

LIMITING DISTRIBUTIONS FOR CRITICAL MULTITYPE BRANCHING PROCESSES WITH DISCRETE TIME⁽¹⁾

BY

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1. Introduction. We shall study a certain class of discrete branching processes with a finite or infinite number of types—a generalized branching process. We shall take as the mathematical representation of such a process a given analytic mapping of the unit sphere of a complex Banach space—a generalized probability-generating function. For a thorough probability-theoretic discussion of general branching processes we refer to the works of Harris [10; 11] and Moyal [17].

A generalized branching process is a generalized, temporally homogeneous, Markov process Z_n , $n = 0, 1, \dots$, where the range of the Z_n is a set of nonnegative integer-valued set functions, and Z_0 is a given nonrandom function. An important classification of branching processes can be made in terms of the spectral radius $\sigma(M)$ of the expectation operator M for one generation (assumed to be a bounded operator), according to whether $\sigma(M) < 1$, $\sigma(M) = 1$, or $\sigma(M) > 1$. For a large class of processes, $\sigma(M)$ is in the point spectrum of M with multiplicity 1, and the expected value of Z_n behaves like $[\sigma(M)]^n$ as $n \rightarrow \infty$. Except when $\sigma(M) = 1$, limiting properties of Z_n have been extensively studied [3; 7; 10; 14; 24].

When $\sigma(M) = 1$ for a process, it is said to be critical. It is known that except in certain special cases, such a process dies with probability 1 after a finite number of generations. Even so, the expected lifetime is infinite, and it is interesting to investigate population size at the n th generation, given that it is not zero.

Population size has been determined for critical discrete processes of one type by Yaglom [24] and for continuous time processes of a finite number of types by Sevast'yanov [23] and Čistyakov [2]. It has been shown that the population (if not empty) of the n th generation, or at time t , tends to contain n , or t , particles which are distributed among the various types in a definite ratio. Random fluctuations from this expected behavior are measured by certain conditional probability distributions defined for the n th generation. Each of these distributions has a limit of exponential type.

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The purpose of this paper is to extend these results to a class of branching processes for discrete time with a finite or an infinite number of types. The n th generation is described by the n th iterate of the analytic mapping alluded to above. With enough assumptions about this mapping, the results we seek follow from an investigation of the iterates of this mapping near a fixed point.

In §2 we define the branching processes we shall study and discuss the notation to be used. Properties of generating functions are extended in §3 to generating functionals. In §4 we study the iterates of the analytic mapping that defines a critical process. In §5 we give probabilistic meaning to the results of §4. Finally, in the last section we discuss two applications which illustrate why the class of processes studied is neither void nor uninteresting. One of these applications is concerned with processes having k types of particles, and the other with the study of the neutron population in an idealized model of a spherical reactor, where the type of a neutron is the spherical shell in which it was produced by fission.

2. Critical, positively regular, branching processes. It is relatively easy to define and analyze a branching process when there are only a finite number of types of particles. In generalizing to processes with an infinite number of types, we select a Banach space and take a particular type of mapping of the unit sphere as the natural extension of the concept of a generating function. We avoid the delicate questions concerned with the derivation of this function from a given set of probabilities [10; 11; 17].

To obtain results analogous to those for a finite number of types, we shall assume compactness and positivity of various mappings. For any specific branching process it may well be that the Banach space we choose here is not the natural one in which to prove compactness. At least we shall attain a certain degree of generality and give a method of proof which can perhaps be modified for other Banach spaces.

Let X denote an abstract set—the set of types—and let Σ denote a σ -field of subsets of X . We designate by $C(X, \Sigma)$ the complex Banach space which is the closure of the linear space of finite combinations of characteristic functions (indicator functions) of sets in Σ with the norm

$$(2.1) \quad \|s\| = \sup_{x \in X} |s(x)|.$$

This is just the space of all bounded Σ -measurable functions.

The conjugate space to $C(X, \Sigma)$ is characterized by the bounded and finitely additive set functions defined on Σ (p.258, Chapter III of [6]). That is, for each continuous linear functional t^* on $C(X, \Sigma)$ there is a unique bounded, finitely additive, set function μ such that t^* evaluated on the function s , written $t^*(s)$, is given by

$$(2.2) \quad t^*(s) = \int_X s(x) d\mu(x).$$

We refer to Chapter III of [6] for a development of the appropriate integration theory.

We denote by $R(X, \Sigma)$ the real Banach space consisting of real-valued functions in $C(X, \Sigma)$. This is a Banach lattice under the obvious partial ordering,

$$(2.3) \quad s \geq t \text{ if and only if } s(x) \geq t(x) \text{ for all } x \text{ in } X.$$

We adopt the following notation:

$$(2.4) \quad s > 0, \quad \text{if } s \geq 0 \quad \text{and } s \neq 0,$$

and

$$(2.5) \quad s \gg 0, \quad \text{if } s(x) \geq \varepsilon > 0 \quad \text{for all } x \text{ in } X.$$

The positive cone P in $R(X, \Sigma)$ consists of those functions which are nonnegative. The dual cone P^* in $R^*(X, \Sigma)$ is characterized by nonnegative and finitely additive set functions defined on Σ . We adopt in P^* the following notation:

$$(2.6) \quad t^* \geq 0 \text{ if } t^*(p) \geq 0 \text{ for all } p \text{ in } P,$$

$$(2.7) \quad t^* > 0 \text{ if } t^* \geq 0 \text{ and } t^*(p) > 0 \text{ for some } p \text{ in } P,$$

and

$$(2.8) \quad t^* \gg 0 \text{ if } t^*(p) > 0 \text{ for all } p \text{ in } P, p \neq 0.$$

A bounded linear operator T on $R(X, \Sigma)$ to $R(X, \Sigma)$ is said to be *positive*, and is written as

$$(2.9) \quad T > 0, \text{ if } Tp \geq 0 \text{ for all } p \text{ in } P.$$

It is *strictly positive*, written

$$(2.10) \quad T \gg 0, \text{ if } Tp \gg 0 \text{ for all } p \text{ in } P, p \neq 0.$$

Some of the theory of positive operators to which we shall appeal depends upon the fact that P has an interior, namely, those functions satisfying $p \gg 0$, and the fact that, by the Jordan Decomposition Theorem [6, p. 98], P^* spans $R^*(X, \Sigma)$.

The open unit sphere in $C(X, \Sigma)$ is defined by

$$(2.11) \quad S = \{s \mid s \text{ is in } C(X, \Sigma) \text{ and } \|s\| < 1\}.$$

For a function s , we define the function $|s|$ in the obvious way:

$$(2.12) \quad |s|(x) = |s(x)|, \text{ all } x \in X.$$

The absolute value $|s|$ of a function should not be confused with the norm $\|s\|$.

There is also the possibility of confusion of the unit scalar 1 with the constant function 1 in $C(X, \Sigma)$. To avoid this, we reserve the symbol $\mathbf{1}$ for the function

$$(2.13) \quad \mathbf{1}(x) = 1, \text{ for all } x \text{ in } X.$$

There is little possibility of confusing the zero function with the scalar 0, so we use the same symbol for both.

We consider now a mapping F of the unit sphere S into $\mathcal{S}$. We require that the following conditions be satisfied:

- (a) F maps Σ -measurable functions into Σ -measurable functions,
 (2.14) (b) $\lim_{s \rightarrow 1} F(s) = 1, \quad \|s\| < 1,$
 (c) F is analytic in S , and all variations computed at $s = 0$ satisfy

$$\delta^n F(0; p_1, \dots, p_n) \geq 0, \text{ all } p_i \in P, \quad n = 0, 1, \dots.$$

We refer to Hille and Phillips [13] for a treatment of analytic functions defined on a Banach space. The assumption of analyticity implies a power series expansion

$$(2.15) \quad F(s) = \sum_{n=0}^{\infty} H_n(s), \quad H_n(s) = \frac{\delta^n F(0; s)}{n!}, \quad \|s\| < 1.$$

Here $\delta^n F(0; s)$ is a bounded, homogeneous polynomial of degree n in s , defined recursively by $H_0(s) = F(0)$ and

$$(2.16) \quad \delta^{n+1} F(0; s) = \lim_{\lambda \rightarrow 0} \frac{\delta^n F(\lambda s; s) - \delta^n F(0; s)}{\lambda}, \quad \lambda \text{ a complex scalar.}$$

A continuous, symmetric, n -linear form is defined by

$$(2.17) \quad H_n(h_1, \dots, h_n) \equiv \frac{\delta^n F(0; h_1, \dots, h_n)}{n!}.$$

This means that H_n is linear in each term, is unchanged by permutations of the h_i , and for some constant K_n ,

$$(2.18) \quad \|H_n(h_1, \dots, h_n)\| \leq K_n \prod_{i=1}^n \|h_i\|.$$

If $h_1 = \dots = h_k = s$ and $h_{k+1} = \dots = h_n = t$, we shall often abbreviate with the notation

$$(2.19) \quad H_n(s^k, t^{n-k}) \equiv H_n(\overbrace{s, \dots, s}^k, \overbrace{t, \dots, t}^{n-k}).$$

The series (2.15) can be differentiated term-wise to give

$$(2.20) \quad \delta^k F(s; h) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} H_n(s^{n-k}, h^k) \text{ for } \|s\| < 1.$$

It should be remarked that $F(s)$ and $H_n(s)$ are functions in $C(X, \Sigma)$. We denote the value at the point x in X of $F(s)$, for example, by

$$(2.21) \quad F(x, s) \equiv F(s)(x).$$

We consider F as defining a generalized branching process for particles with types specified by points in X . This is to be a temporally homogeneous Markov process Z_n , $n = 0, 1, \dots$, where the range of the Z_n is a set L of functionals in $C^*(X, \Sigma)$ defined by additive, nonnegative and integer valued set functions on Σ . The initial state Z_0 is a nonrandom functional which is a finite sum of positive integer multiples of functionals e_x defined by

$$(2.22) \quad e_x(s) = s(x).$$

The functional $F(x, s)$ defined on S for fixed x in X is the *generating functional* for the first generation of progeny of a particle of type x .

