

FAMILIES OF PRINCIPAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS⁽¹⁾

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PART I. INTRODUCTION

Let $A(y, Dy, \dots, D^n y) = 0$ be an algebraic differential equation belonging to Strodt's class (D) [1, p. 5], and let M be a principal monomial [1, §66] for $A = 0$. In search for principal solutions of $A = 0$ (i.e., solutions which are $\sim M$) one substitutes $y = M(1 + z)$. This almost always [2, §121] leads to a differential equation $P(z) = 0$ in which P is an asymptotically quasilinear algebraic differential operator having a nonexceptional factorization sequence $(W_1, \dots, W_n)$ such that P is normal with respect to $(W_1, \dots, W_n, r)$ for a sufficiently large positive integer r . Strodt [2] has shown that $P = 0$ has at least one solution $Z < 1$, and accordingly $A = 0$ has at least one solution $Y = M(1 + Z) \sim M$; but if P is, in addition, uniformly quasilinear, then $P = 0$ has a u -parameter family of solutions < 1 (and $A = 0$ has a u -parameter family of solutions $\sim M$), where u is the number of indices i for which $[IF]^\sim(W_i, \phi) < 0$ (ϕ denoting the direction in which the complex variable approaches infinity). The significance of the number u in the more general a.q.l. situation has hitherto been obscured by the complicated manipulations used to ascertain the mere existence of a solution < 1 for a.q.l. equations.

In the present study, a u -parameter family of solutions < 1 is exhibited for a large class of normal a.q.l. equations $P = 0$. We require that the factorization sequence for P satisfy a condition which resembles (and includes) the classical "distinctness of characteristic roots" in the case of linear differential equations with constant coefficients (§2, below). The virtue of this condition is that it allows us to pass from approximate factorizations of linear operators in terms of the operators $\dot{W}_i = (1 - W_i^{-1}D)$ to exact factorizations in terms of certain $\dot{V}_i$ such that $V_i \sim W_i$. Once the linear part of an algebraic a.q.l. equation has been exactly factored, the situation may be said to be well in hand. Among other things, exact factorization enables us to measure with considerable precision the asymptotic size of our small (< 1) solutions, and even to obtain a sort of asymptotic development for them. Other phenomena used to advantage in this study include the invariance of normality and asymptotic quasilinearity of differential operators

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under small displacements of dependent variable or small displacements of factorization sequences, and the evolution of new a.q.l. operators from old ones under judicious substitutions.

For an index of terminology used in references [1] and [2] and in the present paper, the reader is referred to [2, Part IX, pp. 105-107].

PART II. EXACT FACTORIZATION OF LINEAR DIFFERENTIAL OPERATORS

1. LEMMA (APPROXIMATE FACTORIZATION $\rightarrow$ PARTLY EXACT FACTORIZATION). *Let $W_1, \dots, W_n$ be logarithmic monomials in the divergence class. Let $W_i - W_1 \approx W_i$ for $i = 2, 3, \dots, n$. Let $U_i = W_i - W_1$ ($i = 2, 3, \dots, n$) and let $(U_2, U_3, \dots, U_n)$ be unblocked in (α, ϕ, β) . Let $L = \dot{W}_n \dots \dot{W}_1 + \sum_{i=1}^n E_i \dot{W}_i \dots \dot{W}_1 + E_0$, with each $E_i < 1$ in $F(\alpha, \beta)$. Then there exists a function V_1 such that $V_1 \sim W_1$ and such that $L = \dot{W}_n \dots \dot{W}_2 \dot{V}_1 + \sum_{i=1}^n E_{1i} \dot{W}_i \dots \dot{W}_2 \dot{V}_1$ with each $E_{1i} < 1$ (all relations holding in $F(\alpha, \beta)$).*

Proof. In the notation of [2, §4], we want $R^*L(V_1) = 0$. According to [2, §§32, 33], we have $R^*L(y) = \ddot{W}_n(y) \dots \ddot{W}_1(y) \cdot 1 + \sum_{i=1}^n E_i \ddot{W}_i(y) \dots \ddot{W}_1(y) \cdot 1 + E_0$, where $\ddot{W}(y) = (1 - W^{-1}y - W^{-1}D)$. Under the substitution $y = W_1(1 - z)$ we have $\ddot{W}_i(y) = U_i W_i^{-1}(\dot{U}_i + W_1 U_i^{-1}z)$, where $U_i W_i^{-1} \approx 1$ and $W_1 U_i^{-1} \lesssim 1$ for all relevant i . One proves by induction on i that $\ddot{W}_i(y) \dots \ddot{W}_1(y) \cdot 1 = b_i B_i(z)$, where $b_i \approx 1$ and B_i is a differential operator which is $\sim (s) \dot{U}(i, \dots, 2)$. In carrying out this proof, it is convenient to dispose of terms of the form

$$\dot{U}_j \{cz^{t_0} (\dot{U}_2 z)^{t_1} \dots (\dot{U}_h \dot{U}_{h-1} \dots \dot{U}_2 z)^{t_{h-1}}\}$$

($h < j$) by means of the relations $\dot{U}(AB) = A\dot{U}(B) + B\dot{U}(A) - AB$ and

$$\dot{U}_j(\dot{U}_h \dots \dot{U}_2 z) = U_{h+1} U_j^{-1}(\dot{U}_{h+1} \dot{U}_h \dots \dot{U}_2 z) + (1 - U_{h+1} U_j^{-1}) \dot{U}_h \dots \dot{U}_2 z.$$

Once the properties of b_i and B_i are established, it follows immediately that $R^*L(W_1(1 - z)) = bB(z)$, where $b \approx 1$ and $B \sim (s) \dot{U}(n, \dots, 2)$. Since $(U_2, \dots, U_n)$ is unblocked, Theorem II of [2] implies the existence of a solution Z of $B(z) = 0$ such that $Z < 1$ in $F(\alpha, \beta)$. Define $V_1 = W_1(1 - Z)$. The lemma now follows from the fact that $R^*L(V_1) = 0$ and $(V_1, W_2, \dots, W_n)$ is a factorization sequence for L [2, §54].

2. DEFINITION. We shall call $(W_1, \dots, W_n)$ *separated* when the W_i are $\sim$ logarithmic monomials in the divergence class satisfying

- (1) $W_i - W_j \approx W_i$ for $i > j$,
- (2) each $W_i - W_j$ is unblocked in (α, ϕ, β) .

THEOREM I (APPROXIMATE FACTORIZATION $\rightarrow$ EXACT FACTORIZATION). *Let $(W_1, \dots, W_n)$ be a separated factorization sequence for the linear differential operator L . Then there exist functions $V_1, \dots, V_n$ such that $V_i \sim W_i$ ($i = 1, \dots, n$) and such that $L = u \dot{V}_n \dots \dot{V}_1$ for some $u \sim 1$.*

Proof. By Lemma 1, $L = L_1 \dot{V}_1$ for some function $V_1 \sim W_1$ and a certain operator L_1 such that $L_1 \sim \dot{W}(n, \dots, 2)$. It is clear that L_1 is qualified to play the role which L plays in Lemma 1, so we get $L_1 = L_2 \dot{V}_2$ where $V_2 \sim W_2$ and $L_2 \sim \dot{W}(n, \dots, 3)$. The conclusion of the present theorem is reached after finitely many repetitions of this same argument.

