INVARIANT MANIFOLDS
FOR SOME AUTONOMOUS SYSTEMS([1])

BY

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1. **Introduction.** Consider the real differential system

$$(\mathscr{S}_n) \qquad\qquad dx/dt = f(x,\mu)$$

where μ is a real parameter, x is an n-vector with components x^i, and $f(x,\mu)$ is a real n-dimensional vector valued function of x and μ with components $f^i(x,\mu)$. Assume that the system has a periodic solution $x = \phi(t)$ for all μ and that all the characteristic exponents of $(\mathscr{S}_n)$ based on ϕ have zero real parts for $\mu = \mu_0$. Following Poincaré we call such a value of μ a bifurcation value for the system. Under certain restrictions we show that an invariant $(n-1)$-manifold in the x-space bifurcates (branches off) from the fixed periodic solutions for values of μ near μ_0. This process will be known as bifurcation from a periodic solution. Poincaré [10] has shown that in the plane such behavior may occur.

Various other authors have studied manifolds defined by solutions of ordinary differential equations. However, most of this work is concerned with the perturbation problem. For an extensive bibliography on this topic see W. T. Kyner [6]. The basic theorem of N. Levinson [9] established the existence of an invariant manifold based on $x = \phi(t)$ in the extended phase space. This manifold was topologically a torus, that is, $\prod_1^n S^1 = S^1 \times S^1 \times \cdots \times S^1$ (S^n is the unit n-sphere) in the (x,t)-space. It was assumed that the real parts of $n-1$ of the characteristic exponents based on $\phi(t)$ were negative for all values of the parameter μ. This theorem has been generalized by various authors including Hufford [4], Kyner [5; 7], Diliberto and Hufford [3], and Diliberto [1; 2]. The assumption of stability (the real parts of $n-1$ of the characteristic exponents are negative) has been replaced by the assumption of conditional stability (the real parts of $n-1$ of the characteristic exponents are nonzero). These results differ from ours in three respects. The first is that our invariant manifold lies in the x-space;

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the perturbation problem leads to a manifold in the (x, t)-space. Secondly, we show the existence of a manifold which is topologically $S^{n-2} \times S^1$; the invariant manifold in the perturbation problem is topologically $\prod_1^n S^1$. The last difference is the nature of the assumptions made concerning the characteristic exponents.

The technique used to establish our results consists of first constructing a transversal (see Lefschetz [8]) to the periodic solution $x = \phi(t)$ of $(\mathscr{S}_n)$ at any point $\bar{x} = \phi(\bar{t})$ on the trajectory. Then we consider the mapping of this transversal into itself which is defined by following a solution of $(\mathscr{S}_n)$ starting at a point x on the transversal until it strikes the transversal again at a point y. For points of the transversal near $\bar{x}$ this defines a transformation T_μ of the transversal into itself which has the fixed point $\bar{x}$ corresponding to the periodic solution $x = \phi(t)$. It is shown that under certain conditions there is an $(n-2)$-dimensional manifold M_μ, not containing $\bar{x}$, embedded in the transversal which is left invariant by the transformation T_μ. This is done in §2 by applying the Schauder fixed point theorem to conditionally compact, convex subsets of the Banach spaces of C^0 and C^1 functions defined on the compact differentiable manifold S^{n-2}. This shows the existence of an invariant $(n-1)$-dimensional manifold for the system $(\mathscr{S}_n)$. This manifold is topologically the product of the $(n-2)$-dimensiona manifold M_μ and the one sphere S^1. In fact, M_μ is shown to be topologically equivalent to the $(n-2)$-sphere S^{n-2}.

The mapping T_μ of the transversal into itself is, by the regularity properties of solutions of ordinary differential equations, a homemorphism of a neighborhood of the point $\bar{x}$ into itself. The mapping T_μ will have the form

$$v = D(\mu)u + F(u,\mu),$$

after suitable normalization, where $D(\mu)$ is a constant $(n-1)$ by $(n-1)$ real matrix, u is an $(n-1)$-vector, and $F(u,\mu)$ is an $(n-1)$-dimensional vector valued function of higher order in u. In §2 we show that after placing suitable restrictions on $D(\mu)$ and $F(u,\mu)$ there exists an invariant $(n-2)$-manifold which is topologically S^{n-2} and which contains $u = 0$ $(x = \bar{x})$ in its interior.