To clarify the concept of a generating functional we take

$$s = \sum_{i=1}^k \lambda_i \chi_{A_i},$$

where λ_i are complex numbers, $|\lambda_i| < 1$, and χ_{A_i} are characteristic functions for disjoint sets A_i such that $\bigcup_{i=1}^k A_i = X$. Then

$$(2.23) \quad \begin{aligned} F(x, s) &= \sum_{n=0}^{\infty} H_n(x, s) \\ &= \sum_{n=0}^{\infty} P_n(\lambda_1, \dots, \lambda_k), \end{aligned}$$

where the P_n are homogeneous polynomials of degree n in $\lambda_1, \dots, \lambda_k$, with coefficients which depend only on the χ_{A_i} and which are easily seen by use of (2.14) to be nonnegative. Since for $\lambda_1 = \dots = \lambda_k = 1$, $s = 1$ and $F(x, 1) = 1$, $F(x, \sum_{i=1}^k \lambda_i \chi_{A_i})$ is a generating function which, we assume, describes the progeny of a particle of type x by distinguishing types of progeny only to the extent that they fall in the sets $A_1, \dots, A_k$.

It is further assumed that if $Z_n = \sum_{i=1}^k r_i e_{x_i}$ for positive integers $r_1, \dots, r_k$ and distinct points $x_1, \dots, x_k$, then Z_{n+1} is the sum of $r_1 + \dots + r_k$ independent random variables with generating functional

$$(2.24) \quad \prod_{i=1}^k [F(x_i, s)]^{r_i}.$$

With these assumptions it is natural that the generating functional for the n th generation of progeny of a particle of type x should be given by the n th iterate of the mapping F , i.e.,

$$(2.25) \quad F_n(x, s) = F(x, F_{n-1}(s)),$$

where $F_{n-1}(s)$ is of course a function.

With additional assumptions it will follow that F_n satisfies the conditions imposed on F so that it qualifies for defining a generating functional, but we shall not justify its defining the generating functional for Z_n .

The mapping F need not be analytic at $\mathbf{I}$; however we do require that F satisfy the condition that

$$(2.26) \quad \lim_{\lambda \uparrow 1} \delta^k F(\lambda \mathbf{I}, \mathbf{I}) \text{ exist for } k = 0, 1, 2, \text{ and } 3.$$

We show in the next section that this implies

$$(2.27) \quad F(s) = \mathbf{I} - M(\mathbf{I} - s) + K(\mathbf{I} - s) + O(\|\mathbf{I} - s\|^3), \quad \|s\| < 1.$$

The operator M , the *expectation operator*, is bounded, linear and nonnegative. The operator K is a nonnegative, continuous and homogeneous polynomial of degree 2 [13], i.e.,

$$(2.28) \quad \begin{aligned} K(\lambda h) &= \lambda^2 K(h), \\ K(h + \lambda k) &= P_0(h, k) + \lambda P_1(h, k) + \lambda^2 P_2(h, k). \end{aligned}$$

where λ is any complex number and P_0 , P_1 , and P_2 are bilinear operators independent of λ .

In infinite-dimensional Banach spaces the closed unit sphere S is not compact in the norm topology, and this vitiates an immediate extension of proofs of results in finite dimensions. For some spaces a weaker topology makes the unit sphere compact, but none is known for general $C(X, \Sigma)$. For this reason it will be necessary to make certain assumptions about the mapping M .

We first of all assume continuity of M on bounded monotone sequences. If $\{s_n\}$ is a sequence of functions in $R(X, \Sigma)$ and $s_{n+1} \geq s_n$, $\|s_n\| \leq A$, then, since Σ is a σ -field, s_n will converge pointwise to a bounded measurable function s in $R(X, \Sigma)$ [9]. We assume that

$$(2.29) \quad M(s)(x) = \lim_{n \rightarrow \infty} M(s_n(x)) \text{ for all } x \text{ in } X.$$

This is not an unnatural assumption since we are considering $F(x, \cdot)$ to be a generating functional. If $F(x, s)$ is derived from a set of probabilities on the range L of Z_1 in $C^*(X, \Sigma)$, then $M(s)(x)$ is an expectation, and (2.29) simply states the interchange of limit and expectation. It is perhaps possible to prove (2.29) from the assumptions (2.14), although we have been unable to do so.

The bounded linear transformation M is positive in that it maps the positive cone P into itself. It can be shown, using positivity of M and the fact, noted earlier, that P^* spans $R^*(X, \Sigma)$, that the spectral radius of M is determined by the least upper bound $\sigma(M)$ of the spectrum of M on the positive real axis [15]. Without further conditions on M it is not possible generally to determine additional properties of $\sigma(M)$. Therefore, we make two additional assumptions on M :

- (a) For some integer n_0 , M^{n_0} is a compact operator.
 (2.30)
 (b) For some integer n_1 , M^{n_1} is a strictly positive operator.

Here M^n denotes the n th iterate of the operator M .

These properties of M are sufficient for $\sigma(M)$ to be in the point spectrum of M with unit multiplicity [16, Theorem 6.3]. In addition, for $\sigma(M)$ eigenfunctions ϕ of M and ϕ^* of M^* can be chosen to be strictly positive, i.e., ϕ is uniformly positive and $\phi^*(p) > 0$ for all p in P , $p > 0$. The eigenvalue $\sigma(M)$ dominates the modulus of all other eigenvalues of M , and by the functional calculus [13],

$$(2.31) \quad M^n = [\sigma(M)]^n P + Q_n.$$

Here, if $\phi^*(\phi) = 1$, P is a projection given by

$$(2.32) \quad Ps = \phi^*(s)\phi, \quad P^*s^* = s^*(\phi)\phi^*,$$

and Q_n is an operator satisfying

$$(2.33) \quad \|Q_n\| \leq A\alpha^n \text{ for some fixed } A \geq 0 \text{ and } \alpha, 0 \leq \alpha < 1.$$

We shall say that the mapping F defines a *positively regular branching process*—a PR process—provided that F is *not* a linear mapping and that F and M satisfy the conditions (2.14), (2.26), (2.29) and (2.30).

Certain limiting properties of the process Z_n as $n \rightarrow \infty$ are determined by the sequence of mappings $\{F_n\}$. As is well known, the behavior of this sequence depends primarily upon whether $\sigma(M) < 1$, $\sigma(M) = 1$, or $\sigma(M) > 1$. Limiting distributions for various functions of Z_n have been rather thoroughly investigated when $\sigma(M) \neq 1$. The main results of this paper are given in §4, where we shall consider only the critical case $\sigma(M) = 1$ and extend to critical, positively regular, branching processes—a CPR process, a result known for discrete time processes of one type [24] and for continuous time processes of a finite number of types [2; 23].

3. Properties of generating functionals. In the next section we shall need various properties of generating functionals that are obvious extensions of properties of ordinary probability generating functions. We derive these in the following sequence of lemmas and theorems, the validity of which seems obvious even though some proofs are lengthy.

We first establish some properties of symmetric n -linear forms of which $\delta^n F(0; h_1, \dots, h_n)$ are examples.

LEMMA 1. *Let H_n be a symmetric n -linear form, continuous on the product space $\prod_{i=1}^n C(X, \Sigma)$ to $C(X, \Sigma)$. If H_n is nonnegative on nonnegative functions, then it satisfies*

$$(3.1) \quad |H_n(h_1, \dots, h_n)| \leq H_n(|h_1|, \dots, |h_n|) \leq \left(\prod_{i=1}^n \|h_i\| \right) H_n(1, \dots, 1)$$

and

$$(3.2) \quad H_n(h_1, \dots, h_n) - H(g_1, \dots, g_n) = \sum_{i=1}^n H_n(g_1, \dots, g_{i-1}, h_i - g_i, h_{i+1}, \dots, h_n).$$

Proof. Suppose first that each function h_i is a finite linear combination of m_i characteristic functions. Since H_n is linear in each argument, $H_n(h_1, \dots, h_n)$ can be viewed as an n th degree polynomial in the $\prod_{i=1}^n m_i$ coefficients of the characteristic functions. Since the coefficients of the polynomial are nonnegative functions such as $H_n(\chi_{A_1}, \dots, \chi_{A_n})$, (3.1) follows immediately for such functions h_i . These functions are dense in $C(X, \Sigma)$, and H_n and the norm are continuous functions so that (3.1) follows.