PART III. ON THE EQUATION $\dot{W}_n \dot{W}_{n-1} \cdots \dot{W}_1 y = 0$

3. We present here some results of a computational nature, necessary for the subsequent study of transformations of differential operators under substitutions for the dependent variable.

LEMMA. Let $W_i \sim$ a logarithmic monomial in the divergence class ($i = 1, \dots, n$), and $W_i - W_j \approx W_i$ if $i > j$. Let $\delta > \tau > 0$. Let $k \in (1, \dots, n)$. Let

$$W_k \sim cx^{-1}(\log x)^{-1} \cdots (\log_{p-1} x)^{-1} (\log_p x)^{-1+\tau} (\log_{p+1} x)^{a_1} \cdots (\log_{p+s} x)^{a_s}.$$

Let $F = (\log_p x)^{-\delta}$. Let $G < 1$ and $DG = FW_k$ (see Lemma ζ). Let $H_k \equiv 1$ and $H_s = (1 - W_k W_{s+1}^{-1})^{-1} (1 - W_k W_{s+1}^{-1})^{-1} \cdots (1 - W_k W_{k-1}^{-1})^{-1}$ for $s = k-1, \dots, 1$. Let $K_s = H_s \exp \int_{x_0} W_k$ ($s = 1, \dots, k$). Let z be an admissible function. Then:

$$(3.1) \quad \dot{W}_s(K_s) = u_s K_{s+1}, \text{ with } u_s \sim 1 \quad (s = 1, \dots, k-1).$$

$$(3.2) \quad \dot{W}_s(K_s z) = u_s K_{s+1} \dot{M}_s z, \text{ with } M_s = u_s (W_s - W_k) \quad (s = 1, \dots, k-1).$$

$$(3.3) \quad \dot{M}_s(G) \sim G \quad (s = 1, \dots, k-1).$$

$$(3.4) \quad \dot{W}_s(GK_s) = v_s GK_{s+1}, \text{ with } v_s \sim 1 \quad (s = 1, \dots, k-1).$$

$$(3.5) \quad \dot{W}_s(GK_s z) = v_s GK_{s+1} \dot{U}_s z, \text{ with } U_s = v_s (W_s - W_k) \quad (s = 1, \dots, k-1).$$

$$(3.6) \quad \text{For } s = 1, \dots, k-1, \dot{W}_s \cdots \dot{W}_1(K_1 z) = K_{s+1} \Psi_s \dot{N}_s \cdots \dot{N}_1 z, \text{ with } \Psi_s \sim 1 \text{ and } N_i \sim W_i - W_k \quad (i = 1, \dots, s).$$

$$(3.7) \quad \text{For } s = 1, \dots, k-1, \dot{W}_s \cdots \dot{W}_1(GK_1 z) = GK_{s+1} \Phi_s \dot{V}_s \cdots \dot{V}_1 z, \text{ with } \Phi_s \sim 1 \text{ and } V_i \sim W_i - W_k \quad (i = 1, \dots, s).$$

$$(3.8) \quad \dot{W}_k(z \exp \int_{x_0} W_k) = -W_k^{-1} D z \exp \int_{x_0} W_k.$$

$$(3.9) \quad \dot{W}_k(z G \exp \int_{x_0} W_k) = -F \exp \int_{x_0} W_k \dot{Q} z, \text{ where } Q = -G^{-1} DG.$$

$$\text{Let } \Phi_t = 1 - W_k W_t^{-1} - F^{-1} W_t^{-1} D F \text{ and } \Psi_t = 1 - W_k W_t^{-1}, \text{ for } t = k+1, \dots, n.$$

Φ_t and Ψ_t are easily seen to be ≈ 1 .

$$(3.10) \quad \dot{W}_t(\exp \int_{x_0} W_k) = \Psi_t \exp \int_{x_0} W_k \quad (t = k+1, \dots, n).$$

$$(3.11) \quad \dot{W}_t(F \exp \int_{x_0} W_k) = \Phi_t F \exp \int_{x_0} W_k \quad (t = k+1, \dots, n).$$

$$(3.12) \quad \dot{W}_t(z \exp \int_{x_0} W_k) = \Psi_t \dot{M}_t z \exp \int_{x_0} W_k, \text{ where } M_t = \Psi_t W_t \quad (t = k+1, \dots, n).$$

$$(3.13) \quad \dot{W}_t(z F \exp \int_{x_0} W_k) = \Phi_t \dot{U}_t z F \exp \int_{x_0} W_k, \text{ where } U_t = \Phi_t W_t \quad (t = k+1, \dots, n).$$

$$(3.14) \quad M_t \sim U_t \sim W_t - W_k \quad (t = k+1, \dots, n).$$

$$(3.15) \quad \dot{W}_t \cdots \dot{W}_{k+1}(z \exp \int_{x_0} W_k) = \Psi_t^* \exp \int_{x_0} W_k \dot{N}_t \cdots \dot{N}_{k+1} z, \text{ with } \Psi_t^* \approx 1 \text{ and } N_t \sim W_t - W_k \quad (t = k+1, \dots, n).$$

$$(3.16) \quad \dot{W}_t \cdots \dot{W}_{k+1}(z F \exp \int_{x_0} W_k) = \Phi_t^* F \exp \int_{x_0} W_k \dot{V}_t \cdots \dot{V}_{k+1} z, \text{ with } \Phi_t^* \approx 1 \text{ and } V_t \sim W_t - W_k \quad (t = k+1, \dots, n).$$

Proof. In verifying (3.1)–(3.16), much use may be made of the relation (R): $\dot{W}(Ty) = (\dot{W}(T))(\dot{M}(y))$, where $M = WT^{-1}\dot{W}(T)$ (cf. [2, §113]).

For (3.1): $\dot{W}_s(K_s) = H_s(1 - W_s^{-1}(H_s^{-1}DH_s + W_k)) \exp \int_{x_0} W_k$. $H_s^{-1}DH_s < W_k$ because $\delta_i(H_s) = 0$ ($i = 0, 1, \dots, p-1$), implying $\delta_i(H_s^{-1}DH_s) = -1$ ($i = 0, \dots, p$). Hence $1 - W_s^{-1}(H_s^{-1}DH_s + W_k) \sim 1 - W_s^{-1}W_k$, and the conclusion follows because $(1 - W_s^{-1}W_k)H_s = H_{s+1}$.