The characteristic roots of $D(\mu)$ are known to be the characteristic multipliers (other than the one equal to unity) of the variational system of $(\mathscr{S}_n)$ based on the periodic solution $\phi(t)$ (see Lefschetz [8]). Thus conditions imposed on $D(\mu)$ are relatively easy to ascertain. It is, unfortunately, much more difficult to verify if conditions imposed on $F(u,\mu)$ are satisfied by the transversal map.

In order to fix the above ideas a little more firmly in mind we consider the following example.

1.1. EXAMPLE. Consider the differential system

$(\mathscr{S})$
$$x' = -y + x(r-1)\left[\mu - \mu(r-1) - (r-1)^2\right],$$
$$y' = x + y(r-1)\left[\mu - \mu(r-1) - (r-1)^2\right],$$

where $r^2 = x^2 + y^2$ and μ is a real parameter. The right hand side has continuous first derivatives everywhere, and therefore there exists a unique solution through any given noncritical point. The polar equations for $(\mathscr{S})$ are

$(\mathscr{P})$
$$r' = r(r-1)[\mu - \mu(r-1) - (r-1)^2],$$
$$\theta' = 1.$$

Since $dr/dt < 0$ for r sufficiently large the solutions of $(\mathscr{S})$ are defined for all non-negative time. Also, $(\mathscr{P})$ has $r(t) = 1$, $\theta(t) = t + \theta_0$ as a periodic solution for all μ where θ_0 is a constant. For convenience we take $\theta_0 = 0$. Thus any open interval of the positive x-axis containing the point $(1,0)$ is transverse to this solution at $(1,0)$.

The variational equation of $(\mathscr{S})$ based on this periodic solution is $(\mathscr{V})$: $v' = A(t)v$ where

$$A(t) = \begin{pmatrix} \mu\cos^2 t & -1 + \tfrac{1}{2}u\sin 2t \\ 1 + \tfrac{1}{2}u\sin 2t & \mu\sin^2 t \end{pmatrix}.$$

The characteristic multipliers λ_1 and λ_2 of $(\mathscr{V})$ satisfy

$$\lambda_1\lambda_2 = \exp\left\{ \int_0^{2\pi} \mathrm{Tr}[A(s)]ds \right\} = \exp[2\mu\pi].$$

But as it is already known that one of the λ's is equal to unity, say $\lambda_2 = 1$, we have $\lambda_1 = \exp[2\mu\pi]$.

In this example every solution starting at a point $(x,0)$, $x > 0$, strikes the positive x-axis again after a time 2π seconds. Thus the map T_μ has the form

$$\bar{x} - 1 = \exp[2\mu\pi](x-1) - Q(x,\mu),$$

where Q is of higher order in $(x-1)$. Here x is the x-coordinate of the initial value of the solution of $(\mathscr{S})$ starting at $(x,0)$ at time $t = 0$, and $\bar{x}$ is the position on the x-axis of the solution at time $t = 2\pi$. If we let $\omega = \exp[2\mu\pi] - 1$, then the linear terms of T_μ are exactly of the form discussed in Theorem 2.5 of §2. However, it is not obvious that the nonlinear terms are of the desired form.

Let $\rho = r - 1$ in the equation for r' as given in $(\mathscr{P})$. Then

$$\rho' = \mu\rho - \rho[(1+\mu)\rho^2 + \rho^3], \quad \theta' = 1$$

give the new system. The solution $\rho(t,u,\mu)$ which starts at $u = x - 1$ at time $t = 0$ satisfies the integral equation

$$\rho(t,u,\mu) = ue^{\mu t}\exp\left\{ \int_0^t -[(1+\mu)\rho^2(\tau,u,\mu) + \rho^3(\tau,u,\mu)]d\tau \right\}.$$

From this it is not difficult to show that

$$Q(u,\mu) = u e^{2\mu\pi} \left\{ 1 - \exp \int_0^{2\pi} - [(1 + \mu)\rho^2(\tau) + \rho^3(\tau)]d\tau \right\}$$

satisfies the conditions of Theorem 2.5 (or the conditions of the remark following Theorem 2.5).

The above argument shows that there is a 0-sphere in the transversal which is left invariant by T_μ, that is, in this case, there is an invariant point on either side of $x = 1$, $y = 0$ for small positive values of μ. Thus there are two periodic solutions branching off from the periodic solution $r(t) = 1$, $\theta(t) = t$ for small positive μ and, moreover, these both approach $r = 1$, $\theta = t$ as μ approaches zero.