To prove (3.2), we observe by linearity that

$$(3.3) \quad H_n(h_1, \dots, h_n) = H(h_1 - g_1, h_2, \dots, h_n) + H(g_1, h_2, \dots, h_n).$$

Repeating this n times, we obtain (3.2) and the proof is complete.

Consequences of assumptions about the behavior of F near $\mathbf{1}$ are given in the following theorem.

THEOREM 1. *Let F satisfy (2.14) and suppose that in $C(X, \Sigma)$ the functions $\delta^k F(\lambda \mathbf{1}; \mathbf{1})$, $0 \leq \lambda < 1$, $k=0,1,2,3$, have limits as λ increases to 1. Then the functions $\delta^k F(s; h)$, $k=0,1,2$, for $h \in C(X, \Sigma)$ have limits as $s \rightarrow \mathbf{1}$, $\|s\| < 1$, and*

$$(3.4) \quad \lim_{s \rightarrow \mathbf{1}; \|s\| < 1} \delta^k F(s; h) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} H_n(\mathbf{1}^{n-k}, h^k).$$

If the limit of $\delta F(s; h)$ is denoted by $M(h)$ and that of $\delta^2 F(s; h)$ by $2K(h)$, then

$$(3.5) \quad F(s) = \mathbf{1} - M(\mathbf{1} - s) + K(\mathbf{1} - s) + O(\|\mathbf{1} - s\|^3), \quad \|s\| < 1,$$

where $O(\|\mathbf{1} - s\|^3)$ is in $C(X, \Sigma)$ and satisfies $\|O(\|\mathbf{1} - s\|^3)\| \leq A \|\mathbf{1} - s\|^3$ for a constant A independent of s , $\|s\| < 1$. (The operator M is nonnegative and linear with bound $\|M(\mathbf{1})\|$. The operator K is a nonnegative, continuous, and homogeneous polynomial of degree 2.)

Proof. The function $\delta^k F(\lambda \mathbf{1}; \mathbf{1})$, where λ is a complex scalar satisfying $|\lambda| < 1$, is given by (2.20) as

$$(3.6) \quad \delta^k F(\lambda \mathbf{1}; \mathbf{1}) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} H_n(\mathbf{1}) \lambda^{n-k}.$$

The coefficients in the power series are nonnegative functions, and an application of Abel's lemma shows that divergence of $\sum_{n=k}^{\infty} [n!/(n-k)!] H_n(x, \mathbf{1})$ for any $x \in X$ denies the existence of a limit for $\delta^k F(\lambda \mathbf{1}; \mathbf{1})$ as λ approaches 1 through real values. It follows readily that

$$(3.7) \quad \lim_{\lambda \rightarrow 1} \delta^k F(\lambda \mathbf{1}; \mathbf{1}) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} H_n(\mathbf{1}), \quad k=0,1,2,3.$$

We need also the fact that the sequence of partial sums

$$\{s_N^{(k)} = \sum_{n=k}^N [n!/(n-k)!] H_n(\mathbf{1})\}$$

converges uniformly as $N \rightarrow \infty$, since (3.7) guarantees only that it converges pointwise to a function $s^{(k)}$ in $C(X, \Sigma)$. We prove this for $k = 0, 1$, and 2 by observing that with the series representations of $s^{(k)}$ and $\delta^k F(\lambda \mathbf{1}; \mathbf{1})$, we have

$$\begin{aligned}
 \|s^{(k)} - s_N^{(k)}\| &\leq \|s^{(k)} - \delta^k F(\lambda \mathbf{1}; \mathbf{1})\| + \left\| \sum_{n=k}^N \frac{n!}{(n-k)!} H_n(\mathbf{1}) (1 - \lambda^{n-k}) \right\| \\
 &\quad + \left\| \sum_{n=N+1}^{\infty} \frac{n!}{(n-k)!} H_n(\mathbf{1}) \lambda^{n-k} \right\| \\
 (3.8) \quad &\leq \|s^{(k)} \delta^k F(\lambda \mathbf{1}; \mathbf{1})\| + (1 - \lambda) \left\| \sum_{n=k+1}^{\infty} \frac{n!}{(n-k-1)!} H_n(\mathbf{1}) \right\| \\
 &\quad + \left\| \sum_{n=N+1}^{\infty} \frac{n!}{(n-k)!} H_n(\mathbf{1}) \lambda^{n-k} \right\|.
 \end{aligned}$$

The coefficient of $(1 - \lambda)$ in this last expression is bounded because of (3.7) and the restriction of k to 0, 1, or 2. Now for any given $\varepsilon > 0$, we can choose $\lambda_0(\varepsilon)$ so that the first term on the right of (3.8) is less than $\varepsilon/3$ for $1 \geq \lambda > \lambda_0(\varepsilon)$ because of (3.7). The second term can be made less than $\varepsilon/3$ by choosing λ sufficiently close to 1. Having chosen a $\lambda_0 < 1$ satisfying these two conditions, we can make the last term less than $\varepsilon/3$ for all $N > N_0(\varepsilon, \lambda_0)$ by virtue of the convergence of the power series. Therefore $s_N^{(k)} \rightarrow s^{(k)}$ uniformly.

For arbitrary h in $C(X, \Sigma)$ and $k = 0, 1$, or 2, we have proved, by Lemma 1 and the uniform convergence, that

$$(3.9) \quad \left\| \sum_{n=N}^M \frac{n!}{(n-k)!} H_n(\mathbf{1}^{n-k}, h^k) \right\| \leq \|h\|^k \left\| \sum_{n=N}^M \frac{n!}{(n-k)!} H_n(\mathbf{1}) \right\| \rightarrow 0 \text{ as } N, M \rightarrow \infty.$$

Therefore, the partial sums $\sum_{n=k}^N n!/(n-k)! H_n(\mathbf{1}^{n-k}, h^k)$ form a Cauchy sequence in $C(X, \Sigma)$ and the series converges.

Finally, by the series representation (2.15) of $\delta^k F$ for $\|s\| < 1$ and by the inequalities in Lemma 1, we have

$$\begin{aligned}
 (3.10) \quad &\left\| \delta^k F(s; h) - \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} H_n(\mathbf{1}^{n-k}, h^k) \right\| \\
 &\leq \|1-s\| \|h\|^k \left\| \sum_{n=k+1}^{\infty} \frac{n!}{(n-k-1)!} H_n(\mathbf{1}) \right\| \text{ for } k = 0, 1, 2.
 \end{aligned}$$

This proves (3.4).

To prove (3.5) we observe that, by analyticity of F , the function defined on $0 \leq \lambda < 1$ by $F(s + \lambda(1-s))$, $\|s\| < 1$, is twice continuously differentiable, and by definition of the variations of F ,

$$(3.11) \quad \frac{d^k}{d\lambda^k} F(s + \lambda(1-s)) \equiv \delta^k F(s; 1-s), \quad k = 1, 2.$$

With the standard notion of the Riemann integral of vector valued functions [13], the result (3.4) proved above, and the definition of the operators M and K ,

$$\begin{aligned}
 & M(\mathbf{1} - s) - \delta F(s + \mu(\mathbf{1} - s); \mathbf{1} - s) \\
 (3.12) \quad &= \int_{\mu}^1 \delta^2 F(s + \lambda(\mathbf{1} - s); \mathbf{1} - s) d\lambda \\
 &= (1 - \mu)2K(\mathbf{1} - s) + O(\|\mathbf{1} - s\|^3),
 \end{aligned}$$

where $O(\|\mathbf{1} - s\|^3)$ denotes a function in $C(X, \Sigma)$ such that

$$(3.13) \quad \frac{\|O(\|\mathbf{1} - s\|^3)\|}{\|\mathbf{1} - s\|^3} \leq A < +\infty, \quad A \text{ independent of } s, \quad \|s\| < 1.$$

By (3.10) with $k = 2$ and $h = \mathbf{1} - s$, the last statement in (3.12) is possible; a possible constant A in (3.13) is simply $\|\sum_{n=3}^{\infty} n!/(n-3)! H_n(\mathbf{1})\|$. If we integrate (3.12) from $\mu = 0$ to $\mu = 1$, we obtain (3.5) and thus complete the proof.

It is an immediate consequence of Lemma 1 that F satisfying (2.14) is a *monotone* mapping on positive functions. That is,

$$(3.14) \quad 0 \leq s \leq t \text{ implies } 0 \leq F(s) \leq F(t).$$

Many of the results that follow depend upon other inequalities which are extensions of properties of generating functions.

THEOREM 2. *Let F satisfy the conditions of Theorem 1. Then for $\|s\| \leq 1$*

$$(3.15) \quad |F(s)| \leq F(|s|)$$

and

$$(3.16) \quad \mathbf{1} - F(|s|) \geq M(\mathbf{1} - |s|) - K(\mathbf{1} - |s|).$$

For $\|s\| < 1$ and $\|t\| \leq 1$,

$$(3.17) \quad |F(t) - F(s)| \leq M(|t - s|),$$

and if $|s| \ll |t|$, equality is possible in (3.17) only if F is linear.

Proof. Directly from Lemma 1 and the series representation (2.15) of F , we get (3.15).

