 2 on the reals are finally subsequences of  $\{2^{-n} : n = 0, 1, \dots\}$  and all sequences converging to 0 in  $S$  are finally subsequences of  $\{g^n : n = 1, \dots\}$ . This finishes the proof of the proposition.

**1.22. PROPOSITION.** *Let  $S$  be a l.c.g.z. Then its maximal subgroup  $G$  contains a unique characteristic maximal compact normal subgroup  $C$  and subgroup  $M$  which is either isomorphic to the multiplicative group of all positive reals (or equivalently to the additive group of all reals) or to an infinite cyclic discrete group, e.g. to the group of all real numbers  $\{2^n, n = 0, \pm 1, \pm 2, \dots\}$  under multiplication;  $C \cap M = \{1\}$  and  $G = MC$ .*

**Proof.** Propositions 1.14, 1.19, 1.20, and 1.21 show the existence of a unique maximal compact normal subgroup  $C$  such that  $G/C$  is either a one-parameter group isomorphic to the multiplicative reals greater than zero or an infinite cyclic group. In the latter case, we take any element  $g$  in one of the two cosets modulo  $C$  which generates  $G/C$  and set  $M = G(g)$ ; then clearly  $M \cap C = \{1\}$  since  $M$  has no nontrivial compact subgroups, and since the coset  $(gC)^n$  is equal to  $g^n C$ , we have in fact  $G = MC$ . If, however,  $G/C$  is isomorphic to the additive reals, it is a little more difficult to construct  $M$ . We quote a theorem which was proved for Lie groups by E. Cartan [2] and was generalized to locally compact groups by Iwasawa [9], see also [11, p. 188]: If  $G'$  is a locally compact connected group, then  $G'$  has maximal compact subgroups which are all connected and conjugates; let  $C'$  be such a maximal compact and connected group, furthermore there are  $n$  isomorphisms  $f_1, \dots, f_n$  of the additive group  $R$  of real numbers into  $G'$  such that the mapping  $(r, \dots, r_n, c) \rightarrow f_1(r_1) \dots f_n(r_n) c$  of  $R^n \times C'$  into  $G'$  is a homeomorphism. Let now  $G'$  be the component of 1 in  $G$ . Then  $G/G'$  is totally disconnected; if  $G'$  were compact, then  $G' \subset C$  because  $C$  is the unique greatest compact subgroup, then  $G/C$  were isomorphic to  $(G/G')/(C/G')$  which as a quotient of a totally disconnected group is totally disconnected, since a locally compact group is totally disconnected if and only if it has arbitrarily small open subgroups; but  $G/C$  is a nontrivial one-parameter group and is therefore connected and contains more than one point. Therefore, from Iwasawa's theorem there is at least one isomorphism  $f$  from  $R$  into  $G'$ ; let  $f(R) = M$ , clearly  $M \cap C = \{1\}$

because  $\{1\}$  is the only compact subgroup contained in  $M$ . Therefore  $M$  is mapped in a one-to-one fashion into the quotient  $G/C$ , but this quotient is isomorphic to  $M$  so that the image of  $M$  in  $G/C$  is necessarily all of  $G/C$ . This proves  $G = MC$  and finishes the argument.

It may be remarked that it is not necessary to invoke Iwasawa's theorem. By not trivial but more elementary methods (i.e. methods not involving Lie groups) one can show that there is a connected locally compact abelian group in  $G'$  which is not contained in  $C$ ; one uses the fact that if  $U$  is a compact neighborhood of  $C$ , then arbitrarily close to 1 there are elements in  $S_0$  whose powers converge to 0 hence in particular finally leave  $U$ . This gives rise to a possibly not continuous algebraic homomorphism of  $R$  into  $G'$  which maps certain elements outside  $U$ . The closure  $A$  of the full image of  $R$  in  $G'$  turns out to be connected.

But any connected locally compact abelian group is the direct product of a vector space  $R^n$  and a compact connected abelian group  $C'$ ; since this group  $A$  is not entirely in  $C$ , having points outside  $U$ , the factor  $R^n$  cannot be trivial which yields at least one not compact one-parameter group which proves the proposition as above.

**1.23. EXAMPLE.** Let  $D$  be any compact nontrivial group, e.g. the group with two elements or the circle group. Let  $C$  be a countable product  $\dots \times D_{-1} \times D_0 \times D_1 \times \dots$  of groups  $D_i$  identical with  $D$ . The elements of  $C$  are therefore "vectors"  $(x_i : -\infty < i < \infty)$  with  $x_i \in D$ . Let  $g$  be the automorphism of  $C$  which maps  $(x_i : -\infty < i < \infty)$  onto  $(y_i : -\infty < i < \infty)$  with  $y_i = x_{i-1}$ . Let  $M$  be the infinite cyclic group of automorphisms generated by  $g$ . On the cartesian product  $M \times C$  we define the multiplication  $(m, c)(m', c') = (mm', c^m c')$ ; this defines on  $M \times C$  the structure of a topological group  $G$ . If we identify  $M$  with the subgroup  $M \times \{1\}$  and  $C$  with the subgroup  $\{1\} \times C$ , then  $G = MC$  and  $M \cap C = \{1\}$  in  $G$ . It is easy to see that  $(m', c')^{-1} = (m'^{-1}, (c'^{-1})^{m'})$  so that  $(m', c')^{-1}(m, c)(m'c') = (m, (c'^{-1})^m c^m c')$ . This shows that  $C$  is the unique maximal compact subgroup.  $G$  does not contain arbitrarily small normal subgroups as one can easily see, compare also [11, p. 57]. If  $M'$  is a subgroup of  $M$ , then  $M'C$  is normal in  $G$  and the quotient is finite cyclic. The center of  $G$  is trivial. If we add an element 0 to  $G$  and define a bases of neighborhoods by taking the collection of all sets of the form  $\{0\} \cup \{(g^n, c) : n > n_0\}$ ,  $n_0 = 1, 2, \dots$ , then  $S = G \cup \{0\}$  and with the extended multiplication  $0x = x0$ ,  $x$  is a l.c.g.z. in which 1 and 0 are not in a connected subspace and are not in a central subset of  $S$  (i.e. in a subset whose elements commute with all elements of  $S$ ).  $S$  may or may not be totally disconnected depending on the fact whether  $D$  is totally disconnected or not.

This example shows that in the case where 0 and 1 are not in some connected subspace, the best possible information about  $G$  is available for this case with Proposition 1.20. In the other alternative, however, we are able to show that the one-parameter group can actually be chosen in the center of  $G$ . Because of

Proposition 1.12, or just because of the fact that the complementary factor  $C$  is normal anyway, it is sufficient to prove that  $M$  can be chosen normal.