2. **Invariant manifolds.** We consider here the problem of finding manifolds which are left invariant by a transformation.

PROBLEM. Let the mapping

$$T_\mu: \ y = D(\mu)x - Q(x,\mu)$$

be a homeomorphism of a neighborhood V of the origin in R^{n+1} into R^{n+1}. $D(\mu)$ is an $(n + 1)$ by $(n + 1)$ matrix defined for all $\mu \in [0,\mu_0]$ and $Q(x,\mu)$ is a vector valued function defined on $V \times [0,\mu_0]$. We wish to find conditions which imply the existence of a smooth surface S_μ having the properties:

(P$_1$) S_μ is homeomorphic to S^n,
(P$_2$) S_μ is convex with respect to the origin,
(P$_3$) S_μ is left invariant by T_μ, that is, $T_\mu S_\mu = S_\mu$.

The considerations will be broken into two parts. In the first we consider the case for any n with very restrictive assumptions on the form of T_μ. In the second part we take $n = 1$ and obtain an invariant surface under "less" restrictive conditions on the form of T_μ.

I. GENERAL CONSIDERATIONS, LIPSCHITZ SURFACES.

ASSUMPTIONS. It is assumed that $D(\mu) = (1 + \mu)A(\mu)$ where $A(\mu)$ is an orthogonal matrix for all $\mu \in [0,\mu_0]$, and $Q(x,\mu)$ satisfies the following conditions.

(α_1) $Q(x,\mu) = P(x,\mu) A(\mu)x$ where P is a real valued function of x and μ on $V \times [0,\mu_0]$.

(α_2) $0 < P(x,\mu) \leqq M_0 |x|$ for $0 < |x| \leqq r_0$, and $0 \leqq \mu \leqq \mu_0$.

(α_3) $(\partial P/\partial x^i)(x,\mu)$ is a continuous function of (x,μ) on $\bar{B}(r_0) \times [0,\mu_0]$ for $i = 1, 2, \cdots, n + 1$. ($\bar{B}(r)$ is the closure of the open ball $B(r)$ or radius r about the origin.)

(α_4) The directional derivative of P along any radial line is positive for each fixed $\mu \in [0,\mu_0]$ and $x \neq 0$, that is, $(\mathrm{grad}\, P(x,\mu),x) > 0$ for $x \neq 0$.

We shall show that the above assumptions imply the existence of a Lipschitz surface S_μ having the properties (P$_1$)-(P$_3$). However, we first give two examples.

The first example shows that the condition (α_1) that $Q(x,\mu)$ be factorable is too stringent. Later when we attempt to show the existence of C^1-surfaces this example will show why some of the hypotheses are needed. The second example is the prototype of transformations T_μ which satisfy the above hypotheses.

2.1. EXAMPLE. Consider the mapping T defined by

$$y^i = (1 + \mu)x^i - (x^i)^3 \qquad\qquad (i = 1, 2)$$

where (x^1, x^2) are rectangular coordinates in R^2 and $\mu \geq 0$. This mapping is a homeomorphism of the open set $V : |x^i| < (1 + \mu/3)^{1/2}$ $(i = 1, 2)$ into itself for $0 \leq \mu < \frac{1}{2}$. T leaves invariant a unique continuous simple closed curve S in V satisfying properties (P_1)-(P_3). S is, in fact, the square given by $\max(|x^1|, |x^2|) = \mu^{1/2}$. This curve is clearly of class C^0 and also satisfies a Lipschitz condition. However, the factorization hypothesis (α_1) is *not* satisfied.

2.2. EXAMPLE. (*Prototype*). Let $A(\mu) = I$, the identity matrix, and let $Q(x,\mu) = x|x|^2$. Then $P(x,\mu) = |x|^2$, and $(\operatorname{grad} P(x,\mu),x) = 2|x|^2$. Thus the hypotheses ($\alpha_1$) $-$ (α_4) are clearly satisfied.

2.3. LEMMA. *Let T_μ be defined as above and suppose (α_1)-(α_3) are satisfied. Then there exists a positive number μ_1, $0 < \mu_1 \leq \mu_0$, such that for a fixed $\mu \in (0,\mu_1)$ there are two numbers $r_1(\mu)$ and $r_2(\mu)$ having the properties:*
  (i) $0 < r_1 \leq r_2 \leq r_0$ *where* $\bar{B}(r_0) \subset V$;
  (ii) $r_1(\mu), r_2(\mu) \to 0$ *as* $\mu \to 0$;
  (iii) *the annulus* $A_\mu : r_1(\mu) \leq |x| \leq r_2(\mu)$ *is mapped into itself by the mapping* T_μ.

