

THE MARCEL RIESZ THEOREM ON CONJUGATE FUNCTIONS⁽¹⁾

BY
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1. **Introduction.** 1. If f is a trigonometric polynomial

$$f = \sum a_n e^{inx}$$

the trigonometric polynomial conjugate to f is

$$\tilde{f} = \sum -ie_n a_n e^{inx}$$

where $\varepsilon_0 = 0$ and $\varepsilon_n = n/|n|$ for $n \neq 0$. The operation taking trigonometric polynomials into their conjugates is linear and carries real trigonometric polynomials into real trigonometric polynomials. Associated with the conjugacy operation is the projection T defined on trigonometric polynomials by

$$T(\sum a_n e^{inx}) = \sum_{n>0} a_n e^{inx}.$$

T carries arbitrary trigonometric polynomials into analytic trigonometric polynomials with mean value zero, and is related to the conjugacy operation by

$$2Tf = f + i\tilde{f} - a_0.$$

M. Riesz' theorem on conjugate functions [11, p. 225; 14, p. 253] states that the projection T (or equivalently the conjugacy operation) is bounded in L^p_σ for $1 < p < \infty$:

$$\int |Tf|^p d\sigma \leq K \int |f|^p d\sigma$$

for all trigonometric polynomials f , where K is a constant depending only on p . Here and in the sequel, all integrals are over $[-\pi, \pi)$ and σ denotes normalized Lebesgue measure on this half-open interval: $d\sigma = dx/2\pi$.

The problem we wish to consider in this paper is that of extending the Riesz theorem to measures other than Lebesgue measure.

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To be more precise, let μ be a finite non-negative measure defined on the Borel sets of $[-\pi, \pi)$ and denote by L_μ^p the space of complex valued μ -measurable functions f such that

$$\int |f|^p d\mu < \infty.$$

We will assume throughout $1 \leq p < \infty$ so that L_μ^p is a Banach space with norm

$$\left(\int |f|^p d\mu \right)^{1/p}.$$

Since μ is finite the trigonometric polynomials form a den linear subset of L_μ^p .

The problem we will consider is this. For a fixed exponent p , for which measures μ is the projection T bounded in L_μ^p :

$$\int |Tf|^p d\mu \leq K \int |f|^p d\mu$$

for all trigonometric polynomials f , where K is a constant depending only on p and μ .

2. Before describing the history of this problem and our contributions to it, a few words about conjugate functions are in order [14, Chapter 7]. For any trigonometric series

$$(1) \quad \sum a_n e^{inx}$$

the conjugate series is

$$(2) \quad \sum -i\varepsilon_n a_n e^{inx}.$$

If (1) is the Fourier series of a summable function f , the series (2) converges in the metric of L_σ^r for $0 < r < 1$ to a function $\tilde{f}$. The function $\tilde{f}$ is called the conjugate function of f . Although $\tilde{f}$ is defined whenever f is summable, $\tilde{f}$ may not be summable. If f is real, so is $\tilde{f}$. If f is real and bounded with $\|f\|_\infty < \pi/2$, $e^{\tilde{f}}$ is summable.

3. If μ is absolutely continuous with respect to Lebesgue measure we will write $d\mu = w d\sigma$ and write L_w^p for L_μ^p .

The relation between T and L_μ^p for measures μ other than Lebesgue measure seems to have been first considered by Hardy and Littlewood [7, p. 371]⁽²⁾. They showed T is bounded in L_w^p for w of the form

$$w = |x|^s \quad (-\pi \leq x < \pi)$$

if $1 < p < \infty$ and $-1 < s < p-1$. Later Babenko rediscovered their result [2]. Hirschman [9, p. 30] and Flett [5, p. 136] have also given proofs of this theorem. The proofs given by these authors use real variable methods and depend in an

(2) We are indebted to [8] for most of this history.

essential way on the function $|x|^s$. We wish to point out that the condition on s is necessary as well as sufficient (Lemma 2). This fact is surely known but it does not seem to appear in the literature. In any event this shows that the class of measures μ for which T is bounded in L_μ^p varies with the exponent p .

More recently, Gapoškin [6] has shown T is bounded in L_w^p if w satisfies

$$|\tilde{w}| \leq Cw \quad \text{a.e.}$$

where C is any positive constant if $2 \leq p < \infty$ and C is less than $\tan(p-1)\pi/2$ if $1 < p < 2$. Gapoškin's theorem intersects the Hardy-Littlewood-Babenko theorem but does not contain it. More precisely, Gapoškin's result shows only that T is bounded in $L_{|x|^s}^p$ for $-1 < s < 1$ if $2 \leq p < \infty$, and that T is bounded in $L_{|x|^s}^p$ for $-p+1 < s < p-1$ if $1 < p \leq 2$. Gapoškin's proof uses analytic functions and is an adaption, which is by no means obvious, of the method used by Riesz for the case $w = 1$.

Finally, Helson and Szegő [8] have completely described the measures μ for which T is bounded in L_μ^2 . They show μ must be absolutely continuous, $d\mu = wd\sigma$, and $\log w$ must be summable. Then a necessary and sufficient condition for T to be bounded in L_w^2 is

$$(3) \quad \sup \left| \int f e^{-i \log \tilde{w}} d\sigma \right| < 1$$

where the supremum is taken over all f with mean value zero in the unit ball of H^1 . H^1 is the set of functions in L_σ^1 whose Fourier coefficients vanish for negative indices. (3) states that the norm of $e^{-i \log \tilde{w}}$ as a linear functional on a subspace of L_σ^1 is smaller than one. Using this observation, Helson and Szegő found that (3) holds only for w of the form

$$(4) \quad w = e^{u+\tilde{v}}$$

where u and v are bounded real functions and $\|v\|_\infty < \pi/2$. This class of weights coincides with that considered by Gapoškin for $p = 2$ modulo multiplication by factors bounded from above and below.

The corresponding problem for the real line and the Hilbert transform has been considered by Widom [13].

4. In §2 we gather together most of what we know of a general nature about the problem. Using results from [8] we are able to describe (Theorems 3 and 4), for a given exponent p , some weights w for which T is bounded in L_w^p . These weights include those considered by Hardy and Littlewood and by Gapoškin.

In §3, using the methods of [8], we obtain for each p a condition, similar to (3) (and of course identical if $p = 2$), for T to be bounded in L_w^p . The condition is that the norm of a certain functional, which depends on p , be smaller than one. However, for $p \neq 2$ this condition does not have the linear character which (3) has, and we know little about this condition other than that it exists.

In §4 we study weights w for which $\log \tilde{w}$ is continuous except for a finite number of simple discontinuities. For this class of weights we are able to extend to exponents $p \neq 2$ results obtained in [8] for $p = 2$.