**1.24. PROPOSITION.** *Let  $S$  be a l.c.g.z. without compact open subgroup. Then the maximal subgroup  $G$  is the direct product of a one-parameter group  $M$  isomorphic to the additive reals and a compact group  $C$ .*

**Proof.** Proposition 1.24 will immediately follow from 1.22 and the following lemma:

**1.25. LEMMA.** *Let  $G$  be a locally compact group with a compact normal subgroup  $C$  such that  $G/C$  is isomorphic to the additive group of reals. Then  $G$  contains a subgroup  $M$  isomorphic to  $G/C$  and  $G$  is the direct product of  $C$  and  $M$ .*

**Proof.** Let  $\mathfrak{A}$  be the group of automorphisms of  $C$ , let  $\mathfrak{G}$  be the subgroup of all automorphisms induced by inner automorphisms of  $G$ , and let finally  $\mathfrak{F}$  be the group of all automorphisms induced by inner automorphisms of  $C$ . Then  $\mathfrak{A}/\mathfrak{F}$  is totally disconnected [9] and with it is its subgroups  $\mathfrak{G}/\mathfrak{F}$ . Let  $Z$  be the centralizer of  $C$  in  $G$ . Then  $\mathfrak{G}$  is isomorphic to  $G/Z$  and  $\mathfrak{F}$  is isomorphic to the subgroup  $CZ/Z$  (or, equivalently, to the quotient of  $C$  modulo its center  $C \cap Z$ ). Hence  $\mathfrak{G}/\mathfrak{F}$  is isomorphic to  $(G/Z)/(CZ/Z)$  or to  $G/CZ$ . Hence  $G/CZ$  is totally disconnected. On the other hand  $G/CZ$  is a homomorphic image of  $G/C$  which is isomorphic to the reals and is, therefore, connected. Thus  $G/CZ$  is both connected and totally disconnected which implies  $G = CZ = ZC$ . The component  $Z_0$  of  $Z$  cannot be compact since  $Z/(C \cap Z)$  is isomorphic to  $CZ/C$  and therefore to the reals. If  $Z_0$  were compact, it would thus have to be contained in the maximal compact subgroup of  $Z$ , namely  $C \cap Z$ ; but then  $Z/(C \cap Z)$  would be totally disconnected which is a contradiction. By Iwasawa's theorem [9, p. 549] (or by a conclusion using approximation by Lie groups or by the observation made at the end of the proof to 1.22)  $Z_0$  contains a one-parameter group  $M$  isomorphic to the additive reals. The product  $CM$  is obviously direct since  $M$  is in the centralizer of  $C$ . But  $CM/C$  is isomorphic to  $M$  hence it is all of  $G/M$ . This shows that  $G = CM$  and completes thereby the proof.

**1.26. PROPOSITION.** *Let  $M_0$  be the semigroup isomorphic to the non-negative reals under multiplication. Let  $C$  be any compact topological group and  $\rho$  the equivalence relation on  $M_0 \times C$  which identifies all points of the ideal  $\{0\} \times C$ . Then  $\rho$  is a congruence relation and  $(M_0 \times C)/\rho$  is isomorphic to  $S$ .*

**Proof.** It is obvious that  $\rho$  is a congruence relation with compact cosets. The coset  $\bar{0} = \{0\} \times C$  is clearly a zero for  $(M_0 \times C)/\rho$  which is not isolated. The subset  $M \times C$  where  $M = M_0 \setminus \{0\}$ , is mapped isomorphically onto the quotient space and is therefore a dense subgroup. Since all cosets modulo  $\rho$  are compact, the quotient space is Hausdorff. We have to show that multiplication

is continuous at  $\bar{0}$  in the quotient semigroup. Let  $U$  and  $V$  be the basic neighborhoods of  $\bar{0}$ , which are images of sets  $[0, r] \times C$  and  $[0, s] \times C$  respectively, with positive real numbers  $r$  and  $s$ . Then the product of  $U$  and  $V$  is the image of the set  $[0, rs] \times C$ ; and when  $r$  and  $s$  vary over all positive reals, then the set of all  $UV = ([0, rs] \times C)/\rho$  is a basis for the neighborhoods of  $\bar{0}$ . This proves the continuity of the multiplication  $\times$  at  $\bar{0}$ .

If  $S$  is a l.c.g.z. in which 0 and 1 are connected, then there is a subsemigroup  $M_0$  isomorphic to the non-negative reals, all of whose points commute element-wise with all points of  $S$  and in particular with all points of the maximal compact subgroup  $C$ . The mapping  $f$  which assigns to an element  $\rho(m, c) \in (M_0 \times C)/\rho$  the element  $mc \in S$  is an algebraic isomorphism onto  $S$  which maps  $(M \times C)/\rho$  isomorphically in the topological sense, too. We show that  $f$  is continuous and open in 0. Since  $(M_0 \cap (S_0 \cup S_1) \times C)/\rho$  is a compact neighborhood of  $\bar{0}$  in  $(M_0 \times C)/\rho$  which is mapped by  $f$  onto the compact neighborhood  $S_0 \cup S_1$  of 0, it is sufficient to prove the continuity of  $f$  in 0. Let  $W$  be a neighborhood of 0 in  $S$ . Then there is an element  $m$  in  $M$  such that  $m(S_0 \cup S_1) \subset W$ . If we denote the set  $M_0 \cap (S_0 \cup S_1)$  with  $M_1$ , then  $mM_1$  is still an interval of  $M_0$  containing 0 in its interior; hence  $(mM_1 \times C)/\rho$  is a neighborhood of 0 which is entirely mapped into  $W$  by  $f$ .

**1.27. PROPOSITION.** *Let  $M_0$  be the semigroup isomorphic to the set  $\{0\} \cup M = \{0\} \cup \{2^n : n = 0, \pm 1, \dots\}$  of real numbers under multiplication. Let  $C$  be any compact topological group and  $\alpha$  any continuous automorphism of  $C$ . Let the product  $(2^n, c)(2^m, d)$  of two elements of the cartesian product  $M \times C$  be defined by  $(2^{n+m}, \alpha^m(c)d)$  and let  $0(2^n, c) = (2^n, c)0 = 0$ . Let the union  $\{0\} \cup M \times C$  be topologized so that the sets  $\{0\} \cup \{(2^n, c) : n = m, m-1, m-2, \dots\}$ ,  $m = 0, -1, -2, \dots$  form a basis for the neighborhoods of 0, and any point  $(2^n, c)$  has a neighborhood basis as in the product space  $M \times C$ , then this multiplication turns  $\{0\} \cup M \times C$  into a l.c.g.z.*

*Conversely, if  $S$  is a l.c.g.z. in which 0 and 1 are not in some connected subset, if  $C$  is the maximal compact subgroup and  $g$  the generator of the cyclic subgroup  $M$  such that  $MC = G$  and  $\lim_{n \rightarrow \infty} g^n = 0$ , then  $S$  is isomorphic to the semigroup constructed as described, provided  $\alpha$  is the inner automorphism  $c \rightarrow g^{-1}cg$ .*

**Proof.** Straightforward computation shows that  $\{0\} \cup M \times C$  with the multiplication and the topology as described above is a l.c.g.z. Let now  $S$  be a l.c.g.z. in which 0 and 1 are not connected, let  $C$  be its maximal compact normal subgroup and  $M$  the cyclic infinite group such that  $G = MC$ , whose existence is guaranteed by 1. Let  $g$  be this one of the two generators of  $M$  whose powers converge to 0. If  $g^n c$  and  $g^m d$  are two elements of  $G$ , then  $g^n c g^m d = g^{n+m} (g^{-m} c g^m) d = g^{n+m} \alpha^m(c) d$  with  $\alpha(c) = g^{-1} c g$ . Hence the mapping  $f$  from  $\{2^n : n = 0, \pm 1, \dots\} \times C$  onto  $G$  which assigns to the pair  $(2^n, c)$  the element  $g^n c$  is an algebraic and

topological isomorphism of the group  $M \times C$  with the described multiplication onto  $G$ . It is left to prove that  $f$  can be continuously extended by setting  $f(0) = 0$ ; since  $\{0\} \cup \{2^n : n = 0, -1, -2\} \times C$  and  $S_0 \cup S_1$  are compact neighborhoods of 0 in  $\{0\} \cup M \times C$  and  $S$  respectively, this will prove that  $f$  is actually an isomorphism. But a sequence  $(2^{n(i)}, c(i)), i = 1, 2, \dots$  of elements in  $\{0\} \cup M \times C$  converges to 0 if and only if  $\lim_{n \rightarrow \infty} n(i) = -\infty$ . But in this case  $f(2^{n(i)}, c(i)) = g^{n(i)}c(i)$  converges to 0 since  $C$  is compact (1.3). This finishes the proof.