For (3.2): In applying (R) and (3.1), we have $M_s = W_s u_s K_{s+1} K_s^{-1} = u_s(W_s - W_k)$.

For (3.3): The conditions imply $G \sim (\tau - \delta)^{-1} x (\log x) \cdots (\log_p x) F W_k$. We also have $M_s \approx W_k$; therefore $M_s^{-1}DG \approx F < G$.

For (3.4): Apply (3.2) with z replaced by G .

For (3.5): Apply (R) together with (3.4).

For (3.6): For successive values of s , we show that if $\Psi_{s-1} \sim 1$, then $\dot{W}_s(\Psi_{s-1}K_s z') = K_{s+1}\Psi_s \dot{N}_s z'$ where $z' = \dot{N}_{s-1} \cdots \dot{N}_1 z$, $\Psi_s \sim 1$, and $N_s \sim W_s - W_k$. By (R), $\dot{W}_s(\Psi_{s-1}K_s z') = (\dot{W}_s \Psi_{s-1})(\dot{S}(K_s z'))$, where $\dot{W}_s \Psi_{s-1} \sim 1$ because $\Psi_{s-1} \sim 1$ while $S = vW_s$, where $v = \Psi_{s-1}^{-1}\dot{W}_s(\Psi_{s-1})$ which is ~ 1 . Then $\dot{S}(K_s z') = (1/v)\dot{W}_s(K_s z') + (1 - 1/v)K_s z' = (1/v)(u_s K_{s+1} \dot{M}_s z' + (v-1)K_s z')$, using (3.2), $= K_{s+1}(u_s/v + ((v-1)/v)(K_s/K_{s+1}))(z' - Dz'/M_s(1 + (v-1)K_s/u_s K_{s+1}))$. To obtain the desired result we set $N_s = M_s(1 + (v-1)K_s/u_s K_{s+1})$ and $\Psi_s = (\dot{W}_s(\Psi_{s-1}))(u_s/v + ((v-1)/v)(K_s/K_{s+1}))$. Noting that $K_{s+1} \gtrsim K_s$ and $(v-1)v < 1$, we have $\Psi_s \sim 1$ and $N_s \sim M_s \sim W_s - W_k$ as claimed. To begin the induction, (3.2) may be applied for the case $s = 1$.

For (3.7): Apply (R) and (3.5) repeatedly.

(3.8)–(3.11) follow from easy computations.

For (3.12) and (3.13): Apply (R), (3.10), and (3.11).

(3.14) is obvious.

For (3.15) and (3.16): Apply (R), (3.12), and (3.13) repeatedly.

4. THEOREM II. Let the hypotheses and notation of Lemma 3 be in force. Let $-\pi \leq \alpha < \phi < \beta \leq \pi$. Let $W_1 - W_k, W_2 - W_k, \dots, W_{k-1} - W_k$ be unblocked in (α, ϕ, β) . Let S be an element of $F(\alpha, \beta)$ in which W_k is analytic, and let $x_0 \in S$. Then for every complex number $g \neq 0$, the differential equation

$$(4.1) \quad \dot{W}_n \cdots \dot{W}_1 y = 0$$

has a solution y_g which is $\sim gH_1 \exp \int_{x_0} W_k$.

Proof. Let $y = gH_1 \exp \int_{x_0} W_k z$. (4.1) becomes

$$\dot{W}_n \cdots \dot{W}_k(\Psi_{k-1} \exp \int_{x_0} W_k \dot{N}_{k-1} \cdots \dot{N}_1 z) = 0$$

(cf. (3.6)), or $\dot{W}_n \cdots \dot{W}_{k+1}(-W_k^{-1} \exp \int_{x_0} W_k D(\Psi_{k-1} \dot{N}_{k-1} \cdots \dot{N}_1 z)) = 0$ (cf. (3.8)), and hence it suffices to solve the equation $\dot{N}_{k-1} \cdots \dot{N}_1 z = \Psi_{k-1}^{-1}$ for a solution $z \sim 1$. Setting $z = 1 + w$, we seek a function $w < 1$ such that $\dot{N}_{k-1} \cdots \dot{N}_1 w = \Psi_{k-1}^{-1} - 1$. But $(N_1, \dots, N_{k-1})$ is unblocked in (α, ϕ, β) and $\Psi_{k-1}^{-1} - 1 < 1$ in $F(\alpha, \beta)$, so the existence of a solution $w^* < 1$ may be inferred from [2, §114]. (A direct proof using [2, §78] is also feasible: There exists a

$w_{k-1} < 1$ such that $\dot{N}_{k-1}w_{k-1} = \Psi_{k-1}^{-1} - 1$, there exists a $w_{k-2} < 1$ such that $\dot{N}_{k-2}w_{k-2} = w_{k-1}$, ..., and finally there exists a $w_1 = w^* < 1$ such that $\dot{N}_1w_1 = w_2$.) Taking $y_g = gH_1(1 + w^*)\exp \int_{x_0} W_k$, it is clear that the conclusions of the theorem are satisfied.

5. LEMMA. Let the hypotheses and notation of 4. be in force. Let $y = y_g + GK_1z$. Then

$$\begin{aligned}\dot{W}_s \cdots \dot{W}_1 y &= \dot{W}_s \cdots \dot{W}_1 y_g + GK_{s+1} \Phi_s \dot{V}_s \cdots \dot{V}_1 z \quad (s = 1, \dots, k-1); \\ \dot{W}_k \cdots \dot{W}_1 y &= (-F \bar{\Phi}_{k-1} \exp \int_{x_0} W_k) \dot{Q}' \dot{V}_{k-1} \cdots \dot{V}_1 z, \text{ with } \bar{\Phi}_{k-1} \sim 1; \text{ and} \\ \dot{W}_t \cdots \dot{W}_1 y &= (-F \bar{\Phi}_t \exp \int_{x_0} W_k) \dot{V}'_t \cdots \dot{V}'_{k+1} \dot{Q}' \dot{V}_{k-1} \cdots \dot{V}_1 z, \text{ with } \bar{\Phi}_t \approx 1, Q' \sim Q, \\ &\text{and } V'_t \sim V_t \quad (t = k+1, \dots, n).\end{aligned}$$

Proof. Clearly $\dot{W}_s \cdots \dot{W}_1 y_g = 0$ for $s = k, \dots, n$, and

$$\dot{W}_s \cdots \dot{W}_1 (GK_1 z) = GK_{s+1} \Phi_s \dot{V}_s \cdots \dot{V}_1 z$$

with Φ_s and the V 's as in (3.7), for $s = 1, \dots, k-1$.