2. General results.

1. First some additional notation which will be used throughout. $\mathbf{C}$ will denote the space of all complex-valued, continuous, 2π -periodic functions on $(-\infty, \infty)$. $\mathbf{P}$ and $\mathbf{F}$ will denote respectively the sets of trigonometric polynomials of the form

$$\sum_{n \leq 0} a_n e^{inx}, \quad \sum_{n \geq 0} a_n e^{inx}.$$

For m a positive integer, $\mathbf{F}_m$ will denote the set of trigonometric polynomials of the form

$$\sum_{n \geq m} a_n e^{inx}$$

and $\mathbf{M}_m$ the sum of $\mathbf{P}$ and $\mathbf{F}_m$: $\mathbf{M}_m$ consists of all trigonometric polynomials $f + g$ where $f \in \mathbf{P}$ and $g \in \mathbf{F}_m$. In particular $\mathbf{M}_1$ is the set of all trigonometric polynomials. Notice that $\mathbf{M}_m$ is a linear set and that T , when restricted to $\mathbf{M}_m$, projects $\mathbf{M}_m$ onto $\mathbf{F}_m$. $\mathbf{M}_0$ will denote the set of trigonometric polynomials with mean value zero. Finally, p' will denote the exponent conjugate to p : $1/p + 1/p' = 1$.

It will be useful to consider a somewhat more general problem than was indicated in the Introduction. Accordingly, for a given positive integer m and exponent p we ask for which measures μ is the projection T , when restricted to $\mathbf{M}_m$, bounded in L_μ^p . This is a more general problem: the continuity of T , when restricted to $\mathbf{M}_m$ (for some $m > 1$), does not imply that T , when operating on all trigonometric polynomials, is continuous. For $p = 2$ this is in [8]. For other exponents p this may be seen by comparing Theorems 6 and 7.

2. It is well known T is not bounded in L_σ^1 . Also, T , restricted to $\mathbf{M}_m$, is not bounded in L_σ^1 for any positive integer m . See, for example, [14, p. 253]. This situation continues for other measures.

THEOREM 1. *For any positive integer m , T , restricted to $\mathbf{M}_m$, is not bounded in L_μ^1 .*

Proof. Suppose on the contrary that for some m , T , restricted to $\mathbf{M}_m$, is bounded in L_μ^1 :

$$\int |Tf| d\mu \leq K \int |f| d\mu$$

for all $f \in \mathbf{M}_m$.

For each real y let T_y be the linear operator mapping trigonometric polynomials into trigonometric polynomials defined by

$$(T_y f)(x) = f(x + y).$$

Notice that T_y commutes with T for each y , and that T_y maps $\mathbf{M}_m$ onto $\mathbf{M}_m$. Let ν be any finite non-negative measure defined on the Borel sets of $[-\pi, \pi)$. Then $\mu * \nu$, the convolution of μ and ν , is the non-negative Borel measure defined by

$$\int f d(\mu * \nu) = \int \int f(x + y) d\mu(x) d\nu(y)$$

for all $f \in \mathbf{C}$. From the definition of $\mu * \nu$, the fact that T_y commutes with T and that $\mathbf{M}_m$ reduces T_y , and the Fubini theorem we obtain

$$\int |Tf| d(\mu * \nu) \leq K \int |f| d(\mu * \nu)$$

for all $f \in \mathbf{M}_m$.

We have shown that T , when restricted to $\mathbf{M}_m$, is bounded in $L_{\mu * \nu}^1$ for all non-negative measures ν . Since $\mu * \sigma$ is a positive multiple of σ , T restricted to $\mathbf{M}_m$ is bounded in L_σ^1 , and this is a contradiction.

Our next theorem states that only absolutely continuous measures are relevant. Helson and Szegö have given a proof of this [8, p. 127] based on a theorem of Rudin concerning the boundary values of analytic functions. The proof we give is different and is based on the theorem of F. and M. Riesz [14, p. 285] which states that if ν is a bounded complex Borel measure on $[-\pi, \pi)$ whose Fourier-Stieltjes coefficients vanish for negative indices, then ν is absolutely continuous with respect to Lebesgue measure.

THEOREM 2. *Let m and p be given and suppose T , restricted to $\mathbf{M}_m$, is bounded in L_μ^p . Then μ is absolutely continuous.*

Proof. We assume there is a constant K such that

$$(5) \quad \left(\int |Tf|^p d\mu \right)^{1/p} \leq K \left(\int |f|^p d\mu \right)^{1/p}$$

for all $f \in \mathbf{M}_m$. Define a linear functional ϕ on $\mathbf{M}_m$ by

$$\phi(f) = \int Tf d\mu.$$

Since (Hölder inequality)

$$|\phi(f)| \leq \left(\int d\mu \right)^{1/p'} \left(\int |Tf|^p d\mu \right)^{1/p},$$

we obtain from (5)

$$|\phi(f)| \leq K \left(\int d\mu \right)^{1/p'} \left(\int |f|^p d\mu \right)^{1/p},$$

and therefore

$$|\phi(f)| \leq K \left(\int d\mu \right) \|f\|_\infty$$

for all $f \in \mathbf{M}_m$. This last inequality states that ϕ is a bounded linear functional (in the uniform norm) on $\mathbf{M}_m$. We may extend ϕ as a bounded linear functional to all of $\mathbf{C}$ and we may represent the extended functional as integration with a bounded complex Borel measure ν on $[-\pi, \pi)$. Then

$$(6) \quad \int T f d\mu = \int f d\nu$$

for all $f \in \mathbf{M}_m$. Since $e^{inx} \in \mathbf{M}_m$ for $n \leq 0$, (6) implies

$$\int e^{inx} d\nu = 0$$

for $n \leq 0$. This and the F. and M. Riesz theorem shows ν is absolutely continuous. Since $e^{inx} \in \mathbf{M}_m$ for $n \geq m$, (6) also implies

$$\int e^{inx} d(\mu - \nu) = 0$$

for $n \geq m$, and, again by the F. and M. Riesz theorem, $\mu - \nu$ is absolutely continuous. This completes the proof of the theorem.

Because of the two preceding theorems we will assume from now on that $1 < p < \infty$ and that the measure μ is absolutely continuous: $d\mu = w d\sigma$.