For real-valued s , $0 \leq s \leq \mathbf{1}$, the function $\delta^2 F(s + \lambda(\mathbf{1} - s); \mathbf{1} - s)$, $0 \leq \lambda \leq 1$, is nondecreasing with increasing λ because

$$\begin{aligned}
 & \delta^2 F(s + \lambda(\mathbf{1} - s); \mathbf{1} - s) \\
 (3.18) \quad &= \sum_{n=2}^{\infty} n(n-1) H_n([s + \lambda(\mathbf{1} - s)]^{n-2}, [\mathbf{1} - s]^2),
 \end{aligned}$$

and $H_n([s + \lambda(\mathbf{1} - s)]^{n-2}, [\mathbf{1} - s]^2)$ is a polynomial in λ of degree $n - 2$ with nonnegative coefficients. Therefore from (3.12) we obtain

$$(3.19) \quad M(1 - |s|) - \delta F(|s| + \mu(1 - |s|); 1 - |s|) \leq (1 - \mu)2K(1 - |s|),$$

and an additional integration on μ from 0 to 1 gives (3.16).

The results of Lemma 1 give

$$\begin{aligned}
 (3.20) \quad |F(t) - F(s)| &\leq \sum_{n=0}^{\infty} |H_n(t) - H_n(s)| \\
 &\leq \sum_{n=1}^{\infty} \sum_{i=1}^n H_n(|t|^{n-i}, |s|^{i-1}, |t-s|) \\
 &\leq \sum_{n=1}^{\infty} \sum_{i=1}^n \|s\|^{i-1} \|t\|^{n-i} H_n(\mathbf{1}^{n-1}, |t-s|) \\
 &\leq \sum_{n=1}^{\infty} n H_n(\mathbf{1}^{n-1}, |t-s|) = M(|t-s|).
 \end{aligned}$$

If $|s| < |t|$, then equality in (3.20) implies

$$(3.21) \quad 0 = H_n(\mathbf{1}^{n-1}, |t-s|) \geq H_n(\mathbf{1}^{n-1}, |t| - |s|) \geq 0, \text{ for } n \geq 2,$$

since factors appear in (3.20) which are powers of $\|s\| < 1$. But since $|t| - |s| \geq 0$ this implies that $H_n(\mathbf{1}^{n-1}, h) = 0$, $n \geq 2$, for all h in $C(X, \Sigma)$, and hence that F is linear. This completes the proof.

In the investigation of iterates of generating functionals which define positively regular processes we shall need the following result.

THEOREM 3. *Let F define a PR process. Then*

$$(3.22) \quad \|F(s)\| < 1 \text{ for } \|s\| < 1.$$

Iterates F_n of F are then analytic on S , continuous on $\bar{S}$, and satisfy conditions imposed on F . For $n \geq n_1$ of (2.30) we have

$$(3.23) \quad \|F_n(s)\| < 1 \text{ for } \|s\| \leq 1, \quad s \neq \mathbf{1}.$$

Proof. By Theorem 2, the monotonicity of F , and the quadratic nature of K , we have for $0 \leq \lambda \leq 1$

$$\begin{aligned}
 (3.24) \quad 1 - |F(s)| &\geq 1 - F(|s|) \geq 1 - F(|s| + \lambda(1 - |s|)) \\
 &\geq (1 - \lambda)[M(1 - |s|) - (1 - \lambda)K(1 - |s|)].
 \end{aligned}$$

Now if $1 - |s| \geq 0$ there is an $\alpha > 0$ such that $1 - |s| \geq \alpha \phi \geq 0$, $M\phi = \sigma(M)\phi$, by assumption (2.30). Since $K(1 - |s|)$ is bounded, there clearly exists a $\lambda < 1$ such that

$$(3.25) \quad 1 - |F(s)| \geq (1 - \lambda)[\alpha\sigma(M)\phi - (1 - \lambda)K(1 - |s|)] \geq 0.$$

This proves (3.22).

The mapping F takes the open sphere S into itself and hence into the domain of analyticity of F . It follows readily that

$$(3.26) \quad \begin{aligned} F_1(s) &= F(s), \\ F_{n+1}(s) &= F(F_n(s)) \end{aligned}$$

defines a sequence of analytic mappings of S satisfying the assumptions made on F . In particular, we have the expectation operator M_n given by

$$(3.27) \quad M_n(h) = M^n(h).$$

The mappings F_n are analytic on S , and by Theorem 1 and (3.15) of Theorem 2, continuous on the closed sphere $\bar{S}$. If, for F_n , we denote the limit of $\delta^2 F_n(s; h)$ at $s=1$ by $K_n(h)$, then as in (3.24) we have

$$(3.28) \quad 1 - |F_n(s)| \geq (1 - \lambda) [M^n(1 - |s|) - (1 - \lambda) K_n(1 - |s|)].$$

By (2.30), $M^n(1 - |s|) \gg 0$ if $n \geq n_1$, and hence there exists a $\lambda < 1$ such that

$$(3.29) \quad 1 - |F_n(s)| \gg 0 \text{ for } \|s\| \leq 1, \quad s \neq 1.$$

This completes the proof.

4. Iterates of generating functionals. We shall now prove several results about iterates of the mapping F of S into S , where F satisfies the conditions required to generate a CPR process as defined in §2. One of these results is known and others are motivated by the treatment of branching processes of one type described in Harris [11], where techniques of Fatou [8] are used.

Throughout this section we reserve the symbols ϕ and ϕ^* to denote strictly positive eigenfunctions of the operators M and M^* respectively. Normalization of ϕ and ϕ^* will be specified subject to the condition that $\phi^*(\phi) = 1$.

For completeness we now prove a wellknown result for branching processes [7; 11; 21; 22].

THEOREM 4. *For a CPR process,*

$$(4.1) \quad \lim_{n \rightarrow \infty} F_n(s) = 1, \text{ uniformly in } s, \quad \|s\| \leq 1.$$

Proof. We first show that there is no solution to

$$(4.2) \quad s = F(s), \quad \|s\| < 1.$$

For if such a solution exists, then by Theorem 2 and nonlinearity of F ,

$$(4.3) \quad |1 - s| = |1 - F(s)| < M(|1 - s|).$$

From (4.3) and the positivity of ϕ^* , we have

$$(4.4) \quad \phi^*[M(|1 - s|) - |1 - s|] > 0.$$

But in fact equality must hold in (4.4) since $M^*\phi^* = \phi^*$, and we are led to a contradiction.

In particular, we note that

$$(4.5) \quad F(0) > 0,$$

and since F is monotone, the sequence $\{F_n(0)\}$ is monotone in the positive cone in S . The functions $F_n(0)$ are bounded by 1, and so they will converge pointwise on X to a unique nonnegative function p , $p \neq 0$, in $C(X, \Sigma)$. The monotonicity of F and the inequality $p \geq F_n(0)$ gives $F(p) \geq p$. By this and (3.17) of Theorem 2, iterated n_0 times, we have

$$(4.6) \quad p - F_{n+n_0}(0) \leq M^{n_0}(p - F_n(0)).$$

If n_0 is defined by (2.30), M^{n_0} is a compact operator, and a subsequence of $\{M^{n_0}(p - F_n(0))\}$ converges uniformly to a limit q . We have assumed in (2.29) that M , and hence M^{n_0} , is continuous on monotone sequences. Since $p - F_n(0)$ converges pointwise to 0, so does $M^{n_0}(p - F_n(0))$, and therefore the limit function q must be 0 for any uniformly convergent subsequence.

This shows that $F_n(0)$ converges uniformly to p , and since $F_{n+1}(0) = F(F_n(0))$, the continuity of F gives

$$(4.7) \quad F(p) = p, \quad 0 < p \leq 1.$$

We have already shown that p cannot satisfy (4.7) and $p \leq 1$. It follows readily from

$$(4.8) \quad 1 - p \leq M(1 - p)$$

that equality must hold and that $1 - p = \alpha\phi$, $\alpha \geq 0$. But ϕ is uniformly positive and $p(x) = 1$ for some x in X so that $\alpha = 0$ and $p = 1$.

By Lemma 1, we have the inequality

$$(4.9) \quad |F(s) - F(0)| \leq \sum_{n=1}^{\infty} H_n(|s|) \leq F(|s|) - F(0) \leq 1 - F(0).$$

A similar result holds for each F_n , so that

$$(4.10) \quad |1 - F_n(s)| \leq |1 - F_n(0)| + |F_n(s) - F_n(0)| \leq 2|1 - F_n(0)|.$$

This proves uniformity of convergence in s and completes the proof.

We now show that the sequence $F_n(0)$ tends to 1 along a given direction.

THEOREM 5. *For a CPR process*

$$(4.11) \quad \phi^*(1 - F_n(0)) > 0 \text{ for } n \geq 0,$$

and

$$(4.12) \quad \lim_{n \rightarrow \infty} \frac{1 - F_n(0)}{\phi^*(1 - F_n(0))} = \phi, \quad \phi^*(\phi) = 1.$$

Proof. For brevity we define Δ_n by

$$(4.13) \quad \Delta_n \equiv 1 - F_n(0).$$

By monotonicity of F and (3.22) of Theorem 3, we have

$$(4.14) \quad 0 \ll \Delta_{n+1} \leq \Delta_n,$$

and hence (4.11).