$\dot{W}_k \cdots \dot{W}_1 (GK_1 z) = \dot{W}_k (G \exp \int_{x_0} W_k \Phi_{k-1} \dot{V}_{k-1} \cdots \dot{V}_1 z)$ and we use (3.9) with $(\Phi_{k-1} \dot{V}_{k-1} \cdots \dot{V}_1 z)$ in place of the z in (3.9). This gives

$$-F \exp \int_{x_0} W_k \{ \dot{Q} (\Phi_{k-1} \dot{V}_{k-1} \cdots \dot{V}_1 z) \}.$$

On $\{\dots\}$ we use (R) with $T = \Phi_{k-1}$ and $y = \dot{V}_{k-1} \cdots \dot{V}_1 z$ (see proof of Lemma 3), which gives $\dot{W}_k \cdots \dot{W}_1 (GK_1 z) = (-F \bar{\Phi}_{k-1} \exp \int_{x_0} W_k) \dot{Q}' \dot{V}_{k-1} \cdots \dot{V}_1 z$, where $\bar{\Phi}_{k-1} = \dot{Q}(\Phi_{k-1})$ and $Q' = Q \Phi_{k-1}^{-1} \bar{\Phi}_{k-1}$.

If $t = k+1, \dots, n$ we have

$$\dot{W}_t \cdots \dot{W}_1 (GK_1 z) = \dot{W}_t \cdots \dot{W}_{k+1} (-F \exp \int_{x_0} W_k \bar{\Phi}_{k-1} \dot{Q}' \dot{V}_{k-1} \cdots \dot{V}_1 z),$$

and we use (3.16) with z replaced by $\bar{\Phi}_{k-1} \dot{Q}' \dot{V}_{k-1} \cdots \dot{V}_1 z$. This gives $\dot{W}_t \cdots \dot{W}_1 (GK_1 z) = -F \exp \int_{x_0} W_k \Phi_t^* \{ \dot{V}_t \cdots \dot{V}_{k+1} (\bar{\Phi}_{k-1} \dot{Q}' \dot{V}_{k-1} \cdots \dot{V}_1 z) \}$. On $\{\dots\}$ we use (R) as in the previous paragraph to "permute" $\bar{\Phi}_{k-1}$ past $\dot{V}_t \cdots \dot{V}_{k+1}$ and obtain the stated result.

PART IV. SOLUTIONS OF NONLINEAR EQUATIONS

6. DEFINITION. Let $P \sim (w) \dot{W}(n, \dots, 1)$. We say that P is condensed with respect to $(W_1, \dots, W_n)$, and write $P \sim (c, w) \dot{W}(n, \dots, 1)$, if $[HLP] \sim u \dot{W}_n \cdots \dot{W}_1$ for some $u \sim 1$. If $P \sim (c, w) \dot{W}(n, \dots, 1)$ and P is also normal with respect to $(W_1, \dots, W_n, r)$ [2, §102], we say that P is condensed with respect to $(W_1, \dots, W_n, r)$ and write $P \sim (r, c, w) \dot{W}(n, \dots, 1)$.

7. LEMMA. Let $(W_1, \dots, W_n)$ be separated and unblocked in (α, ϕ, β) . Let P' be normal with respect to $(W_1, \dots, W_n, r)$. Then there exists a function $Y < 1$ and functions $V_1, \dots, V_n$ with $W_i \sim V_i$ ($i = 1, \dots, n$) such that if $P(y) = P'(Y + y)$, then $P \sim (r, c, w) \dot{V}(n, \dots, 1)$ and $P(0) = 0$.

Proof. By Theorem IIIa of [2], there exists a solution of $P'(y) = 0$ which is < 1 . Let Y be one such solution, and let $P(y) = P'(Y + y)$. Then $P(0) = 0$.

By Lemma α , P is normal with respect to $(W_1, \dots, W_n, r)$. Theorem I above shows that the approximate factorization of $[HLP]^\sim$ using $(W_1, \dots, W_n)$ can be converted into an exact factorization in terms of $(V_1, \dots, V_n)$ where $V_i \sim W_i$ for all i . Lemma β shows that P is normal with respect to $(V_1, \dots, V_n, r)$. Hence $P \sim (r, c, w)\check{V}(n, \dots, 1)$.

8. LEMMA. *Let the hypotheses and notation of Lemma 5 be in force. Let $[IF]^\sim(W_k, \phi) < 0$. Let $X_i = V_i$ ($i = 1, \dots, k-1$), $X_k = Q'$, and $X_i = V_i'$ ($i = k+1, \dots, n$). Let $P \sim (r, c, w)\check{W}(n, \dots, 1)$, and let $R(z, \check{X}_1 z, \dots, \check{X}_n \dots \check{X}_1 z) = P(y, \check{W}_1 y, \dots, \check{W}_n \dots \check{W}_1 y)$. Then whenever $z < 1$, $R(z, \check{X}_1 z, \dots, \check{X}_n \dots \check{X}_1 z) < F \exp \int_{x_0} W_k$.*

Proof. Since $\check{W}_n \dots \check{W}_1(y_g) = 0$, and since $\check{X}_n \dots \check{X}_1$ is unimajoral, Lemma 5 immediately implies $\check{W}_n \dots \check{W}_1 y < F \exp \int_{x_0} W_k$ for every $z < 1$. Let $\theta = \theta_*$. (We assume, with no loss of generality, that $r \geq p+1$.) Then $P(y) - [HLP]^\sim(y)$ may be expressed as a polynomial in $y, \theta y, \dots, \theta^n y$ such that each monomial has degree ≥ 2 and each coefficient is $\lesssim 1$. It is easily shown that for every $z < 1$, $P(y) - [HLP]^\sim(y) \lesssim (\theta^t y)^2$ for some $t \in (0, \dots, n)$. In fact, for the functions y being considered, it is readily verified that $\theta^t y = E_t J_t \exp \int_{x_0} W_k$ where $E_t \sim 1$ and J_t is a logarithmic monomial such that $\delta_i(J_t) = 0$ ($i = 0, \dots, p-1$; $t = 2, \dots, n$). Hence there exists a logarithmic monomial J^* such that $\delta_i(J^*) = 0$ ($i = 0, \dots, p-1$) and such that $J^* \exp \int_{x_0} 2W_k > P(y, \check{W}_1 y, \dots, \check{W}_n \dots \check{W}_1 y) - [HLP]^\sim(y)$ whenever $z < 1$. But $J^* \exp \int_{x_0} 2W_k < F \exp \int_{x_0} W_k$ by Lemma ϵ . It follows that $R(z, \check{X}_1 z, \dots, \check{X}_n \dots \check{X}_1 z) < F \exp \int_{x_0} W_k$ whenever $z < 1$.