Proof. Since $0 < P(x,\mu) \leq M_0|x|$ for $|x| \neq 0$, assume r_0 has been chosen sufficiently small that $P(x,\mu) < \frac{1}{2}$ in $\bar{B}(r_0)$.

Let $p_1(r,\mu) = \inf_{|x|=r} P(x,\mu)$ and $p_2(r,\mu) = \sup_{|x|=r} P(x,\mu)$ for $x \in \bar{B}(r_0)$. Then

$$|x|[1 + \mu - p_2(|x|,\mu)] \leq |y| \leq |x|[1 + \mu - p_1(|x|,\mu)].$$

Also, it can be shown that

$$\inf_{|x|=r} (\operatorname{grad} P(x,\mu),x/|x|) \;\leq\; D\,p_1(r,\mu), D\,p_2(r,\mu)$$
$$\leq\; \sup_{|x|=r} (\operatorname{grad} P(x,\mu),x/|x|)$$

where Df stands for any one of the four derivates of a given function f. Thus $D[r(1 + \mu - p_i(r,\mu))] = 1 - O(r)$ as $r \to 0$ uniformly in μ. Hence, for r sufficiently small, $1 - O(r) > 0$ and, as a consequence of the fact that a function having a positive derived number is strictly increasing, the functions $r[1 + \mu - p_i(r,\mu)]$ are strictly increasing. So we need to choose $r_1(\mu)$ and $r_2(\mu)$ such that $r_2[1 + \mu - p_1(r_2,\mu)] \leq r_2$ and $r_1[1 + \mu - p_2(r_1,\mu)] \geq r_1$. We choose these such that we have equality, that is, such that $p_1(r_2,\mu) = \mu$ and $p_2(r_1,\mu) = \mu$. Let $\bar{\mu}_0 = \min[p_1(r_0,\mu) : \mu \in [0,\mu_0]]$. Then $\bar{\mu}_0$ is positive. Let $\mu_1 = \min(\mu_0,\bar{\mu}_0)$. For

fixed $\mu \in (0, \mu_1]$ consider $p_1(r, \mu) - \mu$ as a function of r. Now $p_1(0, \mu) - \mu = -\mu < 0$, and $p_1(r_0, \mu) - \mu \geqq \mu_1 - \mu \geqq 0$. Thus there is a point r' such that $p_1(r', \mu) = \mu$. Let $r_2 = \inf [r' : p_1(r', \mu) = \mu]$. Then $p_1(r_2, \mu) = \mu$ and $0 < r_2 \leqq r_0$. Since $p_1(r_2, \mu) \leqq p_2(r_2, \mu)$ we may, in a similar manner, choose r_1, $0 < r_1 \leqq r_2$, such that $p_2(r_1, \mu) = \mu$.

We next consider the Banach space $C^0(S^n)$ of continuous functions defined on S^n with the uniform norm. Let A_μ denote the invariant annulus of Lemma 2.3. Let

$$\mathscr{F}(\mu) = [f \in C^0(S^n) : r_1(\mu) \leqq f \leqq r_2(\mu)].$$

The map T_μ induces a transformation of $\mathscr{F}(\mu)$ into itself as follows.

(i) There is a 1-1 correspondence between $\mathscr{F}(\mu)$ and the family of closed C^0 surfaces lying in A_μ which are convex with respect to the origin. If $f \in \mathscr{F}(\mu)$ and $u \in S^n$, then $x^i = f(u)u^i$ ($i = 1, 2, \cdots, n$) gives the corresponding point on $S(f)$, the surface associated with the function f.

(ii) T_μ maps $S(f)$ into another surface $T_\mu S(f)$ lying in A_μ. $T_\mu S(f)$ is a C^0 surface given parametrically by $v \in S^n$ as

$$y(v) = f(u)A(\mu)u[(1 + \mu) - P(f(u)u, \mu)]$$

where $v = A(\mu)u$.

(iii) The surface $T_\mu(S(f))$ given above corresponds to the C^0 function $g(v)$ defined on S^n by

$$g(v) = f(A^{-1}(\mu)v)[(1 + \mu) - P(f(A^{-1}(\mu)v)A^{-1}(\mu)v, \mu)].$$

Thus the mapping T_μ induces a mapping $\bar{T}_\mu : f(u) \to g(v)$ of $\mathscr{F}(\mu)$ into itself.