1.28. DEFINITION. If  $C$  is compact and  $M$  is infinite cyclic generated by  $g$ , and if  $\alpha$  is a continuous automorphism of  $C$ , then the locally compact topological group defined on the cartesian product  $M \times C$  by setting  $(g^n, c)(g^m, d) = (g^{n+m}, \alpha^m(c)d)$  will be called the semidirect product of  $M$  and  $C$  with  $\alpha$ , and we denote it with  $M \tilde{\times} C$ . If a zero 0 is topologically added as described in 1.26, then the arising semigroup is called the union of 0 and  $M \times C$ . We conclude this section with

**THEOREM I.** *Let  $S$  be a locally compact group with zero and  $G$  its maximal subgroup  $S \setminus \{0\}$ . Then*

- (i)  *$G$  contains a unique characteristic maximal compact subgroup  $C$ .*
- (ii) *If 0 and 1 lie in some connected subspace, then  $S$  contains in its center a locally compact group with zero  $M_0 = M \cup \{0\}$  which is isomorphic to the multiplicative semigroup of all non-negative real numbers and  $S$  is isomorphic to the quotient semigroup of the direct product  $M_0 \times C$  modulo the congruence relation identifying all points of  $\{0\} \times C$ .*

*If 0 and 1 do not lie in any connected subspace of  $S$ , then  $S$  contains a locally compact group with zero  $M_0 = M \cup \{0\}$  which is isomorphic to the set of real numbers  $\{0\} \cup \{2^n : n = 0, \pm 1, \pm 2, \dots\}$  under multiplication and  $S$  is isomorphic to the union of 0 and the semidirect product  $M \tilde{\times} C$ , where  $\alpha$  is the inner automorphism  $c \rightarrow g^{-1}cg$  with the generator  $g$  of  $M$  whose powers converge to 0.*

- (iii) *If  $C$  is any compact group and  $\alpha$  any automorphism of it, then there exists a locally compact group with zero whose maximal compact group is isomorphic to  $C$  and whose maximal group can be made isomorphic to  $M \times C$  where  $M$  are positive reals under multiplication or to  $M \tilde{\times} C$  where in this case  $M$  is infinite cyclic.*

## II. Groups with compact boundary.

2.1. DEFINITION. Let  $S$  be a topological Hausdorff space which satisfies the following conditions:

- (i)  $S$  is a topological semigroup;
- (ii) there is a nonempty subset  $B$  in  $S$  which is nowhere dense and compact such that  $S \setminus B$  is a group, and  $B \subset \overline{S \setminus B}$ .

Then  $S$  is called a *topological group with compact boundary*. The complement

$S \setminus B$  will always be denoted with  $G$ . The term "locally compact topological group with compact boundary" will be abbreviated as l.c.g.c.b.

**2.2. PROPOSITION.** *In a l.c.g.c.b.  $S$ , the group  $G$  is a locally compact topological group whose identity is a unit for  $S$ ; and  $B$  is an ideal; the equivalence relation which identifies all points of  $B$  is a congruence relation  $\rho$  and the quotient semigroup  $S/\rho$  is a locally compact group with zero  $\bar{0} = \rho(B)$ .*

**Proof.** The subspace  $S \setminus B = G$  is open in the locally compact space  $S$ , hence it is locally compact. But every locally compact topological semigroup with algebraic group structure is a topological group [4]. Since  $G$  is dense in  $S$ , its identity  $1$  is an identity for  $S$ . We prove now that  $B$  is an ideal: Let  $g \in G$ ,  $b \in B$  and suppose that  $gb \in G$ . Then  $g^{-1}(gb)$  is in  $G$ , but  $g^{-1}(gb) = 1b = b \in B$ , which is a contradiction. This shows that  $GB \subset B$ . Because of the continuity of multiplication,  $SB = \bar{GB} \subset \bar{GB} = B$  since  $B$  is closed. Similarly  $BS \subset B$ . It is obvious that the equivalence relation  $\rho$  is compatible with the multiplication so that  $S/\rho$  is a semigroup. Since all cosets modulo  $\rho$  are compact, the quotient space is locally compact Hausdorff, and the restriction of the coset mapping to  $G$  is an isomorphism algebraically and topologically. Because  $\bar{G} \subset B$ , the point  $\bar{0} = \rho(B)$  is not isolated in  $S/\rho$ , for any neighborhood of  $B$  contains a point of  $G$ . In order to finish the proof we have to show that the multiplication of  $S/\rho$  is continuous at  $\bar{0}$ . Let  $U$  be an open neighborhood of  $\bar{0}$ ; then  $U' = \rho^{-1}(U)$  is an open neighborhood of  $B$  in  $S$ . For each  $x$  in  $B$  there is a compact neighborhood  $U_x$  of  $x$  in  $G$  which is contained in  $U'$ . Since  $B$  is compact, there is a finite number of points  $x_1, \dots, x_n$  such that the interiors of the neighborhoods  $U_{x_1}, \dots, U_{x_n}$  cover  $B$ . Let  $V$  be the union of these neighborhoods; as a finite union of compact sets it is a compact neighborhood of  $B$  contained in the open neighborhood  $U'$ . For each  $y$  in  $B$  the set  $Vy$  is a compact subset of  $B$ , hence there is a neighborhood  $W_y$  such that  $VW_y$  is in the neighborhood  $U'$  of  $Vy$ ; for otherwise there were a net  $\{y_i : i \in I\}$  of points converging to  $y$  and a net of points  $\{v_i : i \in I\}$  such that  $v_i y_i \in U'$ . But  $V$  is compact, thus, on passing to convergent subnets if necessary, we may assume that  $\{v_i : i \in I\}$  converges to  $v \in V$ . Then  $\{v_i y_i : i \in I\}$  converges to  $vy \in B$ , the elements  $v_i y_i$ , however, stay outside the open set  $U'$  so that  $vy$  cannot be in  $U'$ , hence a fortiori not in  $B$ . The compact set  $B$  is covered by a finite number of neighborhoods  $W_y$  whose union is a neighborhood  $W$  of  $B$  which satisfies  $VW \subset U'$ . The sets  $\rho(V)$  and  $\rho(W)$  are neighborhoods of  $\bar{0}$  whose product is contained in  $U$ . This shows that multiplication is continuous at  $\bar{0}$ .