9. LEMMA. *Let the hypotheses and notation of Lemma 8 be in force. Let P_s and R_s denote the partial derivatives of P and R with respect to their s th arguments. Then $P_s(y, \check{W}_1 y, \dots, \check{W}_n \dots \check{W}_1 y) < J'' \exp \int_{x_0} W_k$ ($s = 0, \dots, n-1$), where J'' is any sufficiently large logarithmic monomial such that $\delta_i(J'') = 0$ ($i = 0, \dots, p-1$).*

Proof. Let $\bar{P} = P - [HLP]^\sim$. Then $\bar{P}_s = P_s$ ($s = 0, \dots, n-1$), since P is condensed. We have

$$\bar{P}(y, \check{W}_1 y, \dots, \check{W}_n \dots \check{W}_1 y) = \sum p_{\hat{\gamma}} y^{j_0} (\theta y)^{j_1} \dots (\theta^n y)^{j_n}$$

with $j_0 + j_1 + \dots + j_n \geq 2$ and $j_t p_{\hat{\gamma}} \lesssim A_1 \dots A_t$ ($t = 1, \dots, n$) for each j , where $A_t = -(x(\log x) \dots (\log_{r-1} x) W_t)^{-1}$. Also, by [2, § 100], we have

$$\theta^t y = \sum_{h=0}^t G_{th} \check{W}_h \dots \check{W}_1 y$$

with $G_{th} \lesssim 1/A_1 \dots A_t$. Hence

$$P_s(y, \check{W}_1 y, \dots, \check{W}_n \dots \check{W}_1 y) = \sum p_{\hat{\gamma}} y^{j_0} (\theta y)^{j_1} \dots (\theta^n y)^{j_n} C_{\hat{\gamma}},$$

where $C_j = j_s(\theta^s y)^{-1} G_{ss} + j_{s+1}(\theta^{s+1} y)^{-1} G_{s+1,s} + \cdots + j_n(\theta^n y)^{-1} G_{ns}$. But $j_{s+q} p_j G_{s+q,s} \lesssim 1$ ($q = 0, \dots, n-s$). Hence we may write $P_s(y, \dot{W}_1 y, \dots, \dot{W}_n \cdots \dot{W}_1 y)$ as a polynomial in $y, \theta y, \dots, \theta^n y$ in which each monomial is of degree ≥ 1 and each coefficient is $\lesssim 1$. The argument given in proof of Lemma 8 shows that there exists a J'' with the prescribed properties.

10. **LEMMA.** *Let the hypotheses and notation of Lemma 9 be in force. Then $R_s(z, \dot{X}_1 z, \dots, \dot{X}_n \cdots \dot{X}_1 z) < F \exp \int_{x_0} W_k$ for every $z < 1$ and every $s \in (0, 1, \dots, n-1)$.*

Proof. $R_s(z, \dot{X}_1 z, \dots, \dot{X}_n \cdots \dot{X}_1 z) = P_s(y, \dot{W}_1 y, \dots, \dot{W}_n \cdots \dot{W}_1 y) T(s)$ where $T(0) = GK_1$, $T(s) \sim GK_{s+1}$ ($s = 1, \dots, k-1$), and $T(s) \approx -F \exp \int_{x_0} W_k$ ($s = k, \dots, n$) (cf. Lemma 5). Hence there exists a logarithmic monomial J with $\delta_i(J) = 0$ ($i = 0, \dots, p-1$) such that $R_s(z, \dot{X}_1 z, \dots, \dot{X}_n \cdots \dot{X}_1 z) < J \exp \int_{x_0} 2W_k$; but $J \exp \int_{x_0} 2W_k < F \exp \int_{x_0} W_k$ as a consequence of Lemma 8, proving the assertion.

11. **LEMMA.** *Let the hypotheses and notation of Lemma 10 be in force. Then $R_n(z, \dot{X}_1 z, \dots, \dot{X}_n \cdots \dot{X}_1 z) \sim -F \exp \int_{x_0} W_k$ whenever $z < 1$.*

Proof. $R_n(z, \dot{X}_1 z, \dots, \dot{X}_n \cdots \dot{X}_1 z) = P_n(y, \dot{W}_1 y, \dots, \dot{W}_n \cdots \dot{W}_1 y) T(n)$ where $T(n) \sim -F \exp \int_{x_0} W_k$, while by hypothesis $P_n(y, \dot{W}_1 y, \dots, \dot{W}_n \cdots \dot{W}_1 y) \sim 1$ whenever $y < 1$ and a fortiori whenever $z < 1$.

12. **LEMMA.** *Let the hypotheses and notation of Lemma 11 be in force. Let $(\dot{X}_1, \dots, \dot{X}_n)$ be an isotone permutation of $(X_1, \dots, X_n)$. Then S is normal with respect to $(\dot{X}_1, \dots, \dot{X}_n, r)$, where S is the operator $(-F \exp \int_{x_0} W_k)^{-1} R$.*

Proof. Lemmas 8-11 imply $S \sim (w) \dot{X}(n, \dots, 1)$. Hence $S(0) < 1$, $[HLS] \sim \dot{X}(n, \dots, 1)$, and $[HLS] \sim \dot{X}^-(n, \dots, 1)$. Let $y_G = GK_1$.

$$P(y_g + y_g z) = \sum \{ q_{\hat{\gamma}} \prod_{i=0}^n (\theta^i y_g + (\theta^i y_g) z + \iota (\theta^{i-1} y_g) \theta z + \cdots + \iota \theta y_g \theta^{i-1} z + y_g \theta^i z)^{i_t} : i_0 + \cdots + i_n \geq 2 \}.$$

Hence $P(y_g + y_g z)$ may be written as a sum of terms each having the form

$$c q_{\hat{\gamma}} M(y_g, \theta y_g, \dots, \theta^n y_g; y_g, \theta y_g, \dots, \theta^n y_g) N(z, \theta z, \dots, \theta^n z)$$

where M and N are products of integral powers of their arguments, c is a constant, M is of degree ≥ 2 , and the differential order of N is $\leq t^*$, where (relative to a fixed $\hat{\gamma}$) t^* is the largest integer t such that $i_t \neq 0$. Let the differential order of N be s^* . As in the proofs of the previous lemmas, it is verified that $M < M^* \exp \int_{x_0} 2W_k$ where M^* is any sufficiently large logarithmic monomial such that $\delta_i(M^*) = 0$ ($i = 0, \dots, p-1$).