3. Suppose T , restricted to $\mathbf{M}_m$, is bounded in $\mathbf{L}_w^p$:

$$(7) \quad \left(\int |Tf|^p w d\sigma \right)^{1/p} \leq K \left(\int |f|^p w d\sigma \right)^{1/p}$$

for all $f \in \mathbf{M}_m$. (7) implies

$$(8) \quad \inf \left(\int |f + g|^p w d\sigma \right)^{1/p} > 0$$

where the infimum is taken over all $f \in \mathbf{P}$ and $g \in \mathbf{F}_m$ such that

$$\int |f|^p w d\sigma = \int |g|^p w d\sigma = 1.$$

Conversely, if (8) holds, then it is easy to see that there is a constant K such that (7) holds. Indeed, denote the infimum (8) by τ and suppose $\tau > 0$. Write

$$\|f\| = \left(\int |f|^p w d\sigma \right)^{1/p}$$

and let $f \in \mathbf{P} (f \neq 0)$, $g \in \mathbf{F}_m$, and

$$c = (\|g\| / \|f\|) - 1.$$

Then

$$\|f + cf\| = \|g\|$$

and the definition of τ gives

$$\|g\| \leq \tau^{-1} \|f + cf + g\| \leq \tau^{-1} \|f + g\| + \tau^{-1} |c| \|f\|.$$

But

$$|c| \|f\| = \|\|g\| - \|f\|\| \leq \|f + g\|$$

so that

$$\|g\| \leq 2\tau^{-1} \|f + g\|$$

and this is (7) with $K = 2\tau^{-1}$.

The condition (8) is the basis of a large part of what follows as was the case in [8].

4. A theorem of Szegö [1, p. 256; 8, p. 108] states

$$\inf \int |1 + f|^p w d\sigma = \exp \left(\int \log w d\sigma \right)$$

where the infimum is taken over all $f \in F_1$. If $\log w$ is not summable the infimum is zero.

Now (8) implies that 1 must be at positive distance from F_m in the metric of L_w^p , and it follows, as is easily seen, that 1 must also be at positive distance from F_1 in the metric of L_w^p . Hence a necessary condition for T , when restricted to M_m , to be bounded in L_w^p is that $\log w$ be summable.

If T , when operating on all trigonometric polynomials, is bounded in L_w^p , we can place a stronger size condition on w (Lemma 2). For this we need the following lemma which extends a result due to Kolmogorov for $p = 2$ [8, p. 108].

LEMMA 1. For $1 < p < \infty$

$$(9) \quad \inf \left(\int |1 + f|^p w d\sigma \right)^{1/p} = \left(\int w^{-1/(p-1)} d\sigma \right)^{-1/p'}$$

where the infimum is taken over all $f \in M_0$. If $w^{-1/(p-1)}$ is not summable the infimum is zero.

Proof. Since M_0 is a linear subspace of L_w^p , the Hahn-Banach extension theorem shows that the infimum (9) is equal to

$$(10) \quad \sup \left| \int g w d\sigma \right|$$

where the supremum is taken over all $g \in L_w^{p'}$ which annihilate M_0 and satisfy

$$\int |g|^{p'} w d\sigma \leq 1.$$

Denoting by M_0^+ the annihilator in $L_w^{p'}$ of M_0 , we have $g \in M_0^+$ if and only if $g \in L_w^{p'}$ and

$$\int e^{inx} g w d\sigma = 0$$

for $n \neq 0$. Therefore $g \in \mathbf{M}_0^+$ only if $g \in \mathbf{L}_w^{p'}$ and gw is constant a.e. with respect to Lebesgue measure. Let $g \in \mathbf{M}_0^+$ and write $gw = c$. Then

$$(11) \quad \int |g|^{p'} w d\sigma = |c|^{p'} \int w^{-1/(p-1)} d\sigma.$$

If $w^{-1/(p-1)}$ is not summable, then (11) shows that $\mathbf{M}_0^+$ contains only the zero vector, and hence the supremum (10) is zero. If $w^{-1/(p-1)}$ is summable, then

$$\int |g|^{p'} w d\sigma = 1$$

only if

$$|c| = \left(\int w^{-1/(p-1)} d\sigma \right)^{-1/p'}$$

and this gives (9).

LEMMA 2. *If T is bounded in $\mathbf{L}_w^p$, then $w^{-1/(p-1)}$ is summable.*

Proof. Let $f \in \mathbf{M}_0$. Then from

$$1 = (1 + f) - T(1 + f) - \overline{T(1 + f)}$$

and (7) we obtain

$$\left(\int w d\sigma \right)^{1/p} \leq (2K + 1) \left(\int |1 + f|^p w d\sigma \right)^{1/p}.$$

Therefore the infimum (9) is positive, and so $w^{-1/(p-1)}$ must be summable.

LEMMA 3. *T is bounded in $\mathbf{L}_w^p$ if and only if T is bounded in $\mathbf{L}_w^{p' - 1/(p-1)}$.*

Proof. The proof consists of the two following observations. First, the dual space of $\mathbf{L}_w^p$ is isometrically anti-isomorphic to $\mathbf{L}_w^{p' - 1/(p-1)}$ with the duality given by

$$\int f \bar{g} d\sigma$$

where $f \in \mathbf{L}_w^p$ and $g \in \mathbf{L}_w^{p' - 1/(p-1)}$. Second, the operator T is formally self-adjoint:

$$\int T f \bar{g} d\sigma = \int f \overline{T g} d\sigma$$

for all trigonometric polynomials f and g .

5. We remarked at the beginning of this section that the continuity of T in $\mathbf{L}_w^p$ when operating on all trigonometric polynomials does not follow merely from the continuity of T when restricted to some $\mathbf{M}_m$. However, if $w^{-1/(p-1)}$ is summable and for some m , T , restricted to $\mathbf{M}_m$, is bounded in $\mathbf{L}_w^p$, then T is bounded in $\mathbf{L}_w^p$.

For consider the projection of $\mathbf{M}_1$ onto $\mathbf{M}_m$: this is the operation on trigonometric polynomials which suppresses coefficients with indices between 1 and $m - 1$. We always have (Hölder inequality)

$$(12) \quad \int |f| d\sigma \leq \left(\int w^{-1/(p-1)} d\sigma \right)^{1/p'} \left(\int |f|^p w d\sigma \right)^{1/p}.$$

Now if $w^{-1/(p-1)}$ is summable, then (12) implies (as one can easily show) that the projection of $\mathbf{M}_1$ onto $\mathbf{M}_m$ is bounded in $\mathbf{L}_w^p$. Since T may be obtained by first projecting $\mathbf{M}_1$ onto $\mathbf{M}_m$ and then projecting $\mathbf{M}_m$ onto $\mathbf{F}_m$, T will be bounded when operating on $\mathbf{M}_1$ if $w^{-1/(p-1)}$ is summable and T is bounded when restricted to $\mathbf{M}_m$.