With $s = F_n(0)$ in (3.16) we have, by (4.14) and the homogeneous quadratic nature of K ,

$$(4.15) \quad \frac{\Delta_n}{\|\Delta_n\|} \geq \frac{\Delta_{n+1}}{\|\Delta_n\|} \geq M\left(\frac{\Delta_n}{\|\Delta_n\|}\right) - \|\Delta_n\| K\left(\frac{\Delta_n}{\|\Delta_n\|}\right).$$

By assumption (2.30) M^{n_0} is a compact operator for some integer $n_0 \geq 1$. Since M^{n_0} is also a nonnegative operator, we can apply it to (4.15) to obtain a sequence of nonnegative functions $\{p_n\}$,

$$(4.16) \quad p_n \equiv M^{n_0} \left(\frac{\Delta_n}{\|\Delta_n\|} \right),$$

which satisfy (by (4.15))

$$(4.17) \quad M p_n \leq p_n + \|\Delta_n\| M^{n_0} \left[K \left(\frac{\Delta_n}{\|\Delta_n\|} \right) \right]$$

and which lie in a compact subset of S .

Every subsequence of $\{p_n\}$ has a convergent subsequence with a limit p which satisfies

$$(4.18) \quad M p \leq p, \quad \|p\| = 1,$$

since in (4.17) $\|\Delta_n\| \rightarrow 0$, by Theorem 4, and

$$\|M^{n_0} [K(\frac{\Delta_n}{\|\Delta_n\|})]\|$$

is $\|M\|^{n_0} \|K(\mathbf{1})\|$. As shown several times before, equality must hold in (4.18) and therefore

$$(4.19) \quad \lim_{n \rightarrow \infty} M^{n_0} \left(\frac{\Delta_n}{\|\Delta_n\|} \right) = \phi, \quad \|\phi\| = 1, \quad M\phi = \phi.$$

Now by (3.5)

$$(4.20) \quad \Delta_{n+1} = M(\Delta_n) - K(\Delta_n) + O(\|\Delta_n\|^3),$$

or by iteration

$$(4.21) \quad \Delta_{n+n_0} = M^{n_0}(\Delta_n) - \sum_{i=0}^{n_0-1} M^{n_0-i-1} [K(\Delta_{n+i}) + O(\|\Delta_{n+i}\|^3)].$$

Dividing by $\|\Delta_n\|$, we get from (4.14), (4.19), and the quadratic nature of K ,

$$(4.22) \quad \lim_{n \rightarrow \infty} \frac{\Delta_{n+n_0}}{\|\Delta_n\|} = \phi, \quad \|\phi\| = 1.$$

Therefore

$$(4.23) \quad \lim_{n \rightarrow \infty} \frac{\|\Delta_{n+n_0}\|}{\|\Delta_n\|} = 1,$$

and finally

$$(4.24) \quad \lim_{n \rightarrow \infty} \frac{\Delta_n}{\|\Delta_n\|} = \phi, \quad \|\phi\| = 1, \quad M\phi = \phi,$$

as well as

$$(4.25) \quad \lim_{n \rightarrow \infty} \frac{\Delta_n}{\phi^*(\Delta_n)} = \phi, \quad \phi^*(\phi) = 1.$$

This completes the proof.

We are now in a position to prove a more refined result. By applying the linear functional ϕ^* , we reduce the function sequence $\{F_n(s)\}$ to a scalar sequence to which the techniques of Fatou can be applied [11].

THEOREM 6. *For a CPR process,*

$$(4.26) \quad \lim_{n \rightarrow \infty} n(1 - F_n(s)) = \frac{1}{\phi^*[K(\phi)]} \phi$$

for $\|s\| < 1$, $\phi^*(\phi) = 1$.

Proof. Recalling (4.13), we set $s = F_n(0)$ in (3.5) and apply the linear functional ϕ^* to obtain

$$(4.27) \quad \phi^*(\Delta_{n+1}) = \phi^*(\Delta_n) - \phi^*(K(\Delta_n) + O(\|\Delta_n\|^3)).$$

Since K is quadratic this gives

$$(4.28) \quad \lim_{n \rightarrow \infty} \frac{\phi^*(\Delta_{n+1})}{\phi^*(\Delta_n)} = 1.$$

Also by Theorem 5, the quadratic nature of K , and (4.27), we have

$$(4.29) \quad \lim_{n \rightarrow \infty} \frac{1}{\phi^*(\Delta_n)} \left[1 - \frac{\phi^*(\Delta_{n+1})}{\phi^*(\Delta_n)} \right] = \phi^*[K(\phi)].$$

We consider the identity

$$(4.30) \quad \frac{1}{\phi^*(\Delta_{n+1})} \equiv \frac{1}{\phi^*(\Delta_n)} + \frac{1}{\phi^*(\Delta_n)} \left[1 - \frac{\phi^*(\Delta_{n+1})}{\phi^*(\Delta_n)} \right] \frac{\phi^*(\Delta_n)}{\phi^*(\Delta_{n+1})}.$$

For brevity, we denote the last term in (4.30) by A_n and observe from (4.28) and (4.29) that

$$(4.31) \quad \lim_{n \rightarrow \infty} A_n = \phi^*[K(\phi)].$$

Adding (4.30) for values of n from 1 to $N-1$ and dividing by N , we obtain

$$(4.32) \quad \frac{1}{N\phi^*(\Delta_N)} = \frac{1}{N\phi^*(\Delta_1)} + \frac{1}{N} \sum_{n=1}^{N-1} A_n.$$

In view of (4.31), we have

$$(4.33) \quad \lim_{N \rightarrow \infty} \frac{1}{N\phi^*(\Delta_N)} = \phi^*[K(\phi)].$$

Combining this result with (4.12) we find that

$$(4.34) \quad \lim_{n \rightarrow \infty} n\Delta_n = \lim_{n \rightarrow \infty} \frac{\Delta_n}{\phi^*(\Delta_n)} \lim_{n \rightarrow \infty} n\phi^*(\Delta_n) = \frac{1}{\phi^*[K(\phi)]} \phi,$$

which proves our result for $s = 0$.

Next consider real-valued functions s , $0 < s < 1$. Since $\lim_{n \rightarrow \infty} F_n(0) = 1$, there is an integer m for which

$$(4.35) \quad F_m(0) \geq s.$$

Then

$$(4.36) \quad 1 - F_{n+m}(0) \leq 1 - F_n(s) \leq 1 - F_n(0)$$

proves (4.26) for positive real-valued s .

Finally, for any s , $\|s\| < 1$.

$$(4.37) \quad \begin{aligned} \left| \frac{\phi}{\phi^*[K(\phi)]} - n(1 - F_n(s)) \right| &\leq \left| \frac{\phi}{\phi^*[K(\phi)]} - n\Delta_n \right| + n \left| F_n(0) - F_n(s) \right| \\ &\leq \left| \frac{\phi}{\phi^*[K(\phi)]} - n\Delta_n \right| + n \left| F_n(|s|) - F_n(0) \right| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

where we have used the inequality $|F_n(s) - F_n(0)| \leq F_n(|s|) - F_n(0)$ discussed in (4.9) and (4.10). This completes the proof.

We have now shown that the sequence of analytic mappings $n(1 - F_n)$ tends to a constant in the limit. In the next section we shall give probabilistic interpretations to this result. We shall also determine certain limiting distributions by Laplace transform techniques.

In preparation for this, we let p be a nonnegative function in $C(X, \Sigma)$ and we let A_i , $i = 1, \dots, k$, be disjoint sets in Σ such that $\bigcup_{i=1}^k A_i = X$. We denote by p_i the function $p \chi_{A_i}$, i.e.,

$$(4.38) \quad p_i(x) = \begin{cases} p(x) & \text{for } x \text{ in } A_i, \\ 0 & \text{otherwise.} \end{cases}$$

A sequence $\{s_n\}$ is defined in $\mathcal{S}$ for complex scalars λ_i by

$$(4.39) \quad s_n = \exp \left[-\frac{1}{n} \sum_{i=1}^k \lambda_i p_i \right], \quad n \geq 1, \quad \operatorname{Re}(\lambda_i) > 0.$$

To avoid cumbersome notation in the following theorem, we shall often denote the vector $(\lambda_1, \dots, \lambda_k)$ by λ and the function $\sum_{i=1}^k \lambda_i p_i$ by λp .

THEOREM 7. For a CPR process, the function G_n defined by

$$(4.40) \quad G_n(p, \lambda) = n(1 - F_n(s_n))$$

is for $n \geq n_1$ of (2.30), an analytic function of the k complex variables, $\lambda_1, \dots, \lambda_k$, in the domain $\operatorname{Re}(\lambda_i) > 0$. Also

$$(4.41) \quad \lim_{n \rightarrow \infty} G_n(p, \lambda) = \frac{\sum_{i=1}^k \lambda_i \phi^*(p_i)}{1 + \phi^*[K(\phi)] \sum_{i=1}^k \lambda_i \phi^*(p_i)} \phi; \quad \phi^*(\phi) = 1$$

uniformly in any compact subset of $\operatorname{Re}(\lambda_i) > 0$.