**2.3. PROPOSITION.** *Let  $S$  be a l.c.g.c.b. and  $G$  the open dense subgroup. Then  $G$  contains a unique characteristic maximal compact subgroup  $C$  and a non-compact subgroup  $M$  which is either isomorphic to the positive reals or the set  $\{2^n : n = 0, \pm 1, \pm 2, \dots\}$  under multiplication such that  $G$  is isomorphic to the direct product of  $M$  and  $C$  in the first case and to the crossed product of  $M$  and*

$C$  with an automorphism  $\alpha$  of  $C$  (see 1.27). Furthermore, a net  $\{m_i c_i : i \in I\}$  of elements of  $G$  with  $m_i \in M$  and  $c_i \in C$  converges to a point  $b$  of  $B$  only if  $\{m'_i : i \in I\}$  converges to zero, where  $m \rightarrow m'$  is a fixed isomorphism of  $M$  onto the positive reals or the set  $\{2^n : n = 0, \pm 1, \pm 2, \dots\}$  respectively.

**Proof.** Let  $\rho$  be defined as in 2.2.  $G$  is mapped isomorphically onto the maximal subgroup of a l.c.g.z. under the coset mapping. Hence, Theorem I describes the structure of  $G$  fully. Moreover, let the isomorphism  $m \rightarrow m'$  of the subgroup  $M$ , whose existence is guaranteed by Theorem I, be normed such that  $\lim_I \rho(m_i) = \bar{0}$  is equivalent to  $\lim_I m'_i = 0$  in the space of real numbers. Then  $\lim_I m_i c_i = b \in B$  implies  $\lim_I \rho(m_i) \rho(c_i) = \bar{0}$  which in turn implies  $\lim_I m'_i = 0$ . The isomorphism  $m \rightarrow m'$  will be fixed throughout the following.

**2.4. PROPOSITION.** Let  $S$  be a l.c.g.c.b. and  $S \cup \{\infty\}$  its one point compactification. Define the sets  $S_0, S_1$ , and  $S_2$  as follows:

$$S_0 = B \cup \{mc : m \in M, c \in C, \lim_{n \rightarrow \infty} m'' = 0\},$$

$$S_1 = C,$$

$$S_2 = \{mc : m \in M, c \in C, \lim_{n \rightarrow \infty} m'' = \infty\} \cup \{\infty\}.$$

Then  $S_0 \cup S_1$  and  $(S_2 \cup S_1) \setminus \{\infty\}$  are locally compact semigroups in  $S$ ,  $S_0 \cup S_1$  is compact, and  $B$  is in the closure of  $\{mc : m \in M, c \in C, \lim_{n \rightarrow \infty} m'' = 0\} \cup C$ .

**Proof.**  $\rho(S_0 \cup S_1)$  and  $\rho(S_2 \cup S_1) \setminus \{\infty\}$  are subsemigroups of  $S/\rho$  and  $(S_0 \cup S_1)$  is compact; but all cosets modulo  $\rho$  are compact so that  $S_0 \cup S_1$  is indeed compact. Every point in  $B$  is the limit of a net  $\{m_i c_i : i \in I\}$  with  $\lim_I m'_i = 0$  (2.3). Hence there is a  $i_0$  in  $I$  such that  $i > i_0$  implies  $m'_i < 1$ , from which we get  $\lim_{n \rightarrow \infty} m''_i = 0$ . Therefore the net  $\{m_i c_i : i \in I\}$  is finally in  $\{mc : m \in M, c \in C, \lim_{n \rightarrow \infty} m'' = 0\}$ , which proves that  $B$  is in the closure of this set.

**2.5.** Let  $S$  be a l.c.g.c.b., then  $B$  is a compact topological group. If  $e$  is the unit of  $B$ , then  $x \rightarrow xe$  is a continuous homomorphism of  $S$  onto  $B$ , the so-called Clifford-Miller endomorphism.

**Proof.** In order to show that  $B$  is a group, we prove that for all  $a, b$  in  $B$  there is an  $x$  in  $B$  such that  $ax = b$ . A similar procedure will then prove that the equation  $x'a = b$  is solvable. Let  $a = \lim_I m_i c_i$  and  $b = \lim_J m_j c_j$  be some elements of  $B$  with  $\lim_I m'_i = 0$  and  $\lim_J m'_j = 0$ . Then the net  $(m'_{ij} : (i, j) \in I \times J)$  with  $m_i^{-1} m'_j = m'_{ij}$  has 0 and  $+\infty$  as cluster points. We shall find a subnet which converges to 0: Let  $K = I \times J \times \mathfrak{U}$  with the filter  $\mathfrak{U}$  of neighborhoods of 0 on the real line. For each  $k = (i, j, U) \in K$  pick an  $i(k) \in I$  and a  $j(k) \in J$  such that  $i(k) \geq i$ ,  $j(k) \geq j$ , and  $m'_{i(k)j(k)} = m'_{i(k)}^{-1} m'_{j(k)} \in U$ ; this is possible because  $\lim_I m'_j = 0$ . We set  $m'_k = m'_{i(k)j(k)}$ . Then clearly  $\lim_K m'_k = 0$ ,  $\lim_K m'_i = 0$ , and  $\lim_K m'_j = 0$ . Now  $c_i^{-1} m_i^{-1} m_j c_j = m_i^{-1} m_j (m_{ij} c_i m_{ij}) c_j$ . The net  $\{m_k^{-1} c_{i(k)} m_k c_{j(k)} :$



$k \in K\}$  has a convergent subnet on the compact space  $C$ ; on renaming the indices, if necessary, we may assume that this net already converges to some  $c \in C$ . Hence  $x = \lim_K c_{i(k)}^{-1} m_{i(k)}^{-1} m_{j(k)}^{-1} c_{j(k)}^{-1} = \lim_K m_k (m_k^{-1} c_{i(k)} m_k c_{j(k)})$  exists and is contained in  $B$  since  $\lim_K m'_k = 0$ . But now  $ax = \lim_K m_{i(k)} c_{i(k)} \lim_K c_{i(k)}^{-1} m_{i(k)}^{-1} m_{j(k)}^{-1} c_{j(k)}^{-1} = \lim_K m_{j(k)} c_{j(k)} = b$ . This proves that  $B$  is a group. Let  $e$  be the unit of  $B$ . The mapping  $x \rightarrow xe$  is a mapping of  $S$  into  $B$  because  $B$  is an ideal. The identity  $xe = e(xe) = (ex)e = ex$  enables us to show that  $(xy)e = (xe)(ye)$ , for  $(xy)e = x(ye) = e(x(ye)) = ex(ye) = (xe)(ye)$ . Hence  $x \rightarrow xe$  is a continuous algebraic homomorphism of  $S$  into  $B$  which clearly maps  $B$  identically; so this homomorphism is actually a retraction.

Note that this homomorphism is not necessarily open on  $G$  even relatively to  $B$ .