Consider a term for which $s^* > k$. Let $B_i = -(x(\log x) \cdots (\log_{r-1} x) \dot{X}_i)^{-1}$ ($i = 1, \dots, n$). We have $B_1 \cdots B_k \sim B^*$, where B^* is a logarithmic monomial with $\delta_j(B^*) = 0$ ($j = 0, \dots, p-1$). By the normality of P with respect to $(W_1, \dots, W_n, r)$, $q_{\hat{\gamma}} \lesssim A_1 \cdots A_{r^*}$. Therefore $A_1 \cdots A_{r^*} M^* \exp \int_{x_0} 2W_k > c q_{\hat{\gamma}} M$;

$$A_1 \cdots A_{t*} M^* F^{-1} \exp \int_{x_0} W_k > (-F \exp \int_{x_0} W_k)^{-1} c q_{\frac{1}{2}} M;$$

and

$$(12.1) \quad (-F \exp \int_{x_0} W_k)^{-1} c q_{\frac{1}{2}} M(B_1, \dots, B_{s*})^{-1} \\ \lesssim \frac{A_1 \cdots A_{t*}}{B_{k+1} \cdots B_{s*}} \cdot \frac{M^*}{B_1 \cdots B_k F} \cdot \exp \int_{x_0} W_k.$$

Now $\delta_i(M^*/B_1 \cdots B_k F) = 0$ ($i = 0, \dots, p-1$); and, since $\bar{X}_h \approx W_h$ ($h = k+1, \dots, n$), we have $B_h \approx A_h$ ($h = k+1, \dots, n$). Hence the right-hand member of (12.1) is $\lesssim 1$. Therefore we can express $(-F \exp \int_{x_0} W_k)^{-1} \cdot P(y_g + y_g z)$ as a sum of terms of the form $f_{\frac{1}{2}} z^{j_0} (\theta z)^{j_1} \cdots (\theta^n z)^{j_n}$ where $j_{s*} f_{\frac{1}{2}} \lesssim B_1 \cdots B_{s*}$ for every j such that $s^* > k$. The same inequality for $j_{s*} f_{\frac{1}{2}}$ can be verified in very much the same way for terms in which $s^* \leq k$. It follows that S is normal with respect to $(\bar{X}_1, \dots, \bar{X}_n, r)$ by [2, §102].

13. In this lemma we show that if P is an operator which has been "normalized," i.e., $\sim (r, c, w) \dot{W}(n, \dots, 1)$ with $P(0) = 0$, then the equation $P(y) = 0$ has a one-parameter family of solutions $y_g + y_g z$. These solutions are $\approx H_1 \exp \int_{x_0} W_k$, the matching constant $g \neq 0$ being arbitrarily chosen. (For $z < 1$, $y_g + y_g z \sim y_g \sim g H_1 \exp \int_{x_0} W_k$.) This is the case for each $k \in (1, \dots, n)$ such that $[IF] \sim (W_k, \phi) < 0$, i.e., such that $\exp \int_{x_0} W_k \rightarrow 0$ in the direction ϕ . Thus we have the "expected" number of solutions < 1 in a weak sense: There are that many separate one-parameter families of solutions < 1 .

Theorem III, below, proceeds beyond this to show how "quasilinear combinations" of such solutions may be formed to obtain a family of solutions < 1 such that each member of the family corresponds to a simultaneous choice of all available parameters.

LEMMA. *Let the hypotheses and notation of Lemma 12 be in force. Then there exists a function $Z < 1$ such that $y = y_g + y_g Z$ satisfies*

$$P(y, \dot{W}_1 y, \dots, \dot{W}_n \cdots \dot{W}_1 y) = 0.$$

Proof. The related operator S is normal with respect to $(\bar{X}_1, \dots, \bar{X}_n, r)$, so we may invoke Theorem IIIa of [2] to show the existence of $Z < 1$ such that $S(Z) = 0$. Of course $P(y_g + y_g Z) = 0$ for such a Z , q.e.d.

14. **THEOREM III.** *Let $W_1 \lesssim \cdots \lesssim W_n$, with each $W_i \sim$ a logarithmic monomial in the divergence class. Let P be normal with respect to $(W_1, \dots, W_n, r)$. Let $Y < 1$ and let $P(Y) = 0$. Let $(k_1, \dots, k_u) \subset (1, \dots, n)$, and let $[IF] \sim (W_{k_i}, \phi) < 0$ ($i = 1, \dots, u$). Let $(W_1, \dots, W_n)$ be separated. Let $H(k) = (1 - W_1^{-1} W_k)^{-1} \cdots (1 - W_{k-1}^{-1} W_k)^{-1}$. Let (γ, δ) be a subinterval of (α, β) in which $[IF] \sim (W_{k_i} - W_{k_{i+1}}, \theta)$ is either always positive or else always negative ($i = 1, \dots, u-1$). (Such a subinterval exists because of the unblockedness conditions.) Let $W_1, \dots, W_n$ be re-*

labeled, if necessary, so that $[IF] \sim (W_{k_i} - W_{k_{i+1}}, \theta)$ is always positive in (γ, δ) . (This can be achieved while retaining $W_1 \lesssim \dots \lesssim W_n$ and all other relevant structure by virtue of stability under isotone permutations of factorization sequences.) Then there exists a family of functions of one complex variable

$$\{Y^*(g_1, \dots, g_u): (g_1, \dots, g_u) \in (\text{complex numbers})^u\}$$

such that:

(14.1) The map $(g_1, \dots, g_u) \rightarrow Y^*(g_1, \dots, g_u)$ is bijective.

(14.2) For each $(i-1)$ -tuple $(g_1, \dots, g_{i-1})$ let $(g_1, \dots, g_{i-1}, *)$ denote the set of all u -tuples $(h_1, \dots, h_u)$ such that $h_j = g_j$ for $j = 1, \dots, i-1$ ($i = 1, \dots, u$). Then for every $(g_1, \dots, g_{i-1}, *)$ there exists $W((g_1, \dots, g_{i-1}, *)) \sim W_{k_i}$ and a complex number $x((g_1, \dots, g_{i-1}, *)) = x_0$ such that $Y^*(g_1, \dots, g_{i-1}, g_i, \dots, g_u) - Y^*(g_1, \dots, g_{i-1}, q_i, \dots, q_u) \sim (g_i - q_i)H(k_i) \exp \int_{x_0} W((g_1, \dots, g_{i-1}, *))$ in $F(\gamma, \delta)$, whenever $g_i \neq q_i$.

(14.3) For each $(g_1, \dots, g_u)$, $Y^*(g_1, \dots, g_u) < 1$ in $F(\alpha, \beta)$ and $P(Y + Y^*(g_1, \dots, g_u)) = 0$.

Proof. Let $P_1(y) = P(Y + y)$. Then, by Lemma α , P_1 is normal with respect to $(W_1, \dots, W_n, r)$, and it follows that there exist $W'_1, \dots, W'_n$ such that $W'_i \sim W_i$ ($i = 1, \dots, n$) and such that $P_1 \sim (r, c, w) \dot{W}'(n, \dots, 1)$ (cf. Lemma 7). For a fixed choice of such functions W'_i define $W((*)) = W'_{k_i}$; define $x((*))$ as some point belonging to an element of $F(\alpha, \beta)$ in which $W((*))$ is analytic. Then for each nonzero complex number g_1 define $Y(g_1)$ to be some particular solution of $P_1(y) = 0$ which is $\sim g_1 H(k_1) \exp \int_{x_0} W((*))$, where $x_0 = x((*))$, and define $Y(0) \equiv 0$. (The existence of $Y(g_1)$ is guaranteed by Lemma 13.) We now have $P_1(Y(g_1)) = P(Y + Y(g_1)) = 0$.