Proof. Since (4.41) is trivial for $p = 0$, we consider $p > 0$. We select a bounded λ -domain Λ by requiring

$$(4.42) \quad 0 < \varepsilon < \operatorname{Re}(\lambda_i) \text{ and } |\lambda_i| < R, \quad \varepsilon < R < \infty, \quad i = 1, \dots, k.$$

Since p is a fixed nonnegative function, there is a constant c independent of λ in Λ and n such that

$$(4.43) \quad n \|1 - s_n\| \leq C < \infty, \quad n \geq 0.$$

For brevity, we define the function $\Delta_{j,n}$ by

$$(4.44) \quad \begin{aligned} \Delta_{0,n} &= n(1 - s_n), \\ \Delta_{j,n} &= n(1 - F_j(s_n)), \quad 0 < j \leq n. \end{aligned}$$

By Theorem 3, $F_{n_1}(s_n)$ is interior to S and hence in the domain of analyticity of all F_j , $j \geq 1$. To conclude that G_n , or $\Delta_{n,n}$, is an analytic function on Λ to $C(X, \Sigma)$ for $n \geq n_1$, it suffices to show that $F_{n_1}(e^{-\lambda p})$, $\lambda p = \sum_{i=1}^k \lambda_i p_i$, is analytic in $\operatorname{Re}(\lambda_i) > 0$ for any $p > 0$. If we let H_j stand for $\delta^j F_{n_1}(0)$ and take $s = \exp(-\sum_{i=1}^k \lambda_i p_i)$, then by Theorem 3 and Lemma 1,

$$(4.45) \quad \sum_{j=1}^{\infty} j |H_j(s^{j-1}, -p_i s)| \leq \sum_{j=1}^{\infty} j H_j(1, p_i) = M^{n_1}(p_i).$$

This series on the left converges absolutely and uniformly. It follows easily from Lemma 1 and Theorem 1 (restated with F replaced by F_{n_1}) that, with $t = e^{-\Delta p_i} s$, $\operatorname{Re}(\lambda_i + \Delta) > 0$,

$$(4.46) \quad \left\| \frac{F_{n_1}(t) - F_{n_1}(s)}{\Delta} - \sum_{j=1}^{\infty} j H_j(s^{j-1}, -p_i s) \right\| \leq |\Delta| [M^{n_1}(p_i) + 6K_{n_1}(p_i)] \rightarrow 0 \text{ as } |\Delta| \rightarrow 0,$$

where $K_{n_1}(p_i)$ is the limit at $s = 1$ of $\delta^2 F_{n_1}(s; p_i)$. Therefore the partial derivatives of F_{n_1} exist for $\operatorname{Re}(\lambda_i) > 0$ and $p_i \geq 0$, and F is analytic in the complex variables

$\lambda_1, \dots, \lambda_k$ [13, p. 107]. Actually, the same argument applies for any $F_j(s_n)$ and it is not necessary to appeal to (2.23) of Theorem 3.

By Theorem 2 and properties (2.31), (2.32) and $\sigma(M) = 1$ of M , we have

$$(4.47) \quad \begin{aligned} \|\Delta_{j,n}\| &\leq \|nM^j(1-s_n)\| = \|n\phi^*(1-s_n)\phi + nQ_j(1-s_n)\| \\ &\leq n\|1-s_n\|(\|\phi^*\| + A\alpha^j), \end{aligned}$$

where A and α are independent of n and λ , $0 \leq \alpha < 1$. Therefore on Λ we have a double sequence $\{\Delta_{j,n}\}$ of analytic functions, and by (4.42),

$$(4.48) \quad \|\Delta_{j,n}\| \leq B,$$

where B is a constant independent of λ in Λ , and $0 \leq j \leq n$.

By a generalized theorem of Vitali ([13, §3.15] and [1, p. 41]), the convergence of $\{\Delta_{n,n}\}$ on Λ will be proved if convergence is demonstrated on a real neighborhood in Λ . Therefore we enlarge Λ , if necessary, so that λ restricted by

$$(4.49) \quad \phi^*[K(\phi)] \sum_{i=1}^k \lambda \phi^*(p_i) < 1, \quad 0 < \varepsilon < \lambda_i < R,$$

defines a relatively open real set in Λ . We shall prove (4.41) on this set N .

We first of all show that $\phi^*(\Delta_{m,n})$ is positive on the set (4.49) for n sufficiently large. By Theorem 2, with $s = F_j(s_n)$, and by the quadratic nature of K ,

$$(4.50) \quad \Delta_{j+1,n} \geq M\Delta_{j,n} - \frac{1}{n}K[M^j(\Delta_{0,n})].$$

Applying the positive functional ϕ^* , summing on j from 0 to $m-1$, and using $n(1-s_n) < \lambda p$ (recalling that $\lambda p = \sum_{i=1}^k \lambda_i p_i$), we obtain

$$(4.51) \quad \phi^*(\Delta_{m,n}) \geq \phi^*(\Delta_{0,n}) \frac{1}{n} - \sum_{j=0}^n \phi^*(K[M^j(\lambda p)]),$$

where the upper limit on the last sum has been increased from $m-1$ to n . Now by (2.31) for $\sigma(M) = 1$ and by the quadratic nature of K , the sequence

$$\{\phi^*(K[M^j(\lambda p)])\}$$

has the limit $[\phi^*(\lambda p)]^2 \phi^*[K(\phi)]$, and so does the average appearing in (4.51). Therefore, since $\Delta_{0,n} \rightarrow \lambda p$ as $n \rightarrow \infty$, we have (by (4.49) and $\phi^*(\lambda p) > 0$)

$$(4.52) \quad \liminf_{n \rightarrow \infty} \phi^*(\Delta_{m,n}) \geq \phi^*(\lambda p) (1 - \phi^*[K(\phi)]) > 0.$$

Since the right-hand side of (4.51) is independent of m , $\phi^*(\Delta_{m,n})$ can be bounded away from 0 for n sufficiently large, independent of m .

By repeated application of (3.5) we obtain

$$(4.53) \quad \Delta_{j+1,n} = M^j(\Delta_{0,n}) - \frac{1}{n} \sum_{k=0}^j M^{j-k}[K(\Delta_{k,n}) - \frac{1}{n}O(\|\Delta_{k,n}\|^3)].$$

We apply (2.31) and (2.32) with $\sigma(M) = 1$ and $Q_0 = M - P$, and identify the terms containing ϕ as simply $\phi^*(\Delta_{j+1,n})$ to get, from (4.53),

$$(4.54) \quad \begin{aligned} \Delta_{j+1,n} &= \phi^*(\Delta_{j+1,n})\phi + Q_j(\Delta_{0,n}) \\ &\quad - \frac{1}{n} \sum_{k=0}^j Q_{j-k} K \left[(\Delta_{k,n}) - \frac{1}{n} O(\|\Delta_{k,n}\|^3) \right]. \end{aligned}$$

Therefore we have

$$(4.55) \quad \begin{aligned} &\|\Delta_{j+1,n} - \phi^*(\Delta_{j+1,n})\phi\| \\ &\leq A\alpha^j \|\Delta_{0,n}\| + \frac{A}{n} \sum_{k=0}^j \alpha^{j-k} \left[\|\Delta_{k,n}\|^2 K(1) + \frac{1}{n} O(\|\Delta_{k,n}\|^3) \right]. \end{aligned}$$

Since $\alpha < 1$, we have, from (4.48) and $\|\Delta_{0,n}\| \leq R\|p\|$,

$$(4.56) \quad \|\Delta_{j+1,n} - \phi^*(\Delta_{j+1,n})\phi\| \leq A\alpha_j R \|p\| + \frac{AB^2 \left[K(1) + \frac{B}{n} \right]}{n} \frac{1 - \alpha^{j+1}}{1 - \alpha}.$$

This shows that, for all $0 \leq j \leq n$,

$$(4.57) \quad \lim_{\rightarrow \infty} \|\Delta_{j,n} - \phi^*(\Delta_{j,n})\phi\| = 0.$$

Or, since $\phi^*(\Delta_{j,n})$ is positive and bounded below, by (4.52) we have

$$(4.58) \quad \lim_{n \rightarrow \infty} \frac{\Delta_{j,n}}{\phi^*(\Delta_{j,n})} = \phi, \quad \phi^*(\phi) = 1, \quad \text{all } 0 \leq j \leq n,$$

and by (4.56)

$$(4.59) \quad \frac{\Delta_{j,n}}{\phi^*(\Delta_{j,n})} = \phi + O(\alpha^j) + O\left(\frac{1}{n}\right),$$

where $\|\alpha^{-j}O(\alpha^j)\|$ and $\|nO(1/n)\|$ are bounded independently of j and n .