2.6. DEFINITION. The kernel of the restriction of the homomorphism  $x \rightarrow xe$  to  $C$  will be called  $N$ . Clearly  $Ce$  is isomorphic to  $C/N$  [17, p. 17]. The collection of all elements killed by  $e$ , i.e. the set of all  $x \in S$  such that  $xe = ex = e$  is called  $K$ . Clearly  $K$  contains  $N$  and  $K \cap G$  is a closed normal subgroup of  $G$  (it is a subgroup since  $xe = e$  implies  $x_e^{-1} = x^{-1}(xe) = 1e = e$ , it is closed because  $K$  is closed as the counter-image of  $e$  under the continuous mapping  $x \rightarrow xe$ , it is normal since  $xe = e$  implies  $g^{-1}xge = g^{-1}xeg = g^{-1}eg = g^{-1}ge = e$ ). Clearly  $K \cap B = \{e\}$ .

2.7. LEMMA. Let  $G$  be a locally compact group,  $K$  a normal subgroup containing a one-parameter group  $E$  isomorphic to the additive reals and central in  $K$ ,  $C$  a maximal compact normal subgroup of  $G$ , and suppose that  $G = CE$ . Then there is a one-parameter group  $E'$  in  $K$  which is even in the center of  $C$ .

**Proof.** As in the proof of Lemma 1.25, one proves  $G = CZ$  where  $Z$  is the centralizer of  $C$  in  $G$ . All we have to show is that the component  $K_0$  of the intersection  $Z \cap K$  is not compact; for then, by the same methods applied in the proof of Lemma 1.25, we pick a one-parameter group  $E'$  isomorphic to the reals in  $K_0$ , and  $E'$  satisfies the requirements of the lemma.

By way of contradiction we suppose that  $K_0$  is compact. The connected component of the identity in the center of  $K$  is a direct product of a vector group and its maximal compact subgroup  $C_0$ . Since  $C_0$  is characteristic in  $K$ , it is normal in  $G$ . Hence  $K_0 C_0$  is a compact normal subgroup. The quotient of  $G$  modulo this compact normal subgroup satisfies again the hypotheses of the lemma. We call this quotient again  $G$  and observe that  $G$  now has the following additional properties:

- (i)  $Z \cap K$  is totally disconnected,
- (ii) the center of  $K$  contains no nontrivial connected compact subgroup.

The component of the center of  $K$  contains  $E$ , but because of  $G = CE$  it cannot contain a vector group of higher dimension. Hence by (ii)  $E$  is the component of the center of  $K$  and is, therefore, characteristic in  $K$  and thus normal

in  $G$ . But then the product  $CE$  is direct and  $E$  is in the centralizer of  $C$  which contradicts (i). This proves the lemma.

2.8. PROPOSITION. *The following statements are equivalent in a l.c.g.c.b.:*

- (i)  $K$  is connected;
- (ii) there is a central subsemigroup  $M_0$  in  $S$  isomorphic to the non-negative reals under multiplication which contains 1 and  $e$ ;
- (iii)  $S$  contains a connected l.c.g.z.;
- (iv) there is an arc joining 1 and  $e$ ;
- (v)  $K$  is not compact and does not contain a compact open subgroup.

**Proof.** (i) implies (ii): If  $K$  is connected, then  $K$  is a l.c.g.z., for  $K$  is closed as the counter-image of  $e$  under the continuous mapping  $x \rightarrow xe$ ; its intersection with  $B$  is  $\{e\}$ , since  $be = b$  with  $b \in B$  implies  $b = e$ ,  $B$  being a group. As  $K$  is connected,  $e$  is not isolated in  $K$ , but by the definition of  $K$  the element  $e$  is a zero for  $K$ . From Theorem I we know that  $K$  contains a subsemigroup  $M_0''$  in its center such that  $M_0''$  is isomorphic to the non-negative reals under multiplication. By Lemma 2.7 a one-parameter group  $M_0$  can be found in  $K$  such that it lies even in the center of  $G$ : The subsemigroup  $M \cup \{e\} = M_0$  is the subsemigroup wanted, for  $e \in \bar{M}$ , since the powers of an element in  $M \cap S_0$  converge to  $e$ .

(ii) implies (iii): This is trivial.

(iii) implies (iv): This follows from the information Theorem I yields about the structure of a connected l.c.g.z.

(iv) implies (v): Let  $\rho$  be the congruence relation whose cosets ( $s$ ) are the orbits  $Cs$  of the maximal compact subgroup  $C$  of  $G$ . Then  $S' = S/\rho$  is a l.c.g.c.b. without a nontrivial compact subgroup in  $G$  and there is still an arc leading from 1 to  $e$ . Since, from Theorem I, the maximal subgroup  $G'$  of  $S'$  is a one-parameter group,  $S'$  is abelian. Let  $\psi$  be a continuous mapping from the unit interval  $[0, 1]$  into  $S'$  such that  $\psi(0) = 1$  and  $\psi(1) = e$ . Let  $r \in [0, 1]$  be the lowest upper bound of all  $t \in [0, 1]$  such that  $\psi(t) \in G'$ . If we decompose  $S'$  into the disjoint union of  $S'_0, S'_1$ , and  $S'_2$  according to 2.4, we may assume that  $\psi([0, 1])$  is entirely contained in  $S'_0 \cup S'_1$ ; since  $G' \cap (S'_0 \cup S'_1)$  is a half-open interval,  $\psi([0, r])$  is a closed interval, namely the closure of  $G' \cap (S'_0 \cup S'_1)$ ; but from 2.4. we know that the closure of  $G' \cap (S'_0 \cup S'_1)$  is  $S'_0 \cup S'_1$ . This shows that  $S' \setminus G'$  consists only of one point  $e' = \rho(e) = \psi(r)$ , which is then the zero of  $S'$ , and  $S'$  is isomorphic to the multiplicative semigroup of all non-negative reals. Hence  $K' = \{s : s \in S'\}$  such that  $se' = e'$  is equal to  $S'$ . Let now  $G = MC$  with a central one parameter subgroup isomorphic to the positive reals and let  $m \in M$  and  $c \in C$ . Then  $mce = de$  with some  $d \in C$ , because  $se' = e'$  for all  $s \in S'$ . But  $mce = de = ed$  implies  $m(cd^{-1})e = mc(d^{-1}e) = (mce)d^{-1} = (ed)d^{-1} = e1 = e$ . Hence every coset modulo  $C$  of  $G$  contains at least one element of  $K$ . But then

$K$  is not compact. If  $K$  would contain a compact open subgroup  $K_0$ , then  $K_0$  would be contained in  $C \cap K$ , and this intersection would be an open compact subgroup of  $G$ . But  $KC/C$  and  $K/(K \cap C)$  are isomorphic and the latter group would be discrete; on the other hand,  $KC = G$  because  $K$  meets every coset modulo  $C$ , and  $G/C = G'$  is a connected one-parameter group, as we know from above. This contradiction shows that  $K$  cannot contain an open and compact subgroup.

(v) implies (i): If  $K$  is not compact, then  $K \cap G$  is not compact; hence  $e$  is not isolated in  $K$  since the powers of any element in  $(K \setminus \{e\}) \cap S_0$  have a cluster-point in  $B$  which must be  $e$ , since  $K$  is closed in  $S$  and  $K \cap B = \{e\}$ . Thus  $K$  is a l.c.g.z. As  $K$  does not contain a compact open subgroup in  $G$ , it contains a one-parameter group going from 1 to  $e$  (Theorem I). Hence  $K$  is connected (Theorem I).