Regardless of the summability of $w^{-1/(p-1)}$ we may always pass from p to a larger exponent and conclude that T is bounded. More precisely:

LEMMA 4. *If T , when restricted to $\mathbf{M}_m$, is bounded in $\mathbf{L}_w^p$, then T , when operating on all trigonometric polynomials, is bounded in $\mathbf{L}_w^{mp}$.*

Proof. Denote by τ the infimum (8). Then the hypothesis of Lemma 4 implies $\tau > 0$. Suppose now $f \in \mathbf{P}$ and $g \in \mathbf{F}_1$ satisfy

$$\int |f|^{mp} w d\sigma = \int |g|^{mp} w d\sigma = 1.$$

Since $f^m \in \mathbf{P}$ and $g^m \in \mathbf{F}_m$ we have

$$(13) \quad \tau \leq \left(\int |f^m - g^m|^p w d\sigma \right)^{1/p}.$$

From the identity

$$f^m - g^m = (f - g) \sum_{k=1}^m f^{k-1} g^{m-k}$$

and the Hölder inequality applied to the pair of conjugate exponents m and $m/(m-1)$ we obtain

$$(14) \quad \int |f^m - g^m|^p w d\sigma \leq \left(\int |f - g|^{mp} w d\sigma \right)^{1/m} \left(\int \left| \sum_{k=1}^m f^{k-1} g^{m-k} \right|^{mp/(m-1)} w d\sigma \right)^{(m-1)/m}.$$

Now

$$(15) \quad \left(\int \left| \sum_{k=1}^m f^{k-1} g^{m-k} \right|^{mp/(m-1)} w d\sigma \right)^{(m-1)/mp} \leq \sum_{k=1}^m \left(\int |f|^{(k-1)mp/(m-1)} |g|^{(m-k)mp/(m-1)} w d\sigma \right)^{(m-1)/mp}$$

and another application of the Hölder inequality (this time applied to the pair of conjugate exponents $(m-1)/(k-1)$ and $(m-1)/(m-k)$) shows each integral on the right-hand side of (15) does not exceed one and therefore

$$(16) \quad \left(\int \left| \sum_{k=1}^m f^{k-1} g^{m-k} \right|^{mp/(m-1)} w d\sigma \right)^{(m-1)/mp} \leq m.$$

Combining (13), (14), and (16) we obtain

$$\tau/m \leq \left(\int |f-g|^{mp} w d\sigma \right)^{1/mp}$$

which implies the conclusion of the lemma.

We are now able to describe some weights w for which T is bounded in L_w^p .

It is known that T , when restricted to M_m , is bounded in L_w^2 if and only if

$$(17) \quad w = e^u |D|$$

where u is a bounded real function and D belongs to H^1 and satisfies

$$(18) \quad |\arg D(x) - (m-1)x| \leq (\pi/2) - \varepsilon \quad \text{a.e. (modulo } 2\pi)$$

for some $\varepsilon > 0$ ⁽³⁾. For $m=1$ and $m=2$ this is in [8, pp. 123, 132]. The extension to $m=3, 4, \dots$ requires no new idea.

We also have as a corollary of an interpolation theorem of E. Stein [12, p. 485] the following lemma.

LEMMA 5. Let $w_1, \dots, w_n$ be non-negative summable functions and suppose T when operating on all trigonometric polynomials is bounded in $L_{w_j}^{p_j}$ ($j=1, \dots, n$)⁽⁴⁾. Then T is bounded in L_w^p for

$$1/p = \sum_{j=1}^n t_j/p_j, \quad w = \prod_{j=1}^n w_j^{t_j p/p_j}$$

where the t_j are non-negative with

$$\sum_{j=1}^n t_j = 1.$$

Combining (17) and (18), Lemma 4, and Lemma 5, we obtain

THEOREM 3. Let $p \geq 2$, and let $p_1, \dots, p_n$ be positive even integers and $t_1, \dots, t_n$ non-negative with

$$\sum_{j=1}^n t_j = 1, \quad 1/p = \sum_{j=1}^n t_j/p_j.$$

⁽³⁾ For $m=1$, (17) and (18) imply w is of the form (4).

⁽⁴⁾ Here it is essential that T operate on all trigonometric polynomials.

Then T is bounded in L_w^p for

$$(19) \quad w = e^u \prod_{j=1}^n |D_j|^{1/p_j}$$

where u is a bounded real function and the D_j belong to H^1 and satisfy

$$|\arg D_j(x) - ((p_j/2) - 1)x| \leq (\pi/2) - \varepsilon \quad \text{a.e. (modulo } 2\pi)$$

for some $\varepsilon > 0$.

6. LEMMA 6. Let $w, w_1, \dots, w_n$ be non-negative summable functions such that

$$(20) \quad w_j \leq Cw \quad \text{a.e. on } [-\pi, \pi]$$

and suppose there are measurable sets A_j ($j = 1, \dots, n$) whose union is $[-\pi, \pi)$ such that

$$(21) \quad w \leq Cw_j \quad \text{a.e. on } A_j$$

where C is a positive constant.

Then if T is bounded in $L_{w_j}^p$ ($j = 1, \dots, n$), T is also bounded in L_w^p .

Proof. Let K be such that

$$(22) \quad \int |Tf|^p w_j d\sigma \leq K \int |f|^p w_j d\sigma$$

for $j = 1, \dots, n$ and for all trigonometric polynomials f . Then (21), (22), and (20) imply

$$\int_{A_j} |Tf|^p w d\sigma \leq KC^2 \int |f|^p w d\sigma,$$

and since the union of the A_j is $[-\pi, \pi)$ this last inequality implies

$$\int |Tf|^p w d\sigma \leq nKC^2 \int |f|^p w d\sigma.$$

The dual of Lemma 6 is:

LEMMA 7. Let $w, w_1, \dots, w_n$ be non-negative summable functions such that

$$(23) \quad w \leq Cw_j \quad \text{a.e. on } [-\pi, \pi]$$

and suppose there are measurable sets A_j ($j = 1, \dots, n$) whose union is $[-\pi, \pi)$ such that

$$(24) \quad w_j \leq Cw \quad \text{a.e. on } A_j$$

where C is a positive constant.

Then if T is bounded in $L_{w_j}^p$ ($j = 1, \dots, n$), T is also bounded in L_w^p .

Proof. Lemma 3 and Lemma 6.

It is curious that Lemma 6 remains true if we assume only that T is bounded when restricted to $\mathbf{M}_m$ for some m (with a corresponding change in the conclusion), but that Lemma 7 is not true in this setting (Theorem 7 provides counterexamples).