To determine the behavior of $\phi^*(\Delta_{n,n})$, we apply ϕ^* to (3.5) with $s = F_j(s_n)$. We then get

$$(4.60) \quad \phi^*(\Delta_{j+1,n}) = \phi^*(\Delta_{j,n}) - \frac{1}{n} \phi^* \left[K(\Delta_{j,n}) + O\left(\frac{1}{n}\right) \right],$$

where $O(1/n) = (1/n)(O\|\Delta_{j,n}\|^3)$ and, by (4.48), satisfies

$$(4.61) \quad \|nO(1/n)\| \leq D < \infty, \text{ with } D \text{ a constant independent of } n, j, \text{ and } \lambda \text{ in } \Lambda.$$

By (4.52), $\phi^*(\Delta_{n+1,n}) > 0$ for n sufficiently large, independent of j . Therefore we can consider the identity

$$(4.62) \quad \frac{1}{\phi^*(\Delta_{j+1,n})} \equiv \frac{1}{\phi^*(\Delta_{j,n})} + \frac{1}{\phi^*(\Delta_{j,n})} \left[1 - \frac{\phi^*(\Delta_{j+1,n})}{\phi^*(\Delta_{j,n})} \right] \frac{\phi^*(\Delta_{j,n})}{\phi^*(\Delta_{j+1,n})}.$$

For brevity, we denote the last term in this identity by $A_{j,n}$. Using the homogeneous and quadratic nature of K , we compute $A_{j,n}$ from (4.60) as

$$(4.63) \quad A_{j,n} = \frac{1}{n} \phi^* \left[K \left(\frac{\Delta_{j,n}}{\phi^*(\Delta_{j,n})} \right) + \frac{O(1/n)}{[\phi^*(\Delta_{j,n})]^2} \right] \left(1 + O\left(\frac{1}{n}\right) \right).$$

Or, using (4.59),

$$(4.64) \quad A_{j,n} = \frac{1}{n} \phi^* \left(K \left[\phi + O(\alpha^j) + O\left(\frac{1}{n}\right) \right] \right) + O\left(\frac{1}{n}\right)^2.$$

Now K is a continuous quadratic polynomial, and hence

$$(4.65) \quad K \left[\phi + O(\alpha^j) + O\left(\frac{1}{n}\right) \right] = K \left[\phi + O\left(\frac{1}{n}\right) \right] + O(\alpha^j).$$

If we now sum (4.62) on j from 0 to $n-1$ and divide by n , we obtain by the two previous equations

$$(4.66) \quad \begin{aligned} \frac{1}{n\phi^*(\Delta_{n,n})} &= \frac{1}{n\phi^*(\Delta_{0,n})} + \frac{1}{n} \sum_{j=0}^{n-1} A_{j,n} \\ &= \frac{1}{\phi^*(\lambda p) + O\left(\frac{1}{n}\right)} + \phi^* \left(K \left[\phi + O\left(\frac{1}{n}\right) \right] \right) \\ &\quad + \frac{1}{n} \left[O\left(\frac{1-\alpha^n}{1-\alpha}\right) + O(\ln n) \right]. \end{aligned}$$

Therefore, since K is continuous,

$$(4.67) \quad \lim_{n \rightarrow \infty} n\phi^*(\Delta_{n,n}) = \frac{\phi^*(\lambda p)}{1 + \phi^*(\lambda p)\phi^*[K(\phi)]}, \quad \lambda p = \sum_{i=1}^k \lambda_i p_i.$$

In combination with (4.58) for $j = n$, this gives (4.43) for λ in the set (4.49) and hence in all of Λ . Since any $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\text{Re}(\lambda_i) > 0$ is contained in some such compact set Λ , we have completed the proof.

5. Limiting distributions. We have interpreted $F_n(x, s)$, x in X , as defining a probability generating functional for a random variable Z_n with range in the set of functionals in $C^*(X, \Sigma)$ defined by additive, nonnegative and integer-valued set functions on Σ . In certain cases, this is defined for each $n \geq 1$ and x in X , by a measure on the range of Z_n [11]. If X is partitioned into disjoint sets A_i , $\bigcup_{i=1}^k A_i = X$, and $s = \sum_{i=1}^k \lambda_i \chi_{A_i}$, $|\lambda_i| \leq 1$, then $F_n(x, s)$ is a generating function in the λ_i . The coefficient of $\lambda_1^{r_1} \dots \lambda_k^{r_k}$ in a series expansion of $F_n(x, s)$ is the probability that $Z_n(A_i) = r_i$, $i = 1, \dots, k$, given that $Z_0 = e_x$. In particular, $F_n(x, 0)$ is the probability that Z_n is the zero functional, given that $Z_0 = e_x$.

With the independence condition (2.24) we have

$$(5.1) \quad \Pr \left(Z_n = 0 \mid Z_0 = \sum_{i=1}^k r_i e_{x_i} \right) = \prod_{i=1}^k [F_n(x_i, 0)]^{r_i}.$$

From Theorem 4 we have

$$(5.2) \quad \lim_{n \rightarrow \infty} \Pr \left(Z_n = 0 \mid Z_0 = \sum_{i=1}^k r_i e_{x_i} \right) = 1,$$

and the process tends to die.

We are interested in Z_n with the condition that $Z_n \neq 0$. Since finite linear combinations of characteristic functions are dense in $C(X, \Sigma)$, it follows that for given $Z_0 = \sum_{i=1}^k r_i e_{x_i}$ and h in $C(X, \Sigma)$, $Z_n(h)$ is a complex-valued random variable. We study Z_n by investigating its values $Z_n(h)$ on $C(X, \Sigma)$.

THEOREM 8. *For a CPR process the expected value of $Z_n(h)$, h in $C(X, \Sigma)$, given that $Z_n \neq 0$, satisfies*

$$(5.3) \quad \lim_{n \rightarrow \infty} \mathcal{E} \left(\frac{Z_n(h)}{n} \mid Z_n \neq 0 \right) = \phi^*[K(\phi)]\phi^*(h)$$

for all Z_0 of the form $Z_0 = \sum_{i=1}^k r_i e_{x_i}$, where $M\phi = \phi$, $M^*\phi^* = \phi^*$, $\phi^*(\phi) = 1$.

Proof. For a given Z_0 , the generating functional for Z_n , given that $Z_n \neq 0$, is

$$(5.4) \quad \frac{\prod_{i=1}^k [F_n(x_i, s)]^{r_i} - \prod_{i=1}^k [F_n(x_i, 0)]^{r_i}}{1 - \prod_{i=1}^k [F_n(x_i, 0)]^{r_i}}.$$

To compute (5.3), we take the variation of (5.4) with differential h and let s tend to 1. This gives

$$(5.5) \quad \mathcal{E}(Z_n(h) \mid Z_0 \neq 0) = \frac{\sum_{i=1}^k r_i M^n(h)(x_i)}{1 - \prod_{i=1}^k [F_n(x_i, 0)]^{r_i}}.$$

The numerator in (5.5) is just $Z_0(M^n(h))$ and approaches $Z_0(\phi)\phi^*(h)$ as $n \rightarrow \infty$. Dividing (5.5) by n gives

$$(5.6) \quad \lim_{n \rightarrow \infty} \mathcal{E} \left(\frac{Z_n(h)}{n} \mid Z_0 \neq 0 \right) = \frac{Z_0(\phi)\phi^*(h)}{\lim_{n \rightarrow \infty} n \left(1 - \prod_{i=1}^k [F_n(x_i, 0)]^{r_i} \right)}.$$

From Theorem 6 we have

$$(5.7) \quad F_n(x_i, 0) = 1 - \frac{1}{\phi^*[K(\phi)]} \frac{\phi(x_i)}{n} + O\left(\frac{1}{n}\right),$$

where $nO(1/n) \rightarrow 0$ as $n \rightarrow \infty$. Substituting from (5.7) in the denominator of (5.6), we get

$$(5.8) \quad \lim_{n \rightarrow \infty} \mathcal{E} \left(\frac{Z_n(h)}{n} \mid Z_n \neq 0 \right) = \frac{Z_0(\phi) \phi^*(h) \phi^*[K(\phi)]}{\sum_{i=1}^k r_i \phi(x_i)}.$$

Since $\sum_{i=1}^k r_i \phi(x_i) = Z_0(\phi)$, this gives (5.3) and completes the proof.

This theorem suggests that Z_n/n , given that $Z_n \neq 0$, tends to a random scalar multiple of the given linear functional ϕ^* , independent of the initial state Z_0 . If the process is alive at the n th generation, it tends to have a population size proportional to n with the distribution among types being determined by the value of ϕ^* on characteristic functions.

We investigate this more thoroughly by considering the conditional distributions for the nonnegative, vector-valued, random variables, defined for p in $C(X, \Sigma)$, $p \geq 0$, and for disjoint sets A_i , $\bigcup_{i=1}^k A_i = X$, by

$$(5.9) \quad W_n(p) = \left(\frac{(Z_n(p_1), \dots, Z_n(p_k))}{n \phi^*[K(\phi)]} \mid Z_n \neq 0 \right), \quad p_i = p \chi_{A_i}.$$

THEOREM 9. For a CPR process, it follows that if $p_i = p \chi_{A_i}$, $\bigcup_{i=1}^k A_i = X$, $p \geq 0$ in $C(X, \Sigma)$, then for real constants $\alpha_1, \alpha_2, \dots, \alpha_k$

$$(5.10) \quad \lim_{n \rightarrow \infty} \Pr \left(\frac{Z_n(p_i)}{n \phi^*[K(\phi)]} \leq \alpha_i, i = 1, \dots, k \mid Z_n \neq 0 \right) = \begin{cases} 0 & \text{if some } \alpha_i < 0, \\ 1 - \exp \left(-\sqrt{k} \min_i \left[\frac{\alpha_i}{\phi^*(p_i)} \right] \right) & \text{if all } \alpha_i \geq 0, \end{cases}$$

for any Z_0 of the form $Z_0 = \sum_{i=1}^m r_i e_{x_i}$, r_i positive integers.