**2.9. PROPOSITION.** *Let  $S$  be an abelian l.c.g.c.b. whose maximal group  $G$  is isomorphic to the multiplicative group of all positive reals. Let:  $R \rightarrow fG$  be an isomorphism of the additive group of reals onto  $G$  such that  $f([0, \infty)) \subset S_0 \cup S_1$  (2.4), and let  $e$  be the identity of  $B$ . Then  $B$  is a compact connected abelian group and  $g$  defined by  $g(r) = f(r)e$  is a continuous homomorphism onto a dense one-parameter subsemigroup of  $B$ . If conversely  $B$  is a compact abelian group admitting a dense one-parameter subgroup, then there is an abelian l.c.g.c.b. having this group as boundary and having a one-parameter group as maximal subgroup.*

**Proof.** Since  $B$  is a compact group in all instances, it is sufficient to observe that  $r \rightarrow f(r)e$  is a continuous homomorphism on a dense subgroup in  $B$ . Continuity follows from the continuity of  $f$  and the continuity of multiplication. Clearly  $f(r)ef(s)e = f(r)f(s)e^2 = f(r+s)e$ . Let now  $b$  be an element of  $B$ ; then there is a net  $\{r_i : i \in I\}$  on the reals such that  $b = \lim_I f(r_i)$ . But then  $b = be = \lim_I f(r_i)e = \lim_I g(r_i)$  which proves that  $g(R)$  is dense in  $B$ .

Let now  $B$  be a compact abelian group and  $g : R \rightarrow B$  a continuous homomorphism of the additive reals onto a dense subgroup of  $B$ . Let  $M_0$  be the multiplicative semigroup of all non-negative reals and  $f : R \rightarrow M_0$  an isomorphism of  $R$  into  $M_0$  such that  $\lim_{n \rightarrow \infty} f(n) = 0$  (set e.g.  $f(r) = \exp -r$ ). The mapping  $r \rightarrow (f(r), g(r))$  is an isomorphism of  $R$  into the product semigroup  $M_0 \times B$ . We let  $S$  be the closure of its image in  $M_0 \times B$ . Let  $b \in B$ ; then there is an indefinitely increasing net  $\{r_i : i \in I\}$  on  $R$  such that  $b = \lim_I g(r_i)$ . Then  $(0, b) = \lim_I (f(r_i), g(r_i))$  which shows that  $\{0\} \times B$  is the boundary of the maximal group in  $S$ .

**2.10.** Let  $S$  be a l.c.g.c.b. and suppose that 1 is contained in a connected subspace that intersects  $B$ . Then  $G$  contains a central one-parameter group  $M$ , and  $\bar{M}$ , the closure of  $M$  in  $S$ , is an abelian l.c.g.c.b. as described in 2.9, with maximal subgroup  $M$  and boundary  $A = \bar{M} \setminus M$ ; any element of  $\bar{M}$  commutes with every element in  $S$ .

**Proof.** Let  $\rho$  be the congruence relation on  $S$  which has the compact cosets  $\rho(s) = \{s\}$  if  $s \in G$  and  $\rho(b) = B$  if  $b \in B$ . Then  $S/\rho$  is a l.c.g.z. whose maximal subgroup is isomorphic to  $G$ , and in which the identity is connected with the zero. Therefore, by Theorem I, the maximal subgroup contains a central not compact one-parameter group. Hence  $S$  contains a central one-parameter group, because the maximal groups of  $S$  and  $S/\rho$  are isomorphic and because  $G$  is dense in  $S$ . The closure  $\bar{M}$  intersects  $B$  because  $M$  is not compact in  $G$  and  $\bar{M}$  is a l.c.g.c.b. with boundary  $A = \bar{M} \cap B$ . Since any element in  $M$  commutes with every element in  $S$ , clearly all elements in  $\bar{M}$  commute with all elements of  $S$ .

2.11. Let  $S$  be as in 2.10, let  $A$  be the boundary of  $M$  in  $\bar{M}$  and let  $C$ , as before, be the maximal compact subgroup of  $G$ . Then  $B = A(Ce)$ , and  $B$  is isomorphic to the factor group  $(A \times (Ce))/L$ , where  $L = \{(a^{-1}, a) : a \in A \cap Ce\}$  (the inverse  $a^{-1}$  is taken as inverse with respect to  $e$  in  $B$ !) is isomorphic to the intersection of  $A$  and  $Ce$ .

**Proof.** Trivially  $A(Ce) \subset B$ . On the other hand  $B = \bar{G} \setminus G = \overline{MC} \setminus MC$ , but since  $C$  is compact we have  $\overline{MC} = \bar{M}C = (M \cup A)C = MC \cup A(Ce)$ , which shows that  $B \subset A(Ce)$ . As  $A$  is in the center of  $B$ , the mapping  $(a, c) \rightarrow ac$  of  $A \times (Ce)$  onto  $A(Ce)$  is clearly a continuous representation of  $A \times (Ce)$  onto  $A(Ce)$ , and since  $A \times (Ce)$  is compact, it is a homomorphism, i.e. it is a continuous and open homomorphism and its image is isomorphic to the factor group of  $A \times (Ce)$  modulo its kernel: [16, Theorem 12] and [17, p. 11, 21]. The kernel is the collection of all  $(a, c) \in A \times (Ce)$  such that  $ac = e$ ; but this relation is equivalent to  $a = c^{-1} \in A \cap Ce$ .

2.12. PROPOSITION. Let  $S$  be as in 2.10 and in 2.11. Let  $M'_0$  be the multiplicative group of all non-negative reals with the subgroup  $M'$  of all positive reals. Let  $f: M' \rightarrow M$  be an isomorphism of  $M'$  onto a central one-parameter group of  $G$  such that  $f([1, 0)) \subset S_0 \cup S_1$ . Let  $S^*$  be the semigroup  $M'_0 \times C \times A$ . Let furthermore  $f^*$  be the isomorphism  $r \rightarrow (r, 1, f(r)e)$  of  $M'$  into  $S^*$  and set  $G^{**} = f^*(M)(\{1\} \times C \times \{e\})$  and  $S^{**} = \overline{G^{**}}$  in  $S^*$ . Let  $N$  be the subgroup of  $C$  which is the kernel of  $c \rightarrow ce$ , and let  $L^*$  be the set of all elements  $(0, c, a)$  such that  $a \in A \cap Ce$  and  $ac = e$ . Then the following statements are valid:

- (i)  $L^*$  is a normal subgroup of  $B^* = B^{**} = \{0\} \times C \times A$ ;
- (ii) The equivalence  $\rho$  assigning to a  $g \in G$  the coset  $\{g\}$  and to a  $b \in B$  the coset  $bL^*$  is a congruence relation with compact cosets;
- (iii)  $S^{***} = S^{**}/\rho$  is isomorphic to  $S$ .

**Proof.** (i) Let  $(0, c, a)$  be in  $L^*$  and let  $(0, d, b)$  be any element in  $B^*$ . Then  $(0, d, b)^{-1}(0, c, a)(0, d, b) = (0, d^{-1}cd, a)$  since  $A$  is abelian, and  $d^{-1}cda = d^{-1}cad = d^{-1}ed = e$  since  $a$  is in  $A$  and hence in the center of  $B$ . This proves (i).