THEOREM 4. *Let u and v be bounded real functions with $\|v\|_\infty < \pi/2$, and let $x_1, \dots, x_n$ be distinct points modulo 2π . If $p \geq 2$, T is bounded in $\mathbf{L}_w^p$ for*

$$(25) \quad w = e^{u+\tilde{v}} \prod_{j=1}^n |e^{ix_j} - e^{ix}|^{s_j}$$

where $0 \leq s_j \leq p-2$. If $1 < p \leq 2$, T is bounded in $\mathbf{L}_w^p$ for

$$w = e^{u+(p-1)\tilde{v}} \prod_{j=1}^n |e^{ix_j} - e^{ix}|^{s_j}$$

where $p-2 \leq s_j \leq 0$.

Proof. Suppose first p is an even integer ($p = 2m$), $n = 1$, and w is given by (25). If $s_1 = 0$, T is bounded in $\mathbf{L}_w^2$, and therefore T is bounded in $\mathbf{L}_w^p$ (Lemma 4). If $s_1 = p-2$, (17) and (18) with⁽⁵⁾

$$D = e^{\tilde{v}-iv} |e^{ix_1} - e^{ix}|^{p-2} e^{i(m-1)x}$$

show that T , restricted to $\mathbf{M}_m$, is bounded in $\mathbf{L}_w^2$, and therefore T , operating on all trigonometric polynomials, is bounded in $\mathbf{L}_w^p$. Using Lemma 5 to interpolate between $s_1 = 0$ and $s_1 = p-2$ shows T is bounded in $\mathbf{L}_w^p$ for $0 \leq s_1 \leq p-2$.

We have shown the theorem is true whenever p is an even integer and $n = 1$. Using Lemma 5 to interpolate between even integers gives the first part of the theorem with $n = 1$. The restriction $n = 1$ may be removed by using Lemma 7. Indeed, if

$$w_j = e^{u+\tilde{v}} |e^{ix_j} - e^{ix}|^{s_j}$$

and $0 \leq s_j \leq p-2$, the weights $w, w_1, \dots, w_n$ satisfy conditions (23) and (24) and T is bounded in $\mathbf{L}_{w_j}^p$.

The second part of the theorem may be obtained from the first part and Lemma 3.

The weights given by (25) with $n = 1$ are of the form (19). On the other hand we do not know if there are functions given by (19) which are essentially different than those given by (25). Relative to this see [8, p. 133].

With $s_j = 0$ Theorem 4 becomes the theorem of Gapoškin mentioned in the Introduction. Theorem 4 presents Gapoškin's theorem in a different form than that in which it was given in the Introduction, but the two versions are equivalent

⁽⁵⁾ $e^{\tilde{v}-iv}$ belongs to $\mathbf{H}^1$ [14, p. 277] and so does $|e^{ix_1} - e^{ix}|^{p-2} e^{i(m-1)x}$ since this belongs to $\mathbf{F}$.

since the weights involved differ only by multiplication by factors bounded from above and below.

If v is the periodic function whose value is $r(x - \pi)/2$ for $0 < x < 2\pi$ where $-1 < r < 1$, Theorem 4 with $n = 1$ and $x_1 = 0$ becomes the theorem of Hardy-Littlewood-Babenko mentioned in the Introduction (since the function conjugate to v will be $-r \log |1 - e^{ix}|$). Theorem 4 also presents the Hardy-Littlewood-Babenko theorem in a different form than that in which it was given in the introduction. Again the different versions are equivalent since the weights involved differ only by multiplication by factors bounded from above and below.

3. A condition for T to be bounded.

1. Define the following inner products in L_μ^p :

$$(f, g) = \int |f|^{p-2} f \bar{g} d\mu,$$

$$\langle f, g \rangle = \int |fg|^{r-1} f \bar{g} d\mu$$

where $r = p/2$. If $p = 2$, these two inner products coincide and give the usual Hilbert space inner product. Otherwise they are distinct and each possess one of the distinguishing characteristics of an inner product: the first is conjugate-linear in the second variable and the second is conjugate-symmetric.

LEMMA 8 (⁶). Suppose A and B are sets of functions in L_μ^p such that

$$\int |f|^p d\mu = 1$$

for all $f \in A \cup B$, and $f \in A$ ($f \in B$) implies $cf \in A$ ($cf \in B$) for all complex numbers c with modulus one. Then

$$(26) \quad \inf \left(\int |f + g|^p d\mu \right)^{1/p} > 0$$

if and only if

$$(27) \quad \sup |(f, g)| < 1$$

where the infimum and supremum are taken over all $f \in A$, $g \in B$.

Proof. Let $f \in A$, $g \in B$. Then (Hölder inequality)

$$\left| \int (f + \bar{g}) |f|^{p-2} f d\mu \right| \leq \left(\int |f + g|^p d\mu \right)^{1/p}.$$

(⁶) For $p = 2$ this lemma is in [8, p. 129]. For $p = 2$ (26) and (27) are related by $2 - 2\rho = \tau^2$ where ρ is the supremum (27) and τ the infimum (26).

Since the left-hand side of this inequality is $1 + (f, g)$ we have

$$(28) \quad |1 + (f, g)| \leq \left(\int |f + g|^p d\mu \right)^{1/p}.$$

Suppose (27) fails. Then there are functions $f \in \mathbf{A}$, $g \in \mathbf{B}$ such that (f, g) is close to 1, and therefore, because of (28), the norm of $f + g$ will be close to 2. Since $1 < p < \infty$, L_μ^p is uniformly convex [4, p. 403], and therefore the norm of $f - g$ must be close to 0. Thus (26) also fails. We have shown that (26) implies (27).

On the other hand, it is clear from (28) that (27) implies (26).

LEMMA 9. *Let $\mathbf{A}$ and $\mathbf{B}$ be as in Lemma 8. Then the infimum (26) is positive if and only if*

$$(29) \quad \sup |\langle f, g \rangle| < 1$$

where the supremum is taken over all $f \in \mathbf{A}$, $g \in \mathbf{B}$.