Proof. This result is trivial for $p = 0$. For $p \geq 0$, $p \neq 0$, we compute the k -dimensional Laplace transform of the distribution of the nonnegative, vector-valued random variables $W_n(p)$ defined by (5.9).

For $Z_0 = \sum_{i=1}^m r_i e_{x_i}$, the continuity of F_n and the density of finite linear combinations of characteristic functions in $C(X, \Sigma)$ leads to

$$(5.11) \quad \mathcal{E} (e^{-W_n(\lambda p)}) = \frac{\prod_{i=1}^m [F_n(x_i, s_n)]^{r_i} - \prod_{i=1}^m [F_n(x_i, 0)]^{r_i}}{1 - \prod_{i=1}^m [F_n(x_i, 0)]^{r_i}}$$

with $s = \exp[-\lambda p / n \phi^*[K(\phi)]]$, $\lambda p = \sum_{i=1}^k \lambda_i p_i$, and $\text{Re}(\lambda_i) > 0$. With G_n defined by (4.40), this can be written as

$$(5.12) \quad \mathcal{E} (e^{-W_n(\lambda p)}) = 1 - \frac{1 - \prod_{i=1}^m \left[1 - \frac{G_n(x_i, p, \lambda)}{n} \right]^{r_i}}{1 - \prod_{i=1}^m [F_n(x_i, 0)]^{r_i}}.$$

From Theorem 6 and Theorem 7, with λp replaced by $\lambda p/\phi^*[K(\phi)]$, we have after expanding the products in (5.12)

$$(5.13) \quad \lim_{n \rightarrow \infty} \mathcal{E}(e^{-W_n(\lambda p)}) = 1 - \frac{\sum_{i=1}^k \lambda_i \phi^*(p_i) Z_0(\phi)}{\left[1 + \sum_{i=1}^k \lambda_i \phi^*(p_i)\right] \phi^*[K(\phi)]} \frac{\phi^*[K(\phi)]}{Z_0(\phi)},$$

or

$$(5.14) \quad \lim_{n \rightarrow \infty} \mathcal{E}(e^{-W_n(\lambda p)}) = \frac{1}{1 + \sum_{i=1}^k \lambda_i \phi^*(p_i)}, \quad \operatorname{Re}(\lambda_i) > 0.$$

The last term in (5.14) is the k -dimensional Laplace transform of the exponential distribution given in (5.10). This is most easily seen by realizing that this exponential distribution puts unit mass on the ray $\alpha(\phi^*(p_1), \dots, \phi^*(p_k))$, $\alpha \geq 0$, with a distribution $1 - e^{-\alpha}$. Since (4.14) holds for all $\operatorname{Re}(\lambda) > 0$, it follows easily from standard theorems on Fourier transforms of probability distributions [4, p. 102] that (5.10) is true. The proof is thus complete.

For the finite dimensional subspace C_k of $C(X, \Sigma)$ having as basis the functions $p\chi_{A_1}$, A_i disjoint and $\bigcup_{i=1}^k A_i = X$, and $p \geq 0$ in $C(X, \Sigma)$, Z_n defines a random positive vector in the dual space to C_k . This last theorem shows that $Z_n/n\phi^*[K(\phi)]$, given $Z_n \neq 0$, has a distribution in C_k^* which converges with $n \rightarrow \infty$, to a random scalar multiple of the given functional $(\phi^*(\chi_{A_1}), \dots, \phi^*(\chi_{A_k}))$, expressed relative to a basis in C_k^* dual to that in C_k .

6. Applications.

I. FINITE TYPES. Let X be a finite set and Σ the set of all subsets of X . Then $C(X, \Sigma)$ is simply a k -dimensional complex vector space, and F is defined by k generating functions of k complex variables. The range of Z_n is the set of non-negative k -vectors with integer components, a subset of $C^*(X, \Sigma)$. Theorem 9 characterizes the limiting distribution of Z_n/n , given $Z_n \neq 0$, as $n \rightarrow \infty$.

II. A NUCLEAR REACTOR MODEL. We consider a homogeneous sphere of fissionable material. We assume that the velocity of neutrons in this sphere is constant and that motion is interrupted only by a collision with a nucleus. Upon collision, a random number of neutrons is produced with an isotropic angular distribution. The collision process is described by a probability generating function:

$$(6.1) \quad f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n,$$

where λ is a complex parameter and a_n is the constant probability that n neutrons are produced.

A branching process is defined in which a neutron is typed according to the spherical shell in which it was produced by fission [11; 12; 18]. The random variable Z_n is defined by the number of neutrons which can be traced

back to a trigger neutron through n fissions. The discrete time of this branching process is related in a complicated manner to physical time.

We parametrize the spherical shells by a real variable x , $0 \leq x \leq R < \infty$, where R is the radius of the reactor. Then the set of types X is the interval $[0, R]$, and we take the σ -field of Borel sets as the set Σ .

With the assumption of homogeneity and isotropy, a neutron born at s will have a first collision in the spherical shell between y and $y + \Delta y$ with probability $K(x, y)\Delta y$, where (see [5])

$$(6.2) \quad K(x, y) = \frac{y}{2x} \int_{|x-y|}^{x+y} \frac{e^{-t}}{t} dt, \quad 0 < x, y \leq R.$$

The unit of length has been chosen as a mean free path.

The generating functional of our process is

$$(6.3) \quad F(x, s) = 1 - \int_0^R K(x, y) [1 - f(s(y))] dy,$$

where s is any Borel-measurable function in $C(X, \Sigma)$ with $|s(x)| \leq 1$ for all x in X . The term $1 - \int_0^R K(x, y) dy$ in (6.3) measures the probability of escape without collision.

Since f is a continuous function it is Borel measurable, and hence $f(s)$ is a Borel-measurable function on X for measurable s , and so is $F(s)$. It follows easily then that conditions (2.14) are satisfied by F .

The expectation operator M is given by

$$(6.4) \quad M(h)(x) = f'(1) \int_0^R K(x, y) h(y) dy.$$

The operator M can be derived from an operator defined for functions on three-dimensional real space, and the results of [20, p. 29] show that the third iterate of M is defined by a continuous kernel. It follows readily that this is a compact operator in $C(X, \Sigma)$. (See [6, p. 260].) Therefore (2.30) is satisfied. By the Lebesgue Dominated Convergence Theorem [6, p. 151] it follows that (2.29) is satisfied. So F satisfied the conditions imposed in §2 as long as f is not linear and its first three derivatives have limits at $\lambda = 1$.

For a given value of $f'(1) > 1$, there is a unique R for which $\sigma(M) = 1$. (See [18].) A good approximation to R is given [19] by solving for R in

$$(6.5) \quad \frac{1}{f'(1)} = 1 - \sin^2 \gamma + R \left[\ln \left(\frac{\pi}{\gamma} \right) - Ci(2\pi) + Ci(2\gamma) \right],$$

$$\gamma \tan \left(\gamma - \frac{\pi}{2} \right) = R, \quad \frac{\pi}{2} \leq \gamma \leq \pi.$$

Good approximations to ϕ and ϕ^* are given [19] by

$$(6.6) \quad \phi = \frac{\sin\left(\frac{\gamma x}{R}\right)}{x}, \quad \phi^*(f) = c \int_0^R x \sin\left(\frac{\gamma x}{R}\right) f(x) dx,$$

with

$$(6.7) \quad c = \frac{2\sqrt{(R^2 + \gamma^2)}}{R(R + \sqrt{(R^2 + \gamma^2)})}.$$

Theorem 8 shows that any critical reactor which is described by this mathematical model will produce a finite number of neutrons when initiated by a finite number of trigger neutrons.

Fluctuations in the number produced are measured by Theorem 9. For example, let A be any Borel set in $[0, R]$ with nonzero Lebesgue measure. Then $\phi^*(\chi_A) > 0$ determines the importance of this set to the branching process. The probability $P_n(A, \alpha)$ that in the n th generation the number of neutrons produced in the spherical shells defined by A exceeds $\alpha n(f''(1)/2f'(1))\phi^*(\phi^2)\phi^*(\chi_A)$, given that this number is not 0, satisfies

$$(6.8) \quad \lim_{n \rightarrow \infty} P_n(A, \alpha) = e^{-\alpha}, \quad 0 \leq \alpha < \infty.$$

Here we have used the explicit expression for K , i.e.,

$$(6.9) \quad K(h) = f''(1) \int_0^R K(x, y) h^2(y) dy.$$

The results of this paper will apply to more complicated reactor models. In a trivial manner, the assumption of homogeneity can be replaced by the assumption that the reactor is stratified into spherical shells. This will simply result in f of (6.1) being dependent upon the variable x , after a proper scaling of x by the total cross section. Of course, the characterization of criticality and of the eigenfunctions ϕ and ϕ^* will be different. (See [19] for a treatment of core and reflector.)

A problem of considerable interest and difficulty is that of relating the characterization of neutron population by generation to its characterization in real time.

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