By hypothesis (α_4), $(\operatorname{grad} P(x, \mu), x) > 0$ for $x \neq 0$. Thus there is a constant K_1 such that

$$(\operatorname{grad} P(x, \mu), x) \geqq K_1 |x| \quad \text{for} \quad x \in A_\mu.$$

Also, we may assume without loss of generality that

$$r_0 |\operatorname{grad} P(x, \mu)| < \tfrac{1}{2} \quad \text{for} \quad (x, \mu) \in \bar{B}(r_0) \times [0, \mu_0].$$

2.4. LEMMA. *There exists a constant K_2 such that $|f(u_1) - f(u_2)| \leqq K_2 |u_1 - u_2|$ implies that the image function $g(v) = \bar{T}_\mu f(u)$ satisfies $|g(v_1) - g(v_2)| \leqq K_2 |v_1 - v_2|$ for $f \in \mathscr{F}(\mu)$.*

Proof. This follows directly from the mean value theorem and the preceding assumptions with $K_2 \geqq (r_2)^2 / 2 K_1 r_1 r_0$.

We now choose a subset $\mathscr{F}(\mu, K_2)$ of $\mathscr{F}(\mu)$ defined by $\mathscr{F}(\mu, K_2) = [f \in \mathscr{F}(\mu) : f$ satisfies a uniform Lipschitz condition with Lipschitz constant less than or equal to $K_2]$. We have just shown that $\mathscr{F}(\mu, K_2)$ is left invariant by $\bar{T}_\mu$. Moreover, this set is a compact, convex subset of $C^0(S^n)$. Now apply the Schauder fixed

point theorem and obtain the existence of a member f_μ of $\mathscr{F}(\mu,K_2)$ such that $\bar{T}_\mu f_\mu(u) = f_\mu(v)$. Now returning to the mapping $\bar{T}_\mu$ we see that the surface $S(f_\mu)$ is invariant under T_μ as a point set.

We may summarize with the following theorem.

2.5. THEOREM. *Let the mapping*

$$T_\mu : y = (1 + \mu) A(\mu)x - Q(x,\mu)$$

be a homeomorphism of a neighborhood V of the origin in R^{n+1} into R^{n+1}. Suppose $A(\mu)$ is an orthogonal matrix and that $Q(x,\mu)$ satisfies the conditions (α_1)-(α_4). Then for $\mu > 0$ sufficiently small there exists a surface S_μ satisfying properties (P_1)-(P_3). The surface S_μ is Lipschitz continuous and $S_\mu \to 0$ as $\mu \to 0$.

It should be noted that this theorem is also true in case T_μ is a homeomorphism of R^1 into itself. This can be seen by noting that the annulus which is mapped into itself is just two line segments, one on the negative x-axis, the other on the positive x-axis. Also, $\operatorname{sgn} T_\mu(x) = \operatorname{sgn} x$, so each segment is mapped in a continuous manner into itself by T_μ. Thus by the Brouwer fixed point theorem there is an invariant point in each segment, that is, there is an invariant 0-sphere S^0 contained in the annulus. It also should be noted that in this case the assumption (α_4) is not used. Also, the factorability condition (α_1) is no longer essential to the proof. $xQ(x,\mu)$ positive for all μ and nonzero x together with $\partial Q/\partial x$ being of first order in x will suffice to establish the existence of the annulus.

II. C^1 SURFACES FOR MAPPINGS OF R^2 INTO R^2.

Let T_μ be a homeomorphism of a neighborhood V of the origin in R^2 into R^2 mapping the polar coordinates $(\bar{\rho},\bar{\theta})$ into (ρ,θ) where

$$\bar{\rho} = (1 + \mu)\rho - g(\rho,\theta,\mu),$$

$$\bar{\theta} = \theta + \beta + w(\rho,\theta,\mu) \qquad\qquad (\beta = \text{constant}).$$

Let I_θ denote the interval $0 \leqq \theta \leqq 2\pi$, I_ρ the half open interval $0 < \rho \leqq \rho_0$, and I_μ the interval $0 \leqq \mu \leqq \mu_0$. We make the following assumptions on g and w.