Proof. We have

$$|(f, g) - \langle f, g \rangle| \leq \int ||f|^{p-1}|g| - |f|^r|g|^r| d\mu.$$

The right-hand side of this inequality does not exceed

$$\int ||f|^{p-1}|g| - |f|^p| d\mu + \int ||f|^{2r} - |f|^r|g|^r| d\mu.$$

The first integral in this sum is bounded by

$$\left(\int |f|^p d\mu \right)^{1/p'} \left(\int ||g| - |f|^p| d\mu \right)^{1/p}$$

and the second by

$$\left(\int |f|^p d\mu \right)^{1/2} \left(\int ||f|^r - |g|^r|^2 d\mu \right)^{1/2}.$$

Therefore for $f \in \mathbf{A}$, $g \in \mathbf{B}$

$$(30) \quad \begin{aligned} & |(f, g) - \langle f, g \rangle| \\ & \leq \left(\int ||g| - |f|^p| d\mu \right)^{1/p} + \left(\int ||f|^r - |g|^r|^2 d\mu \right)^{1/2}. \end{aligned}$$

If $\langle f, g \rangle$ is close to 1, then

$$\int ||f|^r - |g|^r|^2 d\mu$$

must be close to 0, and this implies that

$$\int ||f| - |g|^p| d\mu$$

is also close to 0. Therefore if $\langle f, g \rangle$ is close to 1, the inequality (30) shows that (f, g) is also close to 1.

On the other hand if (f, g) is close to 1, then

$$\int |f - g|^p d\mu$$

is close to 0 ((28) and uniform convexity), and therefore both

$$\int ||f| - |g|^p| d\mu, \quad \int ||f|^r - |g|^r|^2 d\mu$$

are close to 0. Thus if (f, g) is close to 1, the inequality (30) shows that $\langle f, g \rangle$ is also close to 1.

We have shown (for $f \in \mathbf{A}$, $g \in \mathbf{B}$) that if either (f, g) or $\langle f, g \rangle$ is close to 1, then the other is also. Lemma 9 now follows from Lemma 8.

2. We will assume from now on that $\log w$ is summable since this must be so if T , when restricted to $\mathbf{M}_m$, is bounded in $\mathbf{L}_w^p$.

For $0 < s < \infty$, $\mathbf{H}^s$ will denote the closure in the metric of $\mathbf{L}_\sigma^s$ of $\mathbf{F}$ [14, p. 271]. Thus if $s \geq 1$, $\mathbf{H}^s$ consists of those functions in $\mathbf{L}_\sigma^s$ whose Fourier coefficients vanish for negative indices.

We can now give a condition analogous to (3) for $p > 1$.

THEOREM 5. *T , when restricted to $\mathbf{M}_m$, is bounded in $\mathbf{L}_w^p$ if and only if*

$$(31) \quad \sup \left| \int |f|^{r-1} f e^{imx} e^{-i(1/r) \log w} d\sigma \right| < 1$$

where $r = p/2$ and the supremum is taken over all f which belong to $\mathbf{H}^r$ and satisfy

$$\int |f|^r d\sigma = 1.$$

Proof. Our proof is the same as that given in [8] for $p = 2$. Since $\mathbf{F}_m = e^{imx}\mathbf{F}$ and $\bar{\mathbf{P}} = \mathbf{F}$, we have from (8) and Lemma 9 that T , restricted to $\mathbf{M}_m$, is bounded in $\mathbf{L}_w^p$ if and only if

$$(32) \quad \sup \left| \int |fg|^{r-1} f g e^{imx} w d\sigma \right| < 1$$

where the supremum is taken over all $f, g \in \mathbf{F}$ with

$$\int |f|^p w d\sigma = \int |g|^p w d\sigma = 1.$$

Since $\log w$ is summable, the conjugate function $\log^{\sim} w$ is defined. Let D be the function

$$D = w^{1/p} e^{i(1/p) \log^{\sim} w}.$$

Then D belongs to $\mathbf{H}^p$ [14, p. 277] and, moreover, D is an outer function [3; 10].

A function $F \in \mathbf{H}^p$ is said to be outer if

$$\log \left| \int F d\sigma \right| = \int \log |F| d\sigma.$$

The outer functions $F \in \mathbf{H}^p$ are characterized by the property that the linear set fF , where f ranges over $\mathbf{F}$, is dense in $\mathbf{H}^p$. Proofs of this are in [3] for $p = 2$ and in [10] for $p = 1$. There is no difficulty in adapting these proofs to other exponents p .

Since

$$w = |D|^2 |D|^{r-1} D^2 e^{-i(1/r) \log^{\sim} w}$$

the integral in (32) may be written

$$\int |(fD)(gD)|^{r-1} (fD)(gD) e^{imx} e^{-i(1/r) \log^{\sim} w} d\sigma,$$

and therefore, because D is outer, the supremum (32) is equal to

$$(33) \quad \sup \left| \int |fg|^{r-1} f g e^{imx} e^{-i(1/r) \log^{\sim} w} d\sigma \right|$$

where this supremum is taken over all $f, g \in \mathbf{H}^p$ with

$$\int |f|^p d\sigma = \int |g|^p d\sigma = 1.$$

Here we have also used that the mapping from $\mathbf{L}_\sigma^p$ to $\mathbf{L}_\sigma^2$ which takes f into $|f|^{r-1}f$ is continuous in the metric topologies.

To complete the proof of the theorem we need only to observe that the supremum (31) is the same as (33) since the unit ball of $\mathbf{H}^r$ is equal to the product of the unit ball of $\mathbf{H}^{2r}$ with itself [14, p. 275].

3. If $w = 1$, Theorem 5 states that the condition

$$(34) \quad \sup \left| \int |f|^{r-1} f e^{ix} d\sigma \right| < 1$$

where $r = p/2$ and f is in the unit ball of $\mathbf{H}^r$ is necessary and sufficient for T to be bounded in $\mathbf{L}_\sigma^p$. In this case the proof is simpler since now no appeal to Beurling's theorem on outer functions is necessary. There are two values of p for which it is easy to see that (34) holds. If $p = 2$, the supremum (34) is obviously zero. If $p = 4$, the supremum (34) is bounded by $1/\sqrt{2}$.

Indeed, if fe^{ix} and $|f|$ have Fourier series

$$\sum_{n \geq 1} a_n e^{inx}, \quad \sum_{n \geq 1} b_n e^{inx},$$

then by the Parseval formula

$$\int |f| fe^{ix} d\sigma = \sum_{n \geq 1} a_n \bar{b}_n,$$

and now the Schwarz inequality applied to this last sum gives

$$\left| \int |f| fe^{ix} d\sigma \right| \leq \left(\sum_{n \geq 1} |a_n|^2 \right)^{1/2} \left(\sum_{n \geq 1} |b_n|^2 \right)^{1/2}.$$

Since $|f|$ is real, $b_{-n} = \bar{b}_n$, and hence

$$\sum_{n \geq 1} |b_n|^2 \leq (1/2) \sum |b_n|^2.$$

But

$$1 = \int |f|^2 d\sigma = \sum_{n \geq 1} |a_n|^2 = \sum |b_n|^2$$

and thus

$$\left| \int |f| fe^{ix} d\sigma \right| \leq 1/\sqrt{2}.$$

For other values of p it is not at all obvious that (34) holds, as indeed it must.