(ii) It is a matter of straightforward computation to check that  $\rho$  is a congruence. Its cosets are closed because  $L$  is closed in the compact group  $B$ .

(iii) Let  $\pi$  be the isomorphism  $(r, c, f(r)e) \rightarrow f(r)c$  of  $G^{**}$  onto  $G$ ; we show that it can be extended to a homomorphism of  $S^{**}$  onto  $S$ :

If  $(0, c, a) \in B^*$ , set  $\pi(0, c, a) = ac = a(ce)$ . Because

$$(r, c, f(r)e)(0, c', a') = (0, cc', f(r)a')$$

it follows that for  $r \neq 0$  we have  $\pi((r, c, f(r)e)(0, c', a')) = \pi(0, cc', f(r)a') = a'f(r)cc'$  which is indeed equal to  $\pi(r, c, f(r)e)\pi(0, c', a') = (f(r)c)(a'c') = a'f(r)cc'$  because  $f(r)$  and  $a'$  are in the center of  $S$ . Thus, on observing that the restriction of  $\pi$  to  $B^*$  is clearly a homomorphism, it is proved that  $\pi$  is algebraically a homomorphism which, restricted to  $G$ , is an isomorphism, and which, restricted to  $B$  is a homomorphism (i.e. a continuous and relatively open homomorphism). We have to show that  $\pi$  is continuous at  $(0, c, a) \in B^*$ . Suppose that  $(0, c, a) = \lim_I (r_i, c_i, a_i)$  where  $a_i = f(r_i)e$  for all  $r_i \neq 0$ . Then

$$\pi(r_i, c_i, a_i) = \begin{cases} f(r_i)c_i & \text{if } r_i \neq 0, \\ a_i c_i & \text{if } r_i = 0, \end{cases}$$

and the net of these elements obviously converges to  $ac$  which is equal to  $\pi(0, c, a)$ . Hence  $\pi$  is continuous in all of  $S^{**}$ . The proof will now be finished if we can show that  $sps'$  in  $S^{**}$  is equivalent to  $\pi(s) = \pi(s')$ , and that the quotient space  $S^{**}/\rho$  is homeomorphic to  $S$ . Since  $\pi$  is an isomorphism on  $G$ , it is sufficient to consider the case  $\pi(0, c, a) = \pi(0, c', a')$ ; but this is equivalent to  $ac = a'c'$  or  $(a^{-1}a')(c^{-1}c') = e$ , which shows that  $a^{-1}a'$  is in  $A \cap Ce$  and, therefore, that  $(0, a, c)$  is congruent  $(0, c', a')$  modulo  $L$ , which means that the two elements are  $\rho$ -congruent. Thus the homomorphism can be decomposed in the successive application of the quotient mapping  $\rho$  and a continuous 1-1 representation  $\pi'$ , of  $S^{**}/\rho$  onto  $S$ . We show that  $\pi'$  is open: First,  $\pi'$  is open on the open subgroup  $G^{**}/\rho$  of  $S^{**}/\rho$ , since  $\pi$  is an isomorphism of  $G^{**}$  onto  $G$ ; second,  $\pi'$  is a homeomorphism on the subspace  $(S_0^{**} \cup S_1^{**})/\rho$  because this subspace is compact; hence  $\pi'$  is open on  $S_0^{**}/\rho$  which is an open set in  $S^{**}/\rho$ ; since  $S^{**}/\rho$  is the union of  $G^{**}/\rho$  and  $S_0^{**}/\rho$ , this proves that  $\pi'$  is open on  $S$ , and this observation finishes the proof.

**2.13. PROPOSITION.** *Let  $C$  be a compact group,  $A$  a compact connected abelian group containing a dense one-parameter group,  $N$  a normal subgroup of  $C$  and  $L$  a closed central subgroup of  $C/N$  isomorphic to a closed subgroup of  $A$  with identity  $e$ . Then there exists a l.c.g.c.b.  $S$  such that its maximal subgroup  $G$  is isomorphic to  $M \times C$  where  $M$  is the group of all positive reals, and such that its boundary  $B$  is isomorphic to a factor group of the product  $A \times C/N$  modulo a subgroup isomorphic to  $L$ , and finally such that  $\bar{M} = M \cup A$ .*

The proof of this proposition follows closely the scheme of the previous proposition and is only given in outline:

(a) Let  $M_0$  be the semigroup of all non-negative reals and  $f$  a homomorphism

of  $M$  onto a dense one parameter-group of  $A$ . Then let  $S^{(1)} = M_0 \times C \times A$ . Define  $S^{(2)}$  as a quotient semigroup of  $S^{(1)}$  which is defined by a congruence identifying all cosets modulo  $\{0\} \times N \times \{e\}$  in the ideal  $B^{(1)} = \{0\} \times C \times A$ .

(b) Let  $G^{(3)}$  be the subgroup in  $S^{(2)}$  of all elements  $(r, c, f(r))$ ,  $r \in M$ ,  $c \in C$  and let  $S^{(3)}$  be the closure of  $G^{(3)}$  in  $S^{(2)}$ .

(c) Let  $\phi$  be an isomorphism of  $L \subset C/N$  into  $A$ ; then  $L^{(3)} = \{(0, x, \phi(x)) : x \in L\}$  is a central subgroup of  $B^{(2)} = B^{(3)} = \{0\} \times C/N \times A$  and the quotient semigroup  $S^{(4)}$  of  $S^{(3)}$  modulo the congruence  $\rho$  is well defined, if  $\rho$  identifies all cosets modulo  $L^{(3)}$  in  $B^{(3)}$ .

The semigroup  $S^{(4)}$  has all the required properties.

**2.14. PROPOSITION.** *Let  $S$  be a l.c.g.c.b. in which 1 and  $e$  are not contained in a connected subspace. Let  $M$  be the cyclic group generated by  $m \in S_0$  whose existence is given by 2.3. Then  $\bar{M}$  is an abelian l.c.g.c.b. which is the disjoint union of  $M$  and  $A$  where  $A$  is a compact abelian group with a dense cyclic group; in fact the restriction of the mapping  $s \rightarrow se$  of  $S$  onto  $B$  is a continuous representation of  $M$  onto a dense subgroup of  $A$ .*

**Proof.** The last statement of the proposition is the only one to be proved. Let  $a \in A$ , then  $a = \lim_I m^{n_i}$  with some net  $\{n_i : i \in I\}$  of natural numbers. But  $a = \lim_I m^{n_i} e \in \bar{M}e$ ; thus the cyclic group  $Me$  is dense in  $A$ .

**2.15. PROPOSITION.** *Let  $S$  be as before; then  $B = A(Ce)$  and  $Ce$  is normal in  $B$ .*

**Proof.** As in 2.11, we have  $A(Ce) \subset B = \bar{G} \setminus G = \overline{MC} \setminus MC = \bar{M}C \setminus MC = (M \cup A)C \setminus MC = AC = A(Ce)$ . To prove that  $Ce$  is normal in  $B$ , let  $a \in A$  and  $ce \in Ce$ ; then  $a^{-1}(ce)a = \lim_I m^{-n_i} c m^{n_i} e \in Ce$  since  $C$  is normal in  $G$ .