(α_5) g and w are periodic of period 2π in θ, and are continuous on $\bar{I}_\rho \times I_\theta \times I_\mu$. $\partial g/\partial\rho$, $\partial g/\partial\theta$, $\partial w/\partial\rho$, $\partial w/\partial\theta$ are continuous on $I_\rho \times I_\theta \times I_\mu$.

(α_6) $0 < g(\rho,\theta,\mu) \leqq M_0\rho^2$, $0 < \partial g/\partial\rho \leqq M_1\rho$, $|w| \leqq M_2\rho$, and $|\partial w/\partial\rho| \leqq M_3$ uniformly in θ and μ on $I_\rho \times I_\theta \times I_\mu$.

(α_7) The first partial derivatives of g and w are zero for $\rho = 0$, and satisfy a uniform Lipschitz condition in (ρ,θ) with constant L on $I_\rho \times I_\theta \times I_\mu$ (L is independent of μ).

(α_8) g/ρ^2 is a nondecreasing function of $\rho > 0$ for each fixed θ and μ.

(α_9) For $\rho \in \bar{I}_\rho$ define $G(\rho,\mu) = \sup_\theta g(p,\theta,\mu)$ and $g(\rho,\mu) = \inf_\theta g(\rho,\theta,\mu)$. Assume $\rho^2|\partial g/\partial\theta| \leqq g^2(\rho,\mu)/3C$ and $\rho|\partial w/\partial\theta| \leqq g(\rho,\mu)B/C$ where

$$C = \max \left[16 M_0, 48(M_3 + B) \right]$$

and $B \geq 1$ is a fixed constant.

Hypotheses (α_5)-(α_7) give the order and regularity conditions imposed on g and w. These are seen to be stronger than those imposed in Part I. However, the factorability assumption (α_1) has been deleted. (α_8) is the direct analogue of (α_4), and (α_9) gives conditions which may be interpreted as circulatory conditions on g and w.

We next reconsider the two previous examples. The first indicates the nature of hypothesis (α_9) and shows that some hypothesis of this nature is needed to show the existence of an invariant C^1 surface. The second is a prototype.

2.6. EXAMPLE. Consider the mapping T defined by

$$y^i = (1 + \mu)x^i - (x^i)^3 \qquad\qquad (i = 1,2)$$

of the neighborhood V of the origin R^2 into itself. We have previously shown that in V this transformation has a unique C^0 curve S which is invariant under T and which has properties (P_1)-(P_3). S was clearly not of class C^1. Simple calculations show that T satisfies hypotheses (α_5)-(α_8), but not (α_9).

2.7. EXAMPLE. Let $\bar{\rho} = (1 + \mu)\rho - \rho^3$, $\bar{\theta} = \theta$. Then it follows directly that (α_5)-(α_9) are satisfied since $\partial g / \partial \theta = 0$ and $\partial w / \partial \theta = 0$.

We now proceed to show the existence of an invariant C^1 surface having the desired properties. As before, the Schauder fixed point theorem is applied to an appropriate subset of a Banach space. We begin by stating the following lemma which is analogous to Lemma 2.3.

2.8. LEMMA. *Let T_μ be defined as above, and suppose (α_5), (α_6) and (α_8) are satisfied. Then there exists a positive number μ_1, $0 < \mu_1 \leq \mu_0$ such that for $\mu \in (0, \mu_1]$ there are numbers $\rho_1(\mu)$ and $\rho_2(\mu)$ having properties (i)-(iii) of Lemma 2.3. In fact, ρ_1 and ρ_2 can be chosen so that $\mu \rho_2 = g(\rho_2, \mu)$ and $\mu \rho_1 = G(\rho_1, \mu)$.*

Let $\mathcal{B}$ be the class of all C^1 functions $\rho(\theta)$ such that $\rho(\theta + 2\pi) = \rho(\theta)$ with the usual C^1 norm:

$$\| \rho \| = \sup | \rho(\theta) | + \sup | d\rho/d\theta |.$$

Let $R_\mu = \{ \rho \in \mathcal{B} : \rho_1(\mu) \leq \rho(\theta) \leq \rho_2(\mu), \ \theta \in I_\theta \}$ for $\mu \in (0, \mu_1]$.