4. As indicated in the Introduction, we do not know how to describe, for a given p and m , those weights w for which (31) holds.

We will conclude this section with two lemmas which will be needed in the sequel.

LEMMA 10. *Suppose T is bounded in L_w^p , and let g be any real continuous function. Then $e^{\tilde{g}} w$ is summable and T is bounded in $L_{e^{\tilde{g}} w}^p$.*

Proof. Suppose first that $p \leq 2$. By using Lemma 4 and interpolating it follows that T is bounded in L_w^r for $r \geq p$. In particular T is bounded in L_w^2 , and therefore w must be of the form (4). Since g is continuous we may write $g = g_1 + g_2$ where g_1 is a real trigonometric polynomial and g_2 is a real continuous function with $\|g_2\|_\infty$ as small as we wish. Now $e^{u+\tilde{g}_1}$ is bounded from above and below and $e^{\tilde{g}_2+v}$ is summable if $\|g_2 + v\|_\infty$ is smaller than $\pi/2$, and therefore $e^{\tilde{g}} w$ is summable. Moreover, since T is bounded in L_w^p , (31) must hold (with $m = 1$). Then (31) will continue to hold if $\log \tilde{w}$ is replaced by $g_2 + \log \tilde{w}$ and $\|g_2\|_\infty$ is sufficiently small, and this in turn implies that T is bounded in $L_{e^{\tilde{g}} w}^p$.

If $p > 2$ the conclusion follows from the case just considered and Lemmas 2 and 3.

LEMMA 11. *Suppose T is bounded in L_w^p . Then for s sufficiently close to one, w^s is summable and T is bounded in $L_{w^s}^p$.*

Proof. As in the proof of Lemma 10 we assume first that $p \leq 2$ and conclude that w must be of the form (4). Then w^s is summable for s close to one since $e^{s\tilde{v}}$ is summable as long as $\|sv\|_\infty < \pi/2$. To show that T is bounded in $L_w^{p,s}$ for s close to one it suffices to show that T is bounded in $L_{e^{s\tilde{v}}}^{p,s}$. But this is implied by the condition (31) (with $m = 1$) since v is bounded and T is bounded in $L_{e^{s\tilde{v}}}^{p,s}$.

If $p > 2$ the conclusion follows from the case just considered and Lemmas 2 and 3.

4. A special class of weights.

1. In this section we assume w is such that $\log \tilde{w}$ is continuous except for a finite number of simple discontinuities at $x_1, \dots, x_n$. Let r_j be the normalized jump of $\log \tilde{w}$ at x_j defined by

$$(35) \quad \pi r_j = \log \tilde{w}(x_j -) - \log \tilde{w}(x_j +).$$

For T operating on all trigonometric polynomials we have:

THEOREM 6. T is bounded in L_w^p if and only if

$$(36) \quad -1 < r_j < p - 1$$

for $j = 1, \dots, n$.

When T is restricted to M_2 a curious break appears in the admissible jumps. We have:

THEOREM 7. T , restricted to M_2 , is bounded in L_w^p if and only if

$$-1 < r_j < p - 1$$

for $j = 1, \dots, n$ with at most one exception r_k which satisfies

$$-1 < r_k < p - 1 \quad \text{or} \quad p - 1 < r_k < 2p - 1.$$

For $p = 2$ Theorem 6 is in [8, p. 136] and Theorem 7, in a less general form, is in [8, p. 133](7).

2. We begin with the proof of Theorem 6. Let J_j be the periodic function whose value is $(x - x_j + \pi)/2$ for $x_j - 2\pi < x < x_j$ and whose value is 0 for $x = x_j$. Then J_j is continuous except for a jump at x_j and, moreover, J_j is the conjugate function of $\log |e^{ix_j} - e^{ix}|$. With $\log \tilde{w}$ normalized by

$$2 \log \tilde{w}(x_j) = \log \tilde{w}(x_j -) + \log \tilde{w}(x_j +),$$

the function $\log \tilde{w} - \sum_{j=1}^n r_j J_j$ is continuous, and therefore

$$(37) \quad w = e^{c + \tilde{g}} \prod_{j=1}^n |e^{ix_j} - e^{ix}|^{r_j}$$

where c is a real number and g is continuous.

(7) Flett [5, p. 136] has shown that if $w = |x|^r$ for $-\pi < x < \pi$, then T , when restricted to odd trigonometric polynomials, is bounded in L_w^p for $-1 < r < 2p - 1$.

Suppose now (36) holds. Because of Lemma 3 we may assume $p \geq 2$. Then $r_j = s_j + t_j$ where $0 \leq s_j \leq p-2$ and $-1 < t_j < 1$. This splitting of r_j , the representation (37), and the known sufficiency of the condition (36) for $p = 2$, show that w is of the form (25).

We have shown that the condition (36) is sufficient. To see that this condition is also necessary assume T is bounded in L_w^p . Because of the representation (37) and Lemma 10,

$$(38) \quad \prod_{j=1}^n |e^{ix_j} - e^{ix}|^{r_j}$$

is summable. Therefore each r_j is greater than -1 . Denoting the function (38) by w' , we also have (Lemma 10) that T is bounded in $L_{w'}^p$. Hence

$$\prod_{j=1}^n |e^{ix_j} - e^{ix}|^{-r_j/(p-1)}$$

is summable (Lemma 2) and therefore each r_j is smaller than $p-1$. This completes the proof of Theorem 6.

3. We begin the proof of Theorem 7 with a lemma.

LEMMA 12. Suppose T is bounded in L_w^p and let

$$w = w' |1 + e^{ix}|^p.$$

Then T , restricted to M_2 , is bounded in L_w^p .

Conversely, if T , restricted to M_2 , is bounded in L_w^p and if

$$w' = w |1 + e^{ix}|^{-p}$$

is summable, then T , operating on all trigonometric polynomials, is bounded in L_w^p .

Proof. Since the function conjugate to $\log w$ for $-\pi < x < \pi$ is $\log \tilde{w}' + (px/2)$, the lemma is a corollary of Theorem 5.

4. That the condition of Theorem 7 on the r_j is sufficient is immediate. If $-1 < r_k < p-1$, the conclusion follows from Theorem 6. If $p-1 < r_k < 2p-1$, the conclusion follows from the representation (37), Theorem 6, and first part of Lemma 12 (we may assume $x_k = \pi$ since nothing is altered by a translation).