Let g_1 be fixed throughout this paragraph. Let $P_2(g_1; y) = P_1(Y(g_1) + y)$. Then $P_2(g_1; -)$ is normal with respect to $(W_1, \dots, W_n, r)$ by Lemma α , and it follows from Lemma 7 that there exist $W''_1, \dots, W''_n$ such that $W''_i \sim W_i$ ($i = 1, \dots, n$) and such that $P_2(g_1; -) \sim (r, c, w) \dot{W}''(n, \dots, 1)$. For a fixed choice of such W''_i define $W((g_1, *)) = W''_{k_2}$ and define $x((g_1, *))$ as some point belonging to an element of $F(\alpha, \beta)$ in which $W((g_1, *))$ is analytic. Then for each nonzero complex number g_2 define $Y(g_1, g_2)$ to be some particular solution of $P_2(g_1; y) = 0$ which is $\sim g_2 H(k_2) \exp \int_{x_0} W((g_1, *))$, where $x_0 = x((g_1, *))$, and let $Y(g_1, 0) \equiv 0$. (The existence of $Y(g_1, g_2)$ is guaranteed by Lemma 13.) We now have $P_2(g_1; Y(g_1, g_2)) = P(Y + Y(g_1) + Y(g_1, g_2)) \equiv 0$. And so forth: When $W((g_1, \dots, g_{s-1}, *))$, $x((g_1, \dots, g_{s-1}, *))$, $P_s(g_1, \dots, g_{s-1}; -)$, and $Y(g_1, \dots, g_s)$ have been defined for all $(g_1, \dots, g_s)$, we next define

$$P_{s+1}(g_1, \dots, g_s; y) = P_s(g_1, \dots, g_{s-1}; Y(g_1, \dots, g_s) + y).$$

Then $P_{s+1}(g_1, \dots, g_s; -)$ is normal with respect to $(W_1, \dots, W_n, r)$ by Lemma α , and it follows from Lemma 7 that there exist $W^*_1, \dots, W^*_n$ such that $W^*_i \sim W_i$ ($i = 1, \dots, n$) and such that $P_{s+1}(g_1, \dots, g_s; -) \sim (r, c, w) \dot{W}^*(n, \dots, 1)$.

For a fixed choice of such W_i^* , define $W((g_1, \dots, g_s, *)) = W_{k_{s+1}}^*$, and define $x((g_1, \dots, g_s, *))$ as some point belonging to an element of $F(\alpha, \beta)$ in which $W((g_1, \dots, g_s, *))$ is analytic. Then if $g_{s+1} \neq 0$ define $Y(g_1, \dots, g_{s+1})$ to be a solution of $P_{s+1}(g_1, \dots, g_s; y) = 0$ which is $\sim g_{s+1} H(k_{s+1}) \exp \int_{x_0} W((g_1, \dots, g_s, *))$, where $x_0 = x((g_1, \dots, g_s, *))$, and let $Y(g_1, \dots, g_s, 0) \equiv 0$.

For each u -tuple $(g_1, \dots, g_u)$ we get u functions in this way: $Y(g_1), Y(g_1, g_2), \dots, Y(g_1, \dots, g_u)$. Define

$$Y^*(g_1, \dots, g_u) = \sum_{v=1}^u Y(g_1, \dots, g_v).$$

Then (14.3) is obviously valid. To verify (14.2), we have $Y^*(g_1, \dots, g_u) - Y^*(g_1, \dots, g_{i-1}, q_i, \dots, q_u) = A + B - C$, where $A = Y(g_1, \dots, g_i) - Y(g_1, \dots, g_{i-1}, q_i)$, $B = \sum_{v=i+1}^u Y(g_1, \dots, g_v)$, and $C = \sum_{v=i+1}^u Y(g_1, \dots, g_{v-1}, q_i, \dots, q_v)$. Now we have

$$A \sim (g_i - q_i) H(k_i) \exp \int_{x_0} W((g_1, \dots, g_{i-1}, *)) \quad (x_0 = x((g_1, \dots, g_{i-1}, *))).$$

For the general term in B we have

$$Y(g_1, \dots, g_v) \sim g_v H(k_v) \exp \int_{x_0} W((g_1, \dots, g_{v-1}, *)) \quad (x_0 = x((g_1, \dots, g_{v-1}, *))),$$

and a similar relation for the analogous term in C ($v = 1 + i, \dots, u$). For $v > i$, $H(k_i)/H(k_v) \approx 1$ in $F(\alpha, \beta)$ and hence in $F(\gamma, \delta)$, while by virtue of the hypotheses we have

$$\exp \int_{x_0} W((g_1, \dots, g_{i-1}, *)) / \exp \int_{x'_0} W((g_1, \dots, g_{v-1}, *)) > 1$$

($x_0 = x((g_1, \dots, g_{i-1}, *)), x'_0 = x((g_1, \dots, g_{v-1}, *))$) in $F(\gamma, \delta)$. Hence $A > B$ in $F(\gamma, \delta)$, and similarly $A > C$ in $F(\gamma, \delta)$, so we have $A + B - C \sim A$ in $F(\gamma, \delta)$. This verifies (14.2). (14.1) follows easily from (14.2).

15. With the foregoing notion of a u -parameter family we arrive at

THEOREM IV. Let α, ϕ, β be real numbers with $-\pi \leq \alpha < \phi < \beta \leq \pi$. Let P be a differential polynomial with coefficients in an $\overline{LD}(F(\alpha, \beta))$. Let M be a simple principal monomial for P , and let $(W_1, \dots, W_n)$ be an asymptotically steady type for P at M which satisfies the separation condition (§2). Let $(W_{k_1}, \dots, W_{k_u})$ be the subsequence of $(W_1, \dots, W_n)$ consisting of the W_i for which $[IF]^\sim(W_i, \theta) < 0$ in (α, β) . Then $P = 0$ has a u -parameter family of solutions Y such that $Y \sim M$ in $F(\alpha, \beta)$. I.e., the reduced equation $P(M(1+z)) = 0$ has a family of solutions $Z < 1$ which possess asymptotic developments, in the sense of §14, in terms of u asymptotically distinct types of functions of the form $gH \exp \int_{x_0} W$.

Proof. This is a corollary of [2, Theorem V] and of Lemma 14 above.

PART V. APPENDIX

LEMMA α . Let P' be normal with respect to $(W_1, \dots, W_n, r)$. Let $Y < 1$. Let $P(y) = P'(Y + y)$. Then P is normal with respect to $(W_1, \dots, W_n, r)$.