The information we get in the case that 1 and  $e$  are not connected is not quite as complete as in the contrary case. This is due to the fact that  $M$  need not be central. It should be clear that the construction of 2.12 works for a cyclic group  $M$  instead of a one-parameter group. This will not yield, however, the most general case as the following example may indicate:

**2.16. EXAMPLE.** Let  $A$  be an abelian group and  $f : M \rightarrow A$  be a representation of the cyclic group  $M$  onto a dense subgroup of  $A$ . Let  $\phi$  be a homomorphism of  $A$  into the automorphism group of a compact group  $C$ . Let  $M_0$  be the group  $M$  together with a zero 0. On the product  $S' = M_0 \times A \times C$  we introduce the following multiplication:

$$(r, a, c)(r', a', c') = (rr', aa', c^{\phi(a')}c').$$

This multiplication turns  $S'$  into a l.c.g.c.b. whose maximal subgroup  $\{1\} \times A \times C$  and whose kernel  $\{0\} \times A \times C$  are isomorphic to the holomorphic extension of  $C$  by  $A$ . Let  $G$  be the subgroup consisting of all points  $(r, f(r), c)$ ,  $r \in M$ , and let  $S$  be the closure of  $G$  in  $S'$ . Then  $S$  is a l.c.g.c.b. whose maximal subgroup is isomorphic to the crossed product of  $M$  and  $C$  with  $\phi \circ f$ , and the ideal  $B = \{0\} \times A \times G$  is isomorphic to the holomorphic extension of  $C$  with  $A$ .

These statements are a matter of straightforward verification along the lines of 2.12. It may be observed that the example can be complicated by factoring out a normal subgroup of  $B$ .

We are now in the position to formulate the main theorem in this section. The hypotheses in the following three theorems are graded according to decreasing generality, so that the Theorem II is the most and the Theorem IV is the least general. The conclusions are, of course, more simple to look over in the most restricted case.

**THEOREM II.** *Let  $S$  be a locally compact group with compact boundary; then the complement  $B$  of the maximal subgroup  $G$  is a compact ideal and a group with identity  $e$ .*

(a) *The structure of  $G$ : There is a characteristic maximal compact subgroup  $C$  and a subgroup  $M$ , which is either a one-parameter group isomorphic to the additive reals or to an infinite cyclic group, and  $G = MC$ ,  $M \cap C = \{1\}$ .*

(b) *The connection between  $G$  and  $B$ : The mapping  $s \rightarrow se$  of  $S$  onto  $B$  (the so-called Clifford-Miller endomorphism) is a continuous endomorphism of  $S$  onto  $B$ . The closure  $\bar{M}$  of  $M$  in  $S$  is a disjoint union of  $M$  and an abelian compact subgroup  $A$  of  $B$  in which the continuous homomorphic image  $Me$  of  $M$  is dense.*

(c) *The structure of  $B$ :  $B$  is the product of the compact abelian subgroup  $A$  and the normal subgroup  $Ce$  which is isomorphic to  $C/N$ , where  $N$  is the set of all elements  $c$  in  $C$  for which  $ce = e$ .*

(d) *Global connectivity: The elements  $1$  and  $e$  lie in some connected subspace if and only if  $M$  is connected.*

**THEOREM III.** *Let  $S$  be a locally compact group with compact boundary in which the elements  $1$  and  $e$  lie in some connected subspace.*

(a) *The structure of  $G$ : There is a characteristic maximal compact subgroup  $C$  and a central one-parameter subgroup  $M$  isomorphic to the additive reals and  $G$  is the direct product  $MC$ .*

(b) *The connection between  $G$  and  $B$ :  $\bar{M}$  is the union of  $M$  and a connected compact abelian group  $A$  containing the dense one-parameter subgroup  $Me$ ; all elements of  $\bar{M}$  commute with all elements of  $S$ .*

(c) *The structure of  $B$ :  $B$  is the product of the compact connected central subgroup  $A$  and the normal compact subgroup  $Ce$ . The direct product  $A \times (Ce)$  contains a central subgroup  $L$  isomorphic to the intersection  $A \cap (Ce)$ , and  $B$  is isomorphic to  $(A \times (Ce))/L$ . The group  $Ce$  is isomorphic to  $C/N$  with the set  $N$  of all  $c \in C$  satisfying  $ce = e$ .*

(d) *Global connectivity: Let  $K$  be the set of all  $s$  in  $S$  such that  $se = e$ . Then the following statements are equivalent:*

(i) *There exists an arc joining  $1$  and  $e$ ;*

- (ii)  $K$  is a locally compact connected group with zero;
- (iii)  $K$  is not compact;
- (iv) There is a central subsemigroup  $M_0$  isomorphic to the non-negative reals with a zero.

(e) *Generic semigroups for  $S$* : Let  $M'_0$  be the multiplicative semigroup of all non-negative reals and let  $f$  be an isomorphism of  $M' = M'_0 \setminus \{0\}$  onto  $M$  such that  $f(1/2^n)$  has cluster points in  $B$  as  $n$  tends to infinity (this means fixing an orientation for  $f$  in one of two possibilities).

Let  $S^{(1)} = M'_0 \times C \times A$  and let  $S^{(2)}$  be the closure in  $S^{(1)}$  of the set of all elements  $(r, c, f(re))$ ,  $(r, c) \in M \times C$ . Then  $S^{(2)}$  is a locally compact group with compact boundary whose maximal subgroup is isomorphic to  $G$  and whose minimal ideal is isomorphic to  $A \times C$ . The subgroup  $\{0\} \times C \times A$  contains the normal subgroup  $L^{(2)}$  of all elements  $(0, c, a)$  such that  $(ce)^{-1} = a$  in  $B$ . If  $\rho$  is the congruence relation on  $S^{(1)}$  whose cosets are one point sets on  $M' \times C \times A$ ,  $r \in M'$ , and are the cosets modulo  $L^{(2)}$  on  $\{0\} \times C \times A$ , then  $S^{(3)} = S^{(2)}/\rho$  is algebraically and topologically isomorphic to  $S$ . Conversely, if the following data are given:  $A$  a compact group,  $C$  with a normal subgroup  $N$ , an abelian compact group  $A$  with a dense one-parameter group, an isomorphism of a central subgroup  $L$  of  $C/N$  into  $A$ , then there exists a locally compact group with compact boundary, whose maximal subgroup is isomorphic to  $M' \times C$  and whose kernel is isomorphic to a factor group of the direct product  $A \times (C/N)$  modulo a central subgroup isomorphic to  $L$ .

**THEOREM IV.** *Let  $S$  be a locally compact group with compact boundary in which there exists an arc joining 1 and  $e$ .*

*Structure of  $S$* : There exists a maximal compact subgroup  $C$  containing 1 with a closed normal subgroup  $N$ . Let  $M'_0$  be the multiplicative semigroup of non-negative reals and the congruence relation on  $M'_0 \times C$  which identifies all coset modulo  $\{0\} \times N$  on  $\{0\} \times C$ . Then  $S$  is isomorphic to  $(M'_0 \times C)/\rho$ .

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