5. We now turn to the necessity of the condition on the r_j .

Lemma 10 remains true when T is restricted to M_2 . Indeed, if T , restricted to M_2 , is bounded in L_w^p , then T , operating on all trigonometric polynomials, is bounded in L_w^{2p} (Lemma 4). Therefore $e^{\tilde{g}} w$ is summable, and now (arguing as at the end of the proof of Lemma 10) the condition (31) (with $m = 2$) shows that T , restricted to M_2 , is bounded in $L_{e^{\tilde{g}} w}^p$. Thus because of the representation (37) we may assume

$$(39) \quad w = \prod_{j=1}^n |e^{ix_j} - e^{ix}|^{r_j}$$

where the x_j are distinct modulo 2π . Since w is summable we always have $-1 < r_j$ for $j = 1, \dots, n$.

We are assuming that T , restricted to $\mathbf{M}_2$, is bounded in $\mathbf{L}_w^p$. If $-1 < r_j < p-1$ for $j = 1, \dots, n$, there is nothing to prove. Suppose then one of the r_j , r_1 for example, is greater than or equal to $p-1$.

We assert first that $r_1 < 2p-1$ and $-1 < r_j < p-1$ for $j = 2, \dots, n$.

If $r_1 > p-1$, then

$$w' = w |e^{ix_1} - e^{ix}|^{-p}$$

is summable, and therefore by the second part of Lemma 12 T is bounded in $\mathbf{L}_{w'}^p$, and the assertion now follows from Theorem 6. If $r_1 = p-1$, w' is no longer summable and the argument just given is not valid. However Lemma 11 also remains true when T is restricted to $\mathbf{M}_2$ (use Lemma 4 to show that $w = e^u + \tilde{v}$ where u and v are bounded and then argue exactly as in the proof of Lemma 11). This throws us back to the case just discussed since we may consider w^s where $s > 1$, and the assertion follows.

6. The status of the r_j is now $p-1 \leq r_1 < 2p-1$ and $-1 < r_j < p-1$ for $j = 2, \dots, n$. To complete the proof we must show $r_1 > p-1$. Suppose, on the contrary, $r_1 = p-1$. We will show this leads to a contradiction.

Consider $\mathbf{L}_{w^{-1/(p-1)}}^{p'}$. Since w is summable, it follows from the Hölder inequality that every function in $\mathbf{L}_{w^{-1/(p-1)}}^{p'}$ is summable. We now claim that if g belongs to $\mathbf{L}_{w^{-1/(p-1)}}^{p'}$, then for a suitable constant c , $c + \tilde{g}$ also belongs to $\mathbf{L}_{w^{-1/(p-1)}}^{p'}$.

Indeed, denote by S the operation which maps trigonometric polynomials into their conjugates ($Sf = \tilde{f}$). Recall that $\mathbf{M}_0$ is the set of trigonometric polynomials with mean value zero, and observe that T restricted to $\mathbf{M}_0$ is bounded in $\mathbf{L}_w^p$ since T restricted to $\mathbf{M}_2$ is bounded in $\mathbf{L}_w^p$. Since

$$2Tf = f + iSf$$

for $f \in \mathbf{M}_0$, S when restricted to $\mathbf{M}_0$ is also bounded in $\mathbf{L}_w^p$. Since $w^{-1/(p-1)}$ is not summable (this because $r_1 = p-1$) the linear set $\mathbf{M}_0$ is dense in $\mathbf{L}_w^p$ (Lemma 1), and thus the conjugacy operation S has a unique continuous extension (also denoted by S) to all of $\mathbf{L}_w^p$. Represent the dual space of $\mathbf{L}_w^p$ by $\mathbf{L}_{w^{-1/(p-1)}}^{p'}$ with the duality given by

$$\int f \tilde{g} d\sigma$$

where $f \in \mathbf{L}_w^p$ and $g \in \mathbf{L}_{w^{-1/(p-1)}}^{p'}$, and consider the operator S^* adjoint to S . S^* is a bounded linear operator mapping $\mathbf{L}_{w^{-1/(p-1)}}^{p'}$ into itself, and is defined by

$$(40) \quad \int f \overline{S^* g} d\sigma = \int S f \bar{g} d\sigma$$

for all $f \in L_w^p$, $g \in L_{w^{-1/(p-1)}}^{p'}$. Now if

$$\sum a_n e^{inx}$$

is the Fourier series of the function $g \in L_{w^{-1/(p-1)}}^{p'}$, then it follows from (40) that

$$-c + \sum i \varepsilon_n a_n e^{inx}$$

is the Fourier series of $S^* g$ with

$$(41) \quad c = - \int \overline{S1} g d\sigma.$$

Thus $g \in L_{w^{-1/(p-1)}}^{p'}$ implies $c + \tilde{g} \in L_{w^{-1/(p-1)}}^{p'}$ with c given by (41).

7. Since it is no restriction to assume $x_1 = 0$ we have

$$w^{-1/(p-1)} = \left| 1 - e^{ix} \right|^{-1} \prod_{j=2}^n \left| e^{ix_j} - e^{ix} \right|^{-r_j/(p-1)}.$$

Since the r_j are smaller than $p-1$ for $j = 2, \dots, n$,

$$\prod_{j=2}^n \left| e^{ix_j} - e^{ix} \right|^{-r_j/(p-1)}$$

is summable. Moreover this function is bounded from above and below in a neighborhood of the origin.

The observations which follow appear in [7, p. 372]. See also [14, pp. 186-189]. For $r > 0$, the trigonometric series

$$(42) \quad \sum_{n \geq 2} n^{-1} (\log n)^{-r} \sin nx$$

is the Fourier series of a continuous function g_r such that

$$g_r(x) \cong (\pi/2) \left| \log x \right|^{-r} \quad \text{for } x \rightarrow 0+.$$

Therefore, since $w^{-1/(p-1)}$ in a neighborhood of the origin behaves like $|x|^{-1}$ and is summable off this neighborhood, $g_r \in L_{w^{-1/(p-1)}}^{p'}$ if $rp' > 1$. On the other hand, the function conjugate to g_r is continuous except at the origin, and there

$$\tilde{g}_r(x) \cong -(1/(1-r)) \left| \log x \right|^{1-r} \quad \text{for } x \rightarrow 0+$$

if $r < 1$. Thus if $0 < r < 1$, the integral

$$\int \left| c + \tilde{g}_r \right|^{p'} w^{-1/(p-1)} d\sigma$$

diverges for all constants c . Choosing r such that $0 < r < 1$ and $rp' > 1$ we get a contradiction. This completes the proof of Theorem 7.

8. For $p = 2$, the method used in [8] to show that the value $p - 1$ is not an admissible jump is to actually evaluate the supremum (31) when $w = |1 + e^{ix}|^r$. This is an alternative which does not seem to be available if $p \neq 2$.

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