2.9. LEMMA. *If hypotheses (α_5), (α_6) and (α_9) are satisfied, then there is a number ρ_3, $0 < \rho_3 \leq \rho_0$, such that for $\rho(\theta) \in \mathcal{B}$, $0 < \rho(\theta) \leq \rho_3$, and $| d\rho/d\theta |$ sufficiently small the image curve $\bar{\rho} = \bar{\rho}(\theta) \in R_\mu$.*

Proof. Since $\bar{\theta} = \theta + \beta + w(\rho, \theta, \mu)$ and $\rho = \rho(\theta)$, $\bar{\theta} = f(\theta) = \theta + \alpha + w(\rho(\theta), \theta, \mu)$ and $d\bar{\theta}/d\theta = 1 + (\partial w/\partial \rho)(d\rho/d\theta) + \partial w/\partial \theta$. Thus, for $d\rho/d\theta$ and ρ sufficiently small, we have $d\bar{\theta}/d\theta > 0$, and hence, by the implicit function theorem, we may

solve for θ as a function of $\bar{\theta}$. Denote this function by $\theta = \theta(\bar{\theta})$. Then we have $\bar{\rho} = \bar{\rho}(\bar{\theta}) = (1 + \mu)\rho(\theta(\bar{\theta})) - g(\rho(\theta(\bar{\theta})), \theta(\bar{\theta}), \mu)$. That $\bar{\rho}(\bar{\theta})$ is periodic of period 2π follows from the periodicity of $\rho(\theta)$, $g(\rho,\theta,\mu)$, and $w(\rho,\theta,\mu)$. Since $\theta(\bar{\theta})$ is of class C^1, $\bar{\rho}(\bar{\theta})$ is of class C^1. That $\bar{\rho} \in R_\mu$ follows from Lemma 2.8.

2.10. LEMMA. *Suppose* (α_5), (α_6) *and* (α_9) *are satisfied. Then for* $\mu \in (0,\mu_1]$ *and* $\rho \in R_\mu$ *the inequality* $\left| d\rho/d\theta \right| \leqq \mu/C$ *implies* $\left| d\bar{\rho}/d\bar{\theta} \right| \leqq \mu/C$.

Proof. Without loss of generality we may assume $\mu_0 \leqq 1$, $\rho_0 \leqq 1$, $\rho_3 = \rho_0$ (ρ_3 is as given in Lemma 2.9), and $\partial g/\partial \rho \leqq 1$. Also, by the choice of C, $\left| d\rho/d\theta \right|$ is so small that Lemma 2.9 is applicable.

Let $\mu \in (0,\mu_1]$ and $\rho \in R_\mu$. Then the image curve $\bar{\rho}(\bar{\theta}) \in R_\mu$ and

$$d\bar{\rho}/d\bar{\theta} = (d\theta/d\bar{\theta})\{(d\rho/d\theta)[1 + \mu - \partial g/\partial \rho] - \partial g/\partial \theta\}.$$

Since g/ρ^2 is an increasing function of ρ,

$$\partial g/\partial \rho \geqq 2g(\rho,\mu)/\rho \geqq 2\mu g(\rho_1,\mu)/G(\rho_1,\mu).$$

Also, by the mean value theorem, (α_6) and (α_9),

$$G(\rho,\mu) - g(\rho,\mu) \leqq 2\pi g^2(\rho,\mu)/3C\rho^2,$$

and hence $g(\rho,\mu)/G(\rho,\mu) \geqq 6/7$. Using this, the preceding estimate for $\partial g/\partial \rho$, $\left| d\rho/d\theta \right| \leqq \mu/C$ and (α_9),

$$\left| (d\rho/d\theta)[1 + \mu - \partial g/\partial \rho] - \partial w/\partial \theta \right|$$

$$\leqq (\mu/C)[1 + \mu - 12\mu/7] + g^2(\rho,\mu)/3C\rho^2$$

$$\leqq (\mu/C)(1 - 5\mu/7) + \mu^2/3C \leqq (\mu/C)(1 - 8\mu/21).$$

Next, $d\bar{\theta}/d\theta = 1 + (d\rho/d\theta)(\partial w/\partial \rho) + \partial w/\partial \theta$, and thus

$$\left| d\bar{\theta}/d\theta - 1 \right| \leqq M_3\mu/C + g(\rho,\mu)B/\rho C$$

$$\leqq (\mu/C)(M_3 + B).$$

From this we have $\left| d\theta/d\bar{\theta} \right| \leqq 1 + \mu/24$, and finally,

$$\left| d\bar{\rho}/d\bar{\theta} \right| \leqq (1 + \mu/24)(\mu/C)(1 - 8\mu/21) \leqq \mu/C.$$

For $\mu \in (0,\mu_1]$, let

$$S_\mu(K) = \{\rho \in R_\mu : \left| (d\rho/d\theta)(\theta_1) - (d\rho/d\theta)(\theta_2) \right| \leqq K|\theta_1 - \theta_2|, \theta_1, \theta_2 \in I_\theta\}.$$

Since this set is a compact, convex subset of $\mathscr{B}$, we need only show that there is some K for which $S_\mu(K)$ is mapped into itself by T_μ in order to be able to use the Schauder fixed point theorem.