Proof. [2, §91] shows that $P \sim (w)\dot{W}(n, \dots, 1)$. Now by the normality of P' , $P(y)$ may be expressed as a sum of functions of the form

$$cq_i Y^{i_0-j_0} (\theta_r Y)^{i_1-j_1} \dots (\theta_r^n Y)^{i_n-j_n} y^{j_0} (\theta_r y)^{j_1} \dots (\theta_r^n y)^{j_n}$$

and we must show that $j_t q_i Y^{i_0-j_0} (\theta_r Y)^{i_1-j_1} \dots (\theta_r^n Y)^{i_n-j_n} \gtrsim A_1 \dots A_t$ ($t = 0, \dots, n$), where $A_i = -(x(\log x) \dots (\log_{r-1} x) W_i)^{-1}$. But this is clear from the fact that $0 \leq j_t \leq i_t$, $i_t q_i \lesssim A_1 \dots A_t$, and $Y, \theta_r Y, \dots$, and $\theta_r^n Y$ are < 1 (cf. [1, §17]).

LEMMA β . Let P be normal with respect to $(W_1, \dots, W_n, r)$, and let $V_i \sim W_i$ ($i = 1, \dots, n$). Then P is normal with respect to $(V_1, \dots, V_n, r)$.

Proof. Since $W_i \sim V_i$, the relation $P \sim (w)\dot{W}(n, \dots, 1)$ implies $P \sim (w)\dot{V}(n, \dots, 1)$. The asymptotic inequalities of [2, §102] are readily verified.

LEMMA γ . Let $W \sim a$ logarithmic monomial in the divergence class in $F(\alpha, \beta)$, with $[IF] \sim (W, \theta) < 0$ in (α, β) . Then $\exp \int_{x_0} W < 1$ in $F(\alpha, \beta)$.

Proof. If $P(y) = \dot{W}y$, obviously $P \sim (s)\dot{W}(1)$ (with $W_1 = W$). If x_1 is sufficiently large and g is sufficiently small, the successive approximations of [2, §94] can be made to converge to a function Y such that $Y(x_1) = g$ and $Y < 1$ in $F(\alpha, \beta)$. From standard uniqueness theory, $Y = g \exp \int_{x_1} W$. Since $\exp \int_{x_0} W = (\exp \int_{x_0}^{x_1} W) \exp \int_{x_1} W$, $\exp \int_{x_0} W < 1$.

LEMMA δ . Let $W \sim a$ logarithmic monomial in the divergence class and let $G < 1$ in $F(\alpha, \beta)$. Let $[IF] \sim (W, \theta)$ have no zeros in (α, β) . Then

- (a) if $[IF] \sim (W, \theta) < 0$, every solution of $\dot{W}y = G$ is < 1 in $F(\alpha, \beta)$;
- (b) if $[IF] \sim (W, \theta) > 0$, just one solution of $\dot{W}y = G$ is < 1 in $F(\alpha, \beta)$, and every other solution is > 1 in $F(\alpha, \beta)$.

Proof. (a) Any solution of $\dot{W}y = G$ has the form

$$y(x) = g \exp \int_{x_0}^x W dt - \int_{x_0}^x W G \left(\exp \int_t^x W ds \right) dt$$

for some constant g . Clearly all such y are analytic in one and the same element of $F(\alpha, \beta)$, and $y(x_0) = g$. In view of [2, §§95-97], and standard uniqueness theory, such functions are < 1 if g is sufficiently small and x_0 sufficiently large. Applying Lemma γ one finds that $y < 1$ for any complex g and any relevant x_0 .

(b) [2, §§95-97] show there is one solution Y_0 which is < 1 . Every other solution is obtained by adding to Y_0 a multiple of $\exp \int_{x_0} W$. But $[IF] \sim (-W, \theta) < 0$ wherever $[IF] \sim (W, \theta) > 0$, and $\exp \int_{x_0} W = (\exp \int_{x_0} -W)^{-1}$; and Lemma γ shows

that $\exp \int_{x_0} -W < 1$. By [1, §13b], $\exp \int_{x_0} W > 1$, so every solution other than Y_0 is > 1 .

LEMMA ε . Let c , τ , and δ be numbers such that $c \neq 0$, τ is real and positive, and δ is real. Let $F = (\log_k x)^{-\delta}$, let

$$W \sim cx^{-1}(\log x)^{-1} \cdots (\log_{k-1} x)^{-1} (\log_k x)^{-1+\tau} (\log_{k+1} x)^{a_1} \cdots (\log_{k+s} x)^{a_s},$$

and let $[IF]^\sim(W, \theta) < 0$ in (α, β) . Then $F \exp \int_{x_0} W < 1$ in $F(\alpha, \beta)$.

Proof. $F \exp \int_{x_0} W = F(x_0) \exp \int_{x_0} (W + D(\log F))$, and the integrand is easily seen to be $\sim W$, so the assertion follows from Lemma γ .

LEMMA ζ . Let $(m_0, \dots, m_p)$ be a sequence of real numbers different from $(-1, \dots, -1)$; let i be the least integer such that $m_i \neq -1$. Let $N \sim cx^{m_0}(\log x)^{m_1} \cdots (\log_p x)^{m_p}$ in $F(\alpha, \beta)$, where c is a nonzero constant and $-\pi \leq \alpha < \beta \leq \pi$. Let N be analytic in $V \in F(\alpha, \beta)$ and let $x_0 \in V$. Let $M = (m_i + 1)^{-1} c (\log_i x)^{m_i+1} (\log_{i+1} x)^{m_{i+1}} \cdots (\log_p x)^{m_p}$. Then

- (a) $\int_{x_0}^x N \sim M(x)$ if $N \in$ the divergence class;
- (b) if $N \notin$ the divergence class, $\int_{x_0} N \lesssim 1$ and there exists a unique complex number A such that $A + \int_{x_0} N \sim M$.

Proof. $\int_{x_0} N$ is a solution of $P(y) = Dy - N = 0$, an equation to which the entire Strodt theory may be applied. It is easily seen that M is the (unique) principal monomial for P . Writing $\int_{x_0} N = M(1 + Z)$, we see that Z satisfies $Q(Z) = 0$, where $Q(z) = z - (-DM/M)^{-1} Dz - (N - DM)/DM$. Thus for some complex number g , the general solution

$$z(g, x) = g \exp \int_{x_0}^x (-DM/M) dt + \int_{x_0}^x (((N - DM)/M) \exp \int_t^x (-DM/M) ds) dt$$

becomes $Z(x)$. Now $Q(z) = 0$ may be written $\dot{W}z = G$, where $W = -DM/M \sim -(m_i + 1)x^{-1}(\log x)^{-1} \cdots (\log_i x)^{-1}$, which belongs to the divergence class; and $G = (N - DM)/DM < 1$. Furthermore, $[IF]^\sim(W, \theta) \equiv -1$ in case (a), and $\equiv +1$ in case (b). By [2, §§95-97], accordingly, every choice of g makes $z(g, -) < 1$ in $F(\alpha, \beta)$ in case (a), and just one choice of g makes $z(g, -) < 1$ in case (b). In case (b), if g^* is that choice, we have $z(g, -) = z(g^*, -) + (g - g^*)M(x_0)M^{-1}$. (a) and (b) follow immediately from this and from the relation $\int_{x_0} N = M(1 + Z)$.

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