2.11. LEMMA. *Suppose* (α_5)-(α_9) *are satisfied and* μ *and* ρ *have been chosen*

as in Lemmas 2.8-2.10. Then there exists a K such that T_μ maps $S_\mu(K)$ into itself.

Proof. Let $\rho' = d\rho/d\theta$ and $\bar\rho' = d\bar\rho/d\bar\theta$. Also, let $\delta = |\bar\rho'(\bar\theta_1) - \bar\rho'(\bar\theta_2)|$ where $\bar\theta_i = \theta_i + \beta + w(\rho(\theta_i), \theta_i, \mu)$ $(i = 1,2)$. For convenience, let $\bar\theta = f(\theta)$, $d\bar\theta/d\theta = f'(\theta)$, and

$$F(\theta) = d\rho/d\theta[1 + \mu - (\partial g/\partial \rho)(\rho(\theta), \theta, \mu)] - (\partial g/\partial \theta)(\rho(\theta), \theta, \mu).$$

Then

$$\delta = [f'(\theta_1)]^{-1}\{[F(\theta_1) - F(\theta_2)] + \bar\rho'(\bar\theta_2)[f'(\theta_2) - f'(\theta_1)]\}$$

and

$$|\delta| \leq (1 + \mu/24)\{|(F(\theta_1) - F(\theta_2)| + (\mu/C)|f'(\theta_2) - f'(\theta_1)|\}$$
$$\leq (1 + \mu/24)\{|F(\theta_1) - F(\theta_2)| + (\mu/C)(2L + M_3K)|\theta_1 - \theta_2|\}$$

since, by (α_6) and (α_7),

$$|f'(\theta_2) - f'(\theta_1)| \leq (2L + M_3K)|\theta_1 - \theta_2|$$

if $|\delta| \leq K|\theta_1 - \theta_2|$. Also, using estimates as in the proof of Lemma 2.10 and using (α_7),

$$|F(\theta_1) - F(\theta_2)| \leq |\delta|(1 - 5\mu/7) + (\mu/C)L\{|\rho(\theta_1) - \rho(\theta_2)| + |\theta_1 - \theta_2|\}$$
$$+ L\{|\rho(\theta_1) - \rho(\theta_2)| + |\theta_1 - \theta_2|\}$$
$$\leq |\theta_1 - \theta_2|\{K(1 - 5\mu/7) + L(1 + \mu/C)^2\}.$$

Now choose K so large that $-5K_\mu/7 + L(1 + \mu/C)^2 \leq -\mu K/6$. Then,

$$|\delta| \leq (1 + \mu/24)\{K(1 - \mu/6) + (\mu/C)(2L + M_3K)\}|\theta_1 - \theta_2|.$$

If, in addition to the above restriction on K, we require that $K \geq 12(2L + M_3K)/C$, then

$$|\delta| \leq K(1 - \mu/48)|\theta_1 - \theta_2|.$$

Next, by the mean value theorem and (α_9),

$$|\bar\theta_1 - \bar\theta_2| \geq |\theta_1 - \theta_2|(1 - \mu M_3/C - \mu B/C)$$
$$\geq |\theta_1 - \theta_2|(1 - \mu/48).$$

Combining these, $|\bar\delta| \leq K|\bar\theta_1 - \bar\theta_2|$.

We may summarize by stating the following theorem.

2.12. **Theorem.** *Let T_μ be defined as above, mapping (ρ, θ) into $(\bar\rho, \bar\theta)$, and satisfy assumptions (α_5)-(α_9). Then for sufficiently small $\mu > 0$ there exists a curve C_μ of class C^1 which satisfies properties (P_1)-(P_3). If C_μ is given by $\rho = \rho(\theta)$, then $d\rho/d\theta$ is Lipschitz continuous. Also, $C_\mu \to 0$ as $\mu \to 0$